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Helical Voronoi tilings on the cylinder

Novel Development of Nonlinear Discrete Integrable Systems

By

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Abstract

A helical Voronoi tiling on the cylinder $\mathbb{C}/\mathbb{Z}$ is one of the most simplified mathematical models of phyllotaxis. Its bifurcation structure is described by the set of the generators $z = x + \mathrm{i}y$, $y > 0$, for rectangular Voronoi tilings, which is a family of half-circles in the upper half plane. In relation to the parastichy transitions in phyllotaxis, we consider the limit set $\Omega(x)$ of aspect ratios of the rectangular tiles, by fixing x and taking the limit as $y \to 0$. If x is a quadratic irrational, then $\Omega(x)$ is a finite set. Moreover, if x is linearly equivalent to the golden section, then the shapes of the rectangular tiles tend to the square.

§ 1. Introduction

Beautiful features of plants are observed in the regular arrangements of botanical units such as seeds of a sunflower, florets in the head inflorescence of a daisy, and scales on a pine cone. Symmetry of phyllotaxis is related to the golden section $\tau = (1+\sqrt{5})/2$, Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, \cdots$, and continued fractions. See [1, 14, 15] for the comprehensive overview on this interdisciplinary subject. For the recent progress on the dynamical study of self-organizing processes, see [5, 16, 23].

One of the most basic models on the geometry of phyllotaxis is a helical Voronoi tiling on the cylinder $\mathbb{C}/\mathbb{Z}$ [2, 6, 7, 18], which is periodic with respect to the additive

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group of translations generated by a single element $z = x + \mathrm{i}y$, $y > 0$. It naturally has a linear lattice structure, and is a prototype for other models such as the disk model [17, 18], the conical model [3] and the constant curvature model [19]. The bifurcation diagram of helical Voronoi tilings is described by the family of half-circles in the upper half plane, whereas the parameter spaces of triangular spiral tilings and Voronoi spiral tilings are described by families of real algebraic curves [20, 22].

In the phyllotaxis theory, $x = \mathrm{Re}(z)$ is called the *divergence* and $\mathrm{e}^y = |\mathrm{e}^{-\mathrm{i}z}|$ is called the *plastochrone ratio*. The transition of combinatorial structures depending on the plastochrone ratio, with a divergence x fixed, is called *parastichy transition*. In this paper, we consider the limit set $\Omega(x)$ of the shapes of rectangular tiles by taking the limit as $y \to 0$. If x is a quadratic irrational, then $\Omega(x)$ is a finite set. Moreover, if x is linearly equivalent to the golden section τ, then the shapes of the rectangular tiles tend to the square. This is an extended result of the *shape invariance under compression* observed by Rothen and Koch [18], as shown in Section 5. See also [21, 22] for the shape limits in triangular spiral tilings and quadrilateral Voronoi spiral tilings.

In Section 2, we define helical Voronoi tilings on the cylinder $\mathbb{C}/\mathbb{Z}$ as quotients of planar Voronoi tilings with a lattice site set. It is shown that the tiles are rectangles or hexagons. In Section 3, we describe the space of parameters of rectangular Voronoi tilings with a given opposed parastichy pair. In Section 4, the relationship is shown between the parastichy pair and the continued fraction expansion of the divergence. In Section 5, we study the limit set $\Omega(x)$ of the *shape parameters* of rectangular tiles.

This work was inspired by Hizume's figurative art works on phyllotactic paper-foldings, triangular spirals and aperiodic quasicrystals [10, 11, 12, 13].

§ 2. Helical Voronoi tilings on the cylinder

A *tiling* [9] of a two dimensional manifold X is a family $\mathcal{T} = \{T_j\}_j$ of topological disks $T_j \subset X$ which covers X without gaps or overlaps, that is, $X = \bigcup_j T_j$ and $\mathrm{int}(T_j) \cap \mathrm{int}(T_k) = \emptyset$, $j \neq k$. Each T_j is called a tile. Two distinct tiles T, T' are called *adjacent* if $T \cap T'$ contains at least two points.

Let $z \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, and $\Lambda = \Lambda(z) := z\mathbb{Z} + \mathbb{Z}$ a lattice. The *Voronoi region* [4] of the site $\lambda \in \Lambda(z)$ is given by

$$V(\lambda) = V(\lambda; z) := \{\zeta \in \mathbb{C} : |\zeta - \lambda| \leq |\zeta - \lambda'|,\ \forall \lambda' \in \Lambda(z)\}. \tag{2.1}$$

The family $\mathcal{V}(z) = \{V(\lambda; z)\}_{\lambda \in \Lambda(z)}$ is a tiling of the plane $\mathbb{C}$. It is a *periodic* tiling with respect to the additive group of translations $\Lambda(z)$, since $V(\lambda) = V(0) + \lambda$ for each $\lambda \in \Lambda(z)$. Moreover, we have $\mathcal{V}(z) = \mathcal{V}(z+1) = z \cdot \mathcal{V}(-z^{-1})$ because $z\mathbb{Z} + \mathbb{Z} = (z+1)\mathbb{Z} + \mathbb{Z} = z(\mathbb{Z} - z^{-1}\mathbb{Z})$.

By the canonical projection $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$, the plane $\mathbb{C}$ is a covering space of the cylinder $\mathbb{C}/\mathbb{Z}$. The Euclidean metric of $\mathbb{C}$ induces a canonical distance in $\mathbb{C}/\mathbb{Z}$. The Voronoi regions in $\mathbb{C}/\mathbb{Z}$ with respect to the site set $\pi(\Lambda(z))$ are given by

$$T(\lambda) := \{\zeta \in \mathbb{C}/\mathbb{Z} : \operatorname{dist}(\zeta, \pi(\lambda)) \le \operatorname{dist}(\zeta, \pi(\lambda')),\ \forall \lambda' \in \Lambda(z)\}, \quad \lambda \in \Lambda(z).$$

Note that $T(\lambda) = \pi(V(\lambda))$. The family $\mathcal{T}(z) := \{T(\lambda)\}_{\lambda \in \Lambda(z)}$ admits a transitive action of an additive group of translations $\pi(z\mathbb{Z} + \mathbb{Z}) = \pi(z)\,\mathbb{Z}$, generated by a single element $\pi(z)$.

Let $B_0 = \{z \in \mathbb{H} : |z - \frac{1}{2}| \le \frac{1}{2}\}$. If $z \notin \mathbb{Z} + B_0$, then the two tiles $V(0), V(1) \in \mathcal{V}(z)$ are adjacent to each other, and hence the Voronoi region $T(0) = T(1)$ in the cylinder is not simply connected. If $z \in \mathbb{Z} + B_0$, then $T(\lambda)$, $\lambda \in \Lambda(z)$, are simply connected, and $\mathcal{T}(z)$ is a tiling of the cylinder by convex polygons. It is called a *helical* Voronoi tiling generated by z.

Suppose that $z \in \mathbb{Z} + B_0$, and fix a lattice $\Lambda = \Lambda(z)$. The dual of a Voronoi tiling is called a *Delaunay diagram*. The line segment $\ell(\lambda, \lambda')$ joining two sites $\lambda, \lambda' \in \Lambda$ is called a *Delaunay edge* if $V(\lambda)$ is adjacent to $V(\lambda')$. Two distinct Delaunay edges may have a point in common only at their endpoint. A connected component of the complement $\mathbb{C} \setminus \bigcup \ell(\lambda, \lambda')$, where $\ell(\lambda, \lambda')$ runs through all the Delaunay edges, is called a *Delaunay polygon*. Each Delaunay polygon is inscribed in a circle. That is, a finite subset $\Lambda' \subset \Lambda$ is the set of the corners of a Delaunay polygon if and only if there exists a disk D such that $\partial D \cap \Lambda = \Lambda'$ and $\operatorname{int}(D) \cap \Lambda = \emptyset$.

For three distinct complex numbers $z_1, z_2, z_3 \in \mathbb{C}$, let

$$\angle(z_1, z_2, z_3) = \operatorname{Arg}\left(\frac{z_1 - z_2}{z_3 - z_2}\right),$$

where $-\pi < \operatorname{Arg}(z) \le \pi$ denotes the principal argument of $z \ne 0$.

Lemma 2.1. *Let $z \in \mathbb{Z} + B_0$. For the tiling $\mathcal{V}(z)$ of the plane, there are $\lambda = mz - a, \lambda' = nz - b \in \Lambda(z)$ with $m, n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$, such that the followings hold.*

1. *The tile $V(0)$ is adjacent to $V(\lambda)$ and $V(\lambda')$,*
2. *$\lambda, \lambda', \lambda'/\lambda \in \mathbb{H}$, $mb - na = 1$, $\operatorname{Re}(\lambda') < 0 < \operatorname{Re}(\lambda)$, and*
3. *Either*
   - *(a) $\mathcal{V}(z)$ is a rectangular tiling, or*
   - *(b) $\mathcal{V}(z)$ is a hexagonal tiling such that $V(0)$ is adjacent to $V(\lambda + \lambda')$.*

Proof. Since the site set $\Lambda = \Lambda(z)$ is a lattice, the Delaunay diagram is also periodic with respect to the translation group Λ. Suppose that $V(0)$ is adjacent to $V(\lambda), V(\lambda')$, $\lambda \ne \lambda'$. Then we have either:

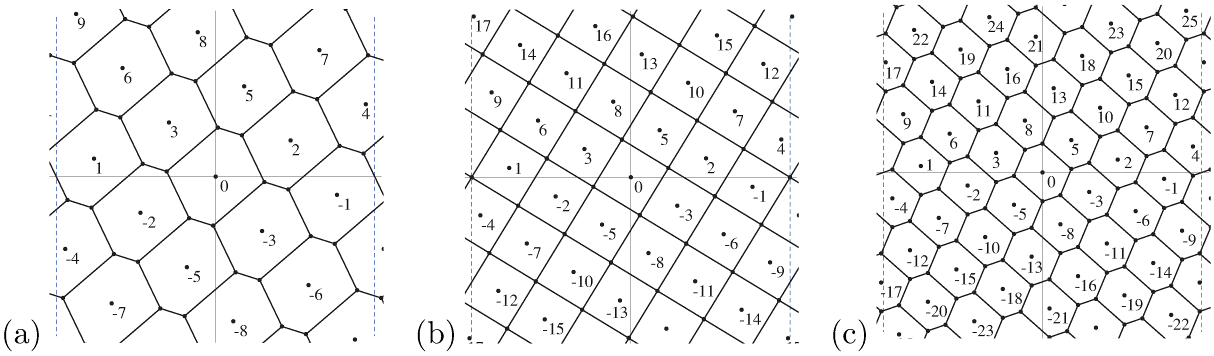

Figure 1. Voronoi tilings $\mathcal{V}((\tau-2)+\mathrm{i}y)$ on the cylinder $\mathbb{C}/\mathbb{Z}$, where $\tau=(1+\sqrt{5})/2$ is the golden section. (a) $y=0.056$, a hexagonal tiling with opposed parastichy pairs $\{2,3\}$, $\{5,3\}$. (b) $y=0.0296149\cdots$, a rectangular tiling with an opposed parastichy pair $\{5,3\}$. (c) $y=0.02$, a hexagonal tiling with opposed parastichy pairs $\{5,3\}$, $\{5,8\}$.

1. the quadrilateral $\square(0,\lambda,\lambda+\lambda',\lambda')$ is a Delaunay polygon, or
2. $\ell(0,\lambda+\lambda')$ or $\ell(\lambda,\lambda')$ is a Delaunay edge.

In the case 1, the quadrilateral $\square(0,\lambda,\lambda+\lambda',\lambda')$ is a parallelogram which is inscribed in a circle. Hence it is a rectangle. Denote by $\lambda=mz-a$, $\lambda'=nz-b$, $a,b,m,n\in\mathbb{Z}$. Since $V(0)$ is also adjacent to $V(-\lambda)$ and $V(-\lambda')$, we may assume that $m,n>0$ without loss of generality, which implies that $\lambda,\lambda'\in\mathbb{H}$. Since $\Lambda=\lambda\mathbb{Z}+\lambda'\mathbb{Z}$, we have $|mb-na|=1$. We may further assume that $\lambda'/\lambda\in\mathbb{H}$, which implies that $\mathrm{Re}(\lambda')<0<\mathrm{Re}(\lambda)$ and $mb-na=1$.

In the case 2, we may assume without loss of generality that $V(0)$ is adjacent to $V(\lambda),V(\lambda'),V(\lambda+\lambda')$, and that $\lambda,\lambda',\lambda'/\lambda\in\mathbb{H}$. Denote by $\lambda=mz-a$, $\lambda'=nz-b$. Then we have $m,n>0$, and $mb-na=1$ because $\Lambda=\lambda\mathbb{Z}+\lambda'\mathbb{Z}$. Since $\triangle(0,\lambda,\lambda+\lambda')$ is a Delaunay polygon, λ' lies outside the circumscribing circle of $\triangle(0,\lambda,\lambda+\lambda')$, whereas $\square(0,\lambda,\lambda+\lambda',\lambda')$ is a parallelogram. This implies that $\angle(\lambda',0,\lambda)>\frac{\pi}{2}$, and hence $\mathrm{Re}(\lambda')<0<\mathrm{Re}(\lambda)$. $\square$

In the phyllotaxis theory, the pair $\{m,n\}$ is called an *opposed parastichy pair* of the tiling $\mathcal{V}(z)$ if $V(0)$ is adjacent to $V(mz-a)$ and $V(nz-b)$, and $\mathrm{Re}(nz-b)\cdot\mathrm{Re}(mz-a)<0$, for some $a,b\in\mathbb{Z}$.

Figure 1 shows the *parastichy transition* of $\mathcal{V}(\tau-2+\mathrm{i}y)$, where τ is the golden section. The parastichy pairs consist of consecutive Fibonacci numbers.

Figure 2 shows the Delaunay diagrams for the Voronoi tilings $\mathcal{V}(z)$ in Figure 1. If $\mathcal{V}(z)$ is a hexagonal tiling, then the Delaunay diagram consists of Delaunay triangles. If $\mathcal{V}(z)$ is a rectangular tiling, then the Delaunay digram is a parallel translate of $\mathcal{V}(z)$.

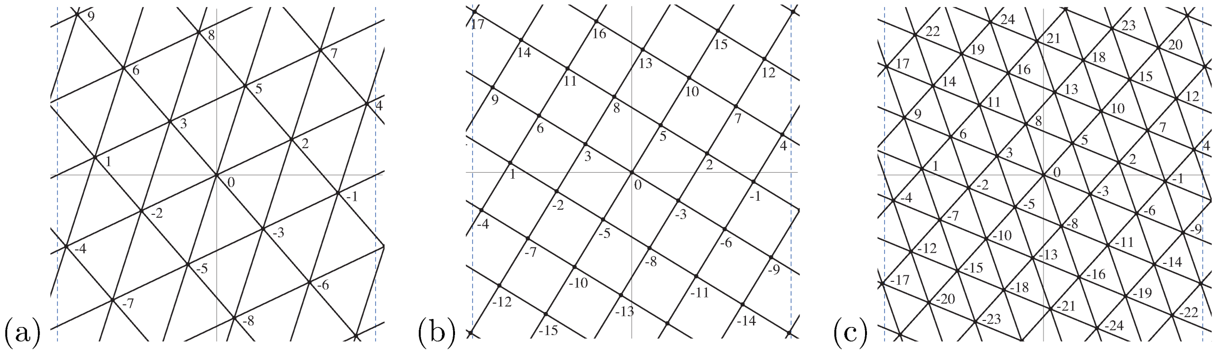

Figure 2. Delaunay diagrams for the Voronoi tilings $\mathcal{V}((\tau-2)+\mathrm{i}y)$ in Figure 1. (a) $y=0.056$, the line segment $\ell(\lambda,\lambda')$ is a Delaunay edge, where $\lambda=3z+1, \lambda'=5z+2$. (b) $y=0.0296149\cdots$, the rectangle $\square(0,\lambda,\lambda+\lambda',\lambda')$ is a Delaunay polygon. (c) $y=0.02$, the line segment $\ell(0,\lambda+\lambda')$ is a Delaunay edge.

§ 3. Helical hexagonal Voronoi tilings

Let $z\in\mathbb{H}$, and consider a tiling $\mathcal{V}(z)$. Suppose that the tile $V(0)$ is adjacent to $V(\lambda), V(\lambda')$, where $\lambda=mz-a$, $\lambda'=nz-b$, $m,n\in\mathbb{N}$, $mb-na=1$. If $V(0)$ is a rectangle, then the angle $\angle(\lambda',0,\lambda)$ is a right angle, and z lies on the half-circle $C(\frac{a}{m},\frac{b}{n})\cap\mathbb{H}$, where

$$C\left(\frac{a}{m},\frac{b}{n}\right)=\left\{z\in\mathbb{C}:\frac{nz-b}{mz-a}\in\mathrm{i}\mathbb{R}\right\}. \tag{3.1}$$

The circle $C(\frac{a}{m},\frac{b}{n})$ is symmetric with respect to the real axis, and passes through the points $\frac{a}{m},\frac{b}{n}\in\mathbb{R}$. This, together with the assumption that $mb-na>0$, implies that $\frac{a}{m}<\mathrm{Re}(z)<\frac{b}{n}$ for $z\in C\left(\frac{a}{m},\frac{b}{n}\right)$.

Lemma 3.1. *Let $z\in\mathbb{Z}+B_0$. Suppose that $\mathcal{V}(z)$ is a hexagonal tiling such that the tile $V(0)$ is adjacent to $V(\lambda), V(\lambda'), V(\lambda+\lambda')$, where $\lambda=mz-a$, $\lambda'=nz-b$, $m,n\in\mathbb{N}$, $mb-na=1$. Then z lies inside the circle (3.1). In particular, we have $\frac{a}{m}<\mathrm{Re}(z)<\frac{b}{n}$.*

Proof. Let $z=x+\mathrm{i}y$, $y>0$. Fix x,m,n,a,b, and consider $\lambda=\lambda(z)=mz-a$, $\lambda'=\lambda'(z)=nz-b$ as functions of y. Since $\lambda(z),\lambda'(z)\in\mathbb{H}$ and $\mathrm{Re}(\lambda'(z))<0<\mathrm{Re}(\lambda(z))$, the angle $\angle(\lambda',0,\lambda)$ is a decreasing function of $y>0$. Since $\ell(0,\lambda+\lambda')$ is a Delaunay edge, we have $\angle(\lambda',0,\lambda)>\frac{\pi}{2}$, which implies that $\mathrm{Re}(\lambda'/\lambda)<0$, and z lies inside the circle (3.1). $\square$

Now suppose that $|\mathrm{Re}(z)|<\frac{1}{2}$ for simplicity. For each pair of relatively prime integers $m,n>0$ with $(m,n)\neq(1,1)$, there exist $a,b\in\mathbb{Z}$ such that $mb-na=1$ and

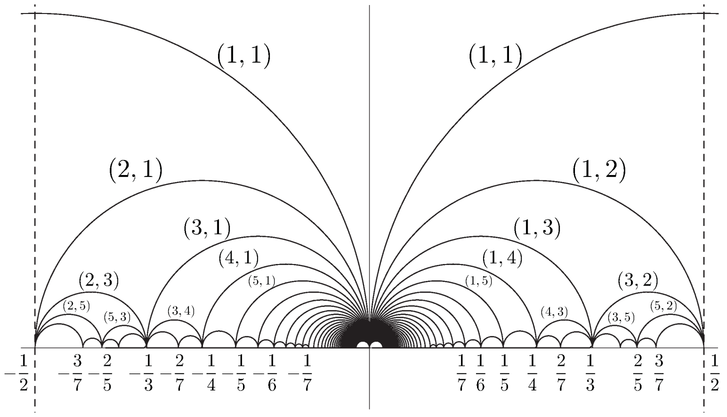

Figure 3. The set of generators of rectangular tilings. For each pair of relatively prime integers $m, n > 0$, there exists a half circle $C(\frac{a}{m}, \frac{b}{n})$, denoted by (m, n), which is the set of generators $z \in \mathbb{H}$, $|\mathrm{Re}(z)| < \frac{1}{2}$, of rectangular tilings with an opposed parastichy pair $\{m, n\}$.

$-\frac{1}{2} < \frac{a}{m} < \frac{b}{n} < \frac{1}{2}$. Denote by $\lambda = \lambda(z) := mz - a$, $\lambda' = \lambda'(z) := nz - b$. Lemma 3.1 implies that for $z \in B_0$ with $|\mathrm{Re}(z)| < \frac{1}{2}$, $\mathcal{V}(z)$ is a hexagonal tiling such that $V(0)$ is adjacent to $V(\lambda)$, $V(\lambda')$, $V(\lambda + \lambda')$ and $\lambda, \lambda', \lambda + \lambda' \in \mathbb{H}$, if and only if z lies inside the circle $C(\frac{a}{m}, \frac{b}{n})$ and outside $C(\frac{a}{m}, \frac{a+b}{m+n})$ and $C(\frac{a+b}{m+n}, \frac{b}{n})$, that is, $z \in W_{m,n}$ where

$$W_{m,n} := \left\{ z \in \mathbb{H} : |\mathrm{Re}(z)| < \frac{1}{2}, \mathrm{Re}\left(\frac{\lambda'}{\lambda}\right) < 0,\ \mathrm{Re}\left(\frac{\lambda + \lambda'}{\lambda}\right), \mathrm{Re}\left(\frac{\lambda'}{\lambda + \lambda'}\right) > 0 \right\}.$$

Figure 3 shows the family of the half-circles $C(\frac{a}{m}, \frac{b}{n}) \cap \mathbb{H}$, denoted by (m, n), in the strip $|\mathrm{Re}(z)| < \frac{1}{2}$.

§ 4. Parastichies and continued fraction expansions

For $x \in \mathbb{R}$, let

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0, a_1, a_2, \cdots],\ a_0 \in \mathbb{Z},\ a_i \in \mathbb{N},\ i \geq 1$$

be a continued fraction expansion of x. Define the sequences $\{p_j\}_{j \geq -1}$, $\{q_j\}_{j \geq -1}$, $\{p_{j,k}\}_{j \geq 0, 0 < k \leq a_{j+1}}$, and $\{q_{j,k}\}_{j \geq 0, 0 < k \leq a_{j+1}}$ as follows: $p_{-1} = 1$, $p_0 = a_0$, $p_1 = a_0 a_1 + 1$, $p_j = p_{j-2} + a_j p_{j-1}$ for $j \geq 2$; $q_{-1} = 0$, $q_0 = 1$, $q_1 = a_1$, $q_j = q_{j-2} + a_j q_{j-1}$ for $j \geq 2$; $p_{j,k} = p_{j-1} + k p_j$; and $q_{j,k} = q_{j-1} + k q_j$. The ratio $\frac{p_j}{q_j} = [a_0, a_1, a_2, \cdots, a_j]$ is called a *principal convergent* of x, and $\frac{p_{j,k}}{q_{j,k}} = [a_0, a_1, a_2, \cdots, a_j, k]$ is called an *intermediate converegent* of x. If $x \in \mathbb{Q}$, there are only finitely many convergents of x.

A pair of rational numbers $\frac{a}{m}, \frac{b}{n}$ is called a *pair of convergents* of $x \in \mathbb{R}$ if $|bm-an| = 1$ and either $\frac{a}{m} < x < \frac{b}{n}$ or $\frac{b}{n} < x < \frac{a}{m}$. It is known that if $\frac{a}{m}, \frac{b}{n}$ are a pair of convergents of x, then either $a = p_j$, $m = q_j$, $b = p_{j,k}$, $n = q_{j,k}$ with j even, or $a = p_{j,k}$, $m = q_{j,k}$, $b = p_j$, $n = q_j$ with j odd, and $0 < k \leq a_{j+1}$.

Lemma 4.1. *Let $z = x+\mathrm{i}y \in \mathbb{Z}+B_0$, and suppose that $\mathcal{V}(z)$ is a hexagonal tiling such that $V(0)$ is adjacent to $V(\lambda), V(\lambda'), V(\lambda+\lambda')$, where $\lambda = mz - a$, $\lambda' = nz - b$, $m, n \in \mathbb{N}$ and $mb - na = 1$. Then $\frac{a}{m}, \frac{b}{n}$ are principal or intermediate convergents of x, at least one of which is principal.*

Proof. We have $mb - na = 1$, and $\frac{a}{m} < x < \frac{b}{n}$ by Lemma 3.1. Hence, $\frac{a}{m}, \frac{b}{n}$ are a pair of convergents of x. □

§ 5. Shape limit of helical rectangular Voronoi tilings

Fix an irrational number x such that $|x| < \frac{1}{2}$, and define the sequences a_j, q_j and $q_{j,k}$, $j \geq 0$, $0 < k \leq a_{j+1}$, as in Section 4. For each $j \geq 0$ and $0 < k \leq a_{j+1}$, let $\frac{a_{j,k}}{m_{j,k}} < \frac{b_{j,k}}{n_{j,k}}$ be a pair of convergents of x such that $\{m_{j,k}, n_{j,k}\} = \{q_j, q_{j,k}\}$. Denote by $C_{j,k}(x) = C(\frac{a_{j,k}}{m_{j,k}}, \frac{b_{j,k}}{n_{j,k}})$. There exists a unique $y_{j,k} > 0$ such that $z_{j,k} := x + \mathrm{i}y_{j,k} \in C_{j,k}(x)$. Let $\lambda_{j,k} = m_{j,k}z_{j,k} - a_{j,k}$, $\lambda'_{j,k} = n_{j,k}z_{j,k} - b_{j,k}$. The ratio

$$R_{j,k}(x) := \frac{\lambda'_{j,k}}{\lambda_{j,k}} = \frac{n_{j,k}(x + \mathrm{i}y_{j,k}) - b_{j,k}}{m_{j,k}(x + \mathrm{i}y_{j,k}) - a_{j,k}} \in \mathrm{i}\mathbb{R}$$

is called a *shape parameter* of the tiling $\mathcal{V}(z_{j,k})$. It is the aspect ratio, or the modulus, of the Delaunay polygon $\square(0, \lambda_{j,k}, \lambda_{j,k} + \lambda'_{j,k}, \lambda'_{j,k})$, which is similar to the rectangular tile $T(0)$ in $\mathcal{V}(z_{j,k})$. Let

$$\Omega(x) := \Omega(\{R_{j,k}(x) : j \geq 0, 0 < k \leq a_{j+1}\})$$

be the limit set of $\{R_{j,k}(x)\}_{j,k}$ as $j \to \infty$, i.e., the set of the accumulation points of $\{R_{j,k}(x)\}_{j,k}$.

Denote by $\langle \xi \rangle \in (-\frac{1}{2}, \frac{1}{2}]$ a fractional part of $\xi \in \mathbb{R}$, such that $[\![\xi]\!] := \xi - \langle \xi \rangle$ is (one of) the closest integer to ξ. We have $\langle x m_{j,k} \rangle = x m_{j,k} - a_{j,k} = \mathrm{Re}(\lambda_{j,k}) > 0$, $\langle x n_{j,k} \rangle = x n_{j,k} - b_{j,k} = \mathrm{Re}(\lambda'_{j,k}) < 0$.

Lemma 5.1. $R_{j,k}(x) = \mathrm{i}\left(-\dfrac{q_{j,k}\langle x q_{j,k} \rangle}{q_j \langle x q_j \rangle}\right)^{\frac{(-1)^j}{2}}$

Proof. Since $\lambda'_{j,k}/\lambda_{j,k} \in \mathrm{i}\mathbb{R}$, we have $\mathrm{Re}(\lambda'_{j,k}/\lambda_{j,k}) = 0$, where

$$|\lambda_{j,k}|^2 \cdot \mathrm{Re}\left(\frac{\lambda'_{j,k}}{\lambda_{j,k}}\right) = (m_{j,k}x - a_{j,k})(n_{j,k}x - b_{j,k}) + m_{j,k}n_{j,k}y_{j,k}^2.$$

So we have

$$y_{j,k} = \left(-\frac{(m_{j,k}x - a_{j,k})(n_{j,k}x - b_{j,k})}{m_{j,k}n_{j,k}} \right)^{\frac{1}{2}},$$
$$|\lambda_{j,k}|^2 = (m_{j,k}x - a_{j,k})^2 + m_{j,k}^2 y_{j,k}^2 = \frac{m_{j,k}x - a_{j,k}}{n_{j,k}},$$

and hence

$$\mathrm{Im}\left(\frac{\lambda'_{j,k}}{\lambda_{j,k}}\right) = \frac{y_{j,k}}{|\lambda_{j,k}|^2} = \left(-\frac{n_{j,k}(n_{j,k}x - b_{j,k})}{m_{j,k}(m_{j,k}x - a_{j,k})} \right)^{\frac{1}{2}} = \left(-\frac{n_{j,k}\langle x n_{j,k}\rangle}{m_{j,k}\langle x m_{j,k}\rangle} \right)^{\frac{1}{2}}.$$

□

Suppose that x is a quadratic irrational. Then it has a periodic continued fraction expansion

$$\begin{aligned} x &= [a_0, a_1, a_2, \dots] \\ &= [a_0, a_1, \dots, a_{j_0}, \overline{b_1, \cdots, b_d}] \\ &= [a_0, a_1, \dots, a_{j_0}, b_1, \dots, b_d, b_1, \cdots, b_d, \dots]. \end{aligned}$$

We may assume that j_0, d are even, by choosing bigger ones if necessary. For each $1 \le s \le d$, let

$$\omega_s = [\overline{b_s, \dots, b_d, b_1, \dots, b_{s-1}}]$$

be a purely periodic continued fraction, and $h_s(X) = X^2 - \alpha_s X - \beta_s \in \mathbb{Q}[X]$ a quadratic polynomial such that $h_s(\omega_s) = 0$. A conjugate ω'_s of ω_s is defined by $h_s(\omega'_s) = 0$ and $\omega'_s \neq \omega_s$. It is written as

$$\omega'_s := -[\overline{b_{s-1}, \dots, b_1, b_d, \dots, b_s}]^{-1},$$

see [8].

Theorem 5.2. *If x is a quadratic irrational, the limit set $\Omega(x)$ is written as*

$$\Omega(x) = \left\{ \mathrm{i}(-h_{s+1}(k))^{\frac{(-1)^s}{2}} : 0 \le s < d,\ 0 < k \le b_{s+1} \right\}.$$

In particular, it is a finite set.

Proof. By using the continued fractions, we have

(5.1) $$\frac{q_{j,k}}{q_j} = [k, a_j, a_{j-1}, \dots, a_1], \quad -\frac{\langle x q_{j,k}\rangle}{\langle x q_j\rangle} = [a_{j+1} - k, a_{j+2}, a_{j+3}, \dots]$$

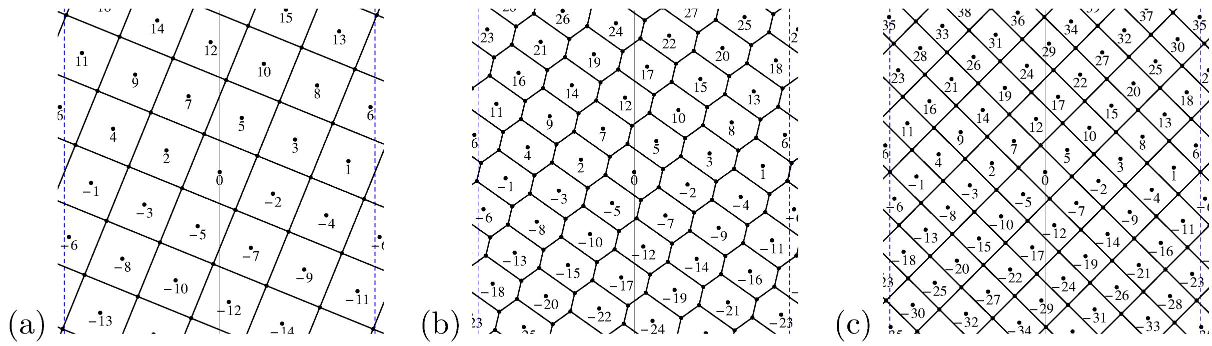

Figure 4. Voronoi tilings $\mathcal{V}((\sqrt{2}-1)+\mathrm{i}y)$. (a) $y = 0.0349189\cdots$, a rectangular tiling with an opposed parastichy pair $\{2,5\}$. (b) $y = 0.02$, a hexagonal tiling with opposed parastichy pairs $\{2,5\}$, $\{7,5\}$. (c) $y = 0.0142855\cdots$, a rectangular tiling with an opposed parastichy pair $\{7,5\}$.

for $j \geq 0$, $0 < k \leq a_{j+1}$. As $j \to +\infty$, they tend to the periodic sequence of continued fractions

$$[k, \overline{b_s, \dots, b_1, b_d, \dots, b_{s+1}}] \text{ and } [b_{s+1} - k, \overline{b_{s+2}, \dots, b_d, b_1, \dots, b_{s+1}}],$$

respectively. However, we have

$$\begin{aligned} &[k, \overline{b_s, \dots, b_1, b_d, \dots, b_{s+1}}] \cdot [b_{s+1} - k, \overline{b_{s+2}, \dots, b_d, b_1, \dots, b_{s+1}}] \\ &= (k - \omega'_{s+1})(-k + \omega_{s+1}) \\ &= -h_{s+1}(k) \end{aligned}$$

for $0 \leq s < d$, $0 < k \leq b_{s+1}$. This completes the proof. $\square$

In the algebraic theory of phyllotaxis [6], it is known that the most common divergences x are the quadratic irrationals such that $a_j = 1$ for sufficiently large j.

Theorem 5.3. *Let x be a quadratic irrational such that $a_j = 1$ for sufficiently large j. Then $\Omega(x) = \{\mathrm{i}\}$.*

Proof. The golden section has the purely periodic continued fraction expansion $\tau = [1, 1, \dots] = [1, \overline{1, 1}]$, and it is a root of a quadratic polynomial $h(x) = x^2 - x - 1$. Thus we have $-h(1) = 1$, and hence $\Omega(x) = \{\mathrm{i}\}$. $\square$

Rothen and Koch [18] observed that if $x = [\overline{a_1}] = [a_1, a_1, \dots]$ for some integer $a_1 > 0$, then $q_{j,k}/q_j$ is close to $q_{j+1,k}/q_{j+1}$ and $\langle xq_{j,k}\rangle/\langle xq_j\rangle$ is close to $\langle xq_{j+1,k}\rangle/\langle xq_{j+1}\rangle$ for $j \geq 0$, $0 < k \leq a_1$.

Figure 4 shows the *parastichy transition* of $\mathcal{V}(x + \mathrm{i}y)$, where $x = \sqrt{2} - 1$ is the silver ratio. The denominators of its convergents, $1, 2, 5, 7, 12, \cdots$, are called silver numbers.

Since we have $\Omega(\sqrt{2}-1) = \{\mathrm{i}, \mathrm{i}\sqrt{2}, \frac{\mathrm{i}}{\sqrt{2}}\}$, the limit shapes of the rectangular tiles are the square and the one with the aspect ratio $1 : \sqrt{2}$.

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