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Dual Systems of algebraic iterated function systems, Rauzy fractals and β-tilings

Dedicated to the 70-th Anniversary of Professor Shunji Ito

By

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Abstract

Algebraic GIFS is a class of graph-directed iterated function systems (IFS) on $\mathbb{R}$ with algebraic parameters. A dual IFS of an algebraic GIFS can be constructed, and the duality between the two systems and has been investigated from various points of view. We review the results of dual IFS concerning the open set condition, purely periodic codings and the Rauzy-Thurston tilings, and their relation with previous studies.

Either a substitution or a numeration system can define an algebraic IFS in a natural way, and both Rauzy fractals and β-tilings can be obtained as dual IFS. The dual IFS provides a unified and simple framework for the theory of Rauzy fractals, β-tilings and related studies.

§ 1. Introduction

Let (V, Γ) be a directed graph with vertex set $V = \{1, \cdots, N\}$ and edge set Γ. Let

$$\mathcal{F} = (f_\gamma : \mathbb{R}^n \mapsto \mathbb{R}^n)_{\gamma \in \Gamma} \tag{1.1}$$

be a sequence of contraction maps. We call $(V, \Gamma, \mathcal{F})$ a *graph-directed iterated function system* ([35]), or an *GIFS* in short. Denote $\Gamma_{i,j}$ the set of edges from vertex i to j. There exists a unique family $(E_j)_{j\in V}$ of non-empty compact sets in $\mathbb{R}^n$ satisfying

$$E_i = \bigcup_{j\in V} \bigcup_{\gamma\in\Gamma_{i,j}} f_\gamma(E_j), \quad i \in V; \tag{1.2}$$

the family $(E_j)_{j\in V}$ is called the *invariant sets* of GIFS (1.1).

In this paper, we are interested in GIFS with algebraic parameters. Let β be an algebraic number with minimal polynomial $P(x) = a_d x^d + \cdots + a_1 x + a_0$. Then β is called an *algebraic integer* if $a_n = \pm 1$, is called an *algebraic unit* if $a_0 = \pm 1$ in addition. Let $\mathbb{Q}(\beta)$ denote the field generated by β and the rational field $\mathbb{Q}$.

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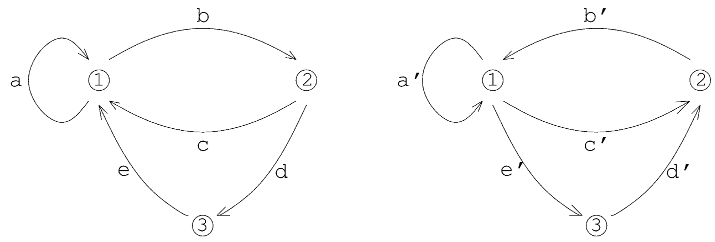

Figure 1. Left: Graph of IFS (2.4) in Example 2.3. Right: Reverse graph.

Definition 1.1. A GIFS $(f_\gamma : \mathbb{R} \mapsto \mathbb{R})_{\gamma\in\Gamma}$ is called an *algebraic GIFS* if there exists an algebraic number $\beta > 1$, such that for each $\gamma \in \Gamma$,

$$f_\gamma(x) = \frac{x + b_\gamma}{\beta} \tag{1.3}$$

with $b_\gamma \in \mathbb{Q}(\beta)$. We call β the expanding factor of the GIFS.

The construction of the dual IFS is motivated by the studies of Rauzy fractals and β-tilings. Tiling systems arising from Pisot substitutions and β-numeration systems are introduced by Rauzy [38] (1982) and Thurston [43] (1989), respectively. Since then, many people work on this field since it is related to different areas such as tiling theory([6, 1, 7, 25, 41]), substitution dynamical system([6, 7, 10, 13, 36, 39]), number theory([1, 3, 10, 26]), spectral theory *etc*([3, 8, 10, 25, 39]). The idea of constructing a dual IFS of an algebraic IFS has appeared implicitly or explicitly in many papers ([38, 43, 21, 17, 9]). We shall see that dual IFS provides a unified and simple framework for the theory of Rauzy fractals, β-tilings and related topics.

Let s be the number of algebraic conjugates of β with modulus less than 1; we assume that $s \geq 1$. Then the dual IFS of (1.3) is defined as $\Big(V, \Gamma', (F_{\gamma'} : \mathbb{R}^s \to \mathbb{R}^s)_{\gamma'\in\Gamma'}\Big)$ where:

(1) (V, Γ') is the reverse graph of (V, Γ), that is, there is an edge $\gamma' \in \Gamma'$ from j to i if and only if there is an edge $\gamma \in \Gamma$ from i to j. (See Figure 1.)

(2) For each edge $\gamma' \in \Gamma'$, $F_{\gamma'}$ is the dual map of $f_\gamma(x) = \frac{x+b_\gamma}{\beta}$, which is defined as

$$F_{\gamma'}(y) = By + (b_\gamma)^*, \tag{1.4}$$

where B is an $s \times s$ matrix which can be regarded as the dual of β, and $(b_\gamma)^*$ is a certain dual of b_γ. For precise definition, see Section 2.

The first part of the paper (Sec. 2-4) is devoted to the definition and examples of dual systems. The dual system is defined in Section 2. In Section 3 and 4, we study algebraic GIFS and dual systems produced by substitutions and by numeration systems, respectively. Either a substitution or a numeration system define a one-dimensional algebraic IFS in a natural way, which we call the *induced IFS*.

In Section 3, we define *Rauzy fractals* to be the invariant sets of the dual IFS of the induced IFS of a substitution σ, no matter the substitution is irreducible Pisot [6, 10, 25, 39], reducible Pisot [7, 14], non-Pisot [20] or non-unimodular [11]. Indeed, using dual IFS, Rauzy fractals can be defined for substitutions which are primitive and that the expanding factor β has at least one conjugate less than 1 in modulus. (See Figure 2.)

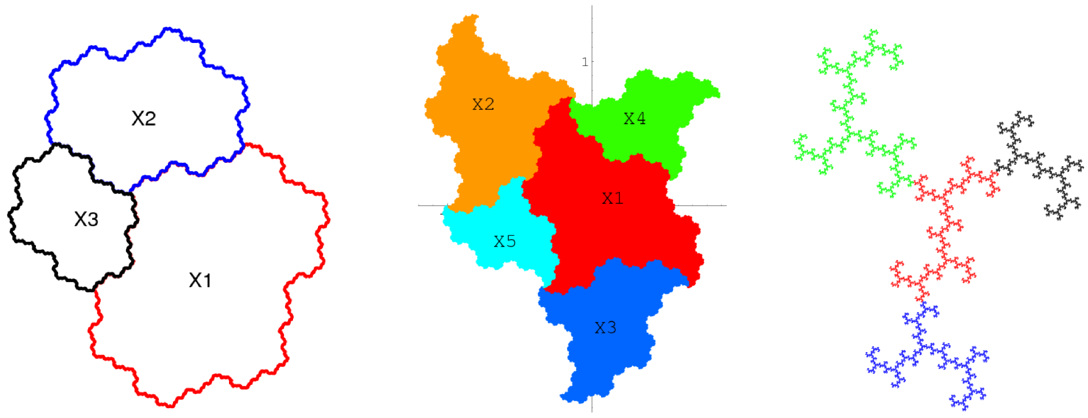

Figure 2. Rauzy fractals of irreducible type, reducible type and non-Pisot type. Left: Rauzy substitution (Example 3.1). Middle: Hokkaido substitution (Example 3.2). Right: A non-Pisot substitution (Example 3.3).

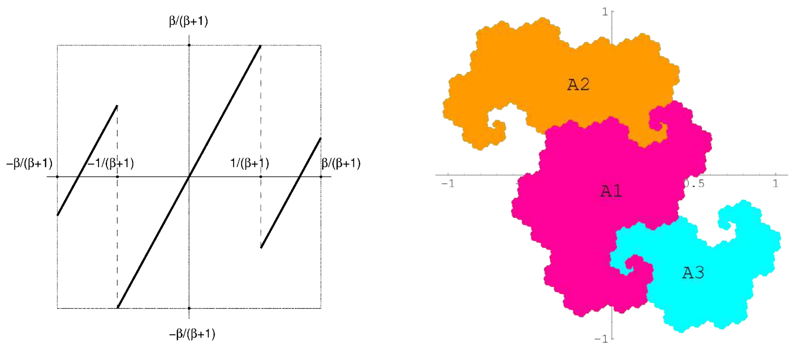

Figure 3. Left: Minimal weight transformation M_β (Example 4.6). Right: The invariant sets of the dual system of M_β.

Recently, Kalle and Steiner [28] studied tiling systems arising from numeration systems. In Section 4, we reformulate the discussion of Kalle and Steiner [28] in terms of dual system. This new formulation avoids the discussion on admissibility of various expansions. In particular, when β is a Pisot unit, we have the following interesting link (see Figure 3):

Piecewise-linear transformation $\rightarrow$ *Induced algebraic IFS* $\rightarrow$ *Dual tiling system.*

The second part of the paper (Sec. 5-6) deals with the open set condition of the dual systems. In section 5, we introduce the notions of *containing ideal* K and *containing lattice* Σ which are important tools.

The *open set condition* (OSC) is the most important separation condition in the theory of GIFS. Rao, Wen and Yang [37] characterizes when a dual system satisfies the open set condition. In particular, it is shown that the OSC of the original system implies the OSC of the dual system, and the converse is also true if β is a Pisot unit. (See Section 6.)

The dual IFS is a useful way to construct higher dimensional self-similar tiling system. A GIFS is called a *self-similar tiling system* (SST system) if the IFS satisfies the open set condition and its invariant sets have non-empty interiors.

Theorem 1.2. *([37]) If an algebraic GIFS $(f_\gamma)_{\gamma\in\Gamma}$ is an SST system, and its expanding factor β is a Pisot unit, then its dual IFS is also an SST system.*

The third part of the paper (Sec. 7-8) concentrates on a special class of algebraic IFS, called feasible Pisot IFS.

Definition 1.3. An algebraic GIFS $\mathcal{F}$ is called a *feasible Pisot system* if $\mathcal{F}$ is a SST, the expanding factor β is a Pisot unit, and the invariant sets of $\mathcal{F}$ are intervals.

For a feasible Pisot system, we shall review results concerning periodic coding and the Rauzy-Thurston tilings, and their relation with previous studies.

The periodic β-expansion has been studied by K. Schmidt [40], Ito and Sano [27], S. Akiyama [3], Ito and Rao [26]. The studies showed that if β is a Pisot unit, then the periodic β-expanions are characterized by the Rauzy box of the associated algebraic GIFS. Kalle and Steiner generalized this result to so-called 'generalized β-transformations'. Then [42] generalizes this result to algebraic GIFS whose expanding factor is a Pisot unit. The Pisot non-unit case has been studied in Akiyama, Berthé and Siegel [11, 4].

For an algebraic IFS, we call $\mathcal{R}=\bigcup_{j\in V}(I_j\times X_j)$ the *Rauzy box*, where $(I_j)_{j\in V}$ and $(X_j)_{j\in V}$ are invariant sets of the original IFS and the dual IFS, respectively.

Theorem 1.4. *([42]) Let $(f_\gamma)_{\gamma\in\Gamma}$ be a feasible Pisot system. Then $x\in\bigcup_{j\in V} I_j$ possesses a periodic coding if and only if $x\in\mathbb{Q}(\beta)$ and $(x,-x^*)\in\mathcal{R}$.*

The Rauzy-Thurston tilings of general algebraic GIFS were constructed in [37]. It is shown that actually there are two tilings of this type, which we denote by $\mathcal{J}^l$ and $\mathcal{J}^r$, and call them the left and right *Rauzy-Thurston (multiple) tiling*, respectively. The collection $\mathcal{J}^l$ is related to the lower codings and $\mathcal{J}^r$ is related to upper codings of $\mathcal{F}$.

Following the arguments in [25, 12, 28], it is not hard to show that Rauzy-Thurston tilings are self-replicating and quasi-periodic, and there is an integer $m\geq 1$ such that almost every point of $\mathbb{R}^{d-1}$ is covered by exactly m tiles. We call m the *multiplicity* of the multiple tiling, or of $\mathcal{F}$. The following theorem illustrates some new properties of the Rauzy-Thurston tiling.

Theorem 1.5. *([37]) Let $\mathcal{J}$ be the left or the right Rauzy-Thurston tiling of a feasible Pisot system. Then*

(i) $\mathcal{J}$ has a decomposition $\mathcal{J}=\bigcup_{j=1}^m \mathcal{J}_j$, where $\mathcal{J}_j$ are tilings of $\mathbb{R}^{d-1}$.

(ii) $\mathcal{L}^d(\mathcal{R})=m\mathcal{N}(K)$, where $\mathcal{L}^d$ denote the d-dimensional Lebesgue measure, $\mathcal{R}$ is the Rauzy box, and $\mathcal{N}(K)$ is the norm of the containing ideal K (see Section 5).

We say a feasible Pisot system is *tight* if the multiplicity m equals 1, or equivalently, $\mathcal{L}^d(\mathcal{R})=\mathcal{N}(K)$. It is natural to ask that: *under what condition, an algebraic GIFS is tight?* (A weaker question is: when $\mathcal{R}=\bigcup_{j\in V}(I_j\times X_j)$ is a disjoint union in measure?)

In the theory of substitution dynamical systems, one major problem is the *Pisot Spectrum Conjecture: Let σ be a unimodular Pisot substitution of irreducible type, then the corresponding substitution dynamical system has purely discrete spectrum* . (See for instance [16].) This

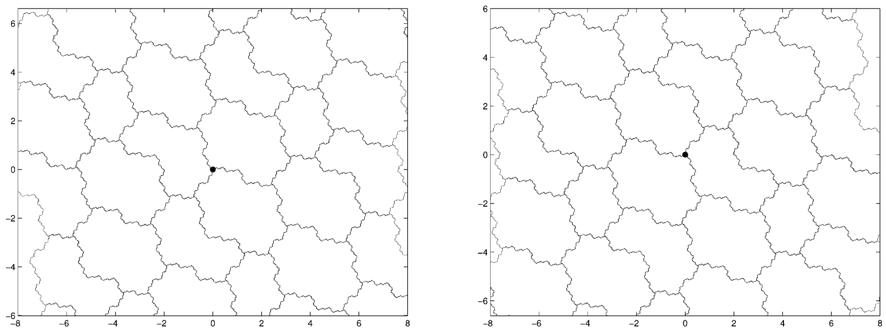

Figure 4. The Rauzy-Thurston tiling of symmetric β-transform has two pages for $\beta : \beta^3 = \beta^2+\beta+1$. ([28]). See also Example 4.5 and Section 8.2.

conjecture is confirmed in two letters case ([8, 23]), but it is widely open in other cases. It is well-known that the Pisot specturum conjecture holds if and only if the induced GIFS of σ is tight ([10, 25]).

Example 1.6. Consider the *Fibonacci substitution*: $1 \mapsto 12,\ 2 \mapsto 1$. Let β be the golden number $(\sqrt{5}+1)/2$ and β' its conjugate. The induced GIFS and its dual IFS are

$$\begin{cases} \beta I_1 = I_1 \cup (I_2+1) \\ \beta I_2 = I_1 \end{cases} \quad \text{and} \quad \begin{cases} X_1 = \beta' X_1 \cup \beta' X_2 \\ X_2 = \beta' X_1 + 1. \end{cases}$$

It is easy to verify that $I_1 = [0,1]$, $I_2 = [0, 1/\beta]$, $X_1 = [-1, 1/\beta]$, $X_2 = [1/\beta, \beta]$. Hence

$$\mathcal{L}^2(\mathcal{R}) = 1 \cdot \beta + 1/\beta \cdot 1 = \beta - \beta',$$

which equals the norm of $K = \mathbb{Z}[\beta]$. Hence $m = 1$ and the spectral type is purely discrete.

In the previous studies, the main method to show the tightness is to confirm a super-coincidence condition ([25, 10]). (It is also called geometrical coincidence, and is called weakly (F)-property for β-tiling [3]). [37] gave a new criterion by using a notion of minimal regular algebraic solution (Theorem 8.5).

Finally, we observe that if $\mathcal{J}^l \neq \mathcal{J}^r$, then a *domain-exchange transformation* can be defined (Theorem 8.8). This generalizes the construction of Arnoux and Ito [6] for unimodular Pisot substitutions.

The paper is organized as follows. In Section 2, we construct the dual system of an algebraic IFS. Algebraic IFS induced by substitutions and numeration systems are discussed in Section 3 and Section 4, respectively. In Section 5, we introduce the notions containing ideal and containing lattice. Section 6 is devoted to the open set condition of dual systems, and Theorem 6.2 is proved there. Section 7 studies the periodic codings and Theorem 7.2 is proved there. Section 8, the last section, is devoted to the Rauzy-Thurston tilings.

§ 2. Construction of dual IFS

Recall that an algebraic IFS $(f_\gamma)_{\gamma\in\Gamma}$ with expanding factor β has the form $f_\gamma(x) = \frac{x+b_\gamma}{\beta}$ with $b_\gamma \in \mathbb{Q}(\beta)$. Assume that at least one conjugate of β has modulus < 1. To define the dual IFS, the crucial step is to define the dual map of f_γ.

Denote the conjugates of β by $\beta_2, \dots, \beta_d$, which are ordered in the manner that

$$|\beta_2| \geq \cdots \geq |\beta_{d-s}| \geq 1 > |\beta_{d-s+1}| \geq \cdots \geq |\beta_d|, \tag{2.1}$$

and each pair of complex conjugates are put next to each other.

Since $1, \beta, \dots, \beta^{d-1}$ is a basis of $\mathbb{Q}(\beta)$, any $x \in \mathbb{Q}(\beta)$ possesses a unique representation $x = \sum_{j=0}^{d-1} x_j \beta^j$ with $x_j \in \mathbb{Q}$. The Galois dual of x in the field $\mathbb{Q}(\beta)$ is

$$x' = \sum_{j=0}^{d-1} x_j (\beta_2^j, \cdots, \beta_d^j)^t \in \mathbb{C}^{d-1},$$

where A^t denotes the transpose of A. To confine our discussion in real space, we replace each pair of complex conjugate components $(w, \bar{w})^t$ in x' by $(\mathrm{Re}\, w, \mathrm{Im}\, w)^t$; we denote the resulted vector in $\mathbb{R}^{d-1}$ by $\hat{x}$, and call it the *real version* of x'. We define the *contractive Galois dual* of x as

$$\sum_{j=0}^{d-1} x_j (\beta_{d-s+1}^j, \cdots, \beta_d^j)^t \in \mathbb{C}^s,$$

and denote its real version by x^*. Next, define a contractive matrix

$$\mathbf{B}' = \mathrm{diag}\ (\beta_{d-s+1}, \dots, \beta_d),$$

then by replacing each block $\begin{pmatrix} w & \\ & \bar{w} \end{pmatrix}$ by $C(w) = \begin{pmatrix} \mathrm{Re}\, w & -\mathrm{Im}\, w \\ \mathrm{Im}\, w & \mathrm{Re}\, w \end{pmatrix}$, we obtain a real matrix and denote it by $\mathbf{B}$. (We call $\mathbf{B}$ the *real version* of B'.)

The *dual map* of $f(x) = \beta^{-1}(x + b)$ with $b \in \mathbb{Q}(\beta)$ is defined to be $F : \mathbb{R}^s \to \mathbb{R}^s$,

$$F(y) = \mathbf{B}y + b^*.$$

Definition 2.1. Let $(V, \Gamma, (f_\gamma)_{\gamma\in\Gamma})$ be an algebraic GIFS, we call $(V, \Gamma', (F_{\gamma'})_{\gamma'\in\Gamma'})$ its *dual GIFS*, where Γ' is the reverse graph of Γ, and $F_{\gamma'}$ is the dual map of f_γ.

The following simple properties of the star operator $* : \mathbb{Q}(\beta) \to \mathbb{R}^s$ will be used frequently.

Lemma 2.2. *([37])(i) Linearity:* $(a + b)^* = a^* + b^*$.
(ii) Scaling property: $(\beta a)^* = \mathbf{B}a^*$. (This property determines the choice of $C(w)$.)

§ 2.1. Associated set equations of dual GIFS

Let $\mathcal{F} = \left(f_\gamma(x) = \frac{x+b_\gamma}{\beta}\right)_{\gamma\in\Gamma}$ be an algebraic IFS. Recall that $\Gamma_{i,j}$ is the set of edges from vertex i to j; set

$$\mathcal{D}_{i,j} := \{b_\gamma; \gamma \in \Gamma_{i,j}\}, \quad i, j \in V.$$

Let $(I_j)_{j\in V}$ be the invariant sets of $\mathcal{F}$, then according to (1.2), we have

$$\beta I_i = \bigcup_{j\in V} (I_j + \mathcal{D}_{i,j}), \quad i \in V. \tag{2.2}$$

We call (2.2) the *set equation form* of $\mathcal{F}$, and call $\mathcal{D} = \bigcup_{i,j} \mathcal{D}_{i,j} = \{b_\gamma;\ \gamma \in \Gamma\}$ the *digit set* of $\mathcal{F}$. Denote $\mathcal{D}^*_{i,j} = \{b^*;\ b \in \mathcal{D}_{i,j}\}$, then the dual system of $\mathcal{F}$ is given by ([17, 37])

$$X_i = \bigcup_{j\in V} \left(\mathbf{B}X_j + \mathcal{D}^*_{j,i}\right). \tag{2.3}$$

Example 2.3. **Rauzy fractals of Rauzy substitution** ([38]). Let β be the Pisot number satisfying $\beta^3 = \beta^2 + \beta + 1$. Consider the GIFS

$$\begin{cases} \beta I_1 = I_1 \cup (I_2 + 1) \\ \beta I_2 = I_1 \cup (I_3 + 1) \\ \beta I_3 = I_1. \end{cases} \tag{2.4}$$

The invariant sets are $I_1 = [0,1], I_2 = [0, 1/\beta + 1/\beta^2], I_3 = [0, 1/\beta]$. The graph Γ of (2.4) and its reverse graph are depicted in Figure 1, where the edges of Γ are labeled by letters a, b, c, d, e, and the corresponding edges in the reverse graph by a', b', c', d', e'. Then

$$f_a(x) = \frac{x}{\beta},\ f_b(x) = \frac{x+1}{\beta},\ f_c(x) = \frac{x}{\beta},\ f_d(x) = \frac{x+1}{\beta},\ f_e(x) = \frac{x}{\beta}.$$

The algebraic conjugates of β are $\beta_2, \beta_3 \approx 0.419 \pm 0.606i$. Hence

$$B \approx \begin{pmatrix} 0.419 & -0.606 \\ 0.606 & 0.419 \end{pmatrix}.$$

The dual of 1 is $1^* = \mathbf{e}_1 = (1,0)^t$. Therefore, the dual IFS is:

$$F_{a'}(y) = By,\ F_{b'}(y) = By + \mathbf{e}_1,\ F_{c'}(y) = By,\ F_{d'}(y) = By + \mathbf{e}_1,\ F_{e'}(y) = By.$$

The set equation form of the dual IFS is:

$$\begin{cases} X_1 = BX_1 \cup BX_2 \cup BX_3 \\ X_2 = BX_1 + \mathbf{e}_1 \\ X_3 = BX_2 + \mathbf{e}_1. \end{cases} \tag{2.5}$$

The invariant sets of (2.5) are the famous Rauzy fractals of the Rauzy Rauzy substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$. (See Figure 2 left.)

§ 3. Algebraic IFS arising from substitutions

Let σ be a primitive substitution over the alphabet $V = \{1, 2, \ldots, N\}$. Let $M_\sigma = (m_{i,j})$ be the incidence matrix of σ, *i.e.*, $m_{i,j}$ counts the number of letter i contained in the word $\sigma(j)$. (σ is said to be *primitive* if M_σ is primitive.) Let β be the maximal eigenvalue of M_σ. σ is a *Pisot* *substitution* if β is a Pisot number; is *unimodular* if $\det M_\sigma = \pm 1$. Let $(v_1, \dots, v_N)$ be the left eigenvector of M_σ w.r.t. β with $v_1 = 1$, then it is a positive vector by Perron-Frobenius Theorem. Let Φ_σ be the characteristic polynomial of M_σ; σ is of *irreducible type* if Φ_σ is irreducible over $\mathbb{Q}$.

Write $\sigma(j) = w_{j,1} \dots w_{j,\ell_j}$, then $(I_j = [0, v_j])_{j \in V}$ are invariant sets of the IFS

$$\beta I_j = I_{w_{j,1}} \cup (v_{w_{j,1}} + I_{w_{j,2}}) \cup \cdots \cup (v_{w_{j,1}} + \cdots + v_{w_{j,\ell_j - 1}} + I_{w_{j,\ell_j}}), \quad j \in V. \tag{3.1}$$

We call GIFS (3.1) the *induced GIFS of* σ. Clearly $v_j \in \mathbb{Q}(\beta)$ for $j \in V$, and the induced GIFS is an algebraic IFS. Clearly the invariant sets of the dual system of (3.1) are Rauzy fractals of σ.

Example 3.1. Rauzy substitution [38].

$$\begin{aligned} \sigma : 1 &\mapsto 12 \\ 2 &\mapsto 13 \\ 3 &\mapsto 1, \end{aligned} \quad M_\sigma = \begin{pmatrix} 1\,1\,1 \\ 1\,0\,0 \\ 0\,1\,0 \end{pmatrix}, \quad \begin{aligned} &\Phi_\sigma = x^3 - x^2 - x - 1, \\ &(v_1, v_2, v_3) = \left(1, \tfrac{1}{\beta} + \tfrac{1}{\beta^2}, \tfrac{1}{\beta}\right). \end{aligned}$$

This substitution is a unimodular Pisot substitution of irreducible type. The induced IFS and its dual IFS are given by (2.4) and (2.5) in Example 2.3, respectively.

Example 3.2. Hokkaido substitution [3].

$$\begin{aligned} \sigma : 1 &\mapsto 12 \\ 2 &\mapsto 3 \\ 3 &\mapsto 4 \\ 4 &\mapsto 5 \\ 5 &\mapsto 1, \end{aligned} \quad M_\sigma = \begin{pmatrix} 1\,0\,0\,0\,1 \\ 1\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0 \\ 0\,0\,1\,0\,0 \\ 0\,0\,0\,1\,0 \end{pmatrix}, \quad \begin{aligned} &\Phi_\sigma = (x^3 - x - 1)(x^2 - x + 1), \\ &\beta^3 - \beta - 1 = 0, \\ &(v_1, v_2, v_3, v_4, v_5) = \left(1, \tfrac{1}{\beta^4}, \tfrac{1}{\beta^3}, \tfrac{1}{\beta^2}, \tfrac{1}{\beta}\right). \end{aligned}$$

This substitution is a unimodular substitution of reducible type. The conjugates of β are approximately $-0.6624 \pm 0.5623\, i$ and

$$B \approx \begin{pmatrix} -0.6624 & 0.5623 \\ -0.5623 & -0.6624 \end{pmatrix}.$$

The induced IFS of σ and its dual IFS are given by

$$\begin{cases} \beta I_1 = I_1 \cup (I_2 + 1) \\ \beta I_2 = I_3 \\ \beta I_3 = I_4 \\ \beta I_4 = I_5 \\ \beta I_5 = I_1. \end{cases} \quad \text{and} \quad \begin{cases} X_1 = BX_1 \cup BX_5 \\ X_2 = BX_1 + \mathbf{e}_1 \\ X_3 = BX_2 \\ X_4 = BX_3 \\ X_5 = BX_4. \end{cases}$$

The Rauzy fractals are depicted by Figure 2 (middle).

Example 3.3. A non-Pisot substitution [20].

$$\begin{aligned} \sigma : 1 &\mapsto 14 \\ 2 &\mapsto 3 \\ 3 &\mapsto 423 \\ 4 &\mapsto 142, \end{aligned} \quad M_\sigma = \begin{pmatrix} 1\,0\,0\,1 \\ 0\,0\,1\,1 \\ 0\,1\,1\,0 \\ 1\,0\,1\,1 \end{pmatrix}, \quad \begin{aligned} &\Phi_\sigma = x^4 - 3x^3 + x^2 + x + 1 = 0, \\ &(v_1, v_2, v_3, v_4) = \left(1, \beta^2 - 2\beta, \beta^3 - 2\beta^2, \beta - 1\right). \end{aligned}$$

This substitution is non-Pisot and unimodular. The conjugates of β are approximately 1.3894, $-0.3391\pm 0.4466i$. Hence

$$B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \approx \begin{pmatrix} -0.3391 & 0.4466 \\ -0.4466 & -0.3391 \end{pmatrix}. \tag{3.2}$$

The induced system is

$$\begin{cases} \beta I_1 = I_1 \cup (I_4 + v_1) \\ \beta I_2 = I_3 \\ \beta I_3 = I_4 \cup (I_2 + v_4) \cup (I_3 + v_4 + v_2) \\ \beta I_4 = I_1 \cup (I_4 + v_1) \cup (I_2 + v_1 + v_4). \end{cases}$$

The dual system is (a, b are given by (3.2))

$$\begin{cases} X_1 = BX_1 \cup BX_4 \\ X_2 = (BX_3 + u_1) \cup (BX_4 + u_2) \\ X_3 = BX_2 \cup (BX_3 + u_3) \\ X_4 = (BX_1 + \mathbf{e}_1) \cup BX_3 \cup (BX_4 + \mathbf{e}_1). \end{cases}$$

where $u_1 = (a-1, b)^t, u_2 = (a, b)^t, u_3 = (a^2 - b^2 - a - 1, 2ab - b)^t$. The Rauzy fractals, which are 'true' fractals, are depicted in Figure 2 (right).

§ 4. Algebraic IFS arising from numeration systems

A numeration system is typically determined by a piecewise linear map T on an interval I with constant slope. The union of orbits of discontinuous points of T determines a finite partition $I = I_1 \cup I_2 \cup \cdots \cup I_N$,. Each $T(I_j)$ is a union of elements in the partition, hence an algebraic IFS is obtained and its dual system can be defined. This section can be regarded as a reformulation of the study of Kalle and Steiner [28], but our approach avoids the discussion on admissibility of various expansions.

Let $T : I = [a, b] \to I$ be a piecewise linear map with constant slope $\beta > 1$. Denote by Ω_0 the set of discontinuous points of T together with the end points a and b. Denote

$$T^+(x) = \lim_{y \to x+} T(y) \quad \text{and} \quad T^-(x) = \lim_{y \to x-} T(y)$$

where we set $T^+(b) = T^-(b)$ and $T^-(a) = T^+(a)$. Let Ω be the smallest set such that $\Omega_0 \subset \Omega$ and

$$T^+(\Omega) \subset \Omega, \quad T^-(\Omega) \subset \Omega.$$

In some sense, Ω is the union of $T^\pm$-orbits of discontinuous points of T. We assume that: *β is an algebraic number, $\Omega \subset \mathbb{Q}(\beta)$ and Ω is a finite set.* Then Ω determines a partition $I = I_1 \cup I_2 \cup \cdots \cup I_N$ (the intervals are from left to right). We call I_j *basic intervals.* Let $V = \{1, \ldots, N\}$. For $j \in V$, since T is continuous on I_j°, the interior of I_j, there exist d_j such that

$$T(x) = \beta x - d_j, \quad x \in I_j^\circ.$$

Since $\beta I_j - d_j$ is an interval with end points in Ω, we have

Theorem 4.1. *$\beta(I_j) - d_j$ is a union of consecutive basic intervals for each $j \in V$. That is, there exists $1 \le l_j \le r_j \le N$ such that*

$$\beta I_j = (I_{l_j} \cup \cdots \cup I_{r_j}) + d_j, \quad j \in V. \tag{4.1}$$

We call GIFS (4.1) the induced GIFS *of T.*

In the following, as examples, we discuss three numeration systems: the classical β-expansion, the symmetric β-expansion and the minimal weight expansion.

§ 4.1. β-numeration systems

For $\beta > 1$, the β-transformation is $T_\beta : [0,1] \to [0,1]$,

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor.$$

Let $T_\beta^+ = T_\beta(x+)$, set $x_n = (T_\beta^+)^{n-1}(1)$, and $c_n = \beta x_n - x_{n+1}$ for $n \ge 1$. Then $1 = 0.c_1c_2\dots$ is called the *characteristic expansion* of β (it is called *carry sequence* in [43]), where $x_{-n}\dots x_{-1}x_0.x_1x_2\dots$ means $\sum_{k=-n}^{\infty} x_j\beta^{-k}$. Denote $\mathbf{c} = c_1c_2\dots$ and $\mathbf{c}_k = c_kc_{k+1}\dots$ for $k \ge 1$. A sequence $(x_k)_{k\ge 1} = x_1x_2\dots$ over $\{0,1,\dots,\lfloor\beta\rfloor\}$ is *weakly admissible* (w.adm.) if $x_nx_{n+1}\cdots \preceq \mathbf{c}$ for all $n \ge 1$; is called *admissible* if $x_nx_{n+1}\cdots \prec \mathbf{c}$ for all $n \ge 1$, where $\prec$ is the lexicographic order.

Let β be a *Parry number*, that is, the characteristic expansion of β

$$1 = 0.c_1c_2\cdots = 0.c_1\dots c_m(c_{m+1}\dots c_{m+p})^\infty \tag{4.2}$$

is eventually periodic [43].

The first IFS: The discontinuous points of T_β are $\Omega_0 = \{0,1,\dots,\lfloor\beta\rfloor\}/\beta \cup \{1\}$, and

$$\Omega = \Omega_0 \cup \{0.\mathbf{c}_n;\ n \ge 1\}.$$

Let $I = I_1 \cup \cdots \cup I_{N'}$ be the partition determined by Ω. Then an algebraic IFS can be obtained by (4.1). However, this is not the β-tiling system introduced by Thurston [43].

The second IFS: To obtain the β-tiling system, we make a slightly modification by setting $\Omega' = \{0\} \cup \{0.\mathbf{c}_n;\ n \ge 1\}$. ($\Omega'$ is the T_β-orbit of 1 together with 0.) Let

$$[0,1] = J_1 \cup \cdots \cup J_N \tag{4.3}$$

be the partition determined by Ω', where $N = m + p$. Extending (4.3) periodically, we obtain a tiling $\mathcal{J}$ of $\mathbb{R}$ with periodic 1. Clearly βJ_k is a union of consecutive tiles in the tiling, and hence a union of translations of the prototiles $J_1,\dots,J_N$. Therefore

$$\beta J_k = \cup\{T \in \mathcal{J};\ T \subset \beta J_k\}, \quad k \in V \tag{4.4}$$

give us an algebraic IFS, where $V = \{1,\dots,N\}$.

We recall the definition of the β-tiling system. Set $\mathbf{s}_0 = 0^\infty$ and let $\mathbf{s}_1 \prec \mathbf{s}_2 \prec \cdots \prec \mathbf{s}_N$ be the ascendant list of the set $\{\mathbf{c}_n;\ n \ge 1\}$. Denote

$$\mathbb{F}[\beta] = \{x_n\dots x_1x_0.0;\ x_n\dots x_1x_00^\infty \text{ is admissible}\}$$

to be the set of admissible β-expansions with 0 fractional part. For $x = x_n \dots x_1 x_0.0 \in \mathbb{F}[\beta]$, we shall use $\bar{x}$ to denote the word $x_n \dots x_0$. Define

$$Y_k = \text{closure}\ \{x^*;\ x \in \mathbb{F}[\beta] \text{ and } \bar{x}.\mathbf{s}_k \text{ is w.adm.}\}, \quad k \in V. \tag{4.5}$$

If β is a Pisot unit, then $(Y_k)_{k\in V}$ are the tiles in the β-tiling system (Thurston [43]).

Theorem 4.2. *If β is a Parry number, then $(Y_k)_{k\in V}$ in (4.5) are invariant sets of the dual IFS of (4.4).*

Proof. We deduce the graph-directed structure of $(Y_k)_{k\in V}$ first.

$$\begin{aligned} Y_k &= \text{closure}\ \textstyle\bigcup_{0\le h\le \lfloor\beta\rfloor}\{(yh)^*;\ y \in \mathbb{F}[\beta] \text{ and } \bar{y}\bar{h}.\mathbf{s}_k \text{ is w.adm.}\} \\ &= \text{closure}\ \textstyle\bigcup_{0\le h\le \lfloor\beta\rfloor}\{By^* + h^*;\ y \in \mathbb{F}[\beta] \text{ and } \bar{y}.\bar{h}\mathbf{s}_k \text{ is w.adm.}\} \end{aligned}$$

If $\bar{y}.\bar{h}\mathbf{s}_k$ is weakly admissible, then $\mathbf{s}_{\ell-1} \prec \bar{h}\mathbf{s}_k \preceq \mathbf{s}_\ell$ for some $1 \le \ell \le N$; in this case, $\bar{y}.\bar{h}\mathbf{s}_k$ is weakly admissible if and only if $\bar{y}.\mathbf{s}_\ell$ is. Therefore $(Y_k)_{1\le k\le N}$ are the invariant sets of the following IFS: $V = \{1,\dots,N\}$,

$$\Gamma' = \{(k,\ell,h) \in V\times V\times\{0,1,\dots,\lfloor\beta\rfloor\};\ \mathbf{s}_{\ell-1} \prec \bar{h}\mathbf{s}_k \preceq \mathbf{s}_\ell\},$$

and $F_{\gamma'}(z) = Bz + h^*$ for $\gamma' = (k,\ell,h)$. (Here (k,ℓ,h) means the h-th edge from k to ℓ.)

Pick $0 \le h \le \lfloor\beta\rfloor$, since $J_k = [0.\mathbf{s}_{k-1}, 0.\mathbf{s}_k]$, $J_k + h \subset \beta J_\ell$ if and only if $\mathbf{s}_{\ell-1} \prec \bar{h}\mathbf{s}_k \preceq \mathbf{s}_\ell$; on the other hand, $J_k + h$ is apparently a tile in $\mathcal{J}$. Hence $(J_k)_{k\in V}$ are the invariant sets of the IFS $(f_\gamma)_{\gamma\in\Gamma}$ where Γ is the reverse graph of Γ', and $f_\gamma(x) = (x+h)/\beta$ for $\gamma' = (k,\ell,h)$. Therefore, $(F_{\gamma'})_{\gamma'\in\Gamma'}$ is the dual IFS of $(f_\gamma)_{\gamma\in\Gamma}$. □

The third IFS: According to the characteristic expansion (4.2), we define a substitution $\sigma = \sigma_\beta$ over $V = \{1,2,\dots,N\}$ (where $N = m+p$) by

$$k \mapsto 1^{c_k}(k+1), \quad j \in V, \tag{4.6}$$

where we identify the symbols $m+p+1$ to the symbol $m+1$. The IFS induced by σ is

$$\beta I_j = \bigcup_{h=0}^{c_j-1}(I_1+h)\bigcup(I_{j+1}+c_j), \quad j \in V, \tag{4.7}$$

where we identify I_{m+p+1} to I_{m+1}. Clearly $I_j = [0, 0.\mathbf{c}_j]$, $\quad j \in V$.

Using a result (Theorem 7.1) on periodic codings which we list in Section 7, we provide a very simple proof of Theorem 4.3. If β be a Pisot unit, this gives the relation between β-tiling and Rauzy fractal tiling.

Theorem 4.3. *Let $(X_j)_{j\in V}$ be the Rauzy fractals of σ_β , and $(Y_j)_{j\in V}$ be the invariant sets of the dual IFS of (4.4). Then*

$$Y_j = \bigcup_{J_j\subset I_k} X_k, \quad j \in V,$$

where $(I_j)_{j\in V}$ and $(J_j)_{j\in V}$ are invariants of the original systems, respectively.

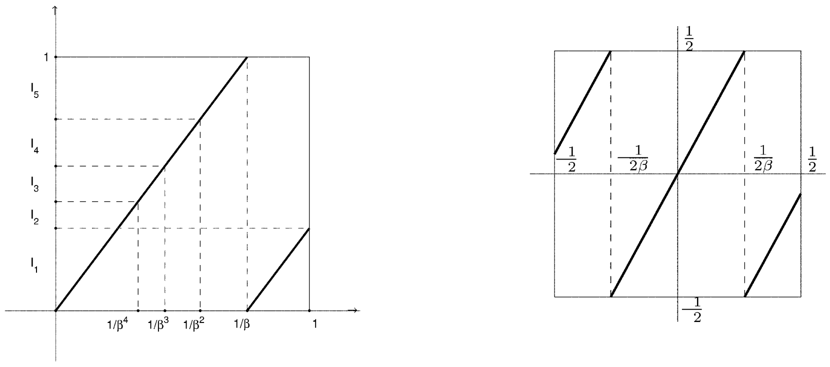

Figure 5. Left: β-transformation system with $\beta^3 = \beta + 1$. Right: Symmetric β-transformation with $\beta^3 = \beta^2 + \beta + 1$.

Proof. First, using a result we shall prove in Section 8, we show the two systems have the same Rauzy box, i.e., $\bigcup_{j\in V} J_j \times Y_j = \bigcup_{k\in V} I_k \times X_k$.

Let $\mathcal{P}_1$ be the set of $x \in \bigcup_{j\in V} I_j$ having a periodic coding w.r.t. the induced IFS of σ_β. It is easy to show that $\mathrm{Per}(T_\beta) \subset \mathcal{P}_1 \subset \mathrm{Per}(T_\beta) \cup \Omega$. The same conclusion holds for the induced IFS (4.4). Moreover, the graphs of the two systems are strongly connected. Hence the Rauzy boxes of the two systems coincide by Theorem 7.1.

For any $j \in V$, the intersections of the Rauzy boxes with $J_j^\circ \times \mathbb{R}^s$ coincides, that is,

$$J_j^\circ \times Y_j = \bigcup_{J_j \subset I_k} J_j^\circ \times X_k.$$

The theorem follows. □

Example 4.4. Let β be the Pisot number satisfying $\beta^3 = \beta + 1$, which has appeared in Hokkaido substitution. The characteristic sequence of β is

$$1 = 0.(10000)^\infty.$$

Hence $\Omega' = \Omega = \{0, 0.0001, 0.001, 0.01, 0.1, 1\}$ and the corresponding partition is $[0,1] = J_1 \cup \cdots \cup J_5$ where

$$J_1 = [0, 0.0001], J_2 = [0.0001, 0.001], J_3 = [0.001, 0.01], J_4 = [0.01, 0.1], J_5 = [0.1, 1].$$

(See Figure 5 (left).) The induced IFS and its dual IFS are given by

$$\begin{cases} \beta J_1 = J_1 \cup J_2 \\ \beta J_2 = J_3 \\ \beta J_3 = J_4 \\ \beta J_4 = J_5 \\ \beta J_5 = J_1 + 1 \end{cases} \quad \text{and} \quad \begin{cases} Y_1 = BY_1 \cup (BY_5 + \mathbf{e}_1) \\ Y_2 = BY_1 \\ Y_3 = BY_2 \\ Y_4 = BY_3 \\ Y_5 = BY_4. \end{cases}$$

The invariant sets are given by

$$\begin{aligned} Y_1 &= X_1 \cup X_5 \cup X_4 \cup X_3 \cup X_2 \\ Y_2 &= X_1 \cup X_5 \cup X_4 \cup X_3 \\ Y_3 &= X_1 \cup X_5 \cup X_4 \\ Y_4 &= X_1 \cup X_5 \\ Y_5 &= X_1, \end{aligned}$$

where X_j's are Rauzy fractals in Example 3.2 (see also Figure 2 (right)). The order is determined by $\mathbf{c}_1 > \mathbf{c}_5 > \mathbf{c}_4 > \mathbf{c}_3 > \mathbf{c}_2$ since $I_k = [0, 0.\mathbf{c}_k]$.

§ 4.2. Symmetric β-expansions

Akiyama and Scheicher [5] introduced symmetric β-expansion. The *symmetric β-transformation* is defined as (see Figure 5 (right))

$$S_\beta(x) = \beta x - \lfloor \beta x + \frac{1}{2} \rfloor, \quad x \in [-1/2, 1/2]. \tag{4.8}$$

Example 4.5. [28] Let β be the Pisot number satisfying $\beta^3 = \beta^2 + \beta + 1$, which has appeared in Rauzy substitution. The union of orbits of discontinuous points of S_β is

$$\Omega = \{-\frac{1}{2}, -\frac{1}{2\beta}, -\frac{1}{2\beta^2}, -\frac{1}{2\beta^3}, \frac{1}{2\beta^3}, \frac{1}{2\beta^2}, \frac{1}{2\beta}, \frac{1}{2}\},$$

which determines a partition $[-1/2, 1/2] = I_1 \cup I_2 \cup I_3 \cup I_0 \cup J_3 \cup J_2 \cup J_1$ (from left to right). It is easy to verify that the induced IFS of S_β and its dual IFS are

$$\begin{cases} \beta I_0 = I_3 \cup I_0 \cup J_3 \\ \beta I_1 = (J_3 \cup J_2 \cup J_1) - 1 \\ \beta I_2 = I_1 \\ \beta I_3 = I_2 \\ \beta J_1 = (I_1 \cup I_2 \cup I_3) + 1 \\ \beta J_2 = J_1 \\ \beta J_3 = J_2. \end{cases} \quad \textit{and} \quad \begin{cases} X_0 = BX_0 \\ X_1 = (BY_1 + \mathbf{e}_1) \cup BX_2 \\ X_2 = (BY_1 + \mathbf{e}_1) \cup BX_3 \\ X_3 = (BY_1 + \mathbf{e}_1) \cup BX_0 \\ Y_1 = (BX_1 - \mathbf{e}_1) \cup BY_2 \\ Y_2 = (BX_1 - \mathbf{e}_1) \cup BY_3 \\ Y_3 = (BX_1 - \mathbf{e}_1) \cup BX_0, \end{cases} \tag{4.9}$$

where B is the same as that in Example 2.3. (Notice that the graph Γ is not strongly connnected.)

From (4.9) (right), it is seen that $X_3 \subset X_2 \subset X_1$, $Y_3 \subset Y_2 \subset Y_1$, and $X_j = -Y_j$ for $j = 1, 2, 3$. The invariant sets are depicted by Figure 6 where $X_1 = A_1 \cup A_2 \cup A_3$, $X_2 = A_1 \cup A_2$, $X_3 = A_1$.

§ 4.3. Minimal weight expansion

The minimal weight expansion is introduced by Frougny and Steiner [19]. Let $\beta > 1$ be a real number, the *minimal weight transformation* is defined on $I = [-\frac{\beta}{\beta+1}, \frac{\beta}{\beta+1}]$ as: $W_\beta(x) =$
$$\begin{cases} \beta x + 1, & \text{if } x \in [-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}] \\ \beta x, & \text{if } x \in \,]-\frac{1}{\beta+1}, \frac{1}{\beta+1}] \\ \beta x - 1, & \text{if } x \in \,]\frac{1}{\beta+1}, \frac{\beta}{\beta+1}]. \end{cases}$$

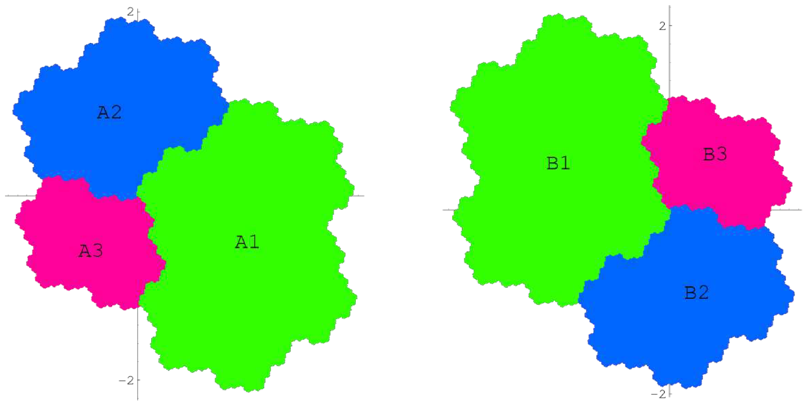

Figure 6. Invariant sets of dual system of symmetric β-transformation.

Example 4.6. [28] Let β be the Pisot number satisfying $\beta^3 = \beta^2 + \beta + 1$, which is the same as Example 4.5. The union of orbits of the discontinuous points of W_β is

$$\Omega = \left\{-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}, -\frac{1/\beta}{\beta+1}, \frac{1/\beta}{\beta+1}, \frac{1}{\beta+1}, \frac{\beta}{\beta+1}\right\},$$

which determines a partition $I = I_1 \cup I_2 \cup I_0 \cup J_2 \cup J_1$. The induced IFS is

$$\begin{cases} W_\beta(I_1) = I_0 \cup J_2 \\ W_\beta(I_2) = I_1 \\ W_\beta(I_0) = I_2 \cup I_0 \cup J_2 \\ W_\beta(J_2) = J_1 \\ W_\beta(J_1) = I_2 \cup I_0. \end{cases} \quad or \quad \begin{cases} \beta I_1 = (I_0 \cup J_2) - 1 \\ \beta I_2 = I_1 \\ \beta I_0 = I_2 \cup I_0 \cup J_2 \\ \beta J_2 = J_1 \\ \beta J_1 = (I_2 \cup I_0) + 1. \end{cases}$$

The dual IFS is (the matrix B is the same as Example 4.5)

$$\begin{cases} X_1 = BX_2 \\ X_2 = BX_0 \cup (BY_1 + \mathbf{e}_1) \\ X_0 = (BX_1 - \mathbf{e}_1) \cup BX_0 \cup (BY_1 + \mathbf{e}_1) \\ Y_2 = BX_0 \cup (BX_1 - \mathbf{e}_1) \\ Y_1 = BY_2. \end{cases}$$

Similar to Example 4.5, we have $X_j = -Y_j$, $X_1 \subset X_2 \subset X_0$, and $Y_1 \subset Y_2 \subset X_0$. The invariant sets are depicted by Figure 3 where $X_0 = A_1 \cup A_2 \cup A_3$, $X_2 = A_1 \cup A_2$, $X_1 = A_1$.

§ 5. Containing lattice and quasi-periodicity

In this section, we provide the tools needed later: the containing ideal and containing lattice. See Meyer [33] or Moody [34].

Let $\beta > 1$ be an algebraic integer of degree $d \geq 2$. Define

$$\mathbb{Z}[\beta] = \left\{ \sum_{j=0}^{d-1} a_j \beta^j;\ a_j \in \mathbb{Z} \right\}.$$

Then $\mathbb{Z}[\beta]$ is a ring. Let $\mathcal{D}$ be a non-empty subset of $\mathbb{Z}[\beta]$ and $\mathcal{D} \neq \{0\}$. We denote

$$K = K(\beta, \mathcal{D})$$

the ideal of $\mathbb{Z}[\beta]$ generated by $\mathcal{D}$, and call it the *containing ideal.* Indeed, K is the smallest module containing $\mathcal{D}$ and satisfying $\beta K \subset K$.

Recall that $\beta_2, \ldots, \beta_d$ are conjugates of β ordered by (2.1), a' is the Galois dual of a in the field $\mathbb{Q}(\beta)$, $\hat{a}$ is the 'real version' of a', and a^* is the real version of the contractive Galois dual of a. Define $\Sigma = \Sigma(\beta, \mathcal{D})$ as

$$\Sigma = \{(a, \hat{a});\ a \in K\} \tag{5.1}$$

and we call it the *containing lattice* determined by β and $\mathcal{D}$.

The following lemma is easy (see [37]).

Lemma 5.1. *(i) K is dense in $\mathbb{R}$. (ii) Σ is a full-rank lattice in $\mathbb{R}^d$. (iii) If β is an algebraic unit, then $\beta K = K$.*

Let $\omega_1, \ldots, \omega_d$ be a basis of the lattice $\Sigma(\beta, K)$. The *norm* of K is defined as $\mathcal{N}(K) = |\det(\omega_1, \ldots, \omega_d)|$.

Remark 1. For an algebraic GIFS with expanding factor β and digit set $\mathcal{D}$, we may assume without loss of generality that $\mathcal{D} \subset \mathbb{Z}[\beta]$; otherwise, we can replace $\mathcal{D}$ by $n\mathcal{D}$ with $n\mathcal{D} \subset \mathbb{Z}[\beta]$. We call $\Sigma(\beta, \mathcal{D})$ the *containing lattice* of the GIFS. This terminology is borrowed from Lagarias and Wang [30].

Let us denote by π and π^* the two natural projections defined on Σ:

$$\pi(a, \hat{a}) = a, \qquad \pi^*(a, \hat{a}) = a^*.$$

Clearly π and π^* are injective.

We shall use $B_n(x, R)$ to denote the ball in $\mathbb{R}^n$ with center x and radius R; sometimes we use $B(x, R)$ if the dimension is implicit.

A set L is said to be a *Delone set* in $\mathbb{R}^s$, if L is relatively dense and uniformly discrete. L is *relatively dense* means that there exists $R > 0$ such that $B(x, R) \cap L \neq \emptyset$ for all $x \in \mathbb{R}^s$; L is *uniformly discrete* means that there exists $r > 0$ such that $|h_1 - h_2| > r$ for any $h_1, h_2 \in L$.

$X \subset \mathbb{R}^n$ is *quasi-periodic*, if for any ball $B(c, r) \subset \mathbb{R}^n$, there exists a $R > 0$ such that a translation of $B(c, r) \cap X$ appears in the ball $B(y, R) \cap X$ for every $y \in \mathbb{R}^n$.

For $A \subset \mathbb{R}^{d-s}$, we denote $\Sigma_A = \Sigma \cap (A \times \mathbb{R}^s)$ the points of Σ in the 'window' A. Notice that if β is a Pisot number, then $s = d - 1$, $\hat{a} = a^*$ and $\pi^*(\Sigma_A) = (K \cap A)^*$.

Theorem 5.2. *([33, 34, 37]) Let A be a bounded subset of $\mathbb{R}^{d-s}$, then*
(i) $\pi^*(\Sigma_A)$ is uniformly discrete.
(ii) If β is a Pisot number and $A \subset \mathbb{R}$ has non-empty interior, then $\pi^*(\Sigma_A) = (K \cap A)^*$ is a Delone set.
(iii) If β be a Pisot number and $A = [u, v)$ is an interval, then $(K \cap [u, v))^*$ is quasi-periodic in $\mathbb{R}^{d-1}$.

§ 6. Open set condition of dual systems

A GIFS $(f_\gamma)_{\gamma\in\Gamma}$ is said to satisfy the OSC, if there exist open sets $(U_j)_{j\in V}$ such that

$$\bigcup_{j\in V} \bigcup_{\gamma\in\Gamma_{i,j}} f_\gamma(U_j) \subset U_i, \ i \in V,$$

and the left-hand sides are disjoint unions. See [35, 15].

§ 6.1. Criterion of the OSC

In general, it is very hard to verify the OSC. But for a special class of GIFS, called single-matrix GIFS, efficient criteria of OSC do exist. A GIFS $(f_\gamma)_{\gamma\in\Gamma}$ is called a *single-matrix GIFS* if f_γ have the form

$$f_\gamma(x) = A^{-1}(x + b_\gamma), \quad \gamma \in \Gamma, \tag{6.1}$$

where A is a $d \times d$ expanding matrix and $b_\gamma \in \mathbb{R}^d$. (A matrix is *expanding* if all its eigenvalues have modulus larger than 1.)

Recall that $\Gamma^n_{i,j}$ is the set of paths from vertex i to j with length n. Denote $f_{\gamma_1\dots\gamma_n} = f_{\gamma_1} \circ \cdots \circ f_{\gamma_n}$. Define

$$\mathcal{D}^n_{i,j} := A^n\{f_{\gamma_1\dots\gamma_n}(0);\ \gamma_1 \dots \gamma_n \in \Gamma^n_{i,j}\}. \tag{6.2}$$

We note that $A^n f_{\gamma_1\dots\gamma_n}(0) = A^{n-1}b_{\gamma_1} + A^{n-2}b_{\gamma_2} + \cdots + Ab_{\gamma_{n-1}} + b_{\gamma_n}$.

Theorem 6.1. *(He-Lau [22], Luo-Yang [32]) GIFS (6.1) satisfies the OSC if and only if the following two conditions hold:*
(i) $\#\mathcal{D}^n_{i,j} = \#\Gamma^n_{i,j}$ for all $i, j \in V$, $n \geq 1$;
(ii) there exists $r > 0$ such that $\mathcal{D}^n_{i,j}$ is r-uniformly discrete for all $i, j \in V$, $n \geq 1$.

§ 6.2. OSC of dual IFS

Let $(f_\gamma)_{\gamma\in\Gamma}$ be an algebraic GIFS with expanding factor β. Recall that $\mathcal{D} = \{b_\gamma;\ \gamma \in \Gamma\}$ is the digit set, $K = K(\beta, \mathcal{D})$ is the containing ideal, and $\Sigma = \Sigma(\beta, \mathcal{D})$ is the containing lattice. In the rest of this section, we assume that: *β is an algebraic unit, and $|\beta_k| \neq 1$ for $2 \leq k \leq d$.* We also assume that $K \subset \mathbb{Z}[\beta]$ without loss of generality. Let $(F_{\gamma'})_{\gamma'\in\Gamma'}$ be the dual IFS.

Theorem 6.2. *([37]) Let $(f_\gamma)_{\gamma\in\Gamma}$ be an algebraic GIFS such that its expanding factor β is an algebraic unit and the conjugates of β do not equal 1 in modulus. Then its dual system satisfies the OSC if and only if*

$$\#\{f_{\gamma_1\ldots\gamma_n}(0);\ \gamma_1\ldots\gamma_n \in \Gamma^n_{i,j}\} = \#\Gamma^n_{i,j}, \quad \forall i,j\in V,\ n\geq 1. \tag{6.3}$$

In particular, the dual system satisfies the OSC provided the original system does.

Corollary 6.3. *([37])Let $(f_\gamma)_{\gamma\in\Gamma}$ be an algebraic GIFS with expanding factor β. If β is a Pisot unit, then the dual system satisfies the OSC if and only if the original IFS does.*

Example 6.4. The dual IFS may satisfies the OSC even if the original GIFS does not. Consider the IFS

$$\beta E = E\cup(E+1)$$

where $\beta\approx 1.8794$ is a zero point of $P(x) = x^3-3x-1$. The above IFS does not satisfy the OSC since $1/\beta > 1/2$. The conjugates of β are $\beta_2 = -1.5321, \beta_3 - 0.3473$. The dual IFS is $X = \beta_3 X\cup(\beta_3 X+1)$ and it satisfies the OSC since $|\beta_3| < 1/2$.

If β is not an algebraic unit, then typically the dual IFS does not satisfy the OSC ([11, 4]).

§ 6.3. Self-similar tiling system

Let $\mathcal{F} = (f_\gamma)_{\gamma\in\Gamma}$ be a single-matrix GIFS on $\mathbb{R}^d$ with expanding matrix A. We say $\mathcal{F}$ is a *self-similar tiling system* if the system satisfies the OSC and its invariant sets have non-empty interiors. It is well-known that a self-similar tiling system provides *dilation and subdivision rules* and hence self-similar tilings can be constructed. Theorem 1.2 can be proved by using Theorem 6.2 and a criterion of Lagarias and Wang [30], see [37].

§ 7. Periodic codings

Let $(f_\gamma)_{\gamma\in\Gamma}$ be an algebraic IFS, let $(F_{\gamma'})_{\gamma'\in\Gamma'}$ be the dual IFS, and $\mathcal{R} = \bigcup_{j\in V} I_j\times X_j$ be the Rauzy box.

For $x\in I_j$, an infinite path $(\gamma_n)_{n\geq 1}$ on Γ starting from j is called a *coding* of x if

$$\{x\} = \bigcap_{n\geq 1} f_{\gamma_1\ldots\gamma_n}(I_{k_n})$$

where k_n is the ending vertex of γ_n.

Let $\mathcal{P}$ be the set of x with periodic coding w.r.t. $(f_\gamma)_{\gamma\in\Gamma}$. Clearly $\mathcal{P}\subset\mathbb{Q}(\beta)$. The set $\mathcal{P}$ is closely related to the Rauzy box $\mathcal{R}$. A graph is *strongly connected* if for any $i,j\in V$, there is a path from i to j.

Theorem 7.1. *([42]) $\{(x,-x^*);\ x\in\mathcal{P}\}$ is a subset of $\mathcal{R}$; it is dense in $\mathcal{R}$ if the graph Γ is strongly connected.*

From now on, let $(f_\gamma)_{\gamma\in\Gamma}$ be an feasible Pisot system, that is, it is an SST system, β is a Pisot unit, the invariant sets are intervals. Let us denote $I_j = [a_j, b_j]$, and set

$$I_j^l = [a_j, b_j), \quad I_j^r = (a_j, b_j], \quad j \in V. \tag{7.1}$$

Then both $(I_j^l)_{j\in V}$ and $(I_j^r)_{j\in V}$ satisfy the equations

$$\beta \tilde{I}_j = \bigcup_{i\in V} \left(\tilde{I}_i + \mathcal{D}_{j,i}\right), \quad j \in V, \tag{7.2}$$

and the right hand side unions are disjoint.

It is well-known that for $x \in I_j$, x may has more than one codings. Set $\tilde{I}_j = I_j^l$ in (7.2), we obtain a unique coding of $x \in I_j^l$ w.r.t. system (7.2), and we call it the *lower coding* of x. Similarly, we can define the *upper coding* of $x \in I_j^r$.

Remark 2. If the algebraic IFS is induced by a β-transformation, then the lower coding of $x \in [0, 1)$ is related to the β-expansion of x, and the upper coding of x is related to the weakly admissible β-expansion.

In [42], it is shown that a coding of $x \in [a_j, b_j]$ is either a lower coding, or an upper coding; $x \in [a_j, b_j)$ has a periodic lower coding starting from j if and only if $(x, -x^*) \in [a_j, b_j) \times X_j$; similarly, $x \in (a_j, b_j]$ has a periodic upper coding starting from j if and only if $(x, -x^*) \in (a_j, b_j] \times X_j$.

Theorem 7.2. *[42] Let $(f_\gamma)_{\gamma\in\Gamma}$ be a feasible Pisot system. Then $x \in \bigcup_{j\in V} I_j$ possesses a periodic coding if and only if $x \in \mathbb{Q}(\beta)$ and $(x, -x^*) \in \mathcal{R}$.*

Proof. $x \in [a_j, b_j]$ has a periodic coding if and only if $(x, -x^*) \in [a_j, b_j) \times X_j$ or $(x, -x^*) \in (a_j, b_j] \times X_j$, that is, $(x, -x^*) \in I_j \times X_j$. □

The following result characterizes the periodic points of generalized β-transformations T when β is a Pisot unit. Denote Per(S) the set of periodic points of a transformation S.

Corollary 7.3. *([28, 42]) Let T be a generalized β-transformation where β is a Pisot unit, then $x \in Per(T^+) \cup Per(T^-)$ if and only if $x \in \mathbb{Q}(\beta)$ and $(x, -x^*) \in \mathcal{R}$, where $\mathcal{R}$ is the Rauzy box of the induced IFS of T.*

§ 8. Rauzy-Thurston Tiling

Recall that an algebraic IFS and its dual system can be written as

$$\beta I_j = \bigcup_{i\in V} (I_i + \mathcal{D}_{j,i}), \quad j \in V. \tag{8.1}$$

$$X_j = \bigcup_{i\in V} \left(BX_i + \mathcal{D}^*_{i,j}\right) \quad j \in V. \tag{8.2}$$

§ 8.1. Rauzy-Thurston Tiling

Let $K = K(\beta, \mathcal{D})$ is the containing ideal defined in Section 5.

Lemma 8.1. *The family $([a_j, b_j[\cap K)_{j\in V}$ (and also $(]a_j, b_j]\cap K)_{j\in V}$) is a solution of the set equation (8.1).*

Proof. Let $I = [a, b[$ be an interval, let $c \in K$. Then it is easy to see that

$$\beta I \cap K = \beta(I \cap K), \quad (I + a) \cap K = (I \cap K) + a. \tag{8.3}$$

Replacing I_j by I_j^l, equation (8.1) still holds; taking an intersection with K on both sides and using (8.3), we obtain the lemma. □

Definition 8.2. Let $\mathcal{F} = (f_\gamma)_{\gamma\in\Gamma}$ be a feasible Pisot system. Set

$$\mathcal{J}^l = \bigcup_{j\in V} \{X_j + a^*;\ a \in [a_j, b_j[\cap K\}, \quad \mathcal{J}^r = \bigcup_{j\in V} \{X_j + a^*;\ a \in]a_j, b_j] \cap K\}.$$

We call $\mathcal{J}^l$ and $\mathcal{J}^r$ the *left and right Rauzy-Thurston tiling* of $\mathcal{F}$, respectively.

We shall only discuss the left Rauzy-Thurston tiling $\mathcal{J}^l$, but all the discussions work for $\mathcal{J}^r$. Set

$$\mathcal{Z}^l = \{T \in \mathcal{J}^l; 0 \in T\}, \tag{8.4}$$

and we call the elements in $\mathcal{Z}^l$ *central tiles*. Clearly $\mathcal{Z}^l$ is a finite set.

Lemma 8.3. *([37]) Let $X_j + a^*$ be a tile in $\mathcal{J}^l$ and denote the lower coding of a by $(\gamma_n)_{n\geq 1}$. Then*

(i) $X_j + a^*$ is a central tile if and only if $(\gamma_n)_{n\geq 1}$ is periodic.

(ii) A tile $X_i + b^*$ belongs to the subdivision of $B^{-1}(X_j + a^*)$ according to (8.2), if and only if there exists $\gamma \in \Gamma_{ij}$ such that the lower coding of b is $\gamma\gamma_1\gamma_2\dots$.

Using Lemma 8.1 and Lemma 8.3, it is easy to show that that $\mathcal{J}^l$ (also $\mathcal{J}^r$) is quasi-periodic and self-replicating ([37]). $\mathcal{J}^l$ is *self-replicating* means that if we subdivide the elements in $B^{-1}\mathcal{J}^l$ according to equations (8.2), the resulting collection is again $\mathcal{J}^l$. $\mathcal{J}^l$ is *quasi-periodic* means that, if for any ball $\mathbb{B}(c, r)$, there exists $R > 0$ such that the pattern

$$\mathbb{B}(c, r) \cap \mathcal{J}^l := \{T \in \mathcal{J}^l;\ T \cap \mathbb{B}(c, r) \neq \emptyset\}$$

appears in $\mathbb{B}(x, R) \cap \mathcal{J}^l$ for every $x \in \mathbb{R}^{d-1}$.

A standard argument shows that a self-replicating and quasi-periodic collection has multiplicity m for some $m \geq 1$ (see Kenyon [29]). In particular, each tile $X_j + a^* \in \mathcal{J}^l$ belongs to the subdivision of B^nT for some central tile T and $n \geq 1$. Obviously the Rauzy-Thurston tiling $\mathcal{J}^l$ has multiplicity 1 if and only if the tiles in $\mathcal{Z}^l$ are non-overlapping.

Moreover, Theorem 1.5 asserts that $\mathcal{J}^l$ can be decomposed into m 'normal' tilings of $\mathbb{R}^{d-1}$.

§ 8.2. Regular algebraic solution

Definition 8.4. A family $(G_j)_{j\in V}$ is called a *regular algebraic solution* of (8.1) if

(i) $(G_j)_{j\in V}$ is a solution of (8.1);

(ii) $G_j \subset [a_j, b_j[\cap K$ for all $j \in V$;

(iii) G_j^* are relatively dense in $\mathbb{R}^{d-1}$.

Clearly $([a_j, b_j[\cap K)_{j\in V}$ is a regular algebraic solution of (8.1). Let $(G_j)_{j\in V}$ and $(G'_j)_{j\in V}$ be two solution of (8.1), we say $(G'_j)_{j\in V}$ is *strictly smaller* than $(G_j)_{j\in V}$ if $G'_j \subset G_j$ for all $j \in V$ and $G'_j \neq G_j$ for at least one j. A regular agebraic solution $(G_j)_{j\in V}$ is *minimal* if no regular algebraic solution is strictly smaller than $(G_j)_{j\in V}$.

We define the *k-th iteration* of $\mathcal{F}$, which we denote by $\mathcal{F}^k$, to be the GIFS

$$(f_{\gamma_1\circ\cdots\circ\gamma_k})_{\gamma_1\circ\cdots\circ\gamma_k\in\Gamma^k}.$$

Clearly $\mathcal{F}^k$ is also a feasible Pisot GIFS, with expanding factor β^k, and with the same invariant sets $(I_j)_{j\in V}$. The dual system of $\mathcal{F}^k$ is the k-th iteration of the dual system of $\mathcal{F}$. Hence $\mathcal{F}$ and $\mathcal{F}^k$ have the same multiplicity.

The following theorem gives a new criterion for tightness.

Theorem 8.5. *([37]) A feasible Pisot system $\mathcal{F}$ is tight, if and only if $([a_j, b_j[\cap K)_{j\in V}$ is the minimal regular algebraic solution of $\mathcal{F}^k$ for any $k \geq 1$.*

The next two examples are systems which are not tight.

Example 8.6. Let $\sigma:\ 1\mapsto 12,\ 2\mapsto 13,\ 3\mapsto 1$ be the Rauzy substitution, and let σ' be a substitution over the alphabet $\{1, 2, 3, 1', 2', 3'\}$ given by

$$\sigma' : \begin{cases} 1\mapsto 12, \quad 2\mapsto 13, \quad 3\mapsto 1' \\ 1'\mapsto 1'2', 2'\mapsto 1'3', 3'\mapsto 1. \end{cases}$$

Let I_1, I_2, I_3 be the invariant sets of the IFS induced by σ, and $J_1, J_2, J_3, J_{1'}, J_{2'}, J_{3'}$ be the invariant sets of the IFS induced by σ'. Then it is easy to see that $J_j = J_{j'} = I_j$ for $1 \leq j \leq 3$.

Let X_1, X_2, X_3 be the Rauzy fractals of σ, and $Y_1, Y_2, Y_3, Y_{1'}, Y_{2'}, Y_{3'}$ be the Rauzy fractals of σ'. It is easy to see that $Y_j = Y_{j'} = X_j$ for $1 \leq j \leq 3$.

Therefore the Rauzy-Thurston tiling of σ' consists of two identical tilings, and the multiplicity is 2.

$\mathbb{Z}[\beta]$ is the containing ideal of the system induced by σ as well as σ'. It is well known that the β-expansion of $x \in \mathbb{Z}[\beta] \cap [0, 1)$ are finite since β satisfies an F-property ([18]). Let

$$K_0 = \{a \in \mathbb{Z}[\beta] \cap [0, 1); \text{'110' appears even times in the } \beta\text{-expansion of } a\},$$

$$K_1 = \{a \in \mathbb{Z}[\beta] \cap [0, 1); \text{'110' appears odd times in the } \beta\text{-expansion of } a\}.$$

Let $G_j = I_j \cap K_0$, $G'_j = I'_j \cap K_1$. It is easy to see that $(G_j)_{1\leq j\leq 3} \cup (G'_j)_{1\leq j\leq 3}$ is a family satisfying the conditions in Theorem 8.5.

Example 8.7. Let us consider the Rauzy-Thurston tiling of symmetric β-transformation S_β with $\beta^3 = \beta^2 + \beta + 1$ (See Example 4.5). Since $X_0 = \{0\}$ and it has no contribution to the Rauzy-Thurstion tiling, we need only consider the systems

$$\begin{cases} \beta I_1 = (J_3 \cup J_2 \cup J_1) - 1 \\ \beta I_2 = I_1 \\ \beta I_3 = I_2 \\ \beta J_1 = (I_1 \cup I_2 \cup I_3) + 1 \\ \beta J_2 = J_1 \\ \beta J_3 = J_2. \end{cases} \quad \textit{and} \quad \begin{cases} X_1 = (BY_1 + \mathbf{e}_1) \cup BX_2 \\ X_2 = (BY_1 + \mathbf{e}_1) \cup BX_3 \\ X_3 = BY_1 + \mathbf{e}_1 \\ Y_1 = (BX_1 - \mathbf{e}_1) \cup BY_2 \\ Y_2 = (BX_1 - \mathbf{e}_1) \cup BY_3 \\ Y_3 = BX_1 - \mathbf{e}_1, \end{cases} \tag{8.5}$$

Kalle and Steiner [28] proved that the multiplicity $m = 2$, where they used a rather complicated transducer to show that $m \geq 2$. Here we give a short proof of the fact $m \geq 2$. Let

$$K_0 = \{c_2\beta^2 + c_1\beta + c_0 \in \mathbb{Z}[\beta];\ c_2 + c_1 + c_0 \text{ is even}\},$$

$$K_1 = \{c_2\beta^2 + c_1\beta + c_0 \in \mathbb{Z}[\beta];\ c_2 + c_1 + c_0 \text{ is odd}\}.$$

Set $G_j = \tilde{I}_j \cap K_0$ and $G'_j = \tilde{J}_j \cap K_1$ for $1 \leq j \leq 3$. Then $G_1, G_2, G_3, G'_1, G'_2, G'_3$ satisfy the conditions in Theorem 8.5 and hence $m > 1$.

§ 8.3. Domain-exchange transformation

Comparing $\mathcal{J}^l$ and $\mathcal{J}^r$, we obtain that

Theorem 8.8. *Let $I_j = [a_j, b_j]$ be invariant sets of a feasible Pisot system, then*

$$\bigcup_{a_j \in K} (X_j + a_j^*) = \bigcup_{b_j \in K} (X_j + b_j^*). \tag{8.6}$$

For the symmetric β-expansion in Example 4.5, all a_j and b_j do not beling to K and hence both sides of (8.6) are empty sets.

Remark 3. If the algebraic system is induced by a Pisot substitution (see Section 3), then $a_j = 0$ and $b_j = |I_j|$ for $j \in V$, and all of them belong to K. In this case, (8.6) becomes

$$\bigcup_{j \in V} X_j = \bigcup_{j \in V} (X_j + b_j^*). \tag{8.7}$$

Let $X = \bigcup_{j \in V} X_j$. If X_j are disjoint in Lebesgue measure, according to (8.7), a *domain-exchange transformation* S can be defined almost everywhere on X as

$$S(z) = z + b_j^*, \quad \text{if } z \in X_j.$$

Arnoux and Ito [6] showed that the dynamical system $(X, S, \mathcal{L}^{d-1}|_X)$ is measure theoretically conjugate to the substitution dynamical system.

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