RIMS Kôkyûroku Bessatsu
B46 (2014), 231–239

Structure of Classes of Circular Words defined by a Quadratic Equivalence

By

Benoît Rittaud*

Abstract

A *circular word* is a finite word such that its last letter is assumed to be followed by its first one. Assuming combinatorial constraints on such circular words gives rise to intersting algebraic structures, as shown in [5] and [4] in the case of the Fibonacci constraint. Here, we consider the case of equivalence classes of circular words defined by an equivalence relation derived from the polynomial $X^2 - kX - 1$ (with $k \geq 1$ integer). We also provide a link with spanning trees of graphs.

§ 1. Introduction

In [5], a *circular word* on an alphabet $\mathcal{A}$ is defined as a finite word whose last letter is assumed to be followed by the first one. A finite word W being given, $\widetilde{W}$ denotes the corresponding circular word. Combinatorial constraints on circular words leads to interesting group properties. The Fibonacci case ($\mathcal{A} = \{0, 1\}$, words with no factor equal to 11) is extensively studied in [5], as well as some applications. In the Fibonacci case, any circular word of even length is equivalent to a (essentially unique) admissible one (under the equivalence relation defined by $\dots \widehat{abc} \dots = \dots (a-1)\widehat{(b-1)}(c+1)\dots$). This fact is helpful to count its elements in an elementary way.

Here, the aim is to investigate the case of an equivalence between circular words derived from the polynomial $X^2 - kX - 1$ with $k \geq 1$ integer (the Fibonacci case corresponding to $k = 1$). Such an equivalence relation provides, for any circular word, an admissible form, essentially unique, in which no factor of the form mk appears for $m > 0$ (Theorem 3.1). We describe the group structure of such circular words, extending in a natural way the results obtained in [5] in the case $k = 1$ (Theorems 4.3 and 4.4). We also show how the assumption for circular words to be of even length, a necessary condition in the context of [5], can be removed (see the remark at the end of section 3.2).

In section 5, we mention a combinatorial interpretation of the sets of circular words with the quadratic equivalence $X^2 - kX - 1$, in terms of spanning trees of a particular family of graphs,

Received 10 October, 2012, Revised 17 November, 2013, Accepted 12 December, 2013.
2010 Mathematics Subject Classification(s): primary: 68R15; secondary: 05C30.
Key Words: finite word; finite abelian group; numeration system; spanning tree
*Université Paris-13, Sorbonne Paris Cité, LAGA, CNRS, UMR 7539, F-93430 Villetaneuse, France.
e-mail: `rittaud@math.univ-paris13.fr`

extending an observation made in the case $k = 1$ in [4]. Eventually, in section 6, we mention some open questions and perspectives.

§ 2. Definitions and notation

An alphabet and an integer $\ell > 0$ being given, a *(dotted) circular word of length ℓ* on this alphabet is an ordered set of letters indexed by $\mathbb{Z}/\ell\mathbb{Z}$. The letter indexed by the neutral element 0 of $\mathbb{Z}/\ell\mathbb{Z}$ is referred as the *initial letter* of the word. To avoid confusion with ordinary words of length ℓ, we write $\widetilde{W} = w_0\widetilde{\ldots w_{\ell-1}}$ for the circular word $\widetilde{W}$ of initial letter w_0 and corresponding to the word $W = w_0 \ldots w_{\ell-1}$. (Hence, $\widetilde{W} = \widetilde{W'}$ iff $W = W'$; note that, on the contrary, $\widetilde{00100}$ is not equal to $\widetilde{10000}$.) The *shift* on circular words is the transformation σ defined by $\sigma(w_0\widetilde{\ldots w_{\ell-1}}) = w_1\widetilde{\ldots w_{\ell-1}}w_\ell = w_1\widetilde{\ldots w_{\ell-1}}w_0$.

With the single exception of the end of section 3.2, all circular words considered in this article are defined on the alphabet $\mathbb{Z}$. Also, the value $k > 0$ is a fixed integer, and we put $\mathcal{A} = \{0, \ldots, k\}$.

Let

$$F_0 = 1 \qquad F_1 = k + 1 \qquad F_n = kF_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

For any circular word $\widetilde{W} = w_0\widetilde{\ldots w_{\ell-1}}$, we define

$$N(\widetilde{W}) = \sum_{0 \leq i < \ell} w_i F_i.$$

We say that a word W or a circular word $\widetilde{W}$ is *k-admissible* (or, simply, *admissible*) iff, for all i, we have $0 \leq w_i \leq k$ and $w_i = k \Rightarrow w_{i-1} = 0$. Recall that N, defined on the set of admissible finite words not ending with a 0, is bijective on $\mathbb{N}^*$. Its inverse function provides the greedy expansion of integers in the numeration system defined by the scale $(F_n)_{n\geq 0}$ (for a presentation of such kind of numeration systems, see the notion of *U-representation* given in [2]).

For any integer i, let $\tau_i = \sigma^{-i}\tau\sigma^i$, where τ is the transformation on circular words such that, for any $\widetilde{W} = w_0\widetilde{\ldots w_{\ell-1}}$ of length at least 3:

$$\tau(\widetilde{W}) = (w_0 + 1)w_1 \ldots w_{\ell-3}\widetilde{(w_{\ell-2} - k)(w_{\ell-1} - 1)}.$$

Write $\widetilde{W} \approx \widetilde{X}$ whenever $\widetilde{W}$ and $\widetilde{X}$ are equivalent under the action of the τ_is. For $\widetilde{W} \approx \widetilde{X}$ and $\widetilde{X}$ admissible, we say that $\widetilde{X}$ is an *admissible form* of $\widetilde{W}$.

§ 3. Admissible forms

Our first aim here is to prove the following result.

Theorem 3.1. *For any circular word $\widetilde{W}$ on $\mathbb{Z}$, there exists a unique admissible form of it, denoted by $Z(\widetilde{W})$, with the only exception of $\widetilde{0^{2\ell}} \approx \widetilde{(0k)^\ell} \approx \widetilde{(k0)^\ell}$.*

The first subsection is devoted to the existence part of the proof, the second subsection to the unicity part. Eventually, the third subsection deals with the difference between this theorem and Proposition 2.1 of [5], which states the same result for the particular case $k = 1$ but in a different way because of a more restrictive definition of circular words.

§ 3.1. Existence of an admissible form

Note first that the application $\tau_0^{-1} \circ \cdots \circ \tau_{\ell-1}^{-1}$ is the application that adds the value k to each letter of a given circular word. Hence, in the sequel, we are allowed to assume that $\widetilde{W}$ is a circular word on $\mathbb{N}$.

To prove the existence, it is sufficient to prove that for any admissible circular word $\widetilde{W} = w_0 \widetilde{\ldots w_{\ell-1}}$ and any i, the word $\widetilde{X} = x_0 \widetilde{\ldots x_{\ell-1}}$ obtained by replacing w_i by $w_i + 1$ in $\widetilde{W}$ has an admissible form. The circular word $\widetilde{X}$ is non-admissible iff $w_{i-1}w_i = 1k'$ or $w_i = k$, where $k' = k - 1$.

In the first case, we have $w_{i+1} < k$ (since $\widetilde{W}$ is admissible), so we consider $\tau_{i+1}(\widetilde{X})$. If $w_{i+2} < k$, then this latter circular word provides an admissible form for $\widetilde{X}$. If $w_{i+2} = k$, then we write $\widetilde{X} \approx \tau_{i+1}(\widetilde{X}) \approx \tau_{i+3}(\tau_{i+1}(\widetilde{X}))$ and, iterating the process while necessary makes the sum $\sum_i w_i$ decreases strictly and the letters remaining nonnnegative, so we eventually get an admisssible form for $\widetilde{X}$.

Consider now the second case, in which $w_i = k$. In this case, we can write $\widetilde{X} \approx \tau_{i+1} \circ \tau_i^{-1}(\widetilde{X}) = \widetilde{Y} = y_0 \widetilde{\ldots y_{\ell-1}}$. Note that we have $y_{i-2}y_{i-1}y_iy_{i+1} = (x_{i-2} + 1)k'0(x_{i+1} + 1)$ (since $x_{i-1} = 0$ by admissibility of $\widetilde{W}$), and that no other letter of $\widetilde{X}$ changes when $\tau_{i+1} \circ \tau_i^{-1}$ is applied to it. Since, again by admissibility of $\widetilde{W}$, we have $x_{i+1} < k$, we then get that the admissibility of $\widetilde{Y}$ only depends on its factors $y_{i-3}y_{i-2}$ and $y_{i+1}y_{i+2}$. The latter one is simple to deal with, since the only possible issue is the case $y_{i+2} = k$, for which we can apply τ_{i+2} (note that $y_{i+1} > 0$), and iterating with τ_{i+4}, τ_{i+6} etc. if necessary (and apply the same reasoning as before on the sum $\sum_i y_i$). The first one, together with the admissibility of $\widetilde{W}$, implies that the only possib ilities for $\widetilde{Y}$ to be non-admissible are the cases $y_{i-2} = k + 1$ and $y_{i-3}y_{i-2} = mk$ with $m > 0$. In the second case, the circular word $\widetilde{Z} = \tau_{i-1}(\widetilde{Y})$ is admissible. In the first case, we iterate the same reasoning as before (replacing i by $i - 2$, then by $i - 4$, etc.) until we get an admissible word or until the position of the problematic factor eventually comes back, by circularity, to the index i. Two possibilities are, then, to be distinguished: if ℓ is even, then, by induction, we get that $\widetilde{W} = \widetilde{(0k)^{\ell/2}}$ (or $\widetilde{(k0)^{\ell/2}}$). If ℓ is odd, then an induction gives that, up to a shift, we have $\widetilde{W} = k\widetilde{(0k)^{(\ell-1)/2}}$, so $\widetilde{W}$ is not admissible, a contradiction.

§ 3.2. Unicity of the admissible form

Note from now that the relations $\widetilde{0^{2\ell}} \approx \widetilde{(0k)^\ell} \approx \widetilde{(k0)^\ell}$ are given by the relation $\tau_1 \circ \tau_3 \circ \cdots \circ \tau_{2\ell-1}(\widetilde{(k0)^\ell}) = \widetilde{0^{2\ell}} = \tau_0 \circ \tau_2 \circ \cdots \circ \tau_{2\ell-2}(\widetilde{(0k)^\ell})$.

We consider two admissible words, $\widetilde{W} = w_0 \widetilde{\ldots w_{\ell-1}}$ and $\widetilde{X} = x_0 \widetilde{\ldots x_{\ell-1}}$, such that, for some integers $a_0, \ldots, a_{\ell-1}$, we have $\widetilde{X} = \tau_0^{a_0} \circ \cdots \circ \tau_{\ell-1}^{a_{\ell-1}}(\widetilde{W})$.

Lemma 3.2. *For all i, we have $|a_i| \leq 1$.*

Proof. For any $i \in \mathbb{Z}/\ell\mathbb{Z}$, we have

$$x_i = w_i + a_i - ka_{i+1} - a_{i+2}. \tag{3.1}$$

We assume, without loss of generality, that $a = a_0 = \max_i(|a_i|)$. Taking $i = \ell - 1$ in Equation (3.1) gives $ka + a_1 - a_{\ell-1} = w_{\ell-1} - x_{\ell-1}$, so $(k-2)a \leq k$.

If $k > 4$, then we get $a \leq 1$ which is the desired result.

If $k = 4$, then $(k-2)a \leq k$ implies $a \leq 2$. Assume $a = 2$: taking $i = \ell - 1$ in Equation (3.1) gives $8 + a_1 - a_{\ell-1} \leq 4$, so $a_1 = -2$ and $a_{\ell-1} = 2$. Taking $i = 0$ in Equation (3.1) then gives $x_0 - w_0 = 2 + 8 - a_2 \geq 10 - 2 = 8$, which is impossible. Hence, $a \leq 1$.

If $k = 3$, then the relation $(k-2)a \leq k$ gives $a \leq 3$. Assume $a = 3$. Then, Equation (3.1) for $i = \ell - 1$ gives $x_{\ell-1} - w_{\ell-1} = a_{\ell-1} - 9 - a_1$. Since $x_{\ell-1} - w_{\ell-1} \geq -3$, we get $a_{\ell-1} = 3 = -a_1$. Taking $i = 0$ in Equation (3.1) gives $x_0 - w_0 = 12 - a_2 \geq 9$, which is impossible. Suppose now $a = 2$. Equation (3.1) for $i = \ell - 1$ becomes $w_{\ell-1} - x_{\ell-1} - a_1 + a_{\ell-1} = 6$, which gives $a_1 = -2$ and/or $a_{\ell-1} = 2$. If, for example, $a_{\ell-1} = 2$, then Equation (3.1) for $i = \ell - 2$ leads to $x_{\ell-2} = w_{\ell-2} + a_{\ell-2} - 8 \leq -6$, which is impossible, so $a \leq 1$. (The case $a_1 = -2$ is similar.)

If $k = 2$, then Equation (3.1) for $i = \ell - 1$ gives $x_{\ell-1} - w_{\ell-1} = a_{\ell-1} - 2a - a_1$. Since $x_{\ell-1} - w_{\ell-1} \geq -2$, we have $a_{\ell-1} - a_1 \in \{2a, 2a-1, 2a-2\}$. If $a_{\ell-1} - a_1 = 2a$ or $2a - 1$, we can assume, for example, that $a_1 = -a$, so Equation (3.1) for $i = 0$ gives $x_0 - w_0 = 3a - a_2 \geq 2a$. Since $x_0 - w_0 \leq 2$, we therefore have $a \leq 1$. Hence, we can assume from now that $a_{\ell-1} - a_1 = 2a - 2$, and also that neither $a_{\ell-1}$ nor a_1 is equal to a in modulus, so $a_{\ell-1} = a - 1$ and $a_1 = 1 - a$. This gives, in Equation (3.1) for $i = 0$, that $x_0 - w_0 = 3a - 2 - a_2 \geq 2a - 2$. Since, again, $x_0 - w_0 \leq 2$, we obtain $a \leq 2$. If $a = 2$, then $a_1 = -1$, $a_{\ell-1} = 1$, and Equation (3.1) for $i = 0$ implies $x_0 = 2$, $w_0 = 0$ and $a_2 = 2$. Equation (3.1) for $i = \ell - 1$ then gives $x_{\ell-1} = w_{\ell-1} - 2$, so $x_{\ell-1} = 2$ and $w_{\ell-1} = 0$, but we have now $x_0 = x_{\ell-1} = 2$, which is impossible since $\widetilde{X}$ is ad missible.

The case $k = 1$ was proved in [5], Lemma 2.2. Note that the difference in the definition of equivalence between circular words in the present paper and in [5] (see the end of section 3.2) does not prevent us from the use of Lemma 2.2 of [5], since the restriction to words in $\mathbb{N}$ is nowhere used in that Lemma. □

Lemma 3.3. *We have*

$$N(\widetilde{X}) = N(\widetilde{W}) + a_0(1 - F_\ell) + a_1(1 - F_{\ell-1}).$$

Proof. Simple verification. □

Lemmas 3.2 and 3.3 enable us now to provide the proof of the desired result. Assume for example that $N(\widetilde{W}) \leq N(\widetilde{X})$. Since we also have $N(\widetilde{X}) < F_\ell$, Lemma 3.2 ensures that the pair (a_0, a_1) belongs to the set $\{(0,0), (-1,0), (0,-1), (-1,1)\}$.

By the unicity of the greedy expansion of integers in the scale $(F_n)_{n\geq 0}$, the case $(a_0, a_1) = (0,0)$ implies $\widetilde{W} = \widetilde{X}$. The case $(a_0, a_1) = (-1, 0)$ forces the equalities $\widetilde{W} = \widetilde{0^\ell}$ and $\widetilde{X} = \widetilde{(0k)^{\ell/2}}$ (so ℓ is even; if ℓ is odd, the circular word $\widetilde{X}$ such that $N(\widetilde{X}) = F_\ell - 1$ is not admissible).

Consider the case $(a_0, a_1) = (-1, 1)$. We have $x_0 = w_0 + a_0 - ka_1 - a_2 = w_0 - (k+1) - a_2$, so, since $x_0 \geq 0$, we must have $w_0 = k$, $a_2 = -1$ and $x_0 = 0$. We then get from the equality $x_1 = w_1 + a_1 - ka_2 - a_3$ (and the fact that $x_1 \leq k$) that $w_1 = 0$, $a_3 = 1$ and $x_1 = k$. An induction eventually gives that $\widetilde{W} = \widetilde{(k0)^{\ell/2}}$ and $\widetilde{X} = \widetilde{(0k)^{\ell/2}}$.

The last remaining case is $(a_0, a_1) = (0, -1)$, which implies $N(\widetilde{X}) = N(\widetilde{W}) + F_{\ell-1} - 1$. We have, in this case, $x_0 = w_0 + k - a_2$, so $w_0 = 0$ or $w_0 = a_2 = 1$. In this latter case, we have

$w_1 < k$ (by admissibility of $\widetilde{W}$), so $x_1 = w_1 - 1 - k - a_3 \leq (k-1) - 1 - k - a_3 < 0$, which is not allowed. Hence, we have $w_0 = 0$, so $x_0 = k - a_2$, so $a_2 = 0$ or 1. Assume $a_2 = 0$. We then have $x_0 = k$, so $x_{\ell-1} = 0$, so $N(\widetilde{X}) < F_{\ell-1}$. Together with the relation $N(\widetilde{X}) = N(\widetilde{W}) + F_{\ell-1} - 1$, this implies $\widetilde{W} = \widetilde{0^\ell}$ and $N(\widetilde{X}) = F_{\ell-1} - 1$, so $\widetilde{X} = \widetilde{(k0)^{\ell/2}}$. Hence, we can assume $a_2 = 1$, so we have $(a_1, a_2) = (-1, 1)$. The same induction as in the case $(a_0, a_1) = (-1, 1)$ then gives the desired conclusion.

Let us end the present section by the following remark. In [5], where only the case $k = 1$ is considered, circular words are defined on the alphabet $\mathbb{N}$ and not, as here, on $\mathbb{Z}$. As a consequence, the τ_is do not define a group action on circular words, hence the statements are a little different. The main difference between them is explained by the following propositions.

Proposition 3.4. *Let $T = \{\tau_i, 0 \leq i < \ell\} \cup \{\tau_i^{-1}, 0 \leq i < \ell\}$. For some $n \geq 1$, let $(t_j)_{1 \leq j \leq n}$ be a finite sequence of elements of T such that $t_1 \circ \cdots \circ t_n(\widetilde{W}) = \widetilde{0^\ell}$ for some $\widetilde{W}$. The circular word $t_2 \circ \cdots \circ t_n(\widetilde{W})$ has at least one negative letter.*

In particular, the words $\widetilde{k^\ell}$ and $\widetilde{0^\ell}$, which are equivalent when considered on the alphabet $\mathbb{Z}$ (see the relation $\tau_0 \circ \cdots \circ \tau_{\ell-1}(\widetilde{k^\ell}) = \widetilde{0^\ell}$) are not equivalent when they are considered on the alphabet $\mathbb{N}$.

Proof. Write $\widetilde{X} = t_2 \circ \cdots \circ t_n(\widetilde{W})$. We have $t_1(\widetilde{X}) = \widetilde{0^\ell}$, which is not possible if $\widetilde{X}$ has only nonnegative letters. □

Proposition 3.5. *Let $\widetilde{W}$ be a circular word of length ℓ, containing only nonnegative letters, and non equivalent to $\widetilde{0^\ell}$. There exists $t_1, \ldots, t_n \in T$ such that $t_1 \circ \cdots \circ t_n(\widetilde{W})$ is admissible and such that, for any i, $t_i \circ \cdots \circ t_n(\widetilde{W})$ has no negative letter.*

Hence, the equivalence class of $\widetilde{0^\ell}$ is the only one which is modified when considering equivalence of circular words on the alphabet $\mathbb{N}$ instead of the alphabet $\mathbb{Z}$.

Proof. We know that, since $\widetilde{W}$ is not equivalent to $\widetilde{0^\ell}$, it has exactly one admissible form. An easy check in the proof in section 3.1 shows that the successive transformations applied to $\widetilde{W}$ and leading to its admissible form never lead to a negative letter. □

§ 4. Group structures

Proposition 4.1. *For any $\ell > 0$, let $\mathcal{G}_\ell$ be the quotient set of the set of circular words of length ℓ by the equivalence relation defined by the τ_is. We embedd $\mathcal{G}_\ell$ with the binary operation, denoted by $+$, in which the sum of two equivalent classes g_1 and g_2 is the equivalence class of the circular word obtained by summing letter-by-letter any element of g_1 and any element of g_2. The set $(\mathcal{G}_\ell, +)$ is an abelian group.*

Proof. Trivial. □

Theorem 3.1 already showed that $\mathcal{G}_\ell$ is a finite set, our aim is now to determine explicitly the form of the finite abelian group $(\mathcal{G}_\ell, +)$. Let us do first the following remark. For any $\widetilde{W} =$

$\widetilde{w_0 \dots w_{\ell-1}}$, let $S(\widetilde{W}) = \sum_i w_i$. For any $(a_0, \dots, a_{\ell-1}) \in \mathbb{Z}^\ell$, we have $S(\tau_0^{a_0} \circ \cdots \circ \tau_{\ell-1}^{a_{\ell-1}}(\widetilde{W})) \equiv S(\widetilde{W}) \bmod k$. In particular, $\mathcal{G}_\ell$ admits $S^{-1}(k\mathbb{Z})$ as a subgroup, so $\mathrm{Card}(\mathcal{G}_\ell)$ is divided by k.

Now, let us consider the question of the cardinality of $\mathcal{G}_\ell$.

Proposition 4.2. *Let $(c_\ell)_\ell$ be the sequence defined by $c_0 = 2$, $c_1 = k$ and, for any $\ell \geq 2$, $c_\ell = kc_{\ell-1} + c_{\ell-2}$. For any $\ell > 0$, we have*

$$\mathrm{Card}(\mathcal{G}_\ell) = \begin{cases} c_\ell & \text{if } \ell \text{ is odd;} \\ c_\ell - 2 & \text{if } \ell \text{ is even.} \end{cases}$$

Proof. Consider the set of words of length ℓ without the circular structure, and define admissibility on it in the natural way. Let $\mathcal{W}_\ell$ be the set of admissible words of length ℓ: its cardinality is equal to F_ℓ. Now, define G_ℓ as the number of elements of $\mathcal{W}'_\ell$, the set of elements of $\mathcal{W}_\ell$ whose initial letter is not k. We split $\mathcal{W}'_\ell$ into two subsets: the first is made by words starting by 0 (it has $F_{\ell-1}$ elements), the second made by words starting by a letter w such that $1 \leq w < k$ (its cardinality is $(k-1)G_{\ell-1}$). Therefore, we have $G_\ell = F_{\ell-1} + (k-1)G_{\ell-1}$. (Note that $G_1 = k$ and $G_2 = k^2 + 2$.)

Now, the set of admissible circular words of length ℓ is written as the union of three sets: the set of admissible words of length ℓ ending with a 0 (cardinality: $F_{\ell-1}$), the set of admissible words ending by some letter w with $1 \leq w < k$ (hence starting by a letter strictly less than k, so of cardinality $(k-1)G_{\ell-1}$), and the set of admissible words ending by $0k$ (hence starting by a letter strictly less than k, so of cardinality $G_{\ell-2}$). From all of this, denoting temporarily by b_ℓ the cardinality of the set of admissible circular words of length ℓ, we get the equality $b_\ell = F_{\ell-1} + (k-1)G_{\ell-1} + G_{\ell-2}$, so $b_\ell = F_{\ell-1} + F_{\ell-3} + (k-1)b_{\ell-1}$. An induction then shows that $b_n = c_n$ for all n, and the conclusion is given by Theorem 3.1. ☐

Theorem 4.3. *Put $\Delta = k^2 + 4$. We have*

$$\mathcal{G}_{2\ell} \textit{ is isomorphic to } \begin{cases} (\mathbb{Z}/\sqrt{c_{2\ell}/\Delta}\mathbb{Z}) \times (\mathbb{Z}/\sqrt{c_{2\ell}\Delta}\mathbb{Z}) & \text{if } \ell = 2n \text{ and } k \text{ odd;} \\ (\mathbb{Z}/\sqrt{4c_{2\ell}/\Delta}\mathbb{Z}) \times (\mathbb{Z}/\sqrt{\Delta c_{2\ell}/4}\mathbb{Z}) & \text{if } \ell = 2n \text{ and } k \text{ even;} \\ (\mathbb{Z}/\sqrt{c_{2\ell} - 2}\mathbb{Z})^2 & \text{if } \ell = 2n+1. \end{cases}$$

Proof. Put $u_0 = 1$, $u_1 = k^2 + 2$, $u_2 = (k^2+2)u_1 - 2$ and $u_{n+1} = (k^2+2)u_n - u_{n-1}$ for any $n \geq 3$. Let $\widetilde{W} = \widetilde{10^{2\ell-1}}$ (but note that what follows would remain true for any circular word). For any positive integer a, we write $a \cdot \widetilde{W}$ for the sum $\widetilde{W} + \cdots + \widetilde{W}$ containing a terms. (For a negative value of a, $a\widetilde{W}$ stands for $(-a) \cdot (-\widetilde{W})$.) A simple verification shows that $u_1 \cdot \widetilde{W} = \sigma^2(\widetilde{W}) + \sigma^{-2}(\widetilde{W})$. Therefore, we get

$$\begin{aligned} u_2 \cdot \widetilde{W} &= (u_1^2 - 2) \cdot \widetilde{W} \\ &= u_1 \cdot (\sigma^2(\widetilde{W}) + \sigma^{-2}(\widetilde{W})) - 2 \cdot \widetilde{W} \\ &= \sigma^2(u_1 \cdot \widetilde{W}) + \sigma^{-2}(u_1 \cdot \widetilde{W}) - 2 \cdot \widetilde{W} \\ &= \sigma^2(\sigma^2(\widetilde{W}) + \sigma^{-2}(\widetilde{W})) + \sigma^{-2}(\sigma^2(\widetilde{W}) + \sigma^{-2}(\widetilde{W})) - 2 \cdot \widetilde{W} \\ &= \sigma^4(\widetilde{W}) + \sigma^{-4}(\widetilde{W}). \end{aligned}$$

An elementary induction then shows that, for any $n \geq 1$, we have $u_n \cdot \widetilde{W} = \sigma^{2n}(\widetilde{W}) + \sigma^{-2n}(\widetilde{W})$.

Put $v_n = \sum_{0\leq i\leq n} ku_i$: what precedes therefore gives that, for $\ell = 2n+1$, the circular word $\widetilde{10^{2\ell-1}}$ satisfies $v_n \cdot \widetilde{10^{2\ell-1}} = \widetilde{(k0)^{\ell}}$, so $\widetilde{10^{2\ell-1}}$ is of order at most v_n. This is true as well for $\widetilde{010^{2\ell-2}}$, and recall that $\mathcal{G}_{2\ell} = \langle \widetilde{010^{2\ell-2}}, \widetilde{10^{2\ell-1}} \rangle$. Eventually, an induction shows that $v_n^2 = c_{2\ell} - 2$ for any n and $\ell = 2n+1$. Hence, the case $\ell = 2n+1$ is proved.

Assume now that $\ell = 2n$ and put $\widetilde{W} = \widetilde{0^{2n-1}10^{2n}}$. We have $v_{n-1}\widetilde{W} \approx \widetilde{(0k)^{2n-1}00}$ and $u_n\widetilde{W} \approx \widetilde{0^{4n-1}2}$, so $v_n\widetilde{W} \approx \widetilde{0^{4n-1}k} \approx -v_{n-1}\widetilde{W}$, so the order of $\widetilde{W}$ divides $v_{n-1} + v_n$. Let $s_n = (v_{n-1}+v_n)/(k^2+4)$. An induction show that, for any n, we have $s_n = (k^2+2)s_{n-1} - s_{n-2}$ with $s_1 = k$ and $s_2 = k^3+2k$, so $s_n \in \mathbb{N}$. Another induction shows that $c_{4n} - 2 = \Delta s_n^2$. Since $\widetilde{W}$ and $\sigma(\widetilde{W})$ are of the same order dividing Δs_n and since $\langle \widetilde{W}, \sigma(\widetilde{W})\rangle = \mathcal{G}_{2\ell}$, there exists δ and δ' such that $\Delta = \delta\delta'$ and such that $\widetilde{W}$ is of order δs_n. The surjective morphism from $(\mathbb{Z}/(\delta s_n)\mathbb{Z})^2$ to $\mathcal{G}_{2\ell}$ defined by $(a,b) \mapsto a\widetilde{W} + b\sigma(\widetilde{W})$ hence gives th at Δ divides δ^2. Since $\Delta = k^2+4$, we therefore have $\delta = \Delta$ in the case k odd, and we are done in this case. If $k = 2k'$, then we get $\delta' = 1$ or 2, and it only remains to prove that $\widetilde{W}$ is of order $(k'^2+1)s_n$, which is easily derived from the observation that $(v_{n-1} + k'u_n)\widetilde{W} \approx \widetilde{(0k)^{2n}}$. □

For any ℓ, the application $\widetilde{W} \mapsto \widetilde{WWWW}$ is an injective morphism of groups from $\mathcal{G}_\ell$ into $\mathcal{G}_{4\ell}$, so the structure of $\mathcal{G}_{2\ell}$ for even values of ℓ is, in a sense, the most significant. It is nevertheless possible to describe the structure of $\mathcal{G}_\ell$ for odd values of ℓ as well. Here is a partial result easy to obtain (we do not consider the general case, which is quite tiresome and, as is the case of $\mathcal{G}_{2\ell}$ with even value of ℓ, depends on whether k is odd or even).

Theorem 4.4. *If ℓ is odd and $\ell \notin 3\mathbb{N}$, then $\mathcal{G}_\ell$ is monogenetic (i.e. isomorphic to $\mathbb{Z}/c_\ell\mathbb{Z}$).*

Proof. The proof of Theorem 4.3 gives that, for the circular word $\widetilde{W} = \widetilde{10^{\ell-1}}$ of length $\ell = 4n+1$ (resp. $4n+3$), we have $u_n(\widetilde{W}) = \sigma^{2n}(\widetilde{W}) + \sigma^{-2n}(\widetilde{W}) = \sigma^{2n}(\widetilde{W}) + \sigma^{2n+1}(\widetilde{W}) = \sigma^{2n-1}(\widetilde{W})$ (resp. $u_{n+1}(\widetilde{W}) = \sigma^{2n+2}(\widetilde{W}) + \sigma^{-2n-2}(\widetilde{W}) = \sigma^{2n+2}(\widetilde{W}) + \sigma^{2n+1}(\widetilde{W}) = \sigma^{2n}(\widetilde{W})$). Hence, multiplying by u_n (resp. u_{n+1}) again and again makes us attain all circular words of the form $\widetilde{0^i10^{4n-i}}$ iff ℓ and $2n-1$ (resp. $2n$) are mutually prime, which is the case iff ℓ is not of the form $6m+3$. Hence, if $\ell \neq 6m+3$ is odd, $\mathcal{G}_\ell$ is monogenetic. □

§ 5. Spanning trees

Classically, the *ℓ-th wheel* $\mathcal{W}_\ell$ is the graph with vertices c, r_1, $\dots r_\ell$, and with edges cr_i for all i and r_ir_{i+1} for all i (this latter being understood modulo ℓ, as in the sequel). Here, we will talk about the ℓ-th wheel *k^2-reinforced*, which means the graph $\mathcal{W}_{\ell,k^2}$ made of the wheel $\mathcal{W}_\ell$ in which each r_i is linked to c by k^2 distinct edges instead of only one. Note that $\mathcal{W}_{\ell,1^2} = \mathcal{W}_\ell$. It is remarked in [4] that, for this case $k=1$, the number of spanning trees of $\mathcal{W}_\ell$ is equal to the cardinality of $\mathcal{G}_{2\ell}$ (thanks to a result of Kenneth Rebman [3]), and that there exists a natural bijection between the two sets. The present section is devoted to the following generalization.

Theorem 5.1. *Let $k \geq 1$ be fixed. For any $\ell \geq 1$, the number of spanning trees of $\mathcal{W}_{\ell,k^2}$ is equal to the cardinality of $\mathcal{G}_{2\ell}$.*

Proof. Let s_ℓ be the number of spanning trees of $\mathcal{W}_{\ell,k^2}$. The value of s_ℓ is given by Kirchhoff's theorem (see [1]): let D_ℓ and A_ℓ be respectively the degree matrix (the diagonal matrix whose i-th diagonal coefficient is the degree of the i-th vertex of the graph) and the adjacency matrix (the matrix whose coefficient at the i-th line and j-th column is the number of edges from i to j) of $\mathcal{W}_{\ell,k^2}$. Then, the modulus of any cofactor of $D_\ell - A_\ell$ is equal to s_ℓ.

Here, writing D_ℓ and A_ℓ with the vertices written in the order $r_1, \ldots, r_\ell, c$, and taking the minor of $D_\ell - A_\ell$ made of the n first rows and column, we easily get that s_ℓ is equal (up to a change of sign) to the determinant of the matrix whose coefficients are equal to $k^2 + 2$ on the diagonal, to -1 on the super- and sub- diagonal, also to -1 at the top-right and bottom-left, and 0 elsewhere. It is now a classical exercice to prove that this determinant satisfies the same induction property as $\mathrm{Card}(\mathcal{G}_\ell)$ and that both values are equal for $\ell = 1$ and 2, hence ending the proof. (For details in the case $k = 1$, see [3], section B, matrix A_n, the general case being a straightforward generalization.) $\square$

§ 6. Open questions and perspectives

The most immediate question to ask concerns the generalization of the study to circular words with a combinatorial constraint defined by a more general polynomial. It appears that, in some cases, the corresponding notion of admissibility does not provide a theorem of existence and unicity as in Theorem 3.1. A simple example is produced by the polynomial $P(X) = X^2 - 2X - 2$, defining admissible circular words as circular words on $\mathcal{A} = \{0, 1, 2\}$ not containing the factor 22. The circular words $\widetilde{1010}$ and $\widetilde{0101}$, both equivalent to $\widetilde{2222}$, are two different admissible forms of the same equivalence class (this one being different of the class of the identity element). Hence, it seems that the notion of admissible circular words, at least in the naive definition here in use, is not enough in itself. Fortunately, an alternative way to count the number of class of circular words is suggested by the use of the determinant of an operato r (see [5], section 3.2). Also, to get the full structure of the groups $\mathcal{G}_\ell$, an interesting approach is given by polynomial algebra. Circular words of length ℓ on the alphabet $\mathbb{Z}$ can be seen as elements of $\mathbb{Z}[X]/(X^\ell - 1)$, and the combinatorial constraint defined by a polynomial P on these circular words leads to consider the set $\mathbb{Z}[X]/(X^\ell - 1, P(X))$. The Euclidean division provides therefore a powerful tool to get the order of a given element of this set. All of this is to be written in a forthcoming paper.

Some computer experiments show that the structure of $\mathcal{G}_\ell$ obtained in Theorem 4.3 is quite particular. For many other choices of P, the groups $\mathcal{G}_\ell$ seems to be monogenetic (apart for the case $P(X) = X^2 - kX + 1$ with $k \geq 3$, for which the group of corresponding circular words of length ℓ seems to be of the form $(\mathbb{Z}/a\mathbb{Z}) \times (\mathbb{Z}/d_\ell\mathbb{Z})^2$, where a depends only on the parity of ℓ). In particular, despite a quite natural guess, the groups of circular words defined by the Tribonacci polynomial $P(X) = X^3 - X^2 - X - 1$ are probably never of the form $(\mathbb{Z}/d\mathbb{Z})^3$, neither close to it, whatever ℓ is. Therefore, one may ask the question of the description of the set of finite abelian groups that can be regarded as the group of circular words of length ℓ quotiented by some combinatorial equivalence given by a polynomial P.

Another question is about a more general choice for the set of indices for a word. As defined in the beginning of section 2, a circular word is a word whose letters are indexed by $\mathbb{Z}/\ell\mathbb{Z}$. How about a set of indices defined, for example, by $(\mathbb{Z}/\ell\mathbb{Z}) \times (\mathbb{Z}\ell'\mathbb{Z})$? Write such a word as an array $\ell \times \ell'$ with a toral structure, and consider for example two combinatorial constraints, the one acting horizontally, the other vertically. It is quite easy to show that this provides a finite abelian group: how can we describe it? In particular, we leave the reader with the following exercise: for horizontal and vertical constraints both defined by the Fibonacci rule (i.e. the polynomial $P(X) = X^2 - X - 1$), the group corresponding to $(\ell, 2)$-circular words quotiented by the natural equivalence relation is the trivial group for any ℓ.

References

[1] Berge C., *The Theory of Graphs and its Applications*, Methuen, London, 1962.

[2] Frougny Ch., Numeration systems, in Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, 2002.

[3] Rebman K., The sequence: 1 5 16 45 121 320... in combinatorics, *Fibonacci Quart.* **13** (1975), 51–55.

[4] Rittaud B. and Vivier L., Circular words and applications, *Proc. 8th Intern. Conf. Words 2011, Prague*, 2011, pp. 31–36.

[5] Rittaud B. and Vivier L., Circular words and three applications: factors of the Fibonacci word, $\mathcal{F}$-adic numbers, and the sequence 1, 5, 16, 45, 121, 302,..., *Functiones et Approximatio* (*to appear*).