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Explicit formulae for sums of products of Cauchy numbers including poly-Cauchy numbers

By

Takao KOMATSU*

Abstract

Recently, K. Kamano studied sums of products of Bernoulli numbers including poly-Bernoulli numbers. A relation among these sums was given, and an explicit expression of sums of two products was also given, reduced to the famous Euler's formula. The concept of poly-Cauchy numbers is given by the author as a generalization of the classical Cauchy number and an analogue of poly-Bernoulli number. In this paper, we investigate sums of products of Cauchy numbers including poly-Cauchy numbers in order to give explicit expressions in any m products.

§ 1. Introduction

Cauchy numbers of the first kind c_n $(n = 0, 1, 2, \dots)$ are defined by the integral of the falling factorial:

$$c_n = \int_0^1 x(x-1)\cdots(x-n+1)dx$$

and the generating function of c_n is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^\infty c_n \frac{x^n}{n!}$$

([4, 16]). On the other hand, Bernoulli numbers B_n $(n = 0, 1, 2, \dots)$ are defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^\infty B_n \frac{x^n}{n!}\,.$$

The following identity on sums of two products of Bernoulli numbers is known as Euler's formula:

$$\sum_{i=0}^n \binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n-1)B_n \quad (n \ge 0)\,.$$

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*Graduate School of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan.
e-mail: komatsu@cc.hirosaki-u.ac.jp

Many generalizations of Euler's formula have been studied. For example, a formula of sums of N products of Bernoulli numbers ([5]) and a formula of sums of N products of Bernoulli polynomials ([3]) have been considered. Other types of sums of products have been also studied (e.g. [1, 2, 6, 15, 17, 18]). M. Kaneko ([11]) introduced the poly-Bernoulli numbers $B_n^{(k)}$ by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^\infty B_n^{(k)} \frac{x^n}{n!}\,,$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^\infty \frac{z^m}{m^k}$$

is the k-th polylogarithm function. If $k=1$, $B_n = (-1)^n B_n^{(1)}$ is the n-th classical Bernoulli number. Kamano ([8]) considered the sums of products of Bernoulli numbers including poly-Bernoulli numbers, $S_m^{(k)}(n)$ for $m \ge 1$, $k \ge 1$, $n \ge 0$

$$S_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}} \binom{n}{i_1,\dots,i_m} \underbrace{B_{i_1}\dots B_{i_{m-1}}}_{m-1} B_{i_m}^{(k)}\,,$$

where $\binom{n}{i_1,\dots,i_m}$ is the multinomial coefficient defined by

$$\binom{n}{i_1,\dots,i_m} = \frac{n!}{i_1!\dots i_m!}\,.$$

The following identity holds ([8, Theorem 1]).

Proposition 1.1. *For $k \in \mathbb{Z}$ and $m \ge 0$, we have*

$$\sum_{l=0}^m (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} S_{m+1}^{(k-l)}(n) = \begin{cases} n(n-1)\cdots(n-m+1)\displaystyle\sum_{l=0}^m \left\{ {m \atop l} \right\} B_{n-m+l}^{(k)} & (n \ge m);\\ 0 & (0 \le n \le m-1)\,. \end{cases}$$

Kamano also gave explicit formulae of $S_2^{(k)}(n)$ and $S_3^{(k)}(n)$.

Recently, the author ([12]) introduced *poly-Cauchy numbers* of the first kind $c_n^{(k)}$ ($n \ge 0$, $k \ge 1$) by

$$\underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1x_2\dots x_k)(x_1x_2\dots x_k - 1)\cdots(x_1x_2\dots x_k - n + 1)dx_1dx_2\dots dx_k\,.$$

The generating function of $c_n^{(k)}$ is given by

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^\infty c_n^{(k)} \frac{x^n}{n!}\,, \tag{1.1}$$

where
$$\mathrm{Lif}_k(z)=\sum_{m=0}^\infty\frac{z^m}{m!(m+1)^k}$$
is the k-th *polylogarithm factorial function*, which is also introduced in [12]. When $k=1$, $c_n=c_n^{(1)}$ is the n-th classical Cauchy number. Note that $c_n^{(k)}$ for $k\le 0$ are also defined by the generating function (1.1). In [14] the author investigated the sums of products of Cauchy numbers, including poly-Cauchy numbers, $T_m^{(k)}(n)$ for $m\ge 1$, $k\ge 1$, $n\ge 0$
$$T_m^{(k)}(n):=\sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}}\binom{n}{i_1,\dots,i_m}\underbrace{c_{i_1}\dots c_{i_{m-1}}}_{m-1}c_{i_m}^{(k)}\,.$$
$T_m^{(k)}(n)$ is an analogue of Kamano's $S_m^{(k)}(n)$. When $k=1$, $T_m^{(1)}(n)$ was studied by Zhao ([20]). The following identity holds ([14, Theorem 1]).

Proposition 1.2. *For $k\in\mathbb{Z}$ and $m\ge 0$, we have*
$$\sum_{l=0}^m(-1)^{m-l}\begin{bmatrix}m+1\\ l+1\end{bmatrix}T_{m+1}^{(k-l)}(n)$$
$$=\begin{cases}\displaystyle\sum_{l=0}^m\sum_{i=0}^{n-m}\frac{n!}{i!}\binom{l}{n-m-i}\left\{{m\atop l}\right\}c_{l+i}^{(k)} & (n\ge m);\\ 0 & (0\le n\le m-1)\,.\end{cases}$$

The author ([14]) also gave explicit formulae of $T_2^{(k)}(n)$ and $T_3^{(k)}(n)$. Kamano ([8]) mentioned that explicit formulae of $S_m^{(k)}(n)$ for $m\ge 4$ might be obtained but seemed to be complicated to describe. The author failed to obtain explicit formulae of $T_m^{(k)}(n)$ as well as $S_m^{(k)}(n)$ for $m\ge 4$ by Kamano's original method. Finally, we ([9]) are successful to obtain explicit formulae of $S_m^{(k)}(n)$ for any general $m\ge 2$ by a different method. In this paper, we give explicit formulae of $T_m^{(k)}(n)$ for any integer $m\ge 2$.

§ 2. Explicit formulae for $T_m^{(k)}(n)$

The generating function of $T_m^{(k)}(n)$ is given by
$$\left(\frac{x}{\ln(1+x)}\right)^{m-1}\mathrm{Lif}_k(\ln(1+x))=\sum_{n=0}^\infty T_m^{(k)}(n)\frac{x^n}{n!}\,.$$
Put
$$G_k(x):=\mathrm{Lif}_k(\ln(1+x))=\sum_{n=0}^\infty c_n^{(k)}\frac{x^n}{n!}\,.$$
Since
$$\mathrm{Lif}_1(z)=\frac{e^z-1}{z},\quad \mathrm{Lif}_0(z)=e^z\quad\text{and}\quad \mathrm{Lif}_{-1}(z)=(z+1)e^z\,,$$

we have

$$G_1(x) = \frac{x}{\ln(1+x)}, \quad G_0(x) = 1 + x \quad \text{and} \quad G_{-1}(x) = (1+x)(\ln(1+x)+1)\,. \tag{2.1}$$

Since

$$x^m \frac{d^l}{dx^l} G_k(x) = \sum_{i=0}^{\infty} c_{l+i}^{(k)} \frac{x^{m+i}}{i!} \quad (m, l \geq 0,\ k \geq 1)\,, \tag{2.2}$$

the coefficient of x^n in

$$x^m \frac{d^l}{dx^l} G_k(x)$$

is equal to

$$\begin{cases} \dfrac{c_{n-m+l}^{(k)}}{(n-m)!} & (n \geq m); \\ 0 & (0 \leq n \leq m-1)\,. \end{cases}$$

We need the following Lemma ([14, Lemma 1]).

Lemma 2.1. *For an integer k and a positive integer m, we have*

$$\left(\left\{ {m \atop m} \right\} \frac{d^m}{dx^m} + \left\{ {m \atop m-1} \right\} \frac{1}{1+x} \frac{d^{m-1}}{dx^{m-1}} + \cdots + \left\{ {m \atop 1} \right\} \frac{1}{(1+x)^{m-1}} \frac{d}{dx} \right) G_k(x)$$
$$= \frac{1}{(1+x)^m (\ln(1+x))^m} \sum_{l=0}^{m} (-1)^{m-l} \left[{m+1 \atop l+1} \right] G_{k-l}(x)\,.$$

In [14] explicit formulae of $T_2^{(k)}(n)$ and $T_3^{(k)}(n)$ were given. By using a different method, it is possible to express $T_m^{(k)}(n)$ for $m = 4, 5, \dots$. However, we can say much more as shown in the next general theorem. In general, we can obtain the following explicit expression of $T_m^{(k)}(n)$ for any general $m \geq 2$.

Theorem 2.2. *For $n \geq 0$ and $k > 0$ we have*

$$T_m^{(0)}(n) = T_{m-1}^{(1)}(n) + n T_{m-1}^{(1)}(n-1)\,, \tag{2.3}$$

$$T_m^{(k)}(n) = \sum_{r=0}^{m-2} (-1)^r \binom{n}{r} \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^i}{(i+1)^k} T_{m-r}^{(0)}(n-r)$$
$$+ \frac{(-1)^{m-1} n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k+m-2 \\ j_1, j_2, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \sum_{\kappa=0}^{m-1} P_{m,\kappa}(n) c_{n-\kappa}^{(j)}\,, \tag{2.4}$$

$$T_m^{(-k)}(n) = \sum_{r=0}^{m-2} (-1)^r \binom{n}{r} \sum_{i=0}^{r} \binom{r}{i} (-1)^i (i+1)^k T_{m-r}^{(0)}(n-r)$$
$$+ \frac{n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k-m+1 \\ j_1, j_2, \dots, j_{m-1} \geq 0}} 2^{j_2} 3^{j_3} \dots (m-1)^{j_{m-1}} \sum_{j=0}^{j_1} \sum_{\kappa=0}^{m-1} P_{m,\kappa}(n) c_{n-\kappa}^{(-j)}\,. \tag{2.5}$$

where

$$P_{m,\kappa}(n)=\sum_{t=0}^{\kappa}\left\{ m-1 \atop m-t-1 \right\}\binom{m-t-1}{m-\kappa-1}\frac{(n-m+1)!}{(n-m-\kappa+t+1)!}\quad(\kappa=0,1,\dots,m-2)$$

and

$$P_{m,m-1}(n)=\sum_{t=0}^{m-2}\left\{ m-1 \atop m-t-1 \right\}\frac{(n-m+1)!}{(n-2m+t+2)!}=(n-m+1)^{m-1}\,.$$

Remark. There are some alternative versions to express $T_m^{(k)}(n)$ explicitly. Since for non-negative integers r and k

$$\left\{ k+1 \atop r+1 \right\}=\frac{1}{r!}\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}(i+1)^k$$

([19]) and

$$\left\{ k \atop m-1 \right\}=\sum_{\substack{j_1+\cdots+j_{m-1}=k-m+1\\ j_1,\dots,j_{m-1}\ge 0}}1^{j_1}2^{j_2}\cdots(m-1)^{j_{m-1}}$$

([4, p.207]), the identities in above Theorem can be also written as

$$\begin{aligned}
T_m^{(k)}(n)&=\sum_{r=0}^{m-2}(-1)^r\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(i+1)^k}T_{m-r}^{(0)}(n-r)\\
&\quad+\frac{(-1)^n n!}{(n-m+1)!}\sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k+m-2\\ j_1,j_2,\dots,j_{m-1}\ge 1}}2^{-j_2}3^{-j_3}\dots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}\\
&\quad\times\sum_{\kappa=0}^{n-m+1}\left[n-m+1 \atop \kappa \right]\frac{(-1)^\kappa}{(m+\kappa)^j}\,,\\
T_m^{(-k)}(n)&=\sum_{r=0}^{m-2}(-1)^r\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}(-1)^i(i+1)^k T_{m-r}^{(0)}(n-r)\\
&\quad+\frac{(-1)^{n-m+1}n!}{(n-m+1)!}\sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k-m+1\\ j_1,j_2,\dots,j_{m-1}\ge 0}}2^{j_2}3^{j_3}\dots(m-1)^{j_{m-1}}\sum_{j=0}^{j_1}\\
&\quad\times\sum_{\kappa=0}^{n-m+1}\left[n-m+1 \atop \kappa \right](-1)^\kappa(m+\kappa)^j\\
&=\sum_{r=0}^{m-2}\frac{n!}{(n-r)!}\left\{ k+1 \atop r+1 \right\}T_{m-r}^{(0)}(n-r)\\
&\quad+\frac{(-1)^{n-m+1}n!}{(n-m+1)!}\sum_{j=0}^{k-m+1}\left\{ k-j \atop m-1 \right\}\sum_{\kappa=0}^{n-m+1}\left[n-m+1 \atop \kappa \right](-1)^\kappa(m+\kappa)^j\,.
\end{aligned}$$

§ 2.1. Proof of Theorem 2.2

Lemma 2.3. *For $r \ge 0$ we have*

$$\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{i+1-y}=\frac{r!}{(1-y)(2-y)\cdots(r+1-y)}\,.$$

Proof. The proof is done by induction on r. See also e.g. [7, (5.41)]. □

Lemma 2.4. *For $n \ge 0$ and $m \ge 1$ we have*

$$\frac{1}{n+1-y}=\sum_{r=0}^{m-1}(-1)^r\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{i+1-y}+\frac{(-1)^m\binom{n}{m}m!}{(1-y)\cdots(m-y)(n+1-y)}\,.$$

Proof. Put

$$f_n(m)=\sum_{r=0}^{m-1}\binom{n}{r}\frac{(-1)^r r!}{(1-y)\cdots(r+1-y)}+\frac{(-1)^m\binom{n}{m}m!}{(1-y)\cdots(m-y)(n+1-y)}\,.$$

When $m=1$, it is easy to see that $f_n(1)=1/(n+1-y)$. For $m \ge 1$

$$\begin{aligned}f_n(m+1)-f_n(m)&=\frac{(-1)^m\binom{n}{m}m!}{(1-y)\cdots(m+1-y)}+\frac{(-1)^{m+1}\binom{n}{m+1}(m+1)!}{(1-y)\cdots(m+1-y)(n+1-y)}\\&\quad-\frac{(-1)^m\binom{n}{m}m!}{(1-y)\cdots(m-y)(n+1-y)}\\&=\frac{(-1)^m\binom{n}{m}m!}{(1-y)\cdots(m+1-y)(n+1-y)}\left((n+1-y)-(m+1)\frac{n-m}{m+1}-(m+1-y)\right)\\&=0\,.\end{aligned}$$

Hence, for any integer $m \ge 1$, $f_n(m)=1/(n+1-y)$. Together with Lemma 2.3, the proof is done. □

Proposition 2.5. *For $k \ge 1$ and $m \ge 2$ we have*

$$\begin{aligned}G_k(x)&=\sum_{r=0}^{m-2}(1+x)\frac{\bigl(-\ln(1+x)\bigr)^r}{r!}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(i+1)^k}\\&\quad+\sum_{\substack{j_1+\cdots+j_{m-1}=k+m-2\\ j_1,\dots,j_{m-1}\ge 1}}2^{-j_2}3^{-j_3}\cdots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}\sum_{l=0}^{m-1}(-1)^l\begin{bmatrix}m\\ l+1\end{bmatrix}G_{j-l}(x)\,.\end{aligned}\tag{2.6}$$

Proof. We shall prove

$$\frac{1}{(n+1)^k}=\sum_{r=0}^{m-1}(-1)^r\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(i+1)^k}+\frac{1}{n+1}\sum_{\substack{j_1+\cdots+j_m\le k-1\\ j_1,\dots,j_m\ge 0}}\frac{(-1)^m\binom{n}{m}}{2^{j_2}\cdots m^{j_m}(n+1)^{j_1}}\,.\tag{2.7}$$

Taking the infinite sum $\sum_{k=1}^{\infty} y^{k-1}$ for small $y>0$, the left-hand side is equal to $1/(n+1-y)$. The right-hand side equals

$$\begin{aligned}
&\sum_{r=0}^{m-1}(-1)^r\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{i+1}\sum_{k=1}^{\infty}\left(\frac{y}{i+1}\right)^{k-1}\\
&\quad+\frac{1}{n+1}\sum_{j_1,\dots,j_m\ge 0}\frac{(-1)^m\binom{n}{m}}{2^{j_2}\cdots m^{j_m}(n+1)^{j_1}}\sum_{l=j_1+\dots+j_m}y^l\\
&=\sum_{r=0}^{m-1}(-1)^r\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{i+1-y}\\
&\quad+\frac{1}{n+1}\sum_{j_1,\dots,j_m\ge 0}\frac{(-1)^m\binom{n}{m}}{2^{j_2}\cdots m^{j_m}(n+1)^{j_1}}\frac{y^{j_1+\dots+j_m}}{1-y}\\
&=\sum_{r=0}^{m-1}(-1)^r\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{i+1-y}\\
&\quad+\frac{(-1)^m\binom{n}{m}m!}{(1-y)\dots(m-y)\cdot(n+1-y)}\,.
\end{aligned}$$

By Lemma 2.4, the identity (2.7) holds.

By taking the summation $\sum_{n=0}^{\infty}\big(\ln(1+x)\big)^n/n!$ on both sides of (2.7), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\big(\ln(1+x)\big)^n}{n!(n+1)^k}&=\sum_{r=0}^{m-1}\frac{(1+x)\big(-\ln(1+x)\big)^r}{r!}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(i+1)^k}\\
&\quad+\sum_{n=0}^{\infty}\sum_{\substack{j_1+\dots+j_m\le k-1\\ j_1,\dots,j_m\ge 0}}\frac{(-1)^m\binom{n}{m}}{2^{j_2}\cdots m^{j_m}(n+1)^{j_1}}\frac{\big(\ln(1+x)\big)^n}{(n+1)!}\,.
\end{aligned}\tag{2.8}$$

Note that $n+r$ is replaced by n and

$$1+x=\sum_{n=0}^{\infty}\frac{\big(\ln(1+x)\big)^n}{n!}\,.$$

Since

$$G_k(x)=\sum_{n=0}^{\infty}\frac{\big(\ln(1+x)\big)^n}{n!(n+1)^k}$$

and

$$\sum_{l=0}^{m}(-1)^l\begin{bmatrix}m+1\\ l+1\end{bmatrix}(n+1)^l=(-1)^m\binom{n}{m}m!$$

(see e.g. [7, (6.13)]), by replacing m by $m-1$, we get the result. □

Proof of Theorem 2.2. By (2.1) we have

$$\left(\frac{x}{\ln(1+x)}\right)^{m-1}G_0(x)=\left(\frac{x}{\ln(1+x)}\right)^{m-2}G_1(x)+x\left(\frac{x}{\ln(1+x)}\right)^{m-2}G_1(x)\,.$$

Hence, the identity (2.3) holds. We shall prove the identity (2.4). The identity (2.5) can be proven similarly.

Replacing m by $m-1$ and k by j in Lemma 2.1, and multiplying both sides by $(1+x)^{m-1}x^{m-1}$, we have

$$
\begin{aligned}
&\left(\frac{x}{\ln(1+x)}\right)^{m-1}\sum_{l=0}^{m-1}(-1)^{m-l-1}\begin{bmatrix} m\\ l+1\end{bmatrix}G_{j-l}(x)\\
&=(1+x)^{m-1}x^{m-1}\sum_{t=0}^{m-2}\left\{\begin{matrix} m-1\\ m-t-1\end{matrix}\right\}\frac{1}{(1+x)^t}\frac{d^{m-t-1}}{dx^{m-t-1}}G_j(x)\\
&=(1+x)^{m-1}x^{m-1}\sum_{t=0}^{m-2}\left\{\begin{matrix} m-1\\ m-t-1\end{matrix}\right\}\frac{1}{(1+x)^t}\sum_{n=0}^\infty c_{m+n-t-1}^{(j)}\frac{x^n}{n!}\\
&=\sum_{t=0}^{m-2}\left\{\begin{matrix} m-1\\ m-t-1\end{matrix}\right\}(1+x)^{m-t-1}\sum_{n=0}^\infty\frac{n!}{(n-m+1)!}c_{n-t}^{(j)}\frac{x^n}{n!}\\
&=\sum_{t=0}^{m-2}\left\{\begin{matrix} m-1\\ m-t-1\end{matrix}\right\}\sum_{r=0}^{m-t-1}\binom{m-t-1}{r}x^r\sum_{n=0}^\infty\frac{n!}{(n-m+1)!}c_{n-t}^{(j)}\frac{x^n}{n!}\\
&=\sum_{t=0}^{m-2}\left\{\begin{matrix} m-1\\ m-t-1\end{matrix}\right\}\sum_{r=0}^{m-t-1}\binom{m-t-1}{r}\sum_{n=0}^\infty\frac{n!}{(n-r-m+1)!}c_{n-r-t}^{(j)}\frac{x^n}{n!}\\
&=\sum_{t=0}^{m-2}\left\{\begin{matrix} m-1\\ m-t-1\end{matrix}\right\}\sum_{\kappa=0}^{m-1}\binom{m-t-1}{m-\kappa-1}\sum_{n=0}^\infty\frac{n!}{(n-\kappa+t-m+1)!}c_{n-\kappa}^{(j)}\frac{x^n}{n!}\quad(\kappa=r+t)\\
&=\sum_{n=0}^\infty\frac{n!}{(n-m+1)!}\sum_{\kappa=0}^{m-1}P_{m,\kappa}(n)c_{n-\kappa}^{(j)}\frac{x^n}{n!}\,.
\end{aligned}
$$

Thus, by Proposition 2.5 we obtain

$$
\begin{aligned}
&\left(\frac{x}{\ln(1+x)}\right)^{m-1}G_k(x)\\
&=\sum_{r=0}^{m-2}\frac{(-x)^r}{r!}\sum_{i=0}^r\binom{r}{i}\frac{(-1)^i}{(i+1)^k}\sum_{n=r}^\infty T_{m-r}^{(0)}(n-r)\frac{x^{n-r}}{(n-r)!}\\
&\quad+(-1)^{m-1}\left(\frac{x}{\ln(1+x)}\right)^{m-1}\sum_{\substack{j_1+\dots+j_{m-1}=k+m-2\\ j_1,\dots,j_{m-1}\ge 1}}2^{-j_2}3^{-j_3}\cdots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}\\
&\qquad\times\sum_{l=0}^{m-1}(-1)^{m-l-1}\begin{bmatrix} m\\ l+1\end{bmatrix}G_{j-l}(x)\,.
\end{aligned}
$$

Comparing the coefficients of $x^n/n!$, we have the identity (2.4). □

§ 2.2. Explicit formulae for $T_4^{(k)}(n)$

By using a little different method from those in [8] and [14], we may continue to obtain the explicit expressions of $T_m^{(k)}(n)$ for $m=4,5,\dots$. However, as special cases of Theorem 2.2, immediately we can get explicit expressions of $T_m^{(k)}(n)$ for any integer $m \ge 2$. Expressions of $T_2^{(k)}(n)$ and $T_3^{(3)}(n)$ are already known ([14]). Here are explicit formulae for $T_4^{(k)}(n)$.

Corollary 2.6. *For $n \ge 0$ and $k > 0$ we have*

$$T_4^{(0)}(n) = T_3^{(1)}(n) + nT_3^{(1)}(n-1)\,, \tag{2.9}$$

$$\begin{aligned} T_4^{(k)}(n) =& T_4^{(0)}(n) - \Bigl(1-\frac{1}{2^k}\Bigr)nT_3^{(0)}(n-1) + \Bigl(\frac{1}{2}-\frac{1}{2^k}+\frac{1}{2\cdot 3^k}\Bigr)n(n-1)T_2^{(0)}(n-2)\\ &- n(n-1)(n-2)\sum_{\mu=1}^{k}3^{\mu-k-1}\sum_{\nu=1}^{\mu}2^{\nu-\mu-1}\\ &\times\sum_{j=1}^{\nu}\bigl((n-3)^3c_{n-3}^{(j)} + (3n^2-15n+19)c_{n-2}^{(j)} + 3(n-2)c_{n-1}^{(j)} + c_n^{(j)}\bigr) \end{aligned} \tag{2.10}$$

$$\begin{aligned} T_4^{(-k)}(n) =& T_4^{(0)}(n) - \Bigl(1-2^k\Bigr)nT_3^{(0)}(n-1) + \Bigl(\frac{1}{2}-2^k+\frac{3^k}{2}\Bigr)n(n-1)T_2^{(0)}(n-2)\\ &+ n(n-1)(n-2)\sum_{\mu=0}^{k-3}2^{k-\mu-3}\sum_{\nu=0}^{\mu}3^{\mu-\nu}\\ &\times\sum_{j=0}^{\nu}\bigl((n-3)^3c_{n-3}^{(-j)} + (3n^2-15n+19)c_{n-2}^{(-j)} + 3(n-2)c_{n-1}^{(-j)} + c_n^{(-j)}\bigr). \end{aligned} \tag{2.11}$$

§ 3. Poly-Cauchy numbers of the second kind

Similarly to $T_m^{(k)}(n)$, define $\hat{T}_m^{(k)}(n)$ by

$$\hat{T}_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}}\binom{n}{i_1,\dots,i_m}\underbrace{\hat{c}_{i_1}\cdots\hat{c}_{i_{m-1}}}_{m-1}\hat{c}_{i_m}^{(k)} \quad (m\ge 1, n\ge 0)\,,$$

where $\hat{c}_n^{(k)}$ are poly-Cauchy numbers of the second kind ([12]), defined by

$$\hat{c}_n^{(k)} = \underbrace{\int_0^1\dots\int_0^1}_{k}(-x_1x_2\dots x_k)(-x_1x_2\dots x_k-1)\dots(-x_1x_2\dots x_k-n+1)dx_1dx_2\dots dx_k.$$

The generating function of $\hat{c}_n^{(k)}$ is given by

$$\hat{G}_k(x) := \mathrm{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty}\hat{c}_n^{(k)}\frac{x^n}{n!}\,.$$

Note that $\hat{c}_n^{(k)}$ for $k \le 0$ are also defined by this generating function.

§ 3.1. Explicit formulae for $\hat{T}_m^{(k)}(n)$

Theorem 3.1. *For $n\ge 0$ and $k>0$ we have*

(3.1)
$$\hat{T}_m^{(0)}(n)=(-1)^n\sum_{l=0}^n(-1)^l\frac{n!}{l!}\hat{T}_{m-1}^{(1)}(l)\,,$$

(3.2)
$$\begin{aligned}\hat{T}_m^{(k)}(n)&=\hat{T}_m^{(0)}(n)+\sum_{r=1}^{m-2}\frac{1}{r!}\sum_{i=0}^r\binom{r}{i}\frac{(-1)^{i+n-r}}{(i+1)^k}\sum_{l=0}^{n-r}\binom{n-l-1}{r-1}(-1)^l\frac{n!}{l!}\hat{T}_{m-r}^{(0)}(l)\\&+(-1)^{m-1}n!\sum_{\substack{j_1+j_2+\dots+j_{m-1}=k+m-2\\ j_1,j_2,\dots,j_{m-1}\ge 1}}2^{-j_2}3^{-j_3}\dots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}\\&\times\left(\frac{\hat{c}_n^{(j)}}{(n-m+1)!}+\sum_{t=1}^{m-2}\left\{{m-1\atop m-t-1}\right\}\sum_{l=0}^{n-m+1}(-1)^{n-m-l+1}\binom{n+t-m-l}{t-1}\frac{\hat{c}_{m+l-t-1}^{(j)}}{l!}\right)\,,\end{aligned}$$

(3.3)
$$\begin{aligned}\hat{T}_m^{(-k)}(n)&=\hat{T}_m^{(0)}(n)+\sum_{r=1}^{m-2}\frac{1}{r!}\sum_{i=0}^r\binom{r}{i}(-1)^{i+n-r}(i+1)^k\sum_{l=0}^{n-r}\binom{n-l-1}{r-1}(-1)^l\frac{n!}{l!}\hat{T}_{m-r}^{(0)}(l)\\&+n!\sum_{\substack{j_1+j_2+\dots+j_{m-1}=k-m+1\\ j_1,j_2,\dots,j_{m-1}\ge 0}}2^{j_2}3^{j_3}\dots(m-1)^{j_{m-1}}\sum_{j=0}^{j_1}\\&\times\left(\frac{\hat{c}_n^{(-j)}}{(n-m+1)!}+\sum_{t=1}^{m-2}\left\{{m-1\atop m-t-1}\right\}\sum_{l=0}^{n-m+1}(-1)^{n-m-l+1}\binom{n+t-m-l}{t-1}\frac{\hat{c}_{m+l-t-1}^{(-j)}}{l!}\right)\,.\end{aligned}$$

Remark. The identities in above Theorem can be also written as

$$\hat{T}_m^{(0)}(n)=(-1)^n\sum_{l=0}^n(-1)^l\frac{n!}{l!}\hat{T}_{m-1}^{(1)}(l)\,,$$

$$\begin{aligned}\hat{T}_m^{(k)}(n)&=\hat{T}_m^{(0)}(n)+\sum_{r=1}^{m-2}\frac{1}{r!}\sum_{i=0}^r\binom{r}{i}\frac{(-1)^{i}}{(i+1)^k}(-1)^{n-r}\sum_{l=0}^{n-r}\binom{n-l-1}{r-1}(-1)^l\frac{n!}{l!}\hat{T}_{m-r}^{(0)}(l)\\&+\frac{(-1)^{n-m+1}n!}{(n-m+1)!}\sum_{\substack{j_1+j_2+\dots+j_{m-1}=k+m-2\\ j_1,j_2,\dots,j_{m-1}\ge 1}}2^{-j_2}3^{-j_3}\dots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}\sum_{\lambda=0}^{n-m+1}\frac{1}{(m+\lambda)^j}\\&\times\sum_{\mu=0}^{n-m-\lambda+1}\binom{\lambda+\mu}{\lambda}\left[{n-m+1\atop \lambda+\mu}\right](m-1)^\mu\,,\end{aligned}$$

$$\hat{T}_m^{(-k)}(n)=\hat{T}_m^{(0)}(n)+\sum_{r=1}^{m-2}(-1)^n\left\{{k+1\atop r+1}\right\}\sum_{l=0}^{n-r}\binom{n-l-1}{r-1}(-1)^l\frac{n!}{l!}\hat{T}_{m-r}^{(0)}(l)$$
$$+\frac{(-1)^n n!}{(n-m+1)!}\sum_{j=0}^{k-m+1}\left\{{k-j\atop m-1}\right\}\sum_{\lambda=0}^{n-m+1}(m+\lambda)^j$$
$$\times\sum_{\mu=0}^{n-m-\lambda+1}\binom{\lambda+\mu}{\lambda}\left[{n-m+1\atop \lambda+\mu}\right](m-1)^\mu\,.$$

Proof. Notice that

$$\begin{aligned}\frac{x^{m-1}}{(1+x)^m\bigl(\ln(1+x)\bigr)^{m-r-1}}&=\left(\frac{x}{1+x}\right)^r\left(\frac{x}{(1+x)\ln(1+x)}\right)^{m-r-1}\frac{1}{1+x}\\&=\left(\frac{x}{1+x}\right)^r\sum_{n=0}^\infty T_{m-r}^{(0)}(n)\frac{x^n}{n!}\\&=x^r\left(\sum_{\mu=0}^\infty(-1)^\mu\binom{\mu+r-1}{\mu}x^\mu\right)\left(\sum_{l=0}^\infty T_{m-r}^{(0)}(l)\frac{x^l}{l!}\right)\\&=x^r\sum_{n=0}^\infty\sum_{l=0}^n(-1)^{n-l}\binom{n+r-l-1}{r-1}\frac{T_{m-r}^{(0)}(l)}{l!}x^n\\&=\sum_{n=0}^\infty\sum_{l=0}^n(-1)^{n-l}\binom{n+r-l-1}{r-1}(n+r)!\frac{T_{m-r}^{(0)}(l)}{l!}\frac{x^{n+r}}{(n+r)!}\\&=\sum_{n=0}^\infty\sum_{l=0}^{n-r}(-1)^{n-r-l}\binom{n-l-1}{r-1}\frac{n!}{l!}T_{m-r}^{(0)}(l)\frac{x^n}{n!}\,.\end{aligned}$$

Hence, similarly to the identity (2.6), for $k\ge 1$ and $m\ge 2$ we have

$$\hat{G}_k(x)=\hat{G}_0(x)+\sum_{r=1}^{m-2}\frac{\bigl(\ln(1+x)\bigr)^r}{(1+x)r!}\sum_{i=0}^r\binom{r}{i}\frac{(-1)^i}{(i+1)^k}\tag{3.4}$$
$$+\sum_{\substack{j_1+\cdots+j_{m-1}=k+m-2\\ j_1,\dots,j_{m-1}\ge 1}}2^{-j_2}3^{-j_3}\cdots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}\sum_{l=0}^{m-1}(-1)^l\left[{m\atop l+1}\right]\hat{G}_{j-l}(x)\,.$$

Now, we have

$$
\begin{aligned}
&\left(\frac{x}{(1+x)\ln(1+x)}\right)^{m-1}\sum_{l=0}^{m-1}(-1)^{m-l-1}\left[{m \atop l+1}\right]\hat G_{j-l}(x)\\
&=x^{m-1}\sum_{t=0}^{m-2}\left\{{m-1 \atop m-t-1}\right\}\frac{1}{(1+x)^t}\frac{d^{m-t-1}}{dx^{m-t-1}}\hat G_j(x)\\
&=x^{m-1}\sum_{t=0}^{m-2}\left\{{m-1 \atop m-t-1}\right\}\frac{1}{(1+x)^t}\sum_{n=0}^\infty \hat c_{m+n-t-1}^{(j)}\frac{x^n}{n!}\\
&=\sum_{t=0}^{m-2}\left\{{m-1 \atop m-t-1}\right\}\frac{1}{(1+x)^t}\sum_{n=0}^\infty \frac{n!}{(n-m+1)!}\hat c_{n-t}^{(j)}\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty \frac{n!}{(n-m+1)!}\hat c_{n}^{(j)}\frac{x^n}{n!}\\
&\quad+\sum_{t=1}^{m-2}\left\{{m-1 \atop m-t-1}\right\}\left(\sum_{\mu=0}^\infty(-1)^\mu\binom{t+\mu-1}{\mu}x^\mu\right)\left(\sum_{\nu=0}^\infty\frac{\nu!}{(\nu-m+1)!}\hat c_{\nu-t}^{(j)}\frac{x^\nu}{\nu!}\right)\\
&=\sum_{n=0}^\infty \frac{n!}{(n-m+1)!}\hat c_{n}^{(j)}\frac{x^n}{n!}\\
&\quad+\sum_{t=1}^{m-2}\left\{{m-1 \atop m-t-1}\right\}\sum_{n=0}^\infty\sum_{\nu=0}^n(-1)^{n-\nu}\binom{t+n-\nu-1}{t-1}\frac{n!}{(\nu-m+1)!}\hat c_{\nu-t}^{(j)}\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty n!\Bigg(\frac{\hat c_n^{(j)}}{(n-m+1)!}\\
&\quad+\sum_{t=1}^{m-2}\left\{{m-1 \atop m-t-1}\right\}\sum_{l=0}^{n-m+1}(-1)^{n-m-l+1}\binom{n+t-m-l}{t-1}\frac{\hat c_{m+l-t-1}^{(j)}}{l!}\Bigg)\frac{x^n}{n!}\,.
\end{aligned}
$$

By multiplying $\big(x/(1+x)\ln(1+x)\big)^{m-1}$ on both sides of (3.5) and comparing the coefficients of $x^n/n!$, we obtain the identity (3.2). The identity (3.3) is similarly proven. The identity (3.1) is clear. □

For Remark, we can insist that

$$
\begin{aligned}
&\frac{\hat c_n^{(j)}}{(n-m+1)!}+\sum_{t=1}^{m-2}\left\{{m-1 \atop m-t-1}\right\}\sum_{l=0}^{n-m+1}(-1)^{n-m-l+1}\binom{n+t-m-l}{t-1}\frac{\hat c_{m+l-t-1}^{(j)}}{l!}\\
&\qquad=\frac{(-1)^n}{(n-m+1)!}\sum_{\lambda=0}^{n-m+1}\frac{1}{(m+\lambda)^j}\sum_{\mu=0}^{n-m-\lambda+1}\binom{\lambda+\mu}{\lambda}\left[{n-m+1 \atop \lambda+\mu}\right](m-1)^\mu\,.
\end{aligned}
$$

By putting $m=4$ in Theorem 3.1, we get explicit formulae for $\hat T_4^{(k)}(n)$.

Corollary 3.2. *For $n \ge 0$ and $k > 0$ we have*

$$\hat{T}_4^{(0)}(n) = (-1)^n \sum_{l=0}^{n} (-1)^l \frac{n!}{l!} \hat{T}_3^{(1)}(l)\,, \tag{3.5}$$

$$\begin{aligned}\hat{T}_4^{(k)}(n) &= \hat{T}_4^{(0)}(n) - \Big(1 - \frac{1}{2^k}\Big)(-1)^{n-1} \sum_{l=0}^{n-1} (-1)^l \frac{n!}{l!} \hat{T}_3^{(0)}(l) \\ &\quad + \Big(\frac{1}{2} - \frac{1}{2^k} + \frac{1}{2 \cdot 3^k}\Big)(-1)^{n-2} \sum_{l=0}^{n-2} (n-l-1)(-1)^l \frac{n!}{l!} \hat{T}_2^{(0)}(l) \\ &\quad - n(n-1)(n-2) \sum_{\mu=1}^{k} 3^{\mu-k-1} \sum_{\nu=1}^{\mu} 2^{\nu-\mu-1} \sum_{j=1}^{\nu} \Big(\hat{c}_n^{(j)} \\ &\quad + 3(-1)^{n-3}(n-3)! \sum_{l=0}^{n-3} (-1)^l \frac{\hat{c}_{l+2}^{(j)}}{l!} + (-1)^{n-3}(n-3)! \sum_{l=0}^{n-3} (n-l-2)(-1)^l \frac{\hat{c}_{l+1}^{(j)}}{l!}\Big)\,,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\hat{T}_4^{(-k)}(n) &= \hat{T}_4^{(0)}(n) + \Big(1 - 2^k\Big)(-1)^{n-1} \sum_{l=0}^{n-1} (-1)^l \frac{n!}{l!} \hat{T}_3^{(0)}(l) \\ &\quad + \Big(\frac{1}{2} - 2^k + \frac{3^k}{2}\Big)(-1)^{n-2} \sum_{l=0}^{n-2} (n-l-1)(-1)^l \frac{n!}{l!} \hat{T}_2^{(0)}(l) \\ &\quad + n(n-1)(n-2) \sum_{\mu=0}^{k-3} 2^{k-\mu-3} \sum_{\nu=0}^{\mu} 3^{\mu-\nu} \sum_{j=0}^{\nu} \Big(\hat{c}_n^{(-j)} \\ &\quad + 3(-1)^{n-3}(n-3)! \sum_{l=0}^{n-3} (-1)^l \frac{\hat{c}_{l+2}^{(-j)}}{l!} + (-1)^{n-3}(n-3)! \sum_{l=0}^{n-3} (n-l-2)(-1)^l \frac{\hat{c}_{l+1}^{(-j)}}{l!}\Big).\end{aligned} \tag{3.7}$$

§ 4. Two kinds of poly-Cauchy numbers

Define $U_m^{(k)}(n)$ by

$$U_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}} \binom{n}{i_1,\dots,i_m} \underbrace{c_{i_1}\cdots c_{i_{m-1}}}_{m-1} \hat{c}_{i_m}^{(k)} \quad (m \ge 1, n \ge 0)\,.$$

Then we obtain the following ([14, Theorem 7]).

Proposition 4.1. *For an integer k and a non-negative integer m, we have*

$$\sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} U_{m+1}^{(k-l)}(n) = \begin{cases} \displaystyle\sum_{l=0}^{m} \sum_{i=0}^{n-m} \frac{n!}{i!} \binom{l}{n-m-i} \left\{ {m \atop l} \right\} \hat{c}_{l+i}^{(k)} & (n \ge m); \\ 0 & (0 \le n \le m-1)\,. \end{cases}$$

Define $V_m^{(k)}(n)$ by

$$V_m^{(k)}(n):=\sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m\ge 0}}\binom{n}{i_1,\dots,i_m}\underbrace{\hat c_{i_1}\cdots\hat c_{i_{m-1}}}_{m-1}c_{i_m}^{(k)}\quad(m\ge 1,n\ge 0)\,.$$

Then we obtain the following ([14, Theorem 10]).

Proposition 4.2. *For an integer k and a non-negative integer m, we have*

$$\sum_{l=0}^{m}(-1)^{m-l}\begin{bmatrix}m+1\\ l+1\end{bmatrix}V_{m+1}^{(k-l)}(n)$$
$$=\begin{cases}(-1)^{n-m}\displaystyle\sum_{l=0}^{m-1}\sum_{i=0}^{n-m}(-1)^i\frac{n!}{i!}\binom{n-l-i-1}{n-m-i}\left\{{m\atop l}\right\}c_{l+i}^{(k)}+\frac{n!}{(n-m)!}c_n^{(k)} & (n\ge m);\\ 0\qquad (0\le n\le m-1)\,. & \end{cases}$$

§ 4.1. Explicit formulae for $U_m^{(k)}(n)$

We can obtain explicit formulae for $U_m^{(k)}(n)$. The proof is similar to the case of $T_m^{(k)}(n)$ and is omitted.

Theorem 4.3. *For $n\ge 0$ and $k>0$ we have*

$$U_m^{(0)}(n)=U_{m-1}^{(1)}(n)\,,\tag{4.1}$$

$$U_m^{(k)}(n)=\sum_{r=0}^{m-2}\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(i+1)^k}U_{m-r}^{(0)}(n-r)\tag{4.2}$$
$$+\frac{(-1)^{m-1}n!}{(n-m+1)!}\sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k+m-2\\ j_1,j_2,\dots,j_{m-1}\ge 1}}2^{-j_2}3^{-j_3}\dots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}\sum_{\kappa=0}^{m-1}P_{m,\kappa}(n)\hat c_{n-\kappa}^{(j)}\,,$$

$$U_m^{(-k)}(n)=\sum_{r=0}^{m-2}\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}(-1)^i(i+1)^kU_{m-r}^{(0)}(n-r)\tag{4.3}$$
$$+\frac{n!}{(n-m+1)!}\sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k-m+1\\ j_1,j_2,\dots,j_{m-1}\ge 0}}2^{j_2}3^{j_3}\dots(m-1)^{j_{m-1}}\sum_{j=0}^{j_1}\sum_{\kappa=0}^{m-1}P_{m,\kappa}(n)\hat c_{n-\kappa}^{(-j)}\,.$$

where $P_{m,\kappa}(n)$ ($\kappa=0,1,\dots,m-2,m-1$) are given as in Theorem 2.2.

Remark. The identities in above Theorem can be also written as

$$U_m^{(k)}(n)=\sum_{r=0}^{m-2}\binom{n}{r}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(i+1)^k}U_{m-r}^{(0)}(n-r)$$
$$+\frac{(-1)^{n-m+1}n!}{(n-m+1)!}\sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k+m-2\\ j_1,j_2,\dots,j_{m-1}\ge 1}}2^{-j_2}3^{-j_3}\dots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}$$
$$\times\sum_{\kappa=0}^{n-m+1}\left[{n-m+1\atop \kappa}\right]\frac{1}{(m+\kappa)^j},$$
$$U_m^{(-k)}(n)=\sum_{r=0}^{m-2}\frac{(-1)^r n!}{(n-r)!}\left\{{k+1\atop r+1}\right\}U_{m-r}^{(0)}(n-r)$$
$$+\frac{(-1)^{n-m}n!}{(n-m+1)!}\sum_{j=0}^{k-m+1}\left\{{k-j\atop m-1}\right\}\sum_{\kappa=0}^{n-m+1}\left[{n-m+1\atop \kappa}\right](m+\kappa)^j.$$

§ 4.2. Explicit formulae for $V_m^{(k)}(n)$

We can obtain explicit formulae for $V_m^{(k)}(n)$. The proof is similar to the case of $\hat{T}_m^{(k)}(n)$ and is omitted.

Theorem 4.4. *For $n\ge 0$ and $k>0$ we have*

(4.4)

$$V_m^{(0)}(n)=V_{m-1}^{(1)}(n),$$

(4.5)

$$V_m^{(k)}(n)=V_m^{(0)}(n)+\sum_{r=1}^{m-2}\frac{1}{r!}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^{i+n}}{(i+1)^k}\sum_{l=0}^{n-r}\binom{n-l-1}{r-1}(-1)^l\frac{n!}{l!}V_{m-r}^{(0)}(l)$$
$$+(-1)^{m-1}n!\sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k+m-2\\ j_1,j_2,\dots,j_{m-1}\ge 1}}2^{-j_2}3^{-j_3}\dots(m-1)^{-j_{m-1}}\sum_{j=1}^{j_1}$$
$$\times\left(\frac{\hat{c}_n^{(j)}}{(n-m+1)!}+\sum_{t=1}^{m-2}\left\{{m-1\atop m-t-1}\right\}\sum_{l=0}^{n-m+1}(-1)^{n-m-l+1}\binom{n+t-m-l}{t-1}\frac{\hat{c}_{m+l-t-1}^{(j)}}{l!}\right),$$

(4.6)

$$V_m^{(-k)}(n)=V_m^{(0)}(n)+\sum_{r=1}^{m-2}\frac{1}{r!}\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+n}(i+1)^k\sum_{l=0}^{n-r}\binom{n-l-1}{r-1}(-1)^l\frac{n!}{l!}V_{m-r}^{(0)}(l)$$
$$+n!\sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k-m+1\\ j_1,j_2,\dots,j_{m-1}\ge 0}}2^{j_2}3^{j_3}\dots(m-1)^{j_{m-1}}\sum_{j=0}^{j_1}$$
$$\times\left(\frac{\hat{c}_n^{(-j)}}{(n-m+1)!}+\sum_{t=1}^{m-2}\left\{{m-1\atop m-t-1}\right\}\sum_{l=0}^{n-m+1}(-1)^{n-m-l+1}\binom{n+t-m-l}{t-1}\frac{\hat{c}_{m+l-t-1}^{(-j)}}{l!}\right).$$

Remark. The identities in above Theorem can be also written as

$$
\begin{aligned}
V_m^{(k)}(n) =& V_m^{(0)}(n) + \sum_{r=1}^{m-2} \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^i}{(i+1)^k} (-1)^n \sum_{l=0}^{n-r} \binom{n-l-1}{r-1} (-1)^l \frac{n!}{l!} V_{m-r}^{(0)}(l) \\
&+ \frac{(-1)^{n-m+1} n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\cdots+j_{m-1}=k+m-2 \\ j_1,j_2,\dots,j_{m-1}\ge 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \\
&\times \sum_{\lambda=0}^{n-m+1} \frac{(-1)^{\lambda+1}}{(m+\lambda)^j} \sum_{\mu=0}^{n-m-\lambda+1} \binom{\lambda+\mu}{\lambda} \left[{n-m+1 \atop \lambda+\mu} \right] (m-1)^{\mu}, \\
V_m^{(-k)}(n) =& V_m^{(0)}(n) + \sum_{r=1}^{m-2} (-1)^{n-r} \left\{ {k+1 \atop r+1} \right\} \sum_{l=0}^{n-r} \binom{n-l-1}{r-1} (-1)^l \frac{n!}{l!} V_{m-r}^{(0)}(l) \\
&+ \frac{(-1)^n n!}{(n-m+1)!} \sum_{j=0}^{k-m+1} \left\{ {k-j \atop m-1} \right\} \sum_{\lambda=0}^{n-m+1} (m+\lambda)^j (-1)^{\lambda+1} \\
&\times \sum_{\mu=0}^{n-m-\lambda+1} \binom{\lambda+\mu}{\lambda} \left[{n-m+1 \atop \lambda+\mu} \right] (m-1)^{\mu}.
\end{aligned}
$$

References

[1] Agoh, T. and Dilcher, K., Convolution identities and lacunary recurrences for Bernoulli numbers, *J. Number Theory*, **124** (2007), 105–122.

[2] Agoh, T. and Dilcher, K., Higher-order recurrences for Bernoulli numbers, *J. Number Theory*, **129** (2009), 1837–1847.

[3] Chen, K.-W., Sums of products of generalized Bernoulli polynomials, *Pacific J. Math.*, **208** (2003), 39–52.

[4] Comtet, L., *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[5] Dilcher, K., Sums of products of Bernoulli numbers, *J. Number Theory*, **60** (1996), 23–41.

[6] Eie, M., A note on Bernoulli numbers and Shintani generalized Bernoulli polynomials, *Trans. Amer. Math. Soc.*, **348** (1996), 1117–1136.

[7] Graham, R. L., Knuth, D. E. and Patashnik, O., *Concrete Mathematics*, Second Edition, Addison-Wesley, Reading, 1994.

[8] Kamano, K., Sums of products of Bernoulli numbers, including poly-Bernoulli numbers, *J. Integer Seq.*, **13** (2010), Article 10.5.2.

[9] Kamano, K. and Komatsu, T., Explicit formulae for sums of products of Bernoulli polynomials, including poly-Bernoulli polynomials, *Ramanujan J.*, (to appear) DOI 10.1007/s11139-013-9509-8.

[10] Kamano, K. and Komatsu, T., Poly-Cauchy polynomials, *Mosc. J. Comb. Number Theory*, **3** (2013), 183–209.

[11] Kaneko, M., Poly-Bernoulli numbers, *J. Th. Nombres Bordeaux*, **9** (1997), 221–228.

[12] Komatsu, T., Poly-Cauchy numbers, *Kyushu J. Math.*, **67** (2013), 143–153.

[13] Komatsu, T., Poly-Cauchy numbers with a q parameter, *Ramanujan J.*, **31** (2013), 353–371.

[14] Komatsu, T., Sums of products of Cauchy numbers, including poly-Cauchy numbers, J. Discrete Math. 2013 (2013), Article ID 373927, (10 pages). available at http://www.hindawi.com/journals/jdm/aip/373927/.

[15] Machide, T., Sums of products of Kronecker's double series, *J. Number Theory*, **128** (2008), 820–834.

[16] Merlini, D, Sprugnoli, R. and Verri, M. C., The Cauchy numbers *Discrete Math.*, **306** (2006), 1906–1920.
[17] Petojević, A., New sums of products of Bernoulli numbers, *Integral transforms Spec. Funct.*, **19** (2008), 105–114.
[18] Petojević, A. and Srivastava, H. M., Computation of Euler's type sums of the products of Bernoulli numbers, *Appl. Math. Lett.*, **22** (2009), 769–801.
[19] Sharp Jr., H., Cardinality of finite topologies, *J. Combinatorial Theory*, **5** (1968), 82–86.
[20] Zhao, F.-Z., Sums of products of Cauchy numbers, *Discrete Math.*, **309** (2009), 3830–3842.