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The Stokes phenomenon for the q-difference equation satisfied by the basic hypergeometric series ${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)$

By

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Abstract

We show the connection formula for the basic hypergeometric series ${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)$ between around the origin and infinity by the using of the q-Borel-Laplace transformations. We also show the limit $q \to 1 - 0$ of the new connection formula.

§ 1. Introduction

In this paper, we show the connection formula for the *divergent* basic hypergeometric series

$$(1.1) \qquad {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x) = \sum_{n\geq 0} \frac{(a_1, a_2, a_3; q)_n}{(b_1; q)_n (q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{-1} x^n$$

between around the origin and around infinity by the using of the q-Borel-Laplace resummation methods.

At first, we review some basic notations. Hereafter, we assume that $0 < |q| < 1$. The q-shifted operator σ_q is given by $\sigma_q f(x) = f(qx)$. The function $(a; q)_n$ is the q-shifted factorial:

$$(a; q)_n := \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \ldots (1 - aq^{n-1}), & n \geq 1, \end{cases}$$

moreover, $(a; q)_\infty := \lim_{n\to\infty} (a; q)_n$ and

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \ldots (a_m; q)_\infty.$$

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The basic hypergeometric series with the base q [4, page 4] is

$$ {}_r\varphi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,x) := \sum_{n\geq 0} \frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s;q)_n(q;q)_n} \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{1+s-r} x^n. $$

The radius of convergence is $\infty, 1$ or 0 according to whether $r-s<1, r-s=1$ or $r-s>1$. Therefore, the function ${}_3\varphi_1(a_1,a_2,a_3;b_1;q,x)$ is the divergent series around the origin.

The series (1.1) satisfy the third order linear q-difference equation

$$ \left(a_1a_2a_3x - \frac{b_1}{q^2}\right)u(q^3x) - \left\{(a_1a_2+a_2a_3+a_3a_1)x - \left(\frac{b_1}{q^2}+\frac{1}{q}\right)\right\}u(q^2x) + \left\{(a_1+a_2+a_3)x - \frac{1}{q}\right\}u(qx) - xu(x) = 0. \tag{1.2} $$

Equation (1.2) also has a fundamental system of solutions around infinity:

$$ v_1(x) := \frac{\theta_q(a_1x)}{\theta_q(x)} {}_3\varphi_2\left(a_1, \frac{a_1q}{b_1}, 0; \frac{a_1q}{a_2}, \frac{a_1q}{a_3}; q, \frac{qb_1}{a_1a_2a_3x}\right) \tag{1.3} $$

$$ v_2(x) := \frac{\theta_q(a_2x)}{\theta_q(x)} {}_3\varphi_2\left(a_2, \frac{a_2q}{b_1}, 0; \frac{a_2q}{a_1}, \frac{a_2q}{a_3}; q, \frac{qb_1}{a_1a_2a_3x}\right) \tag{1.4} $$

$$ v_3(x) := \frac{\theta_q(a_3x)}{\theta_q(x)} {}_3\varphi_2\left(a_3, \frac{a_3q}{b_1}, 0; \frac{a_3q}{a_2}, \frac{a_3q}{a_1}; q, \frac{qb_1}{a_1a_2a_3x}\right). \tag{1.5} $$

In section 2, we show the connection formula between (1.3), (1.4) (1.5) and (1.1).

Here, $\theta_q(x)$ is the theta function of Jacobi, which plays an important role in connection problems on linear q-difference equations. The theta function with the base q is

$$ \theta_q(x) := \sum_{n\in\mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \qquad \forall x \in \mathbb{C}^*. $$

The theta function has the triple product identity

$$ \theta_q(x) = (q, -x, -q/x; q)_\infty . \tag{1.6} $$

The theta function satisfies the q-difference equation $\theta_q(q^kx) = q^{-\frac{n(n-1)}{2}} x^{-k}\theta_q(x)$, $\forall k \in \mathbb{Z}$. The theta function also has the inversion formula $\theta_q(1/x) = \theta_q(x)/x$.

For any fixed $\lambda \in \mathbb{C}^* \setminus q^{\mathbb{Z}}$, the set $[\lambda;q]$-spiral is $[\lambda;q] := \lambda q^{\mathbb{Z}} = \{\lambda q^k; k \in \mathbb{Z}\}$. We remark that $\theta(\lambda q^k/x) = 0$ if and only if $x \in [-\lambda;q]$.

The function $\theta_q(x)/\theta_q(q^\alpha x)$, $\forall \alpha \notin \mathbb{Z}$ satisfies a q-difference equation

$$ u(qx) = q^\alpha u(x), $$

which is also satisfied by the function $u(x) = x^{\alpha}$.

We review the connection problems on the linear q-difference equations. Connection problems on the linear q-difference equations with regular singular points were studied by G. D. Birkhoff [1]. Connection formulae for the second order linear q-difference equations are given by the matrix form

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} C_{11}(x) & C_{12}(x) \\ C_{21}(x) & C_{22}(x) \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}.$$

The pair $(u_1(x), u_2(x))$ is a fundamental system of solutions around the origin and the pair $(v_1(x), v_2(x))$ is a fundamental system of solutions around infinity. The connection coefficients $C_{jk}(x)$ $(1 \le j, k \le 2)$ are given by q-periodic and unique valued functions

$$\sigma_q C_{jk}(x) = C_{jk}(x), \quad C_{jk}(e^{2\pi i}x) = C_{jk}(x),$$

namely, the *elliptic functions*.

The first example of the connection formula was given by G. N. Watson [11] in 1910. Watson gave the connection formula for Heine's basic hypergeometric series

$${}_2\varphi_1(a, b; c; q, x) := \sum_{n \ge 0} \frac{(a, b; q)_n}{(c; q)_n (q; q)_n} x^n$$

around the origin and around the infinity [4, page 117]. Heine's ${}_2\varphi_1(a, b; c; q, x)$ satisfies the q-difference equation

$$\big[(c - abqx)\sigma_q^2 - \{(c + q) - (a + b)qx\}\,\sigma_q + q(1 - x)\big]\, u(x) = 0. \tag{1.7}$$

The equation (1.7) also has a fundamental system of solutions around the infinity:

$$y_\infty^{(a,b)}(x) = x^{-\alpha}{}_2\varphi_1\left(a, \frac{aq}{c}; \frac{aq}{b}; q, \frac{cq}{abx}\right)$$

and

$$y_\infty^{(b,a)}(x) = x^{-\beta}{}_2\varphi_1\left(b, \frac{bq}{c}; \frac{bq}{a}; q, \frac{cq}{abx}\right),$$

provided that $a = q^{\alpha}$ and $b = q^{\beta}$. Watson's connection formula for ${}_2\varphi_1(a, b; c; q, x)$ is given by

$$\begin{aligned} {}_2\varphi_1\left(a, b; c; q; x\right) = & \frac{(b, c/a; q)_\infty \theta_q(-ax)_\infty}{(c, b/a; q)_\infty \theta_q(-x)_\infty} \frac{\theta_q(x)}{\theta_q(ax)} y_\infty^{(a,b)}(x) \\ & + \frac{(a, c/b; q)_\infty \theta_q(-bx)_\infty}{(c, a/b; q)_\infty \theta_q(-x)_\infty} \frac{\theta_q(x)}{\theta_q(bx)} y_\infty^{(b,a)}(x). \end{aligned}$$

We remark that the connection coefficients are given by the q-elliptic functions.

But connection formulae for q-difference equations with irregular singular points had not known for a long time. We remark that A. Duval and C. Mitschi gave connection matrices for degenerated *differential* equations [3]. In the differential case, the divergent solution for the hypergeometric equations converges on the suitable sectors by the Borel-Laplace resummation process.

The irregularity of q-difference equations are studied by the using of the Newton polygons by J.-P. Ramis, J. Sauloy and C. Zhang [9]. In the q-difference case, the resummation of the divergent series converges on $\mathbb{C}^*$ without the points of the suitable $[\lambda; q]$-spirals. This point is essentially different from the differential case.

C. Zhang gave connection formulae for some confluent type basic hypergeometric series [12, 13, 14] where he uses the q-Borel-Laplace transformations. In [6, 7], the author gave the connection formula for the Hahn-Exton q-Bessel function and the q-confluent type function by the q-Borel-Laplace transformations. These resummation methods are powerful tools for connection problems on linear q-difference equations with irregular singular points.

Definition 1.1. We assume that $f(x)$ is a formal power series $f(x) = \sum_{n\in\mathbb{Z}} a_n x^n$, $a_0 = 1$.

1. The q-Borel transformation is

$$\left(\mathcal{B}_q^+ f\right)(\xi) := \sum_{n\in\mathbb{Z}} a_n q^{\frac{n(n-1)}{2}} \xi^n \; (=: \varphi(\xi)).$$

2. For any analytic function $\varphi(\xi)$ around $\xi = 0$, the q-Laplace transformation is

$$\left(\mathcal{L}_{q,\lambda}^+ \varphi\right)(x) := \frac{1}{1-q} \int_0^{\lambda\infty} \frac{\varphi(\xi)}{\theta_q\left(\frac{\xi}{x}\right)} \frac{d_q\xi}{\xi} = \sum_{n\in\mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)}.$$

Here, this transformation is given by Jackson's q-integral [4, page 23].

The definition is a special case of one of the q-Laplace transformations in [2, 12]. The q-Borel transformation is the formal inverse of the q-Laplace transformation as follows:

Lemma 1.2 (Zhang, [12]). *For any entire function $f(x)$, we have*

$$\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ f = f.$$

Thanks to these methods, some connection formulae for the second order q-difference equations were found. However, the connection formulae for more higher order linear

q-difference equations have not known. In this paper, especially we apply the q-Borel-Laplace transformations to the divergent series (1.1) to study the connection problem on the third order q-difference equation. In section 2, we show the following Theorem:

Theorem. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have*

$$\begin{aligned}
{}_3f_1(a_1, a_2, a_3; b_1; q; \lambda, x) &:= \left(\mathcal{L}^+_{q,\lambda} \circ \mathcal{B}^+_q {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)\right)(x) \\
&= \frac{(a_2, a_3, b_1/a_1; q)_\infty}{(b_1, a_2/a_1, a_3/a_1; q)_\infty} \frac{\theta_q(a_1\lambda)}{\theta_q(\lambda)} \frac{\theta_q(a_1qx/\lambda)}{\theta_q(qx/\lambda)} \frac{\theta_q(x)}{\theta_q(a_1x)} v_1(x) \\
&+ \frac{(a_1, a_3, b_1/a_2; q)_\infty}{(b_1, a_1/a_2, a_3/a_2; q)_\infty} \frac{\theta_q(a_2\lambda)}{\theta_q(\lambda)} \frac{\theta_q(a_2qx/\lambda)}{\theta_q(qx/\lambda)} \frac{\theta_q(x)}{\theta_q(a_2x)} v_2(x) \\
&+ \frac{(a_2, a_1, b_1/a_3; q)_\infty}{(b_1, a_2/a_3, a_1/a_3; q)_\infty} \frac{\theta_q(a_3\lambda)}{\theta_q(\lambda)} \frac{\theta_q(a_3qx/\lambda)}{\theta_q(qx/\lambda)} \frac{\theta_q(x)}{\theta_q(a_3x)} v_3(x).
\end{aligned}$$

Here, $\left(\mathcal{L}^+_{q,\lambda} \circ \mathcal{B}^+_q {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)\right)(x)$ is the q-Borel-Laplace transform of the divergent series ${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)$.

The connection coefficients(with the new parameter λ) are given by the q-elliptic functions. Therefore, the values of the connection coefficients change *continuously.* These coefficients are also the new example of the Stokes phenomenon [2] for the q-difference equation (1.2). We remark that the parameter λ corresponds to *"the path"* of the resummation process [12].

In the last section, we give the limit $q \to 1 - 0$ of the new connection formula. We review the q-gamma function to obtain the limit. The q-gamma function $\Gamma_q(x)$ is

$$\Gamma_q(x) := \frac{(q; q)_\infty}{(q^x; q)_\infty}(1 - q)^{1-x}.$$

The limit $q \to 1 - 0$ of $\Gamma_q(x)$ gives the gamma gunction [4, page 20]

$$\lim_{q \to 1-0} \Gamma_q(x) = \Gamma(x). \tag{1.8}$$

By the limit, finally we obtain the limit of the connection formula.

§ 2. The connection formula

In this section, we give the new connection formula for the basic hypergeometric series ${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)$. In section 2.1, we review the connection formula of non-degenerated series ${}_3\varphi_2(a_1, a_2, a_3; b_1, b_2; q, x)$.

§ 2.1. The non-degenerated case

The non-degenerated convergent series

$$ {}_3\varphi_2(a_1,a_2,a_3;b_1,b_2;q,x) := \sum_{n\geq 0} \frac{(a_1,a_2,a_3;q)_n}{(b_1,b_2;q)_n(q;q)_n} x^n \tag{2.1} $$

satisfies the third order q-difference equation

$$ \begin{aligned} &\left[\left(a_1a_2a_3x - \frac{b_1b_2}{q^2}\right)\sigma_q^3 - \left\{(a_1a_2+a_2a_3+a_3a_1)x - \left(\frac{b_1b_2}{q^2}+\frac{b_2}{q}+\frac{b_1}{q}\right)\right\}\sigma_q^2 \right.\\ &\left.\left\{(a_1+a_2+a_3)x - \left(\frac{b_1}{q}+\frac{b_2}{q}+1\right)\right\}\sigma_q - (x-1)\right]u(x) = 0. \end{aligned} \tag{2.2} $$

Equation (2.2) also has a fundamental system of solutions around infinity:

$$ \tilde{v}_1(x) = \frac{\theta_q(a_1x)}{\theta_q(x)} {}_3\varphi_2\left(a_1, \frac{a_1q}{b_1}, \frac{a_1q}{b_2}; \frac{a_1q}{a_2}, \frac{a_1q}{a_3}; q, \frac{qb_1b_2}{a_1a_2a_3x}\right), \tag{2.3} $$

$$ \tilde{v}_2(x) = \frac{\theta_q(a_2x)}{\theta_q(x)} {}_3\varphi_2\left(a_2, \frac{a_2q}{b_1}, \frac{a_2q}{b_2}; \frac{a_2q}{a_1}, \frac{a_2q}{a_3}; q, \frac{qb_1b_2}{a_1a_2a_3x}\right), \tag{2.4} $$

$$ \tilde{v}_3(x) = \frac{\theta_q(a_3x)}{\theta_q(x)} {}_3\varphi_2\left(a_3, \frac{a_3q}{b_1}, \frac{a_3q}{b_2}; \frac{a_3q}{a_2}, \frac{a_3q}{a_1}; q, \frac{qb_1b_2}{a_1a_2a_3x}\right). \tag{2.5} $$

The connection formula between the solutions (2.3), (2.4), (2.5) and (2.1) can be found in [4, page 121]. We remark that the following formula was essentially given by L. J. Slater.

Theorem 2.1 (Slater, [10]). *For any $x \in \mathbb{C}^*$, we have*

$$ \begin{aligned} {}_3\varphi_2(a_1,a_2,a_3;b_1,b_2;q,x) = &\frac{(a_2,a_3,b_1/a_1,b_2/a_1;q)_\infty}{(b_1,b_2,a_2/a_1,a_3/a_1;q)_\infty} \frac{\theta_q(-a_1x)}{\theta_q(-x)} \frac{\theta_q(x)}{\theta_q(a_1x)} \tilde{v}_1 \\ &+ \mathrm{idem}(a_1;a_2,a_3). \end{aligned} $$

Provided that the notation $\mathrm{idem}(a_1;a_2,a_3)$ *after an expression stands for the sum expressions obtained from the preceding expression by interchanging a_1 with each a_2 and a_3.*

This Theorem can be considered as the higher order extension of Watson's formula. By Theorem 2.1, we obtain the following key Lemma.

Lemma 2.2. *For any $x \in \mathbb{C}^*$, we have*

$$ \begin{aligned} &{}_3\varphi_2(a_1,a_2,a_3;b_1,0;q,x) \\ &= \frac{(a_2,a_3,b_1/a_1;q)_\infty}{(b_1,a_2/a_1,a_3/a_1;q)_\infty} \frac{\theta_q(-a_1x)}{\theta_q(-x)} {}_2\varphi_2\left(a_1, \frac{a_1q}{b_1}; \frac{a_1q}{a_2}, \frac{a_1q}{a_3}; q, \frac{q^2b_1}{a_2a_3x}\right) \\ &+ \mathrm{idem}(a_1;a_2,a_3). \end{aligned} $$

Proof. We tale the limit $b_2 \to 0$ in Theorem 2.1, we obtain the conclusion. □

In the next section, we prove our new connection formula by Lemma 2.2 and the q-Borel-Laplace transformations.

§ 2.2. Proof of main Theorem

In this section, we prove the following Theorem.

Theorem 2.3. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have*

$$
\begin{aligned}
&{}_3f_1(a_1, a_2, a_3; b_1; q; \lambda, x) := \left(\mathcal{L}^+_{q,\lambda} \circ \mathcal{B}^+_q {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)\right)(x) \\
&= \frac{(a_2, a_3, b_1/a_1; q)_\infty}{(b_1, a_2/a_1, a_3/a_1; q)_\infty} \frac{\theta_q(a_1\lambda)}{\theta_q(\lambda)} \frac{\theta_q(a_1qx/\lambda)}{\theta_q(qx/\lambda)} \frac{\theta_q(x)}{\theta_q(a_1x)} v_1(x) \\
&+ \frac{(a_1, a_3, b_1/a_2; q)_\infty}{(b_1, a_1/a_2, a_3/a_2; q)_\infty} \frac{\theta_q(a_2\lambda)}{\theta_q(\lambda)} \frac{\theta_q(a_2qx/\lambda)}{\theta_q(qx/\lambda)} \frac{\theta_q(x)}{\theta_q(a_2x)} v_2(x) \\
&+ \frac{(a_2, a_1, b_1/a_3; q)_\infty}{(b_1, a_2/a_3, a_1/a_3; q)_\infty} \frac{\theta_q(a_3\lambda)}{\theta_q(\lambda)} \frac{\theta_q(a_3qx/\lambda)}{\theta_q(qx/\lambda)} \frac{\theta_q(x)}{\theta_q(a_3x)} v_3(x).
\end{aligned}
$$

Proof. We apply the q-Borel transformation to the series ${}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)$.

$$
\left(\mathcal{B}^+_q {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)\right)(\xi) = {}_3\varphi_2(a_1, a_2, a_3; b_1, 0, -\xi) =: \varphi(\xi).
$$

By Lemma 2.2, we have another expression of the function $\varphi(\xi)$. We also apply the q-Laplace transformation $\mathcal{L}_{q,\lambda}$ to the function $\varphi(\xi)$, we obtain the conclusion. □

Remark. We remark that the fundamental system of solutions for equation (1.2) is given by

$$
v_1(x) := \frac{\theta_q(a_1x)}{\theta_q(x)} {}_3\varphi_2\left(a_1, \frac{a_1q}{b_1}, 0; \frac{a_1q}{a_2}, \frac{a_1q}{a_3}; q, \frac{qb_1}{a_1a_2a_3x}\right), \tag{2.6}
$$

$$
v_2(x) := \frac{\theta_q(a_2x)}{\theta_q(x)} {}_3\varphi_2\left(a_2, \frac{a_2q}{b_1}, 0; \frac{a_2q}{a_1}, \frac{a_2q}{a_3}; q, \frac{qb_1}{a_1a_2a_3x}\right), \tag{2.7}
$$

$$
v_3(x) := \frac{\theta_q(a_3x)}{\theta_q(x)} {}_3\varphi_2\left(a_3, \frac{a_3q}{b_1}, 0; \frac{a_3q}{a_2}, \frac{a_3q}{a_1}; q, \frac{qb_1}{a_1a_2a_3x}\right) \tag{2.8}
$$

in the Theorem 2.3.

Remark. By the q-difference equation of the theta function, we can check out that the connection coefficients (with the new parameter λ)

$$
\begin{aligned}
C_1(x) &:= \frac{(a_2, a_3, b_1/a_1; q)_\infty}{(b_1, a_2/a_1, a_3/a_1; q)_\infty} \frac{\theta_q(a_1\lambda)}{\theta_q(\lambda)} \frac{\theta_q(a_1qx/\lambda)}{\theta_q(qx/\lambda)} \frac{\theta_q(x)}{\theta_q(a_1x)}, \\
C_2(x) &:= \frac{(a_1, a_3, b_1/a_2; q)_\infty}{(b_1, a_1/a_2, a_3/a_2; q)_\infty} \frac{\theta_q(a_2\lambda)}{\theta_q(\lambda)} \frac{\theta_q(a_2qx/\lambda)}{\theta_q(qx/\lambda)} \frac{\theta_q(x)}{\theta_q(a_2x)}, \\
C_3(x) &:= \frac{(a_2, a_1, b_1/a_3; q)_\infty}{(b_1, a_2/a_3, a_1/a_3; q)_\infty} \frac{\theta_q(a_3\lambda)}{\theta_q(\lambda)} \frac{\theta_q(a_3qx/\lambda)}{\theta_q(qx/\lambda)} \frac{\theta_q(x)}{\theta_q(a_3x)}
\end{aligned}
$$

are the q-elliptic functions.

§ 3. The limit $q \to 1-0$ of the connection formula

The aim of this section is to give the limit $q \to 1-0$ of the new connection formula as follows:

Theorem 3.1. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have the following limit $q \to 1-0$ of the connection formula*

$$\begin{aligned}
&\lim_{q\to 1-0} {}_3f_1(q^{\alpha_1}, q^{\alpha_2}, q^{\alpha_3}; q^{\beta_1}; q; \lambda, x)\\
&= \frac{\Gamma(\beta_1)\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_3-\alpha_1)}{\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\beta_1-\alpha_1)} x^{-\alpha_1} {}_2F_2\left(\alpha_1, \alpha_1+1-\beta_1; \alpha_1+1-\alpha_2, \alpha_1+1-\alpha_3; \frac{1}{x}\right)\\
&+ \frac{\Gamma(\beta_1)\Gamma(\alpha_1-\alpha_2)\Gamma(\alpha_3-\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_3)\Gamma(\beta_1-\alpha_2)} x^{-\alpha_2} {}_2F_2\left(\alpha_2, \alpha_2+1-\beta_1; \alpha_2+1-\alpha_1, \alpha_2+1-\alpha_3; \frac{1}{x}\right)\\
&+ \frac{\Gamma(\beta_1)\Gamma(\alpha_2-\alpha_3)\Gamma(\alpha_1-\alpha_3)}{\Gamma(\alpha_2)\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_3)} x^{-\alpha_3} {}_2F_2\left(\alpha_3, \alpha_3+1-\beta_1; \alpha_3+1-\alpha_2, \alpha_3+1-\alpha_1; \frac{1}{x}\right),
\end{aligned}$$

provided that $-\pi < \arg x < \pi$.

The following proposition [13] is important to consider the limit $q \to 1-0$ of our connection formula.

Proposition 3.2. *For any $x \in \mathbb{C}^*(-\pi < \arg x < \pi)$, we have*

$$\lim_{q\to 1-0} \frac{\theta_q(q^\beta x)}{\theta_q(q^\alpha x)} = x^{\alpha-\beta} \tag{3.1}$$

and

$$\lim_{q\to 1-0} \frac{\theta_q\left(\dfrac{q^\alpha x}{(1-q)}\right)}{\theta_q\left(\dfrac{q^\beta x}{(1-q)}\right)} (1-q)^{\beta-\alpha} = x^{\beta-\alpha}. \tag{3.2}$$

We give the proof of the Theorem 3.1.

Proof. At first, we put $a_j := q^{\alpha_j}$ $(j = 1, 2, 3)$, $b_1 := q^{\beta_1}$ and $x \mapsto x/(1-q)$. We remark that the limit $q \to 1-0$ of the left hand-side of Theorem 3.1 formally converges the hypergeometric series

$${}_3F_1(\alpha_1, \alpha_2, \alpha_3; \beta_1; x) = \sum_{n\geq 0} \frac{(\alpha_1, \alpha_2, \alpha_3)_n}{(\beta_1)_n n!} x^n.$$

We consider the right hand-side. The connection formula can be rewritten as follows:

$$
\begin{aligned}
&{}_3f_1(q^{\alpha_1},q^{\alpha_2},q^{\alpha_3};q^{\beta_1};q;\lambda,x)\\
&=\frac{(q^{\alpha_2},q^{\alpha_3},q^{\beta_1-\alpha_1};q)_\infty}{(q^{\beta_1},q^{\alpha_2-\alpha_1},q^{\alpha_3-\alpha_1};q)_\infty}\frac{\theta_q(q^{\alpha_1}\lambda)}{\theta_q(\lambda)}\frac{\theta_q\left(\frac{q^{\alpha_1+1}x}{\lambda(1-q)}\right)}{\theta_q\left(\frac{qx}{\lambda(1-q)}\right)}\\
&\times {}_3\varphi_2\left(q^{\alpha_1},q^{\alpha_1+1-\beta_1},0;q^{\alpha_1+1-\alpha_2},q^{\alpha_1+1-\alpha_3};q,\frac{q^{1+\beta_1}(1-q)}{q^{\alpha_1+\alpha_2+\alpha_3}x}\right)\\
&+\mathrm{idem}(q^{\alpha_1};q^{\alpha_2},q^{\alpha_3})\\
&=\frac{\Gamma_q(\beta_1)\Gamma_q(\alpha_2-\alpha_1)\Gamma_q(\alpha_3-\alpha_1)}{\Gamma_q(\alpha_2)\Gamma_q(\alpha_3)\Gamma_q(\beta_1-\alpha_1)}\frac{\theta_q(q^{\alpha_1}\lambda)}{\theta_q(\lambda)}\left\{\frac{\theta_q\left(\frac{q^{\alpha_1+1}x}{\lambda(1-q)}\right)}{\theta_q\left(\frac{qx}{\lambda(1-q)}\right)}(1-q)^{-\alpha_1}\right\}\\
&\times {}_3\varphi_2\left(q^{\alpha_1},q^{\alpha_1+1-\beta_1},0;q^{\alpha_1+1-\alpha_2},q^{\alpha_1+1-\alpha_3};q,\frac{q^{1+\beta_1}(1-q)}{q^{\alpha_1+\alpha_2+\alpha_3}x}\right)\\
&+\mathrm{idem}(q^{\alpha_1};q^{\alpha_2},q^{\alpha_3}).
\end{aligned}
$$

By (3.1), (3.2) and (1.8), we obtain the conclusion. □

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