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# Min-Plus Algebra and Networks

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**Abstract**

The present paper deals with two topics from linear algebra over min-plus algebra. One is the linear equation and the other is the eigenvalue problem. The Bellman equation for the shortest path length is the min-plus analogue of the linear equation and can be solved by Bellman-Ford algorithm, which is the min-plus analogue of the Jacobi iteration algorithm. One can accelerate the Bellman-Ford algorithm by the similar way as in the Gauss-Seidel algorithm for the linear equation. Next, it is proved that the unique eigenvalue of a matrix with entries in min-plus algebra comes from the minimal average weight of circuits in the network associated with the matrix. It is also proved that corresponding eigenvectors arise in the column vectors of the minimal weight matrix of the specified network. Finally, it is proved that unique right eigenvalue coincide with the unique left eigenvalue.

## § 1. Introduction

The purpose of the present paper is to investigate the relation between the network on the digraph and the min-plus algebra. Min-plus algebra is one of many idempotent semirings which have been studied in various fields of mathematics. Many theorems and techniques that we use in usual linear algebra seems to have analogues in linear algebra over min-plus algebra. However, such kind of investigation have not yet exploited sufficiently. First, we focus on the linear equation over the min-plus algebra. The min-plus analogue of the linear equation which is solved by the Jacobi's iteration procedure become the Bellman equation for the shortest path length. The shortest path problem is one of the most important optimization problems in the network on digraphs. There are various combinatorial algorithms for solving the Bellman equation

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and finding the shortest path or the shortest path length. Examples of such algorithms are the Bellman-Ford algorithm, Floyd-Warshall algorithm and Dijkstra algorithm. The Bellman-Ford algorithm is very primitive and simple, and solve the Bellman equation for the shortest path length by iteration, which is considered to be the analogue of the Jacobi iteration algorithm for the conventional linear equation. In the Jacobi iteration algorithm, if we do not use the values obtained in the previous step but use the renewed values obtained in the present step in the renewal of the values in the iteration of the algorithm, we can accelerate the Jacobi algorithm. This acceleration algorithm is known as the Gauss-Seidel algorithm. If one accelerates the Bellman-Ford algorithm on the min-plus algebra by the same way as in the Gauss-Seidel algorithm for the conventional linear equation, one can get a new algorithm for solving the Bellman equation for the shortest path length.

Next, we consider the eigenvalue problem of matrices with entries in min-plus algebra and characterize the eigenvalues and corresponding eigenvectors using the terminology of the network theory on graphs. We define the network $\mathcal{N}(A)$ associated with the matrix $A$ with entries in min-plus algebra. We show that the minimal average weight $\lambda$ of the circuits in the network $\mathcal{N}(A)$ become the eigenvalue of $A$. Also we show that the corresponding eigenvectors appears as the column vectors of the minimal weight matrix of the specified network which is obtained from $\mathcal{N}(A)$ by shifting weights of every edges by the minimal average weight $\lambda$. Further, we prove that the minimal average weight of the network $\mathcal{N}(A)$ is the unique eigenvalue of the matrix $A$. Finally we refer to the coincidence between the unique right eigenvalue and the unique left eigenvalue.

## § 2. Min-Plus Algebra

### § 2.1. Basic Notations and Definitions

Let $\mathbb{R}$ be the field of real numbers. We define the min-plus algebra $\mathbb{R}_{\min}$ by $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$, with the binary operations $\oplus$ and $\otimes$:

$$a \oplus b = \min\{a, b\} \quad , \quad a \otimes b = a + b \ .$$

Both operations $\oplus$ and $\otimes$ are associative and commutative: We have

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c \ \ , \ a \otimes (b \otimes c) = (a \otimes b) \otimes c$$
$$a \oplus b = b \oplus a \ \ , \ a \otimes b = b \otimes a$$

for all $a, b, c \in \mathbb{R}_{\min}$. The algebra $\mathbb{R}_{\min}$ has the identity $\varepsilon = +\infty$ with respect to $\oplus$:

$$a \oplus \varepsilon = \varepsilon \oplus a = \min\{a, +\infty\} = a \ ,$$

and the identity $e = 0$ with respect to $\otimes$:

$$a \otimes e = e \otimes a = a + 0 = a \ .$$

If $x \neq \varepsilon$, there exists the unique inverse $y$ $(= -x)$ of $x$ with respect to $\otimes$:

$$x \otimes y = e \ .$$

The operation $\otimes$ is distributive with respect to $\oplus$:

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z) \ .$$

The identity $\varepsilon = +\infty$ with respect to $\oplus$ is absorbing for $\otimes$:

$$x \otimes \varepsilon = \varepsilon \otimes x = x + \infty = +\infty = \varepsilon \ .$$

The operation $\oplus$ is idempotent:

$$x \oplus x = \min\{x, x\} = x \ .$$

**Definition 2.1.** For $x \in \mathbb{R}_{\min}$ and $k \in \mathbb{N}$, the $k^{\text{th}}$ power of $x$ is defined by

$$x^{\otimes k} = \underbrace{x \otimes x \otimes \ldots \otimes x}_{k \text{ times}} \ .$$

In $\mathbb{R}_{\min}$, the $k^{\text{th}}$ power of $x$ reduces to the conventional multiplication $x^{\otimes k} = kx$.

## § 2.2. Matrix Algebra over Min-Plus Algebra

For positive integers $m$ and $n$, we denote by $\mathbb{R}_{\min}^{m\times n}$ the set of all $m \times n$ matrices with entries in $\mathbb{R}_{\min}$. We define the several operations in $\mathbb{R}_{\min}^{m\times n}$ analogous to those in the conventional matrix algebra as follows.

**Definition 2.2.**

1. For $A = (a_{ij}) \in \mathbb{R}_{\min}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{R}_{\min}^{m\times n}$, we define their sum $A \oplus B \in \mathbb{R}_{\min}^{m\times n}$ by
$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \min\{a_{ij}, b_{ij}\} \ .$$
2. For $A = (a_{ij}) \in \mathbb{R}_{\min}^{m\times k}$ and $B = (b_{ij}) \in \mathbb{R}_{\min}^{k\times n}$, we define their product $A \otimes B \in \mathbb{R}_{\min}^{m\times n}$ by
$$[A \otimes B]_{ij} = \bigoplus_{\ell=1}^{k} (a_{i\ell} \otimes b_{\ell j}) = \min_{\ell=1,2,\ldots,k} \{a_{i\ell} + b_{\ell j}\} \ .$$
3. For $A = (a_{ij}) \in \mathbb{R}_{\min}^{m\times n}$, we define its transpose ${}^tA \in \mathbb{R}_{\min}^{n\times m}$ of $A$ by $[{}^tA]_{ij} = a_{ji}$.

4. Define the matrix $I_n \in \mathbb{R}_{\min}^{n\times n}$ by
$$[I_n]_{ij} = \begin{cases} e & \text{if} \quad i = j \\ \varepsilon & \text{if} \quad i \neq j \ . \end{cases}$$
Then we see that $A \otimes I_n = I_n \otimes A = A$ for $A \in \mathbb{R}_{\min}^{n\times n}$, which means that $I = I_n$ becomes the identity with respect to the matrix multiplication in $\mathbb{R}_{\min}^{n\times n}$.

5. For $A \in \mathbb{R}_{\min}^{n\times n}$ and $k \in \mathbb{N}$, we define the $k^{\text{th}}$ power of $A$ by
$$A^{\otimes k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ times}} \ .$$
For $k = 0$, we set $A^{\otimes 0} = I$.

6. For $A = (a_{ij}) \in \mathbb{R}_{\min}^{m\times n}$ and $\alpha \in \mathbb{R}_{\min}$, we define the scalar multiplication $\alpha \otimes A \in \mathbb{R}_{\min}^{m\times n}$ by
$$[\alpha \otimes A]_{ij} = \alpha \otimes a_{ij} \ .$$

The operation $\oplus$ is commutative in $\mathbb{R}_{\min}^{m\times n}$, but $\otimes$ is not. The operation $\otimes$ is distributive with respect to the operation $\oplus$ in the matrix algebra $\mathbb{R}_{\min}^{m\times n}$. Also $\oplus$ is idempotent in $\mathbb{R}_{\min}^{m\times n}$, that is, we have $A \oplus A = A$.

## § 3. Graph Theory and Min-Plus Algebra

### § 3.1. Graphs

A directed graph or, for short, a digraph $G$ consists of the finite sets $V$ and $E$; an element $v \in V$ is called a vertex and an element $e \in E$ is called an edge of the digraph $G$. An edge $e \in E$ can be expressed as an ordered pair $e = (u, v)$ of vertices $u, v \in V$. We introduce the maps $\partial^- : E \to V$ and $\partial^+ : E \to V$ by $\partial^-(e) = u$, $\partial^+(e) = v$ for $e = (u, v)$; the vertices $u$ and $v$ are called the tail and the head of the edge $e = (u, v)$ respectively; vertices $u$ and $v$ are simply called the end vertices of the edge $e = (u, v)$. If distinct edges $e$ and $e'$ have two end vertices in common, then one of the cases (i) $\partial^-(e) = \partial^-(e')$, $\partial^+(e) = \partial^+(e')$ or (ii) $\partial^-(e) = \partial^+(e')$, $\partial^+(e) = \partial^-(e')$ occurs; in the former case, edges $e$ and $e'$ are called parallel edges and in the latter case, they are called antiparallel edges. An edge with just one end vertex ($\partial^-(e) = \partial^+(e)$) is called a loop. A graph without loops, parallel edges and antiparallel edges is called simple. A path in $G$ is an alternating sequence $P = (v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_s}, v_{i_s})$ of pairwise distinct vertices and edges except the vertices $v_{i_0}$ and $v_{i_s}$ such that each edge $e_{i_j}$ has the tail $v_{i_{j-1}}$ and the head $v_{i_j}$. Vertices $v_{i_0}$ and $v_{i_s}$ are respectively called the initial and the terminal vertex of the path $P$; $P$ is called a path from $v_{i_0}$ to $v_{i_s}$ or $v_{i_0}$-$v_{i_s}$ path. If the initial and the terminal vertex of a path $P$ coincide, $P$ is called a closed path or a circuit. We consider a loop or a pair of antiparallel edges as a circuit.

## § 3.2. Network on Graphs

Let $G = (V, E)$ be a digraph with $n$ vertices and $m$ edges. We assign a real number $w(e)$ to each edge $e \in E$; $w(e)$ is called the weight of the edge $e$. The pair $\mathcal{N} = (G, w)$ is called a network on the digraph $G$.

**Definition 3.1.** Let $\mathcal{N} = (G, w)$ be a network on the digraph $G$. We define the weighted adjacency matrix $A(\mathcal{N}) = A = (a_{ij}) \in \mathbb{R}^{n\times n}$ of $\mathcal{N}$ by

$$a_{ij} = \begin{cases} w((v_i, v_j)) & \text{if} \quad (v_i, v_j) \in E \\ 0 & \text{if} \quad (v_i, v_j) \notin E \ . \end{cases}$$

Let $P = (v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_s}, v_{i_s})$ be a path in $G$. The length $\ell(P)$ of the path $P$ is the number $s$ of edges in $P$; the weight $\omega(P)$ of the path $P$ is the sum of weights of edges in the path:

$$\omega(P) = \sum_{j=1}^{s} w(e_{i_j}) = \sum_{k=0}^{s-1} a_{i_k i_{k+1}} \ .$$

For a circuit $C$, we define the length $\ell(C)$ and the weight $\omega(C)$ of $C$ in the same way as for paths.

**Definition 3.2.** For the circuit $C$, we define the average weight $\mathrm{ave}(C)$ of a circuit $C$ by

$$\mathrm{ave}(C) = \frac{\omega(C)}{\ell(C)} \ .$$

**Definition 3.3.** Let $\mathcal{N}$ be a network on the digraph $G = (V, E)$ with the set of vertices $V = \{v_1, \ldots, v_n\}$. We denote by $a^*_{ij}$ $(i, j = 1, \ldots, n)$ the minimal value of weights of all $v_i$-$v_j$ paths in $G$. We set $a^*_{ij} = \infty$, if there exists no $v_i$-$v_j$ path. Define the minimal weight matrix $A^*(\mathcal{N}) = A^* \in \mathbb{R}^{n\times n}_{\min}$ by $A^* = (a^*_{ij})$.

## § 3.3. Adjacency matrix with values in Min-Plus Algebra

Let $G = (V, E)$ be a digraph with $n$ vertices and $m$ edges and $\mathcal{N} = (G, w)$ be a network on $G$. Let $A(\mathcal{N}) = (a_{ij})$ be the weighted adjacency matrix of the network $\mathcal{N}$.

**Definition 3.4.** We define the weighted adjacency matrix $\widetilde{A} = (\widetilde{a}_{ij})$ with values in $\mathbb{R}_{\min}$ of the network $\mathcal{N}$ by

$$\widetilde{a}_{ij} = \begin{cases} a_{ij} & \text{if} \quad (v_i, v_j) \in E \\ +\infty & \text{if} \quad (v_i, v_j) \notin E \ . \end{cases}$$

Since the conventional addition $+$ becomes the operation $\otimes$ in $\mathbb{R}_{\min}$, we compute the weight $\omega(P)$ of the path $P = (v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_s}, v_{i_s})$ in the digraph $G$ as follows:

$$\omega(P) = \bigotimes_{k=0}^{s-1} \widetilde{a}_{i_k i_{k+1}} \ .$$

For an arbitrary matrix $A \in \mathbb{R}_{\min}^{n\times n}$, we can define the network $\mathcal{N} = (G, w)$ on the digraph $G$ whose weighted adjacency matrix with values in $\mathbb{R}_{\min}$ coincides with $A$. We denoted such network by $\mathcal{N}(A)$ and call the network associated with the matrix $A \in \mathbb{R}_{\min}^{n\times n}$.

**Proposition 3.5** ([5, 7]). *Given a matrix $A \in \mathbb{R}_{min}^{n\times n}$. Assume that the network $\mathcal{N}(A)$ has no circuits of negative weight. Then the power sum*

$$A^{(k)} = I \oplus A \oplus A^{\otimes 2} \oplus \cdots \oplus A^{\otimes k}$$

*become stable for $k = n-1$ : that is, we have $A^{(n-1)} = A^{(n)} = \cdots$. Further, the minimal weight matrix $A^*$ of the network $\mathcal{N}(A)$ is given by $A^* = A^{(n-1)}$*

## § 4. Shortest Path Problem and Min-Plus Algebra

Let $G = (V, E)$ be a digraph with the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and the set of edges $E = \{e_1, e_2, \ldots, e_m\}$, and $\mathcal{N} = (G, w)$ be a network on $G$. Fix the initial vertex $v_1$ and denote by $y_j$ $(j = 1, \ldots, n)$ the minimal value of weights of the $v_1$-$v_j$ paths. Then it is well known that $y_1, y_2, \ldots, y_n$ satisfy the so-called Bellman equation:

**Proposition 4.1** (Bellman equation, [2]). *Let $A = (a_{ij})$ be the usual weighted adjacency matrix of the network on the graph with $n$ vertices and $m$ edges. We assume that the network has no circuits of negative weight. Then the minimal weight $y_1, \ldots, y_n$ satisfy the following equation:*

$$\begin{cases} y_1 = 0 \\ y_j = \min\{y_j, \min\limits_{k=1,\ldots,n}\{y_k + a_{kj}\}\} \ . \end{cases} \tag{4.1}$$

*The second expression in* (4.1) *is called the Bellman equation.*

The solution of the Bellman equation (4.1) is unique if the network has no circuits of negative weight. In order to investigate into the Bellman equation from the min-plus algebra view point, we consider the linear equation on min-plus algebra with respect to the unknown min-plus row vector $\boldsymbol{y}$ as follows:

$$\boldsymbol{y} = \boldsymbol{y} \otimes A \oplus \boldsymbol{b} \tag{4.2}$$

where $A \in \mathbb{R}_{\min}^{n\times n}$ is a min-plus square matrix of order $n$ and $\boldsymbol{b}$ is a min-plus row-vector of order $n$. This equation (4.2) is a min-plus analogue of the equation on the conventional linear algebra:

$$\boldsymbol{y} = \boldsymbol{y}A + \boldsymbol{b} \ . \tag{4.3}$$

Rewriting Eq.(4.3) as

$$\boldsymbol{y}(I - A) = \boldsymbol{b},$$

we see that the solution of Eq.(4.3) can be written formally in terms of the formal power series as

$$\boldsymbol{y} = \boldsymbol{b}(I - A)^{-1} = \boldsymbol{b}(I + A + A^2 + \cdots) \ .$$

If the power series of matrices $I + A + A^2 + \cdots$ converges, then the above formula actually gives the solution of the Eq.(4.3). Such a solution can be computed by the iteration $\boldsymbol{y}^{(k+1)} = \boldsymbol{y}^{(k)}A + \boldsymbol{b}$ ($\boldsymbol{y}^{(0)} = \boldsymbol{b}$) and this iteration algorithm is well-known as the Jacobi algorithm.

Let $A \in \mathbb{R}_{\min}^{n\times n}$ be a weighted adjacency matrix of the network $\mathcal{N}$ with values in $\mathbb{R}_{\min}$ and let $I \in \mathbb{R}_{\min}^{n\times n}$ be the identity matrix with values in $\mathbb{R}_{\min}$. Then it is easily verified that the Bellman equation is equivalent to the min-plus linear equation

$$\boldsymbol{y} = \boldsymbol{y} \otimes (I \oplus A) \tag{4.4}$$

with values in $\mathbb{R}_{\min}$ subject to the condition $y_1 = 0$. We must note that the equation (4.4) is equivalent to the linear equation (4.2) for the weighted adjacency matrix $A$ of the network without circuits of negative weight under the conditions $y_1 = 0$ and $\boldsymbol{b} = (0, +\infty, \ldots, +\infty)$. In order to solve the linear equation (4.4), for the minimal weights $y_1 \ldots, y_n$, we can apply the min-plus analogue of the Jacobi algorithm, which is called the Bellman-Ford algorithm:

**Algorithm 4.2** (Bellman-Ford algorithm, [2, 4])**.** Let $A$ be the weighted adjacency matrix of the network with $n$ vertices and $m$ edges. We assume that the network has no circuits of negative weight. The minimal weight values $y_1, \ldots, y_n$ are computed by the following procedure:

1. Set $y_1 := 0$ and $y_j := +\infty$ for $j = 2, \ldots, n$.
2. Set $y'_i := y_i$ for $i = 1, \ldots, n$ and compute

$$y_j := \min\{y'_j, \min_{k=1,\ldots,n} \{y'_k + a_{kj}\}\} \ .$$

3. If $y_i = y'_i$ for $i = 1, \ldots, n$ then return $\boldsymbol{y}$ else go to step 2.

It is easy to check that the Bellman-Ford algorithm perform the iteration:

$$\boldsymbol{y}^{(0)} = \boldsymbol{b}, \quad \boldsymbol{y}^{(k+1)} = \boldsymbol{y}^{(k)} \otimes (I \oplus A) \ .$$

In order to verify the validity of the algorithm, we compute $\boldsymbol{y}^{(n-1)} = \boldsymbol{b} \otimes (I \oplus A)^{\otimes(n-1)}$. Using the idempotency of the $\oplus$, we obtain

$$(I + A)^{\otimes k} = I \oplus A^{\otimes 2} \oplus \cdots \oplus A^{\otimes k} \qquad \text{for } k = 1, 2, \ldots, \ .$$

It follows from Proposition 3.5 that we have $\boldsymbol{y}^{(n-1)} = \boldsymbol{b} \otimes A^*$, which shows that $\boldsymbol{y}^{(n-1)}$ gives the minimum weight vector from the initial vertex $v_1$. In the above proof of the validity of the algorithm, one finds the power sum expression of the solution of the equation (4.4), from which one see that the Bellman-Ford iteration is nothing but a min-plus Jacobi iteration for the solution of the linear equation (4.2).

Next, we consider the acceleration of the Jacobi algorithm. As an example of such accelerating algorithms, we consider the famous algorithm named the Gauss-Seidel algorithm. Min-plus version of the Gauss-Seidel algorithm accelerating the Bellman-Ford algorithm is described formally as follows:

**Algorithm 4.3** (Gauss-Seidel algorithm over min-plus algebra, [5])**.** Consider the Bellman-Ford equation (4.4). Let $B = I \oplus A$ be the matrix defined by min-plus sum of the identity matrix $I$ and the weighted adjacency matrix $A \in \mathbb{R}_{\min}^{n\times n}$ with values in $\mathbb{R}_{\min}$. We can obtain $B$ from $A$ by substituting the diagonal elements of $A$ by 0. We decompose the matrix $B = I \oplus A$ as the min-plus sum $B = U \oplus L$ of an upper triangular matrix $U$ and a lower triangular matrix $L$. Matrices $U = (u_{ij})$ and $L = (l_{ij})$ are defined respectively by

$$u_{ij} = \begin{cases} b_{ij} = a_{ij} & \text{if } i < j \\ +\infty & \text{if } i \leq j \end{cases} \quad \text{and} \quad l_{ij} = \begin{cases} +\infty & \text{if } i > j \\ 0 & \text{if } i = j \\ b_{ij} = a_{ij} & \text{if } i > j \end{cases}$$

Then the solution of (4.4) can be computed by the following iteration:

1. Set $\boldsymbol{b} := (0, +\infty, \ldots, +\infty)$ and $\boldsymbol{y} := (0, +\infty, \ldots, +\infty)$.
2. Set $\boldsymbol{y}' := \boldsymbol{y}$ and compute
$$\boldsymbol{y} := \boldsymbol{y} \otimes U \oplus \boldsymbol{y}' \otimes L \ .$$
3. If $\boldsymbol{y} = \boldsymbol{y}'$ then return $\boldsymbol{y}$ as the solution of (4.4) else go to step 2.

The above description of the min-plus Gauss-Seidel algorithm is formal and not easy to understand. So we explain the iteration algorithm more explicitly and make clear the difference between the Bellman-Ford algorithm. In the Bellman-Ford iteration, the old values $\boldsymbol{y}$ are substituted to the variable $\boldsymbol{y}'$ and values of $\boldsymbol{y}$ are renewed by the formula:

$$y_j := \min\{y'_j, \min_{k=1,\dots,n}\{y'_k + a_{kj}\}\} \ .$$

in terms of old values $y'_j (j = 1, \dots, n)$. On the other hand, in the Gauss-Seidel iteration, the values $\boldsymbol{y}$ are renewed by the formula:

$$\begin{aligned} y_j &= \min\{\min_{k=1,\dots,j-1}\{y_k + u_{kj}\}, \min_{k=j,\dots,n}\{y'_k + l_{kj}\}\} \\ &= \min\{y'_j, \min_{k=1,\dots,j-1}\{y_k + a_{kj}\}, \min_{k=j+1,\dots,n}\{y'_k + a_{kj}\}\} \end{aligned}$$

We must note that the renewal of the value $y_j$ is executed in terms of the previously renewed values $y_1, \dots, y_{j-1}$ and the old values $y'_{j+1}, \dots, y'_n$ and this perform the the acceleration of the Bellman-Ford algorithm.

In some references, one see the description that the Gauss-Seidel acceleration algorithm of the Bellman-Ford algorithm is the Floyd-Warshall algorithm [3] for computing the minimal weight matrix $A^*$. We recognize that there are some kind of similarities between two algorithms, but we have not yet obtained the sufficient reason to assert that they are the same. We will leave the clarification of the similarities of the two algorithms for the future study.

## § 5. Eigenvalue Problem over Min-Plus Algebra

In this section, we show that the min-plus eigenvalues and eigenvectors admit a graph theoretical interpretation. Throughout this section, we consider the digraph $G = (V, E)$ with the set of $n$ vertices $V = \{1, 2, \dots, n\}$ and $m$ edges. Then, an edge $e \in E$ can be expressed as a pair $e = (i, j),\ i, j \in V$.

**Definition 5.1.** Given a matrix $A \in \mathbb{R}_{\min}^{n\times n}$, we say that $\lambda \in \mathbb{R}_{\min}$ is a right eigenvalue of $A$ if there exists $\boldsymbol{x} \in \mathbb{R}_{\min}^n$ such that $\boldsymbol{x} \neq {}^t(\varepsilon, \varepsilon, \dots, \varepsilon)$ and

$$A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x} \ .$$

The vector $\boldsymbol{x}$ is called the right eigenvector of $A$ belonging to the right eigenvalue $\lambda$. Similarly, we say that $\lambda' \in \mathbb{R}_{\min}$ is a left eigenvalue of $A$ if there exists $\boldsymbol{y} \in \mathbb{R}_{\min}^n$ such that $\boldsymbol{y} \neq {}^t(\varepsilon, \varepsilon, \dots, \varepsilon)$ and

$${}^tA \otimes \boldsymbol{y} = \lambda' \otimes \boldsymbol{y} \quad (\text{ or } {}^t\boldsymbol{y} \otimes A = \lambda' \otimes {}^t\boldsymbol{y}) \ .$$

The vector $\boldsymbol{y}$ is called the left eigenvector of $A$ belonging to the left eigenvalue $\lambda'$.

We allow a right eigenvalue and a left eigenvalue to have the value $\varepsilon$. First, we characterize matrices $A$ having the right or the left eigenvalue $\varepsilon$. A matrix $A \in \mathbb{R}_{\min}^{n\times n}$ is said to have $\varepsilon$-columns if it has at least one column whose all entries are $\varepsilon$. Similarly, if a matrix $A$ has at least one row whose all entries are $\varepsilon$ then it is said to have $\varepsilon$-rows.

**Proposition 5.2.** *The identity $\varepsilon$ of $\oplus$ is a right eigenvalue of $A$ if and only if $A$ has $\varepsilon$-columns. Similarly, $\varepsilon$ is a left eigenvalue of $A$ if and only if $A$ has $\varepsilon$-rows.*

*Proof.* We prove the assertion for the right eigenvalues. Let $\boldsymbol{x}$ be a right eigenvector of $A$ belonging to the right eigenvalue $\lambda = \varepsilon$. From the definition, the right eigenvector $\boldsymbol{x}$ has at least one entry $x_j \neq \varepsilon$. Then we will prove that all entries of the $j^{\text{th}}$ column of $A$ are equal to $\varepsilon$. Suppose that one entry $a_{ij}$ of the $j^{\text{th}}$ column of $A$ satisfies $a_{ij} \neq \varepsilon$ for some $i$, then we have $a_{ij} \otimes x_j \neq \varepsilon$. On the other hand, we have $\lambda \otimes x_i = \varepsilon$ since $\lambda = \varepsilon$, which lead to the contradiction. Thus we have proved that the all entries of the $j^{\text{th}}$ column of $A$ are $\varepsilon$. This completes the proof of the if part. Next we prove the only if part. We assume that all entries of $j^{\text{th}}$ column of $A$ are $\varepsilon$. Then it is easy to show that the vector $\boldsymbol{x} = {}^t(x_1, \ldots, x_j, \ldots, x_n)$ with $x_j = a \neq \varepsilon$ and $x_i = \varepsilon$ $(i \neq j)$ becomes a right eigenvector of $A$ belonging to the right eigenvalue $\varepsilon$. We have proved the only if part. If we note that the expression ${}^t\boldsymbol{y} \otimes A = \lambda' \otimes {}^t\boldsymbol{y}$ is equivalent to the expression ${}^tA \otimes \boldsymbol{y} = \lambda' \otimes \boldsymbol{y}$, we can easily prove the assertion for the left eigenvalue $\lambda'$ from the assertion for the right eigenvalues. Thus we have completed the proof of the proposition. □

Next, we characterize matrices $A \in \mathbb{R}_{\min}^{n\times n}$ having the right eigenvalue $\lambda \neq \varepsilon$. We consider the case where matrices $A$ do not have $\varepsilon$-columns.

**Lemma 5.3.** *Assume that $A \in \mathbb{R}_{min}^{n\times n}$ does not have $\varepsilon$-columns. Then the network $\mathcal{N}(A)$ associated with the matrix $A$ has at least one circuit.*

*Proof.* The assumption is equivalent to the fact that any vertices in the network $\mathcal{N}(A)$ become the head of at least one edge. That is, for any $v_1 \in V$, there exist at least one edge $e_1$ with $e_1 = (v_2, v_1), v_2 \in V$. Applying the same procedure, we can find a sequence of vertices, $v_1, \ldots, v_i, v_{i+1}, \ldots$ such that $e_i = (v_{i+1}, v_i) \in E$ and in the sequence we find a circuit $C$ since the number of vertices is finite. □

**Definition 5.4.** Let $\lambda(\neq \varepsilon)$ be an element of $\mathbb{R}_{\min}$. We define the matrix $A_\lambda$ by $[A_\lambda]_{ij} = [A]_{ij} - \lambda$.

We assume that the matrix $A \in \mathbb{R}_{\min}^{n\times n}$ does not have $\varepsilon$-columns. Then it follows from Lemma 5.3 that the network $\mathcal{N}(A)$ associated with the matrix $A$ has circuits. Let $\lambda$ be the minimal value of the average weight of circuits in $\mathcal{N}(A)$ and consider the network $\mathcal{N}(A_\lambda)$ associated with the matrix $A_\lambda$. Since the network $\mathcal{N}(A_\lambda)$ does not have circuits with negative weights, we can compute by Proposition 3.5 the minimal weight matrix $A_\lambda^*$ by the power sum : $A_\lambda^* = I \oplus A_\lambda \oplus A_\lambda^{\otimes 2} \oplus \cdots$.

**Theorem 5.5.** *Let $A \in \mathbb{R}_{min}^{n\times n}$ be a matrix without $\varepsilon$-columns and let $\lambda \neq \varepsilon$ be the minimal average weight of circuits in the network $\mathcal{N}(A)$. Let $C = ((v_1, v_2), (v_2, v_3), \ldots, (v_k, v_1))$ be the circuit in $\mathcal{N}(A)$ expressed as a sequence of edges and having the minimal average weight $\lambda$. Then the column vectors $[A_\lambda^*]_{v_1}, \ldots, [A_\lambda^*]_{v_k}$ of the minimal weight matrix $A_\lambda^*$ of the network $\mathcal{N}(A_\lambda)$ become the right eigenvectors of $A$ belonging to the right eigenvalue $\lambda$.*

*Proof.* We represent by $\nu$ one of the vertices in $\{v_1, v_2, \ldots, v_k\}$. Then it is enough to prove the following equality:

$$A \otimes [A_\lambda^*]_\nu = \lambda \otimes [A_\lambda^*]_\nu \tag{1}$$

First, we compute the left hand side of the equality (1). From the definition of $A_\lambda$, we have $A = \lambda \otimes A_\lambda$ and if we use the natation: $A_\lambda^+ = A_\lambda \oplus A_\lambda^{\otimes 2} \oplus \cdots$, we have

$$\begin{aligned} A \otimes [A_\lambda^*]_\nu &= \lambda \otimes A_\lambda \otimes [A_\lambda^*]_\nu \\ &= \lambda \otimes A_\lambda \otimes [I \oplus A_\lambda \oplus A_\lambda^{\otimes 2} \oplus \cdots]_\nu \\ &= \lambda \otimes [A_\lambda \oplus A_\lambda^{\otimes 2} \oplus \cdots]_\nu \\ &= \lambda \otimes [A_\lambda^+]_\nu \end{aligned}$$

where $I$ is the identity matrix in $\mathbb{R}_{\min}^{n\times n}$. Thus the equality (1) is rewritten as:

$$\lambda \otimes [A_\lambda^+]_\nu = \lambda \otimes [A_\lambda^*]_\nu \ .$$

So it is enough to prove $[A_\lambda^+]_\nu = [A_\lambda^*]_\nu$. The entries $[A_\lambda^*]_{i\nu}$ $(i = 1, 2, \ldots, n)$ are given by:

$$[A_\lambda^*]_{i\nu} = \begin{cases} \varepsilon \oplus [A_\lambda^+]_{i\nu} \text{ if } i \neq \nu \\ e \oplus [A_\lambda^+]_{i\nu} \text{ if } i = \nu \end{cases}$$

So the identity for the $i \neq \nu$ case is trivial. Consider the identity for $i = \nu$ case. The entries $[A_\lambda^+]_{\nu\nu}$ indicate the minimal weight of $\nu$-$\nu$ path in the network $\mathcal{N}(A_\lambda)$. Since $\lambda$ is the minimal average weight of the circuit $C$ in $\mathcal{N}(A)$ and $\nu$ is an arbitrary vertex in the circuit $C$, the minimal weight of $\nu$-$\nu$ path is equal to $e$. Thus we have proved $[A_\lambda^+]_\nu = [A_\lambda^*]_\nu$. This completes the proof of the theorem. □

Let $A \in \mathbb{R}_{\min}^{n\times n}$ be a matrix without $\varepsilon$-columns. Then we have proved that $A$ has an right eigenvalue $\lambda \neq \varepsilon$ which is the minimal average weight of the circuits in the network $\mathcal{N}(A)$ associated with $A$. Next we will prove that this right eigenvalue is the only right eigenvalue of the matrix $A$.

**Proposition 5.6.** *If the matrix $A \in \mathbb{R}_{min}^{n\times n}$ has a right eigenvalue $\lambda \neq \varepsilon$, there exists a circuit in the network $\mathcal{N}(A)$ whose average weight is equal to $\lambda$.*

*Proof.* Let $\lambda \neq \varepsilon$ be the right eigenvalue of $A$. By the definition, a right eigenvector $\boldsymbol{x}$ belonging to the right eigenvalue $\lambda$ has at least one entry $x_{v_1} \neq \varepsilon$. Hence we have $[A \otimes \boldsymbol{x}]_{v_1} = \lambda \otimes x_{v_1} \neq \varepsilon$. Thus we can find a vertex $v_2$ with $a_{v_1 v_2} \otimes x_{v_2} = \lambda \otimes x_{v_1}$. This implies that $a_{v_1 v_2} \neq \varepsilon$, $x_{v_2} \neq \varepsilon$ and $(v_1, v_2) \in E$. By the same argument we can find $v_3 \in V$ with $a_{v_2 v_3} \otimes x_{v_3} = \lambda \otimes x_{v_2}$ and $(v_2, v_3) \in E$. Applying the same procedure, we find the sequence of vertices $v_1, v_2, \ldots, v_i, \ldots$ such that $(v_{i-1}, v_i) \in E$. Since the number of vertices is finite, we can find the subsequence $(v_h, v_{h+1}, \ldots, v_{h+k})$, in which the vertices are pairwise distinct except $v_h = v_{h+k}$. Then the sequence of edges

$$C = ((v_h, v_{h+1}), (v_{h+1}, v_{h+2}), \ldots (v_{h+k-1}, v_h))$$

express the circuit $C$. The circuit $C$ has the length $\ell(C) = k$ and the weight $\omega(C) = \bigotimes_{j=0}^{k-1} a_{v_{h+j} v_{h+j+1}}$, where $v_h = v_{h+k}$. By the construction of sequence of vertices, we have

$$\bigotimes_{j=0}^{k-1} (a_{v_{h+j} v_{h+j+1}} \otimes x_{v_{h+j+1}}) = \lambda^{\otimes k} \otimes \bigotimes_{j=0}^{k-1} x_{v_{h+j}}.$$

Converting $\otimes$ to $+$ in conventional algebra, we have

$$\sum_{j=0}^{k-1} (a_{v_{h+j} v_{h+j+1}} + x_{v_{h+j+1}}) = k\lambda + \sum_{j=0}^{k-1} x_{v_{h+j}}.$$

Using the fact that

$$\sum_{j=0}^{k-1} x_{v_{h+j+1}} = \sum_{j=0}^{k-1} x_{v_{h+j}},$$

we obtain

$$\bigotimes_{j=0}^{k-1} a_{v_{h+j} v_{h+j+1}} = k\lambda,$$

which means that $w(C) = k\lambda$. Therefore we have proved that the average weight of the circuit $C$ is

$$\frac{\omega(C)}{\ell(C)} = \frac{k\lambda}{k} = \lambda$$

This completes the proof of the proposition. □

Proposition 5.6 shows that an arbitrary right eigenvalue of a matrix $A$ comes from the average weight of circuits in $\mathcal{N}(A)$.

**Theorem 5.7.** *Assume that $A \in \mathbb{R}_{min}^{n\times n}$ has a right eigenvalue $\lambda \neq \varepsilon$. Then the right eigenvalue $\lambda$ of $A$ is uniquely determined, and $\lambda$ is equal to the minimal average weight of circuits in $\mathcal{N}(A)$ associated with the matrix $A$.*

*Proof.* Let $\boldsymbol{x} = {}^t(x_1, \ldots, x_n)$ be an eigenvector belonging to the eigenvalue $\lambda$. We see from Proposition 5.6 that the network $\mathcal{N}(A)$ contains at least one circuit. Let $C = ((v_1, v_2), (v_2, v_3), \ldots, (v_k, v_1))$ be any one of circuits in $\mathcal{N}(A)$. Then we have

$$\bigotimes_{j=1}^{k} a_{v_j v_{j+1}} \otimes x_{v_{j+1}} \geq \lambda^{\otimes k} \otimes \bigotimes_{j=1}^{k} x_{v_j} \quad (v_{k+1} = v_1)$$

Using the same argument as in the proof of Proposition 5.6, we have

$$\frac{\omega(C)}{\ell(C)} \geq \frac{k\lambda}{k} = \lambda$$

This inequality holds for an arbitrary right eigenvalue $\lambda \neq \varepsilon$ of $A$ and for an arbitrary circuit $C$ in $\mathcal{N}(A)$. So the right eigenvalue $\lambda$ of $A$ has to be smaller or equal to the minimal average weight of the circuit in $\mathcal{N}(A)$. By Proposition 5.6, any right eigenvalue comes from the average weight of a circuit in $\mathcal{N}(A)$. Therefore the right eigenvalue $\lambda \neq \varepsilon$ of $A$ is uniquely determined and equal to the minimal average weight of the circuit in $\mathcal{N}(A)$. □

Here, we are concerned only with the right eigenvalue. We show that the unique right eigenvalue coincide with the unique left eigenvalue.

**Corollary 5.8.** *Assume that $A \in \mathbb{R}_{min}^{n\times n}$ has a right eigenvalue $\lambda \neq \varepsilon$ and a left eigenvalue $\lambda' \neq \varepsilon$. Then the unique right eigenvalue of $A$ is identical with the unique left eigenvalue of $A$.*

*Proof.* By the definition, $\lambda'$ is the left eigenvalue of $A$ if and only if it is the right eigenvalue of the transpose ${}^tA$ of $A$. Let $G = (V, E)$ be the digraph which defines the network $\mathcal{N}(A)$. We define the new digraph ${}^tG = (V, {}^tE)$ with the set of vertices $V$ and the set of edges ${}^tE$: The set of edges ${}^tE$ is defined by $(i, j) \in {}^tE$ if and only if $(j, i) \in E$. Let $w$ be the weight function on $E$ defined by the matrix $A$. We define the weight function $\bar{w}$ on ${}^tE$ by $\bar{w}((i, j)) = w((j, i))$. Thus we can define the network $({}^tG, \bar{w})$. It is easy to verify the weighted adjacency matrix with values in $\mathbb{R}_{\min}$ of the network $({}^tG, \bar{w})$ coincides with the matrix ${}^tA$. It follows from the definition that the minimal average weight of circuits in the network $(G, w)$ become the minimal average weight of the network $({}^tG, \bar{w})$ and vice versa. By the assumption of the corollary, $A$ and ${}^tA$ have the unique eigenvalue which is the minimal average weight of circuits in $(G, w)$

or $({}^tG, \bar{w})$ respectively. Then we see that the unique eigenvalue of $A$ and ${}^tA$ coincide. Thus we have completed the proof of the corollary. $\square$

## § 6. Conclusion

In the present paper, first we focus on the shortest path problem. Bellman's equation for the shortest path problem can be solved by the Bellman-Ford algorithm, which is the min-plus analogue of Jacobi's iteration algorithm for linear equations. It is well known that Jacobi's algorithm shift to the Gauss-Seidel algorithm by some kind of acceleration. We have proved that the Bellman-Ford algorithm shift to the new algorithm for the shortest path problem by the similar process as in the process from Jacobi's algorithm to Gauss Seidel algorithm. This result gives an example to suggest that the algorithms for the shortest path problems can be obtained as min-plus analogues of the algorithms for linear equations. Then are there any algorithms for linear equations whose min-plus analogues are the celebrated algorithms by Dijkstra or by Floyd-Warshall for the shortest path problem? This remains as interesting problems for the future study. Second, we focus on the eigenvalue problem of a matrix with entries in min-plus algebra. We characterize such eigenvalues as the minimal average weight of the circuits in the network associated with the matrix, and further prove that the eigenvectors appears in the column vectors of the minimal weight matrix of the certain specified network. We have not yet made clear the reason why non-minimal average weights of circuits do not appear as the eigenvalues. If we find the relation between all average weights of circuits in the given network and the eigenvalue problem of matrices with entries in min-plus algebra, we will find a new approach for enumeration of circuits of the network.

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