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Fully Subtractive Algorithm, Tribonacci numeration and connectedness of discrete planes

By

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Abstract

We investigate connections between a well known multidimensional continued fraction algorithm, the so-called fully subtractive algorithm, the finiteness property for β-numeration, and the connectedness of arithmetic discrete hyperplanes. A discrete hyperplane is said to be critical if its thickness is equal to the infimum of the set of thicknesses for which discrete hyperplanes with the same normal vector are connected. We focus on particular planes the parameters of which belong to the cubic extension generated by the Tribonacci number, we prove connectedness in the critical case, and we exhibit an intriguing tree structure rooted at the origin.

§ 1. Introduction

Discrete geometry attempts to provide a rigorous approach to the investigation of discrete objects of $\mathbb{R}^d$. In the present paper, we deal with a common class of discrete objects, namely the class of discrete hyperplanes which are, in some sense, the discrete analogues of Euclidean hyperplanes. Discrete hyperplanes are the most basic primitives in discrete geometry, and are defined as sets of integral points which satisfy a double Diophantine inequality expressed in terms of an underlying Euclidean plane. Since discrete hyperplanes depend on a thickness parameter (see Definition 2.1 below), it becomes natural to study the behaviour of the topology of theses objects when this parameter changes. This study has been initiated in [3], see also [8]. It has been proved in [4, 9, 10] that the computation of the infimum of the set of thicknesses for which a discrete hyperplane remains connected is provided by a multidimensional continued fraction algorithm, namely, the fully subtractive algorithm (for more details, see for instance [14]).

When reducing the thickness of a discrete hyperplane, we loose connectedness. We focus here on the transition case, that is, when the thickness is equal to the infimum of the set of

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thicknesses for which discrete hyperplanes with the same normal vector are connected. We call this thickness *critical*. The fully subtractive algorithm (see Section 2) acts as a (piecewise) linear map on a vector. In this paper, we thoroughly investigate Tribonacci planes, which is a special case where the normal vector is an eigenvector of the linear map, so that its direction is unchanged throughout the process. More precisely, the normal vector of the discrete plane is $(\alpha, \alpha+\alpha^2, 1)$, where α is the inverse of the Tribonacci number, i.e., $\alpha+\alpha^2+\alpha^3 = 1$ and $\alpha \in \mathbb{R}$. This corresponds to one of the simplest periodic orbits of the fully subtractive algorithm.

Our approach relies on a generation method for the discrete planes under study. To this aim, we build an increasing sequence $(\mathbb{P}_n)_{n\geqslant 0}$ of subsets of $\mathbb{Z}^3$, and we show that $\mathbb{P}_n$ is connected for all n. The key point relies on the proof of the fact that their limit $\mathbb{P}_\infty$ is actually equal to the full discrete plane with critical thickness. For this purpose, we introduce a suitable representation for the points of $\mathbb{P}_\infty$ closely related to the β-numeration associated with the Tribonacci number (for more details on β-numeration, see e.g. [2]). The fact that $\lim \mathbb{P}_n = \mathbb{P}_\infty$ is proved to be a consequence of the so-called finiteness property for the Tribonacci numeration [7].

The ergodic properties of the fully subtractive algorithm have been thoroughly investigated (see [11, 12, 6]) motivated by considerations issued from percolation theory. In particular, in [12], a dependent percolation model on $\mathbb{Z}^2$ is considered closely related to the connectedness problem we investigated in the present paper. Given the same double Diophantine inequalities as the ones of discrete hyperplanes, the number of infinite clusters, that is, the number of infinite connected components of this set, is shown to be equal to zero, one or infinity. Furthermore, it is also shown that for almost all choices of parameters, the critical value, that is the value for which there exists at least one infinite cluster, can be calculated in a finite number of iterations of the fully subtractive algorithm.

Tijdeman and Zamboni [15] applied a very similar algorithm to a generalisation of Fine and Wilf's theorem. The fully subtractive algorithm is used to compute words of maximal length with given periods $p_1, \ldots, p_r$ and such that $\gcd(p_1, \ldots, p_r)$ is not a period.

The contents of this paper can be sketched as follows. After recalling basic notions and definitions in Section 2, we focus on the Tribonacci case in Section 3. Topological properties of the Tribonacci discrete plane with critical thickness are investigated in Section 4: it has exactly 3 connected components when deprived of the origin, and is proved to be a tree rooted at $\mathbf{0}$. It is shown in Section 5 that the connectedness for the critical thickness depends on the shift parameter of the plane.

§ 2. Basic notions and preliminaries

Given an integer $d \geqslant 1$, $(\mathbf{e_1}, \ldots, \mathbf{e_d})$ is the canonical basis of $\mathbb{R}^d$. The standard scalar product in $\mathbb{R}^d$ is denoted by $\langle\, .\, ,\, .\, \rangle$. Given $\mathbf{x} \in \mathbb{R}^d$, x_i stands for $\langle \mathbf{x}, \mathbf{e_i} \rangle$. Let us now introduce the main definition of the present work.

Definition 2.1 ([1, 13]). Given a non-zero vector $\mathbf{v} \in \mathbb{R}^d$ and $\mu, \omega \in \mathbb{R}$, the *discrete hyperplane with normal vector* $\mathbf{v}$*, shift* μ *and thickness* ω, denoted by $\mathbb{P}(\mathbf{v}, \mu, \omega)$, is the subset of $\mathbb{Z}^d$ defined by:

$$\mathbb{P}(\mathbf{v}, \mu, \omega) = \{\mathbf{x} \in \mathbb{Z}^d \mid 0 \leqslant \langle \mathbf{v}, \mathbf{x} \rangle + \mu < \omega\}$$

If $d = 2$ (resp. $d = 3$), one refers to the hyperplane $\mathbb{P}(\mathbf{v}, \mu, \omega)$ as a *discrete line* (resp. a *discrete plane*).

One commonly represents in discrete/digital geometry integer points $\mathbf{x} \in \mathbb{Z}^d \subset \mathbb{R}^d$ as voxels, that is, as unit cubes centred at $\mathbf{x}$ (see Figure 2 and 3). One can define several notions of connectivity, each of them being linked to some neighbourhood relation. In practice, in the literature, the most used ones are vertex and facet connectivity, that is, the ones where unit cubes share respectively, at least, one vertex or one facet. In the present paper, we deal only with the latter: two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{Z}^d$ are *facet neighbours* (or *neighbours* for short) if $\sum_{i=1}^{d} |x_i - y_i| = 1$ or, equivalently, if $\mathbf{x} - \mathbf{y} = \pm \mathbf{e_i}$ for some $i \in \{1, \ldots, d\}$.

A *path* in $\mathbb{Z}^d$ is a sequence $(\mathbf{u_1}, \ldots, \mathbf{u_n})$ where $\mathbf{u_i}$ and $\mathbf{u_{i+1}}$ are neighbours for all $i = 1, \ldots, n-1$. A subset S of $\mathbb{Z}^d$ is *connected* if it is not empty, and for each pair $(\mathbf{x}, \mathbf{y}) \in S^2$, there exists in S a *path* $(\mathbf{u_1}, \ldots, \mathbf{u_n})$ such that $\mathbf{u_1} = \mathbf{x}$ and $\mathbf{u_n} = \mathbf{y}$. Two subsets S and S' of $\mathbb{Z}^d$ are *adjacent* if they are disjoint, and there exist $\mathbf{x} \in S$ and $\mathbf{x}' \in S'$ such that $\mathbf{x}$ and $\mathbf{x}'$ are neighbours.

Given a non-zero vector $\mathbf{v} \in \mathbb{R}^d$ and a shift $\mu \in \mathbb{R}$, our first interest is in the values of ω for which $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected.

Lemma 2.2 ([5]). *Let $\mathbf{v} \in \mathbb{R}^d$ be a non-zero vector and let $\mu \in \mathbb{R}$. The set $\{\omega \in \mathbb{R} \mid \mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected$\}$ is a right-unbounded interval.*

Definition 2.3 (Connecting thickness). Given a non-zero vector $\mathbf{v} \in \mathbb{R}^d$ and $\mu \in \mathbb{R}$, the *connecting thickness* of $\mathbf{v}$ with shift μ, denoted by $\Omega(\mathbf{v}, \mu)$, is the infimum of the values ω for which $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected:

$$\Omega(\mathbf{v}, \mu) = \inf\{\omega \in \mathbb{R} \mid \mathbb{P}(\mathbf{v}, \mu, \omega) \text{ is connected}\}.$$

We may note that for any $\lambda \neq 0$, we have $\mathbb{P}(\lambda\, \mathbf{v}, \mu, \omega) = \operatorname{sign}(\lambda)\, \mathbb{P}(\mathbf{v}, \mu/|\lambda|, \omega/|\lambda|)$. Hence $\mathbb{P}(\lambda\, \mathbf{v}, \mu, \omega)$ is connected if and only if so is $\mathbb{P}(\mathbf{v}, \mu/|\lambda|, \omega/|\lambda|)$. Therefore, $\Omega(\lambda\, \mathbf{v}, \mu) = |\lambda|\, \Omega(\mathbf{v}, \mu/|\lambda|)$. In order to compute $\Omega(\mathbf{v}, \mu)$, we may assume without loss of generality that $0 \leqslant v_1 \leqslant \cdots \leqslant v_d$. Indeed, given a signed permutation $\mathbf{M}_{\sigma,u}$:

$$\begin{aligned} \mathbf{M}_{\sigma,u} : \mathbb{Z}^d &\to \mathbb{Z}^d \\ \mathbf{x} &\mapsto ((-1)^{u_1} x_{\sigma(1)}, \ldots, (-1)^{u_d} x_{\sigma(d)}), \end{aligned}$$

where $u \in \{1, 2\}^d$ and σ is a permutation of $\{1, \ldots, d\}$, one checks that $\Omega(\mathbf{v}, \mu) = \Omega(\mathbf{M}_{\sigma,u}(\mathbf{v}), \mu)$. More precisely, $\mathbb{P}(\mathbf{M}_{\sigma,u}(\mathbf{v}), \mu, \omega) = {}^t\mathbf{M}_{\sigma,u}^{-1}(\mathbf{v})\, (\mathbb{P}(\mathbf{v}, \mu, \omega))$ and $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected if and only if so is $\mathbb{P}(\mathbf{M}_{\sigma,u}(\mathbf{v}), \mu, \omega)$. In particular, $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ is connected if and only if so is $\mathbb{P}(\mathbf{M}_{\sigma,u}(\mathbf{v}), \mu, \Omega(\mathbf{v}, \mu))$.

Therefore, in the sequel, we restrict ourselves to the set of parameters

$$\mathcal{O}_d^+ = \left\{\mathbf{v} \in \mathbb{R}^d \mid 0 \leqslant v_1 \leqslant \cdots \leqslant v_d \text{ and } v_d > 0\right\}.$$

It is shown in [4, 10] how to compute $\Omega(\mathbf{v}, \mu)$ from the expansion of the vector $\mathbf{v}$ according to the *ordered fully subtractive algorithm* [14]: $\mathbf{F} : \mathcal{O}_d^+ \to \mathcal{O}_d^+$ defined by

$$\mathbf{F}(\mathbf{v}) = \operatorname{sort}(v_1, v_2 - v_1, \ldots, v_d - v_1)$$

where sort($\mathbf{v}$) orders the coordinates of $\mathbf{v}$ in non-decreasing order.

Theorem 2.4 ([4, 10]). *Let $\mathbf{v} \in \mathcal{O}_d^+$ and $\mu \in \mathbb{R}$. The discrete hyperplane $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected if and only if so is $\mathbb{P}(\mathbf{F}(\mathbf{v}), \mu, \omega - v_1)$. Consequently, $\Omega(\mathbf{v}, \mu) = \Omega(\mathbf{F}(\mathbf{v}), \mu) + v_1$.*

Given a non-zero vector $\mathbf{v} \in \mathcal{O}_d^+$ and a shift $\mu \in \mathbb{R}$, we reduce the computation of $\Omega(\mathbf{v}, \mu)$ to the computation of $\Omega(\mathbf{v}, 0)$ thanks to the following equalities [5]:

$$\Omega(\mathbf{v}, \mu) = \begin{cases} \Omega(\mathbf{v}, 0) + (\mu \bmod \gcd(v_1, \ldots, v_d)) & \text{if } \dim_{\mathbb{Q}}(v_1, \ldots, v_d) = 1, \\ \Omega(\mathbf{v}, 0) & \text{if } \dim_{\mathbb{Q}}(v_1, \ldots, v_d) > 1, \end{cases}$$

where $\gcd(v_1, \ldots, v_n) = \max\{\alpha \in \mathbb{R} \mid \forall i \in \{1, \ldots, d\},\ v_i/\alpha \in \mathbb{Z}\}$.

We are then left to compute $\Omega(\mathbf{v}, 0)$ that we simply write as $\Omega(\mathbf{v})$. Note that the case $d = 1$ corresponds to a halting condition for this algorithm. One has:

$$\Omega(\mathbf{v}) = \begin{cases} 0 & \text{if } d = 1 \\ \Omega((v_2, \ldots, v_d)) & \text{if } d \geqslant 2 \text{ and } v_1 = 0 \\ v_1 + \Omega(\mathbf{F}(\mathbf{v})) & \text{if } d \geqslant 2 \text{ and } v_1 > 0 \end{cases} \tag{1}$$

From these equations, one deduces an algorithm to compute $\Omega(\mathbf{v})$ (see Algorithm 1). This algorithm is derived from the *ordered fully subtractive algorithm*.

Input: a vector $\mathbf{v} \in \bigcup_{i=1}^{d} \mathcal{O}_d^+$
Output: $\Omega(\mathbf{v})$
```
begin
    ω ← 0;
    while #{i | v_i ≠ 0} ⩾ 2 do
        if v_1 = 0 then
            /* projection step                         */
            v ← (v_2, ..., v_d);
        else
            ω ← ω + v_1;
            /* reduction step                          */
            v ← F(v) ;
        end
    end
    return ω;
end
```

Algorithm 1: Computation of $\Omega(\mathbf{v})$

Let $\mathbf{v} \in \mathcal{O}_d^+$ and let $(\mathbf{v}^{(n)})_{n \geqslant 0}$ be the sequence of elements of $\bigcup_{i=1}^{d} \mathcal{O}_i^+$ defined as follows:

$\mathbf{v}^{(0)} = \mathbf{v}$ and, for $n \geqslant 1$,

$$\mathbf{v}^{(n+1)} = \begin{cases} \pi_1(\mathbf{v^{(n)}}), & \text{if } v_1 = 0 \quad \texttt{/* projection step */} \\ \mathbf{F}(\mathbf{v^{(n)}}), & \text{otherwise} \quad \texttt{/* reduction step */} \end{cases}$$

where $\pi_1 : \mathbb{Z}^d \longrightarrow \mathbb{Z}^{d-1}$ is the orthogonal projection onto the coordinate plane with normal vector $\mathbf{e_1}$ of $\mathbb{Z}^d$.

For any $\mathbf{v} \in \mathcal{O}_d^+$, the `projection step` applies finitely many times. Moreover, Algorithm 1 terminates if and only if $\dim_{\mathbb{Q}}(v_1, \dots, v_d) = 1$. If $\dim_{\mathbb{Q}}(v_1, \dots, v_d) \geqslant 2$, the algorithm does not terminate and after a finite number of steps, only the `reduction step` applies. Hence, with no loss of generality, in the sequel, we deal with vectors $\mathbf{v} \in \mathcal{O}_d^+$ for which only the `reduction step` applies, that is, such that $v_1^{(n)} \neq 0$ for all $n \geqslant 0$. Thus $\mathbf{v}^{(n+1)} = \mathbf{F}(\mathbf{v^{(n)}})$ for all n. Then, for all $n \in \mathbb{N}$, one has:

$$\Omega(\mathbf{v}) = \sum_{i=0}^{n-1} v_1^{(i)} + \Omega(\mathbf{v}^{(n)}). \tag{2}$$

Proposition 2.5 ([1]). *Let $\mathbf{v} \in \mathbb{R}^d$ be a non-zero vector, $\mu \in \mathbb{R}$ and $\omega \in \mathbb{R}$. One has $\|\mathbf{v}\|_\infty \leqslant \Omega(\mathbf{v}) \leqslant \|\mathbf{v}\|_1$.*

By (2), the sequence $\left(\sum_{i=1}^{n-1} v_1^{(i)}\right)_{n\geqslant 0}$ is bounded, and since it is also increasing, it tends to a finite limit as n tends to infinity. Thus, we get

$$\Omega(\mathbf{v}) = \sum_{n=0}^{\infty} v_1^{(i)} + \lim_{n\to\infty} \Omega(\mathbf{v}^{(n)}).$$

Moreover, $\left(\mathbf{v}^{(n)}\right)_{n\geqslant 0}$ is a bounded componentwise decreasing sequence which thus tends to a finite limit $\mathbf{v}^{(\infty)}$ as n tends to infinity; one has $v_i^{(\infty)} < v_i^{(n)}$ for all n and all i. The map $\Omega : \mathbb{R}^d \longrightarrow \mathbb{R}$ is not continuous and, in general, we have:

$$\Omega(\mathbf{v}) \neq \sum_{n=0}^{\infty} v_1^{(i)} + \Omega(\lim_{n\to\infty} \mathbf{v}^{(n)}) = \sum_{n=0}^{\infty} v_1^{(i)} + \Omega(\mathbf{v}^{(\infty)}).$$

Theorem 2.6 ([5]). *Given a non-zero vector $\mathbf{v} \in \mathcal{O}_+^d$ such that $v_1^{(n)} \neq 0$ for all $n \geqslant 0$, we have:*

$$\Omega(\mathbf{v}) = \frac{\|\mathbf{v}\|_1 - \|\mathbf{v}^{(\infty)}\|_1}{d-1} + \|\mathbf{v}^{(\infty)}\|_\infty.$$

Theorem 2.6 shows that one can possibly provide a close formula for $\Omega(\mathbf{v})$, as soon as one can explicitly compute $\mathbf{v}^{(\infty)}$. A priori, it would seem to be natural to expect $\lim_{n\to\infty} \mathbf{v}^{(n)} = \mathbf{0}$. Nevertheless, by taking triples for which $v_1 + \cdots + v_{d-1} < (d-2)\, v_d$, one obtains vectors for which $\lim_{n\to\infty} \mathbf{v}^{(n)} \neq \mathbf{0}$. More precisely, given $\mathbf{v} \in \mathcal{O}_d^+$, two cases occur:

$$\exists n_0 \in \mathbb{N}, \forall n \geqslant n_0,\ v_1^{(n)} + \cdots + v_{d-1}^{(n)} \leqslant (d-2)\, v_d^{(n)} \tag{3}$$

$$\forall n \in \mathbb{N},\ v_1^{(n)} + \cdots + v_{d-1}^{(n)} > (d-2)\, v_d^{(n)}. \tag{4}$$

It has been shown in [11] that the set of vectors satisfying $\forall n \in \mathbb{N},\ v_1^{(n)} + \cdots + v_{d-1}^{(n)} > (d-2)\, v_d^{(n)}$ coincides with the set of totally irrational vectors for which $\lim_{n\to\infty} \mathbf{v}^{(n)} = \mathbf{0}$, and that this subset is Lebesgue-negligible (see Figure 1). Consequently, for almost all vectors of $\mathcal{O}_d^+$, Condition (3) is satisfied and $\lim_{n\to\infty} \mathbf{v}^{(n)} \neq \mathbf{0}$. When Condition (4) is satisfied, we get $\Omega(\mathbf{v}) = \frac{\|\mathbf{v}\|_1}{d-1}$.

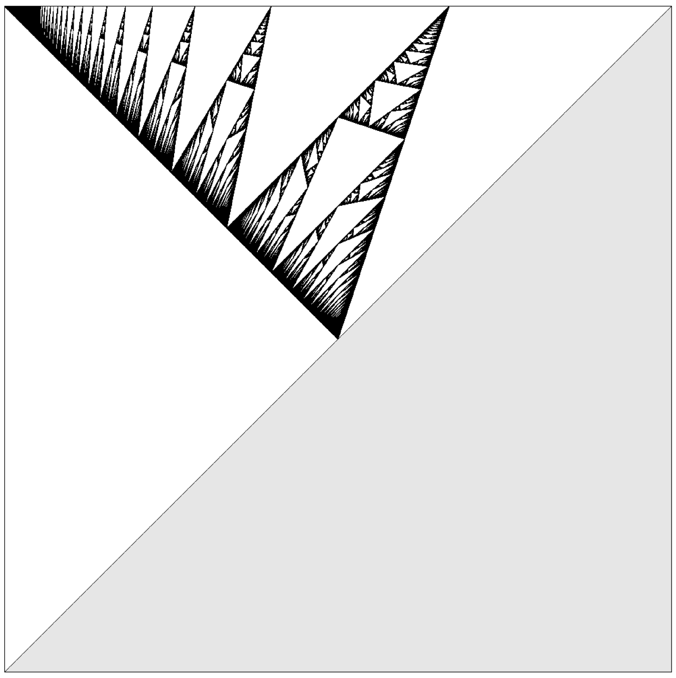

Figure 1. Pairs $(v_1/v_3, v_2/v_3)$ for vectors $\mathbf{v} = (v_1, v_2, v_3)$ with $0 < v_1 \leqslant v_2 \leqslant v_3$, satisfying Condition (4).

§ 3. The critical case

By definition of $\Omega(\mathbf{v})$, the hyperplane $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is not connected for any $\omega < \Omega(\mathbf{v}, \mu)$ while it is connected for any $\omega > \Omega(\mathbf{v}, \mu)$. The question which arises naturally is whether $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ is connected or not, or equivalently, whether the set $\{\omega \in \mathbb{R} \mid \mathbb{P}(\mathbf{v}, \mu, \omega) \text{ is connected}\}$ is an open or closed interval.

This question may be answered in most cases.

Theorem 3.1 ([5]).

- *If* $d = 2$ *then* $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ *is connected if and only if* $\dim_{\mathbb{Q}}\{v_1, v_2\} = 2$*, whatever* μ.
- *If* $d \geqslant 3$ *and Condition (3) holds, then* $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ *is not connected, whatever* μ.

The remaining cases are the ones where $d \geqslant 3$ and Condition (3) does not hold.

§ 3.1. A generation process

An archetype of the vectors satisfying Condition (4) is the vector $\mathbf{v} = (\alpha, \alpha + \alpha^2, \ldots, \alpha + \alpha^2 + \cdots + \alpha^{d-1}, 1)$ where α is the positive real root of $X + X^2 + \cdots + X^d = 1$. When $d = 3$, α is the inverse of the Tribonacci number. For all $d \geqslant 3$, we have $\alpha > 1/2$ so that $1 - \alpha < \alpha$. Thus, applying Algorithm (1) with $\mathbf{v}$ as input, we get:

$$\begin{aligned}\Omega(\mathbf{v}) &= \Omega(\alpha, \alpha+\alpha^2, \ldots, \alpha+\alpha^2+\cdots+\alpha^{d-1}, 1)\\ &= \alpha + \Omega(\alpha+\alpha^2-\alpha, \ldots, \alpha+\alpha^2+\cdots+\alpha^{d-1}-\alpha, 1-\alpha, \alpha)\\ &= \alpha + \Omega(\alpha^2, \alpha^2+\alpha^3, \alpha^2+\cdots+\alpha^d, \alpha)\\ &= \alpha + \Omega(\alpha\,\mathbf{v}) = \alpha + \alpha\,\Omega(\mathbf{v})\end{aligned}$$

Hence, $\Omega(\mathbf{v}) = \frac{\alpha}{1-\alpha}$ and $\mathbf{v}^{(n)} = \alpha^{n-1}\mathbf{v}$. Since $\dim_{\mathbb{Q}}(v_1, \ldots, v_d) > 1$, we have $\Omega(\mathbf{v}, \mu) = \Omega(\mathbf{v})$ for all $\mu \in \mathbb{R}$. Note also that $\mathbf{F}(\mathbf{v}) = \alpha\mathbf{v}$.

Let us now deal with the three-dimensional case. In the sequel, we show that for $\mathbf{v} = (\alpha, \alpha+\alpha^2, 1)$, where α is the inverse of the Tribonacci number, $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected and $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}, \mu))$ may be connected or not, depending on the value of μ.

To this aim, we build an increasing sequence $(\mathbb{P}_n)_{n\geqslant 0}$ of subsets of $\mathbb{Z}^3$, and we show that $\mathbb{P}_n$ is connected for all n (see Lemma 3.4 below). Then, it will be shown in Section 3.2 that its limit $\mathbb{P}_\infty = \lim_{n\to\infty} \mathbb{P}_n$ is actually equal to the whole plane $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$. This proves that $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected (Theorem 3.8). We also exhibit in Section 5 a value of μ for which $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is not connected.

Let $(\mathbf{t_n})_{n\geqslant 1}$ be the sequence of vectors of $\mathbb{Z}^3$ defined by $\mathbf{t_1} = \mathbf{e_1}$, $\mathbf{t_2} = \mathbf{e_2} - \mathbf{e_1}$, $\mathbf{t_3} = \mathbf{e_3} - \mathbf{e_2}$ and, for $n > 3$

$$\mathbf{t_n} = \mathbf{t_{n-3}} - \mathbf{t_{n-2}} - \mathbf{t_{n-1}}. \tag{5}$$

An easy induction shows that $\langle \mathbf{v}, \mathbf{t_n} \rangle = \alpha^n$ for all $n \geqslant 1$. Now we define a sequence $(\mathbb{P}_n)_{n\geqslant 0}$ of subsets of $\mathbb{Z}^3$ by $\mathbb{P}_0 = \{\mathbf{0}\}$ and $\mathbb{P}_n = \mathbb{P}_{n-1} \cup (\mathbb{P}_{n-1} + \mathbf{t_n})$ for $n \geqslant 1$ (see Figure 2). Since $(\mathbb{P}_n)_{n\geqslant 0}$ is an increasing sequence, it makes sense to consider its limit $\mathbb{P}_\infty$ as n tends to ∞.

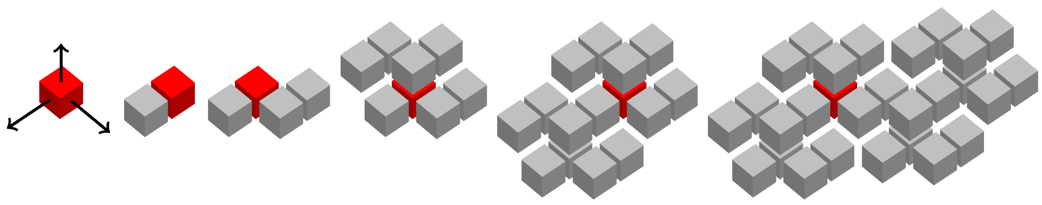

Figure 2. The sets P_0 through P_5. Points of $\mathbb{Z}^3$ are depicted as unit cubes (voxels).

The following lemma directly follows by induction from the definitions of the sequences $(\mathbb{P}_n)_{n\geqslant 0}$ and $(\mathbf{t_n})_{n\geqslant 0}$. The notation $\mathbb{N}^\star$ stands below for the set of positive integers.

Lemma 3.2. *One has*

$$\mathbb{P}_n = \left\{ \sum_{i\in I} \mathbf{t_i} \mid I \subset \{1, \ldots, n\} \right\} \text{ and } \mathbb{P}_\infty = \left\{ \sum_{i\in I} \mathbf{t_i} \mid I \subset \mathbb{N}^\star,\ |I| < \infty \right\}.$$

We need a first technical lemma.

Lemma 3.3. *For each $r \in \{1, 2, 3\}$ and all $q \geqslant 0$, we have*

$$\mathbf{e_r} = \sum_{i=1}^{r-1} \mathbf{t_i} + \sum_{k=1}^{q} (\mathbf{t_{3k+r-2}} + \mathbf{t_{3k+r-1}}) + \mathbf{t_{3q+r}}\,.$$

Proof. The proof works by induction. It is easily checked that the formula is valid for $q = 0$, *i.e.*, $\mathbf{e_r} = \mathbf{t_1} + \cdots + \mathbf{t_r}$. Let $q \geqslant 1$, and assume that the formula is valid for $q - 1$. From the definition of $\mathbf{t_n}$, we get

$$\begin{aligned}\mathbf{e_r} &= \sum_{i=1}^{r-1} \mathbf{t_i} + \sum_{k=1}^{q-1} (\mathbf{t_{3k+r-2}} + \mathbf{t_{3k+r-1}}) + \mathbf{t_{3q-3+r}} \\ &= \sum_{i=1}^{r-1} \mathbf{t_i} + \sum_{k=1}^{q-1} (\mathbf{t_{3k+r-2}} + \mathbf{t_{3k+r-1}}) + \mathbf{t_{3q-2+r}} + \mathbf{t_{3q-1+r}} + \mathbf{t_{3q+r}} \\ &= \sum_{i=1}^{r-1} \mathbf{t_i} + \sum_{k=1}^{q} (\mathbf{t_{3k+r-2}} + \mathbf{t_{3k+r-1}}) + \mathbf{t_{3q+r}},\end{aligned}$$

which ends the proof. □

Lemma 3.4. *The sets $\mathbb{P}_n$, for $n \in \mathbb{N}$, and $\mathbb{P}_\infty$ are connected.*

Proof. The proof works again by induction. One checks that $\mathbb{P}_0$, $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ are connected (see Figure 2). Let $n \geqslant 4$, and assume that $\mathbb{P}_{n-1}$ is connected. This implies that $\mathbb{P}_{n-1} + \mathbf{t_n}$ is also connected. Let us write $n = 3m + r$, with $r \in \{1, 2, 3\}$, and let us first prove that $\mathbf{e_r} \in \mathbb{P}_{n-1} \cap (\mathbb{P}_{n-1} + \mathbf{t_n})$. As soon as $n \geqslant 3$, we have $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\} \subset \mathbb{P}_n$. It thus remains to prove that $\mathbf{e_r} \in \mathbb{P}_{n-1} + \mathbf{t_n}$ or, equivalently, $\mathbf{e_r} - \mathbf{t_n} \in \mathbb{P}_{n-1}$. According to Lemma 3.3, we have $\mathbf{e_r} - \mathbf{t_n} = \mathbf{e_r} - \mathbf{t_{3m+r}} = \sum_{i=1}^{r-1} \mathbf{t_i} + \sum_{k=1}^{m} (\mathbf{t_{3k+r-2}} + \mathbf{t_{3k+r-1}})$ which belongs to $\mathbb{P}_{3m+r-1} = \mathbb{P}_{n-1}$ by Lemma 3.2, hence $\mathbf{e_r} \in \mathbb{P}_{n-1} \cap (\mathbb{P}_{n-1} + \mathbf{t_n})$. The sets $\mathbb{P}_{n-1}$ and $\mathbb{P}_{n-1} + \mathbf{t_n}$ are both connected, and they have a non-empty intersection (namely, it contains at least $\mathbf{e}_r$). One deduces that $\mathbb{P}_n = \mathbb{P}_{n-1} \cup (\mathbb{P}_{n-1} + \mathbf{t_n})$ is connected, which ends the induction proof. This immediately implies that $\mathbb{P}_\infty$ is also connected. □

Lemma 3.5. *One has $\mathbb{P}_\infty \subset \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$.*

Proof. We recall that $\Omega(\mathbf{v}) = \frac{\alpha}{1-\alpha}$. From Lemma 3.2, any $\mathbf{x} \in \mathbb{P}_\infty$ may be written as $\mathbf{x} = \sum_{i \in I} \mathbf{t_i}$ where I is a finite subset of $\mathbb{N}^\star$. Then $\langle \mathbf{v}, \mathbf{x} \rangle = \sum_{i \in I} \langle \mathbf{v}, \mathbf{t_i} \rangle = \sum_{i \in I} \alpha^i \in [0; \frac{\alpha}{1-\alpha}[$. Therefore $\mathbf{x} \in \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$. □

§ 3.2. Finite β-expansions and connectedness of $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$

In order to prove that $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected, we first prove that $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v})) = \mathbb{P}_\infty$. Since we already obtained $\mathbb{P}_\infty \subset \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$, it remains to prove $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v})) \subset \mathbb{P}_\infty$.

Given an element $\mathbf{x} \in \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$, we have $\langle \mathbf{v}, \mathbf{x} \rangle \in \mathbb{Z}[\alpha] \cap [0; \frac{\alpha}{1-\alpha}[$. Moreover, since α is an algebraic number of degree 3, we have $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 3$, so that $\langle \mathbf{v}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{y} \rangle \iff \mathbf{x} = \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$.

If we are able to write $\langle \mathbf{v}, \mathbf{x}\rangle$ as $\sum_{i\in I} \alpha^i$, where I is a finite subset of $\mathbb{N}^\star$, then we have $\langle \mathbf{v}, \mathbf{x}\rangle = \sum_{i\in I}\langle \mathbf{v}, \mathbf{t_i}\rangle = \langle \mathbf{v}, \sum_{i\in I} \mathbf{t_i}\rangle$. Hence $\mathbf{x} = \sum_{i\in I} \mathbf{t_i} \in \mathbb{P}_{\max(I)} \subseteq \mathbb{P}_\infty$. This type of representation (namely, as $\sum_{i\in I} \alpha^i$) is natural in the framework of the Tribonacci numeration, that is, when considering β-expansions in β-numeration, with $\beta = 1/\alpha$, by noticing that $\sum_{i\in I} \alpha^i = \sum_{i\in I} \beta^{-i}$ (the set I is finite). In fact, the following finiteness property (see Theorem 3.6 below) that is known to hold for the Tribonacci numeration will prove to be crucial for the following: the finiteness property states indeed that the set of real numbers having a finite β-expansion coincides with $\mathbb{Z}[\beta^{-1}] \cap \mathbb{R}^+ = \mathbb{Z}[\alpha] \cap \mathbb{R}^+$. For more details on β-numeration, the reader is referred for instance to [2].

Theorem 3.6 ([7]). *Let β be an algebraic integer with minimal polynomial $X^d - a_{d-1} X^{d-1} - \cdots - a_1 X - a_0$, and assume that $a_{d-1} \geqslant a_{d-2} \geqslant \cdots \geqslant a_1 \geqslant a_0 > 0$. Then,*

$$\left\{\sum_{i\in I} x_i\, \beta^i \,\middle|\, I \subset \mathbb{Z},\ |I| < \infty,\ \forall i\ x_i \in \{0, \ldots, \lceil \beta \rceil - 1\}\right\} = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}^+.$$

The minimal polynomial of $1/\alpha$ is $X^3 - X^2 - X - 1$ and satisfies the conditions of Theorem 3.6. Since $\alpha \in]1/2; 1[$, we have $\lceil 1/\alpha \rceil - 1 = 1$, that is, the digits belong here to $\{0, 1\}$. Let

$$\widetilde{\mathrm{Fin}}(1/\alpha) = \left\{\sum_{i\in I} \alpha^i \,\middle|\, I \subset \mathbb{Z},\ |I| < \infty\right\}.$$

Thus one has $\widetilde{\mathrm{Fin}}(1/\alpha) = \mathbb{Z}[\alpha] \cap \mathbb{R}^+$. Note that β-expansions correspond to sequences of digits where no three 1's in a row are allowed. Here, we do not have to consider any such restriction. Note also that the sets $\mathbb{P}_n$ provide approximations of Rauzy fractals (see Figure 3 below for an illustration). For more on Rauzy fractals, see for instance [2].

We are left to prove the following lemma:

Lemma 3.7. *One has*

$$\widetilde{\mathrm{Fin}}(1/\alpha) \cap \left[0; \frac{\alpha}{1-\alpha}\right[= \left\{\sum_{i\in I} \alpha^i \,\middle|\, I \subset \mathbb{N}^\star,\ |I| < \infty\right\}.$$

Proof. Let $\lambda \in \mathrm{Fin}(1/\alpha) \cap \left[0; \frac{\alpha}{1-\alpha}\right[$. We have $\lambda = \sum_{i=-m}^{n} \lambda_i\, \alpha^i$ where $\lambda_i \in \{0, 1\}$ for all $i \in \{-m, \ldots, n\}$. Since $1/\alpha > \frac{\alpha}{1-\alpha}$ we must have $\lambda_i = 0$ for all $i \leqslant -1$. Thus $\lambda = \sum_{i=0}^{n} \lambda_i\, \alpha^i$. If $\lambda_0 = 0$ then we are done. Otherwise, we may always write $\lambda = 1 + \sum_{j=1}^{k} \alpha^{3j} + \sum_{i=3k+1}^{n} \lambda_i\, \alpha^i$ for some $k \geqslant 0$. Consider the largest possible such k. Then $\lambda = \frac{1-\alpha^{3k+3}}{1-\alpha^3} + \sum_{i=3k+1}^{n} \lambda_i\, \alpha^i$. Since $\alpha^{3k+1} > \alpha^{3k+2}$ and $\frac{1-\alpha^{3k+3}}{1-\alpha^3} + \alpha^{3k+2} = (1 + \alpha^{3k+4})\frac{\alpha}{1-\alpha} > \frac{\alpha}{1-\alpha}$, we must have $\lambda_{3k+1} = \lambda_{3k+2} = 0$. Then we have also $\lambda_{3k+3} = 0$ otherwise k is not maximal. Finally, we get

$$\lambda = \frac{1-\alpha^{3k+3}}{1-\alpha^3} + \sum_{i=3k+4}^{n} \lambda_i\, \alpha^i = \alpha\frac{1-\alpha^{3k+3}}{1-\alpha} + \sum_{i=3k+4}^{n} \lambda_i\, \alpha^i = \sum_{i=1}^{3k+3} \alpha^i + \sum_{i=3k+4}^{n} \lambda_i\, \alpha^i.$$

□

We can now state the main result of the present section:

Theorem 3.8. *Let $\mathbf{v} = (\alpha, \alpha+\alpha^2, 1)$, where α is the real root of X^3+X^2+X-1. The discrete plane $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected.*

§ 4. On topological properties of $\mathbb{P}(\mathbf{v}, \mathbf{0}, \boldsymbol{\Omega}(\mathbf{v}))$

In the present section, we investigate two topological properties of $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$, with $\mathbf{v} = (\alpha, \alpha+\alpha^2, 1)$, where α is the real root of X^3+X^2+X-1. We first show that $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v})) \setminus \{\mathbf{0}\}$ has exactly 3 connected components, and then, we deduce that $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is a tree rooted at $\mathbf{0}$.

For $r \in \{1,2,3\}$ and $n \geqslant 0$, let us set:

$$\mathbb{K}_n^{(r)} = \left\{ \sum_{i \in I} \mathbf{t_i} \in \mathbb{Z}^d \,\middle|\, I \subset \{1,\ldots,n\},\ I \neq \emptyset,\ \max(I) \equiv r \pmod 3 \right\}.$$

For each $r \in \{1,2,3\}$, the sequence $(\mathbb{K}_n^{(r)})_{n\geqslant 0}$ is non-decreasing. Let $\mathbb{K}_\infty^{(r)} = \bigcup_{n=0}^{\infty} \mathbb{K}_n^{(r)}$ be the limit of the sequence $(\mathbb{K}_n^{(r)})_{n\geqslant 0}$. From the definition of $\mathbb{K}_n^{(r)}$ and from Lemma 3.2, we get:

Lemma 4.1.

- *For all $n \geqslant 0$, $\mathbb{P}_n = \{\mathbf{0}\} \cup \mathbb{K}_n^{(1)} \cup \mathbb{K}_n^{(2)} \cup \mathbb{K}_n^{(3)}$.*
- *Furthermore, $\mathbb{P}_\infty = \{\mathbf{0}\} \cup \mathbb{K}_\infty^{(1)} \cup \mathbb{K}_\infty^{(2)} \cup \mathbb{K}_\infty^{(3)}$.*

Figure 3 depicts $\mathbb{P}_n$, $\mathbb{K}_n^{(1)}$, $\mathbb{K}_n^{(2)}$ and $\mathbb{K}_n^{(3)}$ for $n = 11$.

Lemma 4.2.

1. *For each $r \in \{1,2,3\}$ and all $n \geqslant 0$, $\mathbb{K}_n^{(r)}$ is either empty, or connected and adjacent to $\{\mathbf{0}\}$.*
2. *For each $r \in \{1,2,3\}$, $\mathbb{K}_\infty^{(r)}$ is connected and adjacent to $\{\mathbf{0}\}$.*

Proof. We first prove, for $n \geqslant 1$, that if $n \not\equiv r \pmod 3$, then $\mathbb{K}_n^{(r)} = \mathbb{K}_{n-1}^{(r)}$, and if $n \equiv r \pmod 3$, then $\mathbb{K}_n^{(r)} = \mathbb{P}_{n-1} + \mathbf{t_n}$. Let us assume that $n \not\equiv r \pmod 3$. We obviously have $\mathbb{K}_{n-1}^{(r)} \subset \mathbb{K}_n^{(r)}$. Conversely, let $\mathbf{x} \in \mathbb{K}_n^{(r)}$. One has $\mathbf{x} = \sum_{i\in I} \mathbf{t_i}$ for some non-empty $I \subset \{1,\ldots,n\}$ such that $\max(I) \equiv r \pmod 3$. Since $n \not\equiv r \pmod 3$, we have $\max(I) < n$, so that $I \subset \{1,\ldots,n-1\}$, and $\mathbf{x} \in \mathbb{K}_{n-1}^{(r)}$. Hence $\mathbb{K}_n^{(r)} \subset \mathbb{K}_{n-1}^{(r)}$.

Assume now that $n \equiv r \pmod 3$. Let us prove that $\mathbb{K}_n^{(r)} = \mathbb{P}_{n-1} + \mathbf{t_n}$. Let $\mathbf{x} \in \mathbb{K}_n^{(r)}$. We have $\mathbf{x} = \sum_{i\in I} \mathbf{t_i}$ for some non-empty $I \subset \{1,\ldots,n\}$ such that $\max(I) \equiv r \equiv n \pmod 3$. Hence $\max(I) = n - 3p$ for some $p \geqslant 0$, and we may write $\mathbf{x} = \sum_{i=1}^{n-3p-1} \varepsilon_i \mathbf{t_i} + \mathbf{t_{n-3p}}$, where $\varepsilon_i \in \{0,1\}$ for all i. According to (5), we get $\mathbf{t_{n-3p}} = \sum_{k=0}^{p-1} (\mathbf{t_{n-3p+3k+1}} + \mathbf{t_{n-3p+3k+2}}) + \mathbf{t_n}$. Thus $\mathbf{x} = \mathbf{x}' + \mathbf{t_n}$ where $\mathbf{x}' = \sum_{i=1}^{n-3p-1} \varepsilon_i \mathbf{t_i} + \sum_{k=0}^{p-1} (\mathbf{t_{n-3p+3k+1}} + \mathbf{t_{n-3p+3k+2}}) \in \mathbb{P}_{n-1}$. Hence $\mathbb{K}_n^{(r)} \subset \mathbb{P}_{n-1} + \mathbf{t_n}$. Now, let $\mathbf{x} \in \mathbb{P}_{n-1} + \mathbf{t_n}$. We have $\mathbf{x} = \sum_{i\in I} \mathbf{t_i} + \mathbf{t_n}$ for some $I \subset \{1,\ldots,n-1\}$. Thus $\mathbf{x} = \sum_{i\in I'} \mathbf{t_i}$ where $I' = I \cup \{n\}$. We have $I' \subset \{1,\ldots,n\}$, $I' \neq \emptyset$ and $\max(I') = n \equiv r \pmod 3$. Hence $\mathbf{x} \in \mathbb{K}_n^{(r)}$, which concludes the proof of $\mathbb{K}_n^{(r)} = \mathbb{P}_{n-1} + \mathbf{t_n}$.

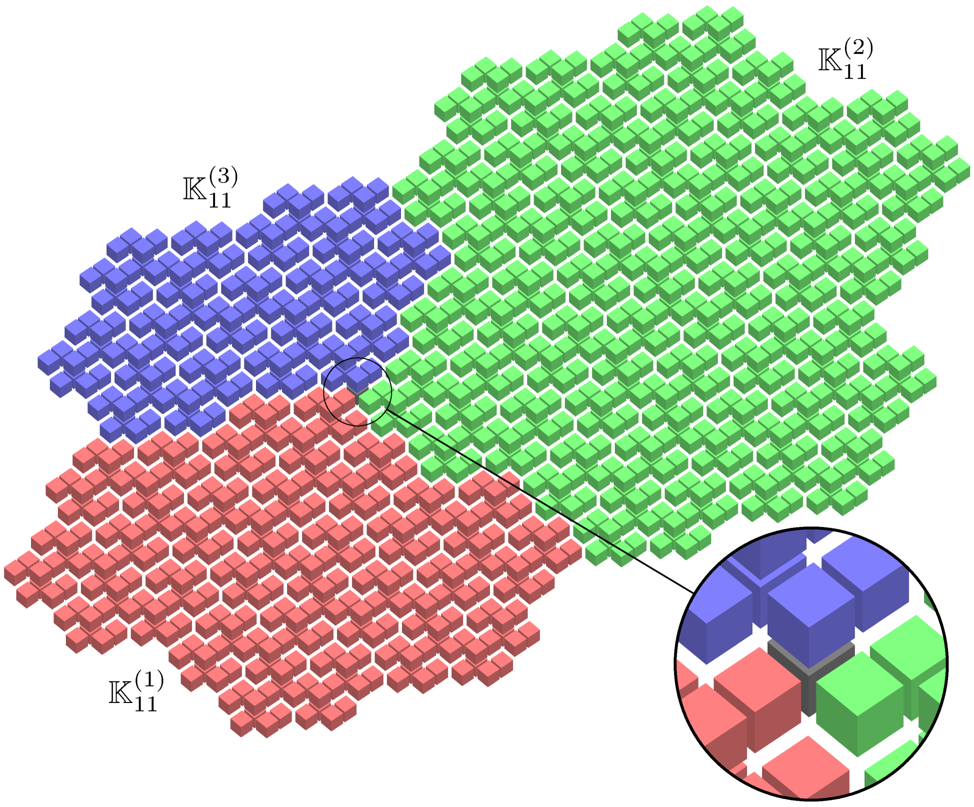

Figure 3. $\mathbb{P}_{11} = \{\mathbf{0}\} \cup \mathbb{K}_{11}^{(1)} \cup \mathbb{K}_{11}^{(2)} \cup \mathbb{K}_{11}^{(3)}$. The voxel which connects $\mathbb{K}_{11}^{(1)}$, $\mathbb{K}_{11}^{(2)}$ and $\mathbb{K}_{11}^{(3)}$ is the origin.

We now prove by induction on n the connectedness of $\mathbb{K}_n^{(r)}$. The result holds for $n = 0$. Let $n \geqslant 1$ and assume that the induction property holds for $\mathbb{K}_{n-1}^{(r)}$. If $n \not\equiv r \pmod 3$, then $\mathbb{K}_n^{(r)} = \mathbb{K}_{n-1}^{(r)}$, and the property holds for n by induction. If $n \equiv r \pmod 3$, then $\mathbb{K}_n^{(r)} = \mathbb{P}_{n-1} + \mathbf{t_n}$. Since $\mathbb{P}_{n-1}$ is connected, so is $\mathbb{P}_{n-1} + \mathbf{t_n}$, and the property holds again for n by induction.

As an immediate consequence we get the connectedness of $\mathbb{K}_\infty^{(r)}$.

Let us prove now that the $\mathbb{K}_n^{(r)}$ are adjacent to $\{\mathbf{0}\}$. We first show that $\mathbf{0} \notin \mathbb{K}_\infty^{(r)}$. Let $\mathbf{x} \in \mathbb{K}_\infty^{(r)}$. We have $\mathbf{x} = \sum_{i\in I} \mathbf{t_i}$ for some finite non-empty $I \subset \mathbb{N}^\star$. Then $\langle \mathbf{v}, \mathbf{x} \rangle = \sum_{i \in I} \langle \mathbf{v}, \mathbf{t_i} \rangle = \sum_{i\in I} \alpha^i > 0$. Thus $\mathbf{x} \neq \mathbf{0}$ which implies $\mathbf{0} \notin \mathbb{K}_n^{(r)}$ for all n.

We already proved in the proof of Lemma 3.4 that $\mathbf{e_r} \in \mathbb{P}_{n-1} + \mathbf{t_n}$ if $n \geqslant 1$ and $n \equiv r \pmod 3$. Since $\mathbf{e_r}$ is a neighbour of 0, $\mathbb{K}_n^{(r)}$ is adjacent to $\{\mathbf{0}\}$ as soon as it is non-empty. As a consequence, $\mathbb{K}_\infty^{(r)}$ is adjacent to $\{\mathbf{0}\}$. □

It remains to prove that, given n, the sets $\mathbb{K}_n^{(r)}$, with $r \in \{1, 2, 3\}$, are pairwise disjoint and non-adjacent (see Lemma 4.4 below). The disjointness will be provided by the following characterization: $\mathbf{x} \in \mathbb{K}_n^{(r)}$ if and only if there exists $p \in \mathcal{B}[X]$, with $\mathcal{B} = \{0, 1\}$, such that $p(\alpha) = \langle \mathbf{x}, \mathbf{v} \rangle$ and $\mathrm{d}^\circ p \equiv r \mod 3$. As for the non-adjacency, it will be given by the following fact: for $\mathbf{x} \in \mathbb{K}_n^{(r)}$, if $\mathbf{x} \pm \mathbf{e_i} \in \mathbb{K}_n^{(r')}$ with $r' \in \{1, 2, 3\}$, then $r = r'$.

Thus, we need the following technical lemma.

Lemma 4.3. *Let $\mathcal{B} = \{0, 1\}$ and $p, q \in \mathcal{B}[X]$. If $p(\alpha) = q(\alpha)$ then either $p = q = 0$ or*

$p \neq 0$, $q \neq 0$ *and* $\mathrm{d}^\circ p \equiv \mathrm{d}^\circ q \pmod 3$.

Proof. If $p = 0$ or $q = 0$, then obviously $p = q = 0$. Otherwise, we cannot have $p(\alpha) = q(\alpha)$. Now, let us assume $p \neq 0$ and $q \neq 0$. We prove the result by induction on $\max(\mathrm{d}^\circ p, \mathrm{d}^\circ q)$.

If $p(0) = q(0)$ then $p(\alpha) = q(\alpha) \iff p'(\alpha) = q'(\alpha)$ where $p' = (p - p(0))/X$ and $q' = (q - q(0))/X$. By the induction hypothesis, we have either $p' = q' = 0$, in which case $p = q = 1$, or $\mathrm{d}^\circ p' \equiv \mathrm{d}^\circ q' \pmod 3$, hence $\mathrm{d}^\circ p \equiv \mathrm{d}^\circ q \pmod 3$ since $\mathrm{d}^\circ p - \mathrm{d}^\circ q = \mathrm{d}^\circ p' - \mathrm{d}^\circ q'$.

We assume now $p(0) \neq q(0)$. By symmetry, we may assume $p(0) = 1$ and $q(0) = 0$. Let $p = 1 + \delta_1 X + \delta_2 X^2 + \delta_3 X^3 + X^4 p_0$ and $q = \varepsilon_1 X + \varepsilon_2 X^2 + \varepsilon_3 X^3 + X^4 q_0$ where $\delta_1, \delta_2, \delta_3, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathcal{B}$ and $p_0, q_0 \in \mathcal{B}[X]$. Then $p(\alpha) \geqslant 1 + \delta_1 \alpha + \delta_2 \alpha^2 + \delta_3 \alpha^3$ while $q(\alpha) < \varepsilon_1 \alpha + \varepsilon_2 \alpha^2 + \varepsilon_3 \alpha^3 + \alpha^4/(1-\alpha)$. Since $p(\alpha) = q(\alpha)$, we must have $\varepsilon_1 \alpha + \varepsilon_2 \alpha^2 + \varepsilon_3 \alpha^3 + \alpha^4/(1-\alpha) > 1 + \delta_1 \alpha + \delta_2 \alpha^2 + \delta_3^3 \alpha^3$, which is equivalent to $(\delta_1 + 1 - \varepsilon_1) + (\delta_2 + 1 - \varepsilon_2)\alpha + (\delta_3 + 1 - \varepsilon_3)\alpha^2 < \alpha^3/(1-\alpha)$. This implies $\delta_1 + 1 - \varepsilon_1 = 0$, $\delta_2 + 1 - \varepsilon_2 = 0$ and $\delta_3 + 1 - \varepsilon_3 \leqslant 1$. Therefore, $\delta_1 = \delta_2 = 0$, $\varepsilon_1 = \varepsilon_2 = 1$ and $\delta_3 \leqslant \varepsilon_3$. Thus $p = 1 + \delta_3 X^3 + X^4 p_0$ and $q = X + X^2 + \varepsilon_3 X^3 + X^4 q_0$. Now

$$\begin{aligned} p(\alpha) = q(\alpha) &\iff 1 + \delta_3 \alpha^3 + \alpha^4 p_0(\alpha) = \alpha + \alpha^2 + \varepsilon_3 \alpha^3 + \alpha^4 q_0(\alpha) \\ &\iff \alpha + \alpha^2 + \alpha^3 + \delta_3 \alpha^3 + \alpha^4 p_0(\alpha) = \alpha + \alpha^2 + \varepsilon_3 \alpha^3 + \alpha^4 q_0(\alpha) \\ &\iff (1 + \delta_3 - \varepsilon_3) + \alpha\, p_0(\alpha) = \alpha\, q_0(\alpha). \end{aligned}$$

Let $p' = (1 + \delta_3 - \varepsilon_3) + X p_0$ and $q' = X q_0$. We have $p', q' \in \mathcal{B}[X]$, $\mathrm{d}^\circ q \geqslant 2$, $\mathrm{d}^\circ q' < \mathrm{d}^\circ q$ and either $\mathrm{d}^\circ p' = \mathrm{d}^\circ p = 0$ or $\mathrm{d}^\circ p' < \mathrm{d}^\circ p$. In all cases, we have $\max(\mathrm{d}^\circ p', \mathrm{d}^\circ q') < \max(\mathrm{d}^\circ p, \mathrm{d}^\circ q)$. By the induction hypothesis, either $p' = q' = 0$, or $p' \neq 0$, $q' \neq 0$ and $\mathrm{d}^\circ p' \equiv \mathrm{d}^\circ q' \pmod 3$.

If $p' = q' = 0$, then $\delta_3 = 0$, $\varepsilon_3 = 1$ and $p_0 = q_0 = 0$. Therefore $p = 1$ and $q = X + X^2 + X^3$, so that $\mathrm{d}^\circ q = \mathrm{d}^\circ p + 3$ and $\mathrm{d}^\circ p \equiv \mathrm{d}^\circ q \pmod 3$.

In the second case, we have $q_0 \neq 0$ so that $\mathrm{d}^\circ q' = \mathrm{d}^\circ q_0 + 1$ and $\mathrm{d}^\circ q = \mathrm{d}^\circ q_0 + 4 = \mathrm{d}^\circ q' + 3 \equiv \mathrm{d}^\circ q' \pmod 3$. If $p_0 \neq 0$, then $\mathrm{d}^\circ p' = \mathrm{d}^\circ p_0 + 1$ and $\mathrm{d}^\circ p = \mathrm{d}^\circ p_0 + 4$, hence $\mathrm{d}^\circ p \equiv \mathrm{d}^\circ p' \pmod 3$. If $p_0 = 0$ and $\delta_3 = 1$, then $\varepsilon_3 = 1$. Thus $p = 1 + X^3$ and $p' = 1$. Again, we have $\mathrm{d}^\circ p \equiv \mathrm{d}^\circ p' \pmod 3$. At last, if $p_0 = 0$ and $\delta_3 = 0$, then $p = 1$ and $p' = 1 - \varepsilon_3$. We must have $\varepsilon_3 = 0$ because otherwise we would have $p = 1$, $q = X + X^2 + X^3 + X^4 q_0$ and $q(\alpha) = 1 + \alpha^4 q_0(\alpha) > 1 = p(\alpha)$, since $q_0 \neq 0$. Hence $p' = 1$ and $\mathrm{d}^\circ p = \mathrm{d}^\circ p'$. In all cases, we have $\mathrm{d}^\circ p \equiv \mathrm{d}^\circ p' \pmod 3$ and $\mathrm{d}^\circ q \equiv \mathrm{d}^\circ q' \pmod 3$, so that $\mathrm{d}^\circ p \equiv \mathrm{d}^\circ q \pmod 3$ by the induction hypothesis. □

Lemma 4.4.

1. *For all* $n \geqslant 0$, *the sets* $\mathbb{K}_n^{(1)}$, $\mathbb{K}_n^{(2)}$ *and* $\mathbb{K}_n^{(3)}$ *are pairwise disjoint and non-adjacent.*
2. *The sets* $\mathbb{K}_\infty^{(1)}$, $\mathbb{K}_\infty^{(2)}$ *and* $\mathbb{K}_\infty^{(3)}$ *are pairwise disjoint and non-adjacent.*

Proof. It is sufficient to prove the second assertion. The first one is an immediate consequence.

Since $\langle \mathbf{v}, \mathbf{t_i} \rangle = \alpha^i$ for all $i \geqslant 1$, from the definition of $\mathbb{K}_n^{(r)}$, if $\mathbf{x} \in \mathbb{K}_n^{(r)}$ then $\langle \mathbf{v}, \mathbf{x} \rangle = p(\alpha)$ where $p \in \mathcal{B}[X]$, $p \neq 0$, $p(0) = 0$ and $\mathrm{d}^\circ p \equiv r \pmod 3$. The fact that $\mathbb{K}_\infty^{(1)}$, $\mathbb{K}_\infty^{(2)}$ and $\mathbb{K}_\infty^{(3)}$ are pairwise disjoint is an immediate consequence of Lemma 4.3.

Now, we want to prove that $\mathbb{K}_\infty^{(1)}$, $\mathbb{K}_\infty^{(2)}$ and $\mathbb{K}_\infty^{(3)}$ are non-adjacent, that is, if $\mathbf{x} \in \mathbb{K}_\infty^{(r)}$, then $\mathbf{x}$ has no neighbour in $\mathbb{K}_\infty^{(r')}$ with $r' \neq r$. The neighbours of $\mathbf{x}$ are $\mathbf{x} \pm \mathbf{e_i}$ for $i = 1, 2, 3$. By symmetry, we may consider only neighbours of the form $\mathbf{x}+\mathbf{e_i}$. We prove that $\mathbf{x}\in\mathbb{K}_\infty^{(r)}$ and $\mathbf{x}+\mathbf{e_i}\in\mathbb{P}(\mathbf{v},0,\frac{\alpha}{1-\alpha})$ implies $\mathbf{x}+\mathbf{e_i}\in\mathbb{K}_\infty^{(r)}$.

If $\mathbf{x}\in\mathbb{K}_\infty^{(r)}$, then $\langle\mathbf{v},\mathbf{x}\rangle=p(\alpha)$ for some $p\in\mathcal{B}[X]$ such that $p\neq 0$, $p(0)=0$ and $\mathrm{d}^\circ p\equiv r \pmod 3$. If $\mathbf{x}+\mathbf{e_i}\in\mathbb{P}(\mathbf{v},0,\frac{\alpha}{1-\alpha})$ then $\langle\mathbf{v},\mathbf{x}+\mathbf{e_i}\rangle=p(\alpha)+\langle\mathbf{v},\mathbf{e_i}\rangle<\frac{\alpha}{1-\alpha}$. From Lemma 3.3, we have $\mathbf{e_i}=\sum_{j=1}^i\mathbf{t_j}$ so that $\langle\mathbf{v},\mathbf{e_i}\rangle=\sum_{j=1}^i\alpha^j=\alpha\,\frac{1-\alpha^i}{1-\alpha}$. Hence $p(\alpha)<\frac{\alpha}{1-\alpha}-\alpha\,\frac{1-\alpha^i}{1-\alpha}=\frac{\alpha^{i+1}}{1-\alpha}$. Let $p=\varepsilon_1 X+\cdots+\varepsilon_i X^i+X^{i+1}p'$ where $\varepsilon_1,\ldots,\varepsilon_i\in\mathcal{B}$ and $p'\in\mathcal{B}[X]$. Since $\alpha^{i-1}>\frac{\alpha^{i+1}}{1-\alpha}$ we must have $\varepsilon_j=0$ for $j<i$. Thus $p=\varepsilon_i X^i+X^{i+1}p'$.

If $\varepsilon_i=0$, then we are done. Indeed, $\langle\mathbf{v},\mathbf{x}+\mathbf{e_i}\rangle=q(\alpha)$ where $q=X+\cdots+X^i+X^{i+1}p'$. Since $p\neq 0$, we have $p'\neq 0$ and $\mathrm{d}^\circ q=\mathrm{d}^\circ p\equiv r \pmod 3$. Therefore, $\mathbf{x}+\mathbf{e_i}\in\mathbb{K}_\infty^{(r)}$.

If $\varepsilon_i=1$, then let us write p as $p=X^i\,(\sum_{j=0}^k X^{3j}+\delta_0 X^{3k+1}+\delta_1 X^{3k+2}+\delta_2 X^{3k+3}+X^{3k+4}p'')$ for some $k\geqslant 0$, $\delta_0,\delta_1,\delta_2\in\mathcal{B}$ and $p''\in\mathcal{B}[X]$. Consider the maximum suitable k. We get

$$p(\alpha)<\frac{\alpha^{i+1}}{1-\alpha}\iff \delta_0+\delta_1\,\alpha+\delta_2\,\alpha^2+\alpha^3p''(\alpha)<\frac{\alpha^3}{1-\alpha},$$

which implies $\delta_0=\delta_1=0$. We must also have $\delta_2=0$ because otherwise k is not maximal. Finally, we have

$$\begin{aligned}\langle\mathbf{v},\mathbf{x}+\mathbf{e_i}\rangle&=\alpha^i\left(\sum_{j=0}^k\alpha^{3j}+\alpha^{3k+4}\,p''(\alpha)\right)+\alpha\,\frac{1-\alpha^i}{1-\alpha}\\&=\alpha\,\frac{1-\alpha^i}{1-\alpha}+\alpha^i\left(\frac{1-\alpha^{3k+3}}{1-\alpha^3}+\alpha^{3k+4}\,p''(\alpha)\right)\\&=\alpha\,\frac{1-\alpha^i}{1-\alpha}+\alpha^i\left(\alpha\frac{1-\alpha^{3k+3}}{1-\alpha}+\alpha^{3k+4}\,p''(\alpha)\right)\\&=\alpha\,\frac{1-\alpha^{3k+i+3}}{1-\alpha}+\alpha^{3k+i+4}\,p''(\alpha)\\&=q(\alpha),\end{aligned}$$

where $q=\sum_{j=1}^{3k+i+3}X^j+X^{3k+i+4}\,p''$. If $p''=0$ then $\mathrm{d}^\circ p=3k+i$ and $\mathrm{d}^\circ q=3k+i+3$. If $p''\neq 0$ then $\mathrm{d}^\circ p=\mathrm{d}^\circ q=\mathrm{d}^\circ p''+3k+i+4$. In all cases, we have

$$\mathrm{d}^\circ q\equiv\mathrm{d}^\circ p\equiv r \pmod 3,$$

so that $\mathbf{x}+\mathbf{e_i}\in\mathbb{K}_\infty^{(r)}$. □

Hence we deduce that

Theorem 4.5. *The set $\mathbb{P}_\infty\setminus\{\mathbf{0}\}$ has exactly 3 connected components.*

From Theorem 4.5, one also deduces the following corollary which is illustrated by Figure 4.

Corollary 4.6. *The set $\mathbb{P}_\infty$ is a tree rooted at $\mathbf{0}$,* i.e., *$\mathbb{P}_\infty$ is cycle-free.*

Proof. The proof works by contradiction. Assume that $\mathbb{P}_\infty$ contains a cycle. Let $\mathcal{C}$ be such a cycle of minimal length, and n be the smallest index such that $\mathcal{C}$ appears in $\mathbb{P}_n$.

The cycle $\mathcal{C}$ cannot be contained in some $\mathbb{K}_n^{(r)}$. Indeed, if $n \not\equiv r \pmod 3$ then $\mathbb{K}_n^{(r)} = \mathbb{K}_{n-1}^{(r)} \subset \mathbb{P}_{n-1}$. This implies that $\mathcal{C}$ is contained in $\mathbb{P}_{n-1}$, which contradicts the minimality of n. Thus we have $n \equiv r \pmod 3$, and $\mathbb{K}_n^{(r)} = \mathbb{P}_{n-1} + \mathbf{t_n}$, according to the proof of Lemma 4.2. Hence, $\mathcal{C} - \mathbf{t_n}$ is contained in $\mathbb{P}_{n-1}$, which again contradicts the minimality of n.

Since $\mathcal{C}$ cannot be contained in some $\mathbb{K}_n^{(r)}$ it must go at least twice through some $\mathbf{e_r}$ which is the unique neighbour of $\mathbf{0}$ in $\mathbb{K}_\infty^{(r)}$. Then a shortest cycle exists, which contradicts the minimality of the length of $\mathcal{C}$. □

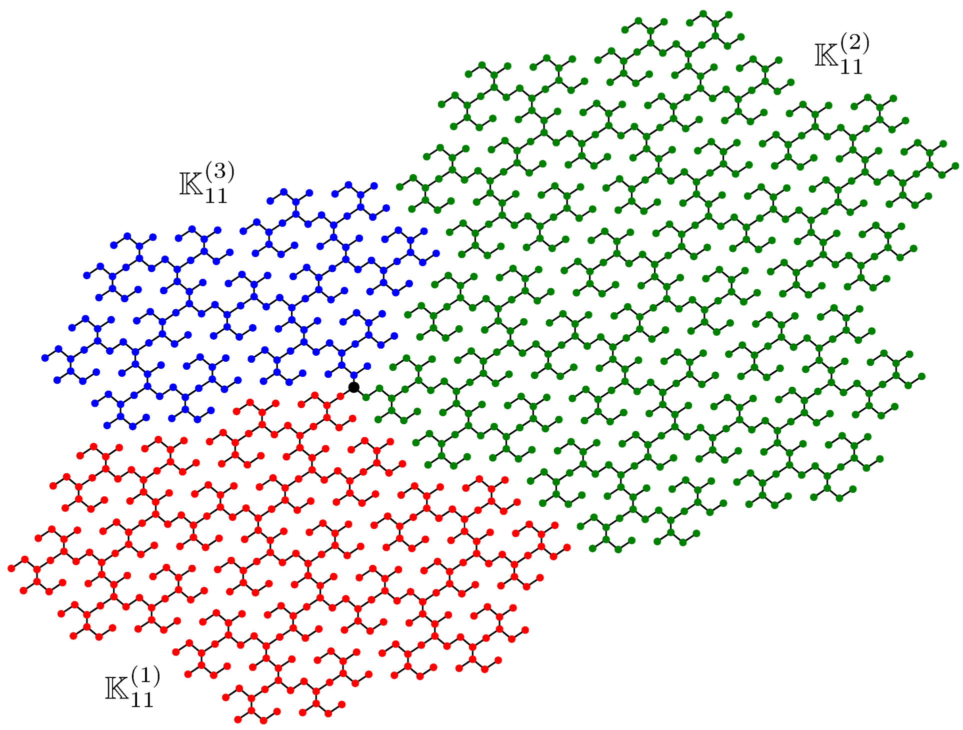

Figure 4. The set $\mathbb{P}_{11}$. In order to enlight the tree structure the point 0 is shown in black and adjacent points are connected.

§ 5. Discrete planes with non-zero shift

Until now, in the present paper, we have only considered discrete planes with shift $\mu = 0$. In particular, we have shown that if $\mathbf{v} = (\alpha, \alpha^2 + \alpha, 1)$, where α is the (positive) real root of $X + X^2 + X^3 = 1$, then the discrete plane $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is connected. The following question arises naturally: given $\mu \neq 0$, what about the connectedness of $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$?

If $\mu = \langle \mathbf{v}, \mathbf{t} \rangle$ with $\mathbf{t} \in \mathbb{Z}^3$, then $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is connected since $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v})) = \mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v})) - \mathbf{t}$. The following theorem shows that $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ may be not connected for some μ.

Theorem 5.1. *Let $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$, where α is the real root of $X^3 + X^2 + X - 1$, and let $\mu = \frac{\alpha}{1-\alpha}$. Then, the discrete plane $\mathbb{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ is not connected.*

Proof. We have

$$\begin{aligned}\mathbb{P}(\mathbf{v},\mu,\Omega(\mathbf{v})) &= \left\{x \in \mathbb{Z}^3 \mid 0 \leqslant \langle \mathbf{v},\mathbf{x}\rangle + \frac{\alpha}{1-\alpha} < \frac{\alpha}{1-\alpha}\right\} \\ &= \left\{x \in \mathbb{Z}^3 \mid -\frac{\alpha}{1-\alpha} \leqslant \langle \mathbf{v},\mathbf{x}\rangle < 0\right\} \\ &= -\left\{x \in \mathbb{Z}^3 \mid 0 < \langle \mathbf{v},\mathbf{x}\rangle \leqslant \frac{\alpha}{1-\alpha}\right\}.\end{aligned}$$

We have already seen (see Section 3.1) that $\Omega(\mathbf{v}) = \frac{\alpha}{1-\alpha}$. According to Theorem 2.6, we have $\Omega(\alpha) = \frac{\|\mathbf{v}\|_1}{2}$. Hence, since $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 3$, we deduce $\Omega(\mathbf{v}) \notin \{\langle \mathbf{x},\mathbf{v}\rangle \mid \mathbf{x} \in \mathbb{Z}^3\}$.

Hence

$$\mathbb{P}(\mathbf{v},\mu,\Omega(\mathbf{v})) = -\left\{x \in \mathbb{Z}^3 \mid 0 < \langle \mathbf{v},\mathbf{x}\rangle < \frac{\alpha}{1-\alpha}\right\} = -(\mathbb{P}(\mathbf{v},0,\Omega(\mathbf{v})) \setminus \{\mathbf{0}\})$$

and $\mathbb{P}(\mathbf{v},0,\Omega(\mathbf{v})) \setminus \{\mathbf{0}\}$ is not connected by Corollary 4.6. □

Conjecture 5.2. *Let* $\mathbf{v} = (\alpha, \alpha+\alpha^2, 1) \in \mathbb{R}^3$ *where* α *is the (positive) real root of* $X + X^2 + X^3 = 1$*. Then,* $\mathbb{P}(\mathbf{v},\mu,\Omega(\mathbf{v}))$ *is connected if and only if* $\mu = \langle \mathbf{v},\mathbf{t}\rangle$ *for some* $\mathbf{t} \in \mathbb{Z}^3$*, that is,* $\mathbf{v} \in \alpha^2\mathbb{Z} + \alpha\mathbb{Z} + \mathbb{Z}$*.*

§ 6. Conclusion

In the present paper, we have considered discrete planes with critical thickness, that is, the thickness that corresponds to the limit case between connectedness and non-connectedness. For almost all parameters in dimension $d = 3$, critical planes are known to be not connected. We have focused here on a non-generic case, namely the Tribonacci case. We have shown that in the case of the Tribonacci discrete plane, depending on the value of the shift, the plane is connected or not. Moreover, we have shown that when the shift is zero, the neighbouring relation on the points of such a plane forms a tree.

This tree structure implies that there exists a unique path connecting any pair of points in this critical plane. Since this tree is formed by the three connected components of the plane deprived of the origin meeting at $\mathbf{0}$, all paths going from a connected component to another one must pass through the origin $\mathbf{0}$. As a consequence, the length of such a path may not be bounded by a function of the distance between its two ends.

Finally, throughout this work we have considered the connectedness of discrete planes with respect to one particular type of neighbourhood relation. It appears that the ordered fully subtractive continued fraction algorithm is an appropriate tool for this study. A natural question that arises is whether another choice of neighbourhood relation leads to similar results using another multidimensional continued fraction algorithm. Reversely, given a multidimensional continued fraction algorithms, does there exist a corresponding neighbourhood relation such that the critical thickness of discrete planes may be studied via this algorithm?

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