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Surface topology and involutive bimodules

By

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Abstract

We remark some basic facts on homological aspects of involutive Lie bialgebras and their involutive bimodules, and present some problems on surface topology related to these facts.

Introduction

The notion of a Lie bialgebra was originated by Drinfel'd in the celebrated paper [5]. There he observed that any bialgebra structure on a fixed Lie algebra $\mathfrak{g}$ is regarded as a 1-cocycle of $\mathfrak{g}$ with values in the second exterior power $\Lambda^2\mathfrak{g}$, and that the coboundary of any element in $\Lambda^2\mathfrak{g}$ satisfying the Yang-Baxter equation defines a Lie bialgebra structure on the Lie algebra $\mathfrak{g}$. It can be regarded as a deformation of the Lie bialgebra structure on $\mathfrak{g}$ with the trivial coalgebra structure.

It was Turaev [22] who discovered a close relation between surface topology and the notion of a Lie bialgebra. Let S be a connected oriented surface, and $\mathbb{Q}\hat{\pi}(S)$ the (rational) Goldman Lie algebra of the surface S [6], which is the $\mathbb{Q}$-free vector space over the homotopy set $\hat{\pi}(S) = [S^1, S]$ of free loops on the surface S equipped with the Goldman bracket. The constant loop 1 is in the center of $\mathbb{Q}\hat{\pi}(S)$, so that the quotient $\mathbb{Q}\hat{\pi}'(S) := \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}1$ has a natural Lie algebra structure. He introduced a natural cobracket, the Turaev cobracket, on $\mathbb{Q}\hat{\pi}'(S)$, and proved that it is a Lie bialgebra. Later Chas [2] proved that it satisfies the involutivity. See Appendix for the definition of these operations.

On the other hand, Schedler [20] introduced a natural involutive Lie bialgebra structure on the necklace Lie algebra associated to a quiver. Let H be a symplectic $\mathbb{Q}$-vector

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space of dimension $2g$, $g \geq 1$, and $\widehat{T} := \prod_{m=0}^{\infty} H^{\otimes m}$ the completed tensor algebra over H. We denote by $\mathfrak{a}_g^- = \mathrm{Der}_\omega(\widehat{T})$ the Lie algebra of continuous derivations on $\widehat{T}$ annihilating the symplectic form $\omega \in H^{\otimes 2}$. It includes Kontsevich's "associative wor(l)d" a_g as a Lie subalgebra. The Lie algebra $\mathfrak{a}_g^-$ is the necklace Lie algebra associated to some quiver. Hence it is an involutive Lie bialgebra by Schedler's cobracket. Massuyeau [16] introduced the notion of a symplectic expansion of the fundamental group of $\Sigma_{g,1}$, a compact connected oriented surface of genus g with 1 boundary component. Kuno and the author [9] [10] proved that a natural completion of the Lie algebra $\mathbb{Q}\hat{\pi}'(\Sigma_{g,1})$ is isomorphic to the Lie algebra $\mathfrak{a}_g^-$ by using a symplectic expansion. In particular, the Turaev cobracket defines an involutive Lie bialgebra structure on the Lie algebra $\mathfrak{a}_g^-$, which depends on the choice of a symplectic expansion, and does not coincide with Schedler's cobracket. In §4 we present some problems related to these cobrackets.

Now we go back to an arbitrary connected oriented surface S. Suppose that its boundary ∂S is non-empty. Then choose two (not necessarily distinct) points $*_0$ and $*_1$ in ∂S. We denote by $\Pi S(*_0, *_1)$ the homotopy set of paths from $*_0$ to $*_1$, namely $[([0,1], 0, 1), (S, *_0, *_1)]$. In [9] and [10] Kuno and the author discovered that $\mathbb{Q}\Pi S(*_0, *_1)$, the $\mathbb{Q}$-free vector space over the set $\Pi S(*_0, *_1)$, is a nontrivial $\mathbb{Q}\hat{\pi}'(S)$-module in a natural way. Moreover, inspired by [21], they [11] introduced a natural operation

$$\mu : \mathbb{Q}\Pi S(*_0, *_1) \to \mathbb{Q}\Pi S(*_0, *_1) \otimes \mathbb{Q}\hat{\pi}'(S).$$

It should satisfy some natural properties analogous to the defining conditions of an involutive Lie bialgebra. So, in [11], they introduced the defining conditions of an involutive $\mathbb{Q}\hat{\pi}'(S)$-module, and proved that μ satisfies all the conditions. See also Appendix for details. As applications of the compatibility condition among them, they [11] obtain a criterion for the non-realizability of generalized Dehn twists [15], and a geometric constraint of the (geometric) Johnson homomorphism of the (smallest) Torelli group.

The purpose of the present paper is to explain a homological background of the defining conditions of an involutive Lie bialgebra and its involutive bimodule, and to present some problems on surface topology related to this background. Our key observation is the classical fact: *the Jacobi identity for a Lie algebra* $\mathfrak{g}$ *is equivalent to the integrability condition* $\partial\partial = 0$ *on the exterior algebra* $\Lambda^*\mathfrak{g}$. Throughout this paper we work over the rationals $\mathbb{Q}$ for simplicity. But all the propositions in this paper hold good over any field of characteristic 0. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{Q}$, $\partial : \Lambda^p\mathfrak{g} \to \Lambda^{p-1}\mathfrak{g}$, $p \geq 1$, the standard boundary operator. See, for example, [1]. Moreover let $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ be a $\mathbb{Q}$-linear map. The map δ has a natural extension $d : \Lambda^p\mathfrak{g} \to \Lambda^{p+1}\mathfrak{g}$ for any $p \geq 0$. Then we have

Proposition 0.1. *The pair* $(\mathfrak{g}, \delta)$ *is an involutive Lie bialgebra, if and only if* $dd = 0$ *and* $d\partial + \partial d = 0$ *on* $\Lambda^*\mathfrak{g}$.

This is an easy exercise. But, to complete our argument, we prove it in §1. The proposition implies the homology group $H_*(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is a cochain complex with the coboundary operator $d(\delta) := H_*(d)$, if $\mathfrak{g}$ is an involutive Lie bialgebra.

Problem 0.2. Find a meaning of the cohomology group $H^*(H_*(\mathfrak{g}), d(\delta))$ for any involutive Lie bialgebra $(\mathfrak{g}, \delta)$.

Suppose $\mathfrak{g}$ is an involutive Lie bialgebra. Let M be a $\mathfrak{g}$-module. Then we can consider the standard chain complex $(M \otimes \Lambda^*\mathfrak{g}, \partial)$ of the Lie algebra $\mathfrak{g}$ with values in M [1]. Any $\mathbb{Q}$-linear map $\mu : M \to M \otimes \mathfrak{g}$ has a natural extension $d = d^M : M \otimes \Lambda^p\mathfrak{g} \to M \otimes \Lambda^{p+1}\mathfrak{g}$ for any $p \geq 0$. Then we have

Proposition 0.3. *The pair (M, μ) is an involutive $\mathfrak{g}$-bimodule in the sense of [11], if and only if $dd = 0$ and $d\partial + \partial d = 0$ on $M \otimes \Lambda^*\mathfrak{g}$.*

Similarly to $H_*(\mathfrak{g})$, the homology group $H_*(\mathfrak{g}; M)$ of $\mathfrak{g}$ with values in M admits the coboundary operator $d(\delta, \mu) := H_*(d)$ if M is an involutive $\mathfrak{g}$-bimodule.

Problem 0.4. Let $(\mathfrak{g}, \delta)$ be an involutive Lie bialgebra. Then find a meaning of the cohomology group $H^*(H_*(\mathfrak{g}; M), d(\delta, \mu))$ for any involutive $\mathfrak{g}$-bimodule (M, μ).

In §3 we study Drinfel'd's deformation of a Lie bialgebra structure by a 1-coboundary stated above. We can consider an analogous deformation of an involutive bimodule. We prove that such a deformation does not affect the coboundary operators $d(\delta)$ and $d(\delta, \mu)$ on $H_*(\mathfrak{g})$ and $H_*(\mathfrak{g}; M)$ (Lemma 3.1 and Proposition 3.4). In §4 we discuss some relation among these homological facts and surface topology, in particular, a tensorial description of the Turaev cobracket and Kontsevich's non-commutative symplectic geometry. In Appendix we briefly review some operations of loops on a surface [6] [22] [9] [11].

The referee kindly informed us that two articles by Conant-Vogtmann [3] and Hamilton [7] would be related to the results in §1 of this paper. In ll.3-7, p.173 [7], Hamilton proves that the cobracket on any involutive Lie bialgebra $\mathfrak{g}$ defines a canonical differential operator $\delta : \Lambda^*\mathfrak{g} \to \Lambda^{*+1}\mathfrak{g}$, which is the same as the operator d in this paper. But neither involutive bimodules nor Drinfel'd deformations were discussed there. While the operator δ provides a deformation of the boundary operator ∂ on $\Lambda^*\mathfrak{g}$ but not an operator on the homology group $H_*(\mathfrak{g})$ in [7], Conant and Vogtmann [3] introduce a differential operator ∂_H on the $\mathcal{O}$-graph homology for any cyclic operad $\mathcal{O}$, which comes from the homology groups of the Lie algebras associated with the operad $\mathcal{O}$. The operator ∂_H does not equal our operator d in the chain level, since ∂_H is not compatible with the standard coproduct Δ of $\Lambda^*\mathfrak{g}$. But the author does not know whether they coincide with each other in the homology level or not.

We conclude the introduction by listing our convention of notation in this paper. For a $\mathbb{Q}$-vector space V and $p \geq 1$, the p-th symmetric group $\mathfrak{S}_p$ acts on the tensor space $V^{\otimes p}$ by permuting the components. In particular, we denote $T := (12) \in \mathrm{Aut}(V^{\otimes 2})$ and $N := 1 + (123) + (123)^2 \in \mathrm{End}(V^{\otimes 3})$. We regard the p-th exterior power $\Lambda^p V$ as a linear subspace of $V^{\otimes p}$ in an obvious way $\Lambda^p V := \{u \in V^{\otimes p}; \sigma(u) = (\mathrm{sgn}\,\sigma)u\}$. For $X_i \in V$, $1 \leq i \leq p$, we identify $X_1 \wedge \cdots \wedge X_p = \sum_{\sigma \in \mathfrak{S}_p} (\mathrm{sgn}\,\sigma) X_{\sigma(1)} \cdots X_{\sigma(p)} \in \Lambda^p V \subset V^{\otimes p}$. Here and throughout this paper we omit the symbol $\otimes$, if there is no fear of confusion. In particular, we have

$$\wedge = (1 - T) : V^{\otimes 2} \to \Lambda^2 V, \quad XY \mapsto X \wedge Y = (1 - T)(XY). \tag{0.1}$$

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Contents

§ 1. Lie bialgebras

In this section we recall the definitions of a Lie algebra, a Lie coalgebra and a Lie bialgebra, and prove Proposition 0.1.

§ 1.1. Lie algebras

Let $\mathfrak{g}$ be a $\mathbb{Q}$-vector space equipped with a $\mathbb{Q}$-linear map $\nabla : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying **the skew condition**

$$\nabla T = -\nabla : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}. \tag{1.1}$$

Following the ordinary terminology, we denote $[X, Y] := \nabla(X \otimes Y)$ for any X and $Y \in \mathfrak{g}$. Then we define $\mathbb{Q}$-linear maps $\sigma : \mathfrak{g} \otimes \Lambda^p\mathfrak{g} \to \Lambda^p\mathfrak{g}$ and $\partial : \Lambda^p\mathfrak{g} \to \Lambda^{p-1}\mathfrak{g}$ by

$$\sigma(Y)(X_1 \wedge \cdots \wedge X_p) := \sum_{i=1}^{p} X_1 \wedge \cdots \wedge X_{i-1} \wedge [Y, X_i] \wedge X_{i+1} \wedge \cdots \wedge X_p,$$
$$\partial(X_1 \wedge \cdots \wedge X_p) := \sum_{i<j} (-1)^{i+j}[X_i, X_j] \wedge X_1 \wedge \overset{\hat{i}}{\cdots}\overset{\hat{j}}{\cdots} \wedge X_p,$$

for X_i and $Y \in \mathfrak{g}$. It is easy to show

$$\partial(X_1 \wedge \cdots \wedge X_p \wedge Y) = \partial(X_1 \wedge \cdots \wedge X_p) \wedge Y + (-1)^{p+1}\sigma(Y)(X_1 \wedge \cdots \wedge X_p). \tag{1.2}$$

Lemma 1.1. *We have $\partial\partial = 0 : \Lambda^*\mathfrak{g} \to \Lambda^*\mathfrak{g}$, if and only if ∇ satisfies* **the Jacobi identity**

$$\nabla(\nabla \otimes 1)N = 0 : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}. \tag{1.3}$$

Proof. For X, Y and $Z \in \mathfrak{g}$, we have

$$\partial\partial(X \wedge Y \wedge Z) = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y].$$

Hence $\partial\partial = 0$ implies the Jacobi identity.

Assume the Jacobi identity. Then, by some straight-forward computation, we have

$$\sigma(Y)\partial(X_1 \wedge \cdots \wedge X_p) = \partial\sigma(Y)(X_1 \wedge \cdots \wedge X_p) \tag{1.4}$$

for any X_i and $Y \in \mathfrak{g}$. This proves $\partial\partial = 0 : \Lambda^p\mathfrak{g} \to \Lambda^{p-2}\mathfrak{g}$ by induction on $p \geq 2$. In the case $p = 2$, $\partial\partial = 0$ is trivial. Assume $\partial\partial = 0 : \Lambda^p\mathfrak{g} \to \Lambda^{p-2}\mathfrak{g}$ for $p \geq 2$. Then, using (1.2) and (1.4) for $\xi \in \Lambda^p\mathfrak{g}$ and $Y \in \mathfrak{g}$, we compute

$$\begin{aligned}\partial\partial(\xi \wedge Y) &= \partial((\partial\xi) \wedge Y + (-1)^{p+1}\sigma(Y)\xi)\\ &= (\partial\partial\xi) \wedge Y + (-1)^p\sigma(Y)\partial\xi + (-1)^{p+1}\partial(\sigma(Y)\xi) = (\partial\partial\xi) \wedge Y = 0\end{aligned}$$

by the inductive assumption. This proves the lemma. $\square$

The pair $(\mathfrak{g}, \nabla)$ is called a **Lie algebra** if the map ∇ satisfies the Jacobi identity (1.3). The map ∇ is called the **bracket** of the Lie algebra. Then the p-th homology group of the chain complex $\Lambda^*\mathfrak{g} = \{\Lambda^p\mathfrak{g}, \partial\}_{p\geq 0}$ is denoted by

$$H_p(\mathfrak{g}) = H_p(\Lambda^*\mathfrak{g})$$

and called the p-th homology group of the Lie algebra $\mathfrak{g}$. See, for example, [1].

For any Lie algebra $\mathfrak{g}$, by some straight-forward computation, one can prove the following, which will be used in §1.3.

Lemma 1.2. *For $\xi = X_1\wedge\cdots\wedge X_p \in \Lambda^p\mathfrak{g}$ and $\eta = Y_1\wedge\cdots\wedge Y_q \in \Lambda^q\mathfrak{g}$, $X_i, Y_j \in \mathfrak{g}$,*

$$\partial(\xi\wedge\eta) - (\partial\xi)\wedge\eta - (-1)^p\xi\wedge\partial\eta = \sum_{i=1}^{p}(-1)^i X_1\wedge\overset{\hat{i}}{\cdots}\wedge X_p\wedge\sigma(X_i)(\eta).$$

§ 1.2. Lie coalgebras

Next we consider a $\mathbb{Q}$-vector space equipped with a $\mathbb{Q}$-linear map $\delta : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ satisfying **the coskew condition**

$$T\delta = -\delta : \mathfrak{g} \to \mathfrak{g}^{\otimes 2}. \tag{1.5}$$

We may regard $\delta(\mathfrak{g}) \subset \Lambda^2\mathfrak{g}$. Then we define a $\mathbb{Q}$-linear map $d : \Lambda^p\mathfrak{g} \to \Lambda^{p+1}\mathfrak{g}$, $p \geq 0$, by $d|_{\Lambda^0\mathfrak{g}} := 0$ and

$$d(X_1\wedge\cdots\wedge X_p) := \sum_{i=1}^{p}(-1)^i(\delta X_i)\wedge X_1\wedge\overset{\hat{i}}{\cdots}\wedge X_p$$

for any $p \geq 1$ and $X_i \in \mathfrak{g}$. In particular, $dX = -\delta X$ for $X \in \mathfrak{g}$. If $\xi \in \Lambda^p\mathfrak{g}$ and $\eta \in \Lambda^q\mathfrak{g}$, then

$$d(\xi\wedge\eta) = (d\xi)\wedge\eta + (-1)^p\xi\wedge(d\eta). \tag{1.6}$$

Lemma 1.3. *We have $dd = 0 : \Lambda^*\mathfrak{g} \to \Lambda^*\mathfrak{g}$, if and only if δ satisfies* **the coJacobi identity**

$$N(\delta\otimes 1)\delta = 0 : \mathfrak{g} \to \mathfrak{g}^{\otimes 3}. \tag{1.7}$$

Proof. If we denote $\delta X = \sum_i X_i'\wedge X_i''$, $X_i', X_i'' \in \mathfrak{g}$, then we have

$$\begin{aligned}
&(\delta X)\wedge Y = \sum X_i'\wedge X_i''\wedge Y \\
&= \sum X_i'X_i''Y + X_i''YX_i' + YX_i'X_i'' - X_i''X_i'Y - X_i'YX_i'' - YX_i''X_i' \\
&= N((\delta X)Y).
\end{aligned}$$

This implies $d(X \wedge Y) = -(\delta X) \wedge Y + (\delta Y) \wedge X = -N((\delta X)Y) + N((\delta Y)X) = -N(\delta \otimes 1)(XY - YX) = -N(\delta \otimes 1)(X \wedge Y)$. Since $\delta \mathfrak{g} \subset \Lambda^2 \mathfrak{g}$, we obtain

$$dd = N(\delta \otimes 1)\delta : \mathfrak{g} \to \mathfrak{g}^{\otimes 3}. \tag{1.8}$$

Hence $dd = 0$ implies the coJacobi identity.

Assume the coJacobi identity. We prove $dd = 0 : \Lambda^p \mathfrak{g} \to \Lambda^{p+2} \mathfrak{g}$ by induction on $p \geq 1$. In the case $p = 1$, $dd = 0$ is equivalent to the coJacobi identity. Assume $dd = 0 : \Lambda^p \mathfrak{g} \to \Lambda^{p+2} \mathfrak{g}$ for $p \geq 1$. Then, for $\xi \in \Lambda^p \mathfrak{g}$ and $Y \in \mathfrak{g}$, we have $dd(\xi \wedge Y) = d((d\xi) \wedge Y + (-1)^p \xi \wedge dY) = (dd\xi) \wedge Y + (-1)^{p+1}(d\xi) \wedge dY + (-1)^p d\xi \wedge dY + \xi \wedge ddY = (dd\xi) \wedge Y + \xi \wedge ddY = 0$ by the inductive assumption. This proves the lemma. □

The pair $(\mathfrak{g}, \delta)$ is called a **Lie coalgebra** if the map δ satisfies the coJacobi identity (1.7). The map δ is called the **cobracket** of the Lie coalgebra. Then the p-th cohomology group of the cochain complex $\Lambda^* \mathfrak{g} = \{\Lambda^p \mathfrak{g}, d\}_{p \geq 0}$ is denoted by

$$H^p(\mathfrak{g}) = H^p(\Lambda^* \mathfrak{g})$$

and called the p-th cohomology group of the Lie coalgebra $\mathfrak{g}$. In view of the formula (1.6), $H^*(\mathfrak{g})$ is a graded commutative algebra.

Assume $\mathfrak{g}$ is a complete filtered $\mathbb{Q}$-vector space, i.e., there exists a decreasing filtration $\mathfrak{g} = F_0\mathfrak{g} \supset F_1\mathfrak{g} \supset \cdots \supset F_n\mathfrak{g} \supset F_{n+1}\mathfrak{g} \supset \cdots$ such that the completion map $\mathfrak{g} \to \widehat{\mathfrak{g}} := \varprojlim_{n \to \infty} \mathfrak{g}/F_n\mathfrak{g}$ is an isomorphism. Then we can consider a $\mathbb{Q}$-linear map $\delta : \mathfrak{g} \to \mathfrak{g} \widehat{\otimes} \mathfrak{g}$, whose target is the completed tensor product of two copies of $\mathfrak{g}$. Then the pair $(\mathfrak{g}, \delta)$ is a complete Lie coalgebra if the map δ satisfies the coskew condition (1.5) and the coJacobi identity (1.7), where $\mathfrak{g}^{\otimes 2}$ and $\mathfrak{g}^{\otimes 3}$ are replaced by the completed tensor product $\mathfrak{g}^{\widehat{\otimes} 2}$ and $\mathfrak{g}^{\widehat{\otimes} 3}$, respectively. In this case we consider the p-th complete exterior power, i.e., the alternating part of $\mathfrak{g}^{\widehat{\otimes} p}$, instead of $\Lambda^p \mathfrak{g}$ for any $p \geq 0$.

§ 1.3. Involutive Lie bialgebras

Let $(\mathfrak{g}, \nabla)$ be a Lie algebra, and $(\mathfrak{g}, \delta)$ a Lie coalgebra with the same underlying vector space $\mathfrak{g}$. We look at the operator $d\partial + \partial d : \Lambda^p \mathfrak{g} \to \Lambda^p \mathfrak{g}$ for $p \geq 0$. It is clear $d\partial + \partial d = 0$ for $p = 0$.

Lemma 1.4. *We have $d\partial + \partial d = 0 : \Lambda^p \mathfrak{g} \to \Lambda^p \mathfrak{g}$ for $p = 1$ and 2, if and only if ∇ and δ satisfy* **the compatibility condition**

$$\forall X, \forall Y \in \mathfrak{g}, \quad \delta[X, Y] = \sigma(X)(\delta Y) - \sigma(Y)(\delta X), \tag{1.9}$$

and **the involutivity**

$$\nabla \delta = 0 : \mathfrak{g} \to \mathfrak{g}. \tag{1.10}$$

Proof. From the definition, the involutivity is equivalent to $d\partial + \partial d = 0$ for $p = 1$. Assume the involutivity. Then, for X and $Y \in \mathfrak{g}$, we have $(d\partial + \partial d)(X \wedge Y) = -d[X, Y] + \partial((dX) \wedge Y - X \wedge (dY)) = \delta[X, Y] + (\partial dX) \wedge Y - \sigma(Y)(dX) - (\partial dY) \wedge X - \sigma(X)(dY) = \delta[X, Y] + \sigma(Y)(\delta X) - \sigma(X)(\delta Y)$. Hence $d\partial + \partial d = 0$ for $p = 2$ is equivalent to the compatibility condition. This proves the lemma. □

When the compatibility condition holds, $\mathfrak{g}$ is called a **Lie bialgebra**. This is the definition given by Drinfel'd in [5]. A Lie bialgebra $\mathfrak{g}$ is called **involutive**, if it satisfies the involutivity.

Lemma 1.5. *If $\mathfrak{g}$ is a Lie bialgebra, we have*

$$\partial(\xi \wedge dY) - (\partial\xi) \wedge dY - (-1)^p \xi \wedge \partial dY = d\sigma(Y)\xi - \sigma(Y)d\xi$$

for $\xi \in \Lambda^p\mathfrak{g}$ and $Y \in \mathfrak{g}$.

Proof. It suffices to show the lemma for $\xi = X_1 \wedge \cdots \wedge X_p$, $X_i \in \mathfrak{g}$. By the compatibility condition, we have

$$d\sigma(Y)\xi - \sigma(Y)d\xi = \sum_{i=1}^{p} (-1)^{i-1} X_1 \wedge \cdots \wedge (d[Y, X_i] - \sigma(Y)dX_i) \wedge \cdots \wedge X_p$$

$$= \sum_{i=1}^{p} (-1)^i X_1 \wedge \cdots \wedge \sigma(X_i)dY \wedge \cdots \wedge X_p = \sum_{i=1}^{p} (-1)^i X_1 \wedge \overset{\hat{i}}{\cdots} \wedge X_p \wedge \sigma(X_i)dY,$$

which equals $\partial(\xi \wedge dY) - (\partial\xi) \wedge dY - (-1)^p \xi \wedge \partial dY$ from Lemma 1.2. This proves the lemma. □

Proposition 1.6. *If $\mathfrak{g}$ is a Lie bialgebra, then we have*

$$(d\partial + \partial d)(X_1 \wedge \cdots \wedge X_p) = \sum_{i=1}^{p} X_1 \wedge \cdots \wedge X_{i-1} \wedge (\partial dX_i) \wedge X_{i+1} \wedge \cdots \wedge X_p$$

for $X_i \in \mathfrak{g}$.

Proof. It is clear for $p = 1$. Assume it holds for $p \geq 1$. Denote $\xi = X_1 \wedge \cdots \wedge X_p$ and $Y = X_{p+1}$. Then, from Lemma 1.5, $(d\partial + \partial d)(\xi \wedge Y) = d((\partial\xi) \wedge Y + (-1)^{p+1}\sigma(Y)\xi) + \partial((d\xi) \wedge Y + (-1)^p \xi \wedge dY) = (d\partial Y) \wedge Y + (-1)^{p+1}(\partial\xi) \wedge dY + (-1)^{p+1} d\sigma(Y)\xi + (\partial d\xi) \wedge Y + (-1)^{p+2}\sigma(Y)d\xi + (-1)^p \partial(\xi \wedge dY) = ((d\partial + \partial d)\xi) \wedge Y + \xi \wedge \partial dY$. This proceeds the induction. □

Corollary 1.7. *A Lie bialgebra $\mathfrak{g}$ satisfies $d\partial + \partial d = 0 : \Lambda^p\mathfrak{g} \to \Lambda^p\mathfrak{g}$ for any $p \geq 0$, if and only if $\mathfrak{g}$ is involutive.*

This completes the proof of Proposition 0.1 stated in Introduction.

For an involutive Lie bialgebra $\mathfrak{g}$, the operator d induces the coboundary operator

$$d = d(\delta) : H_p(\mathfrak{g}) \to H_{p+1}(\mathfrak{g}), \quad [u] \mapsto [du] \tag{1.11}$$

on the homology group $H_*(\mathfrak{g})$. Hence one can define the cohomology of the homology $H^*(H_*(\mathfrak{g}))$.

When the pair $(\mathfrak{g}, \delta)$ is a complete Lie coalgebra, we have to assume that the bracket ∇ is continuous with respect to the filtration of $\mathfrak{g}$, and to replace the exterior algebra $\Lambda^*\mathfrak{g}$ by the complete exterior algebra of $\mathfrak{g}$ in the three propositions in this subsection. Then all of them hold good. In particular, we can consider a complete Lie bialgebra and a complete involutive Lie bialgebra. Similarly we can consider a complete comodule and a complete (involutive) bimodule in the next section.

§ 2. Bimodules

We discuss a homological background of the defining conditions of an involutive bimodule introduced by Kuno and the author in [11]. In other words, we prove Proposition 0.3 stated in Introduction.

§ 2.1. Modules

Let $\mathfrak{g}$ be a Lie algebra, M a $\mathbb{Q}$-vector space equipped with a $\mathbb{Q}$-linear map $\sigma : \mathfrak{g}\otimes M \to M$, $X\otimes m \mapsto Xm$. We define a $\mathbb{Q}$-linear map $\Gamma_\sigma = \Gamma : M\otimes\Lambda^p\mathfrak{g} \to M\otimes\Lambda^{p-1}\mathfrak{g}$ by $\Gamma(m\otimes X_1\wedge\cdots\wedge X_p) := \sum_{i=1}^p (-1)^i (X_i m)\otimes X_1\wedge\overset{\hat{i}}{\cdots}\wedge X_p$ for $p \geq 1$, $m \in M$ and $X_i \in \mathfrak{g}$, and a $\mathbb{Q}$-linear map $\partial^M = \partial : M\otimes\Lambda^p\mathfrak{g} \to M\otimes\Lambda^{p-1}\mathfrak{g}$ by $\partial(m\otimes\xi) := \Gamma(m\otimes\xi) + m\otimes\partial(\xi)$ for $m \in M$ and $\xi \in \Lambda^p\mathfrak{g}$. Here $\partial : \Lambda^p\mathfrak{g} \to \Lambda^{p-1}\mathfrak{g}$ is the operator introduced in §1.1. By some straight-forward computation, we have

$$\Gamma(m\otimes\xi\wedge\eta) = \Gamma(m\otimes\xi)\wedge\eta + (-1)^{pq}\Gamma(m\otimes\eta)\wedge\xi \tag{2.1}$$

for any $m \in M$, $\xi \in \Lambda^p\mathfrak{g}$ and $\eta \in \Lambda^q\mathfrak{g}$. Furthermore we define a $\mathbb{Q}$-linear map $\sigma : \mathfrak{g}\otimes M\otimes\Lambda^p\mathfrak{g} \to M\otimes\Lambda^p\mathfrak{g}$ by $\sigma(Y)(m\otimes\xi) := (Ym)\otimes\xi + m\otimes\sigma(Y)(\xi)$ for $Y \in \mathfrak{g}$, $m \in M$ and $\xi \in \Lambda^p\mathfrak{g}$. Then it is easy to show

$$\partial(m\otimes\xi\wedge Y) = \partial(m\otimes\xi)\wedge Y + (-1)^{p+1}\sigma(Y)(m\otimes\xi). \tag{2.2}$$

Lemma 2.1. *We have $\partial^M\partial^M = 0 : M\otimes\Lambda^*\mathfrak{g} \to M\otimes\Lambda^*\mathfrak{g}$, if and only if the condition*

$$\forall X, \forall Y \in \mathfrak{g}, \forall m \in M, \quad [X,Y]m = X(Ym) - Y(Xm) \tag{2.3}$$

holds.

Proof. For $X, Y \in \mathfrak{g}$ and $m \in M$, we have

$$\partial\partial(m \otimes X \wedge y) = [X, Y]m - X(Ym) + Y(Xm).$$

Hence $\partial^M \partial^M = 0$ implies the condition (2.3).

Assume the condition (2.3). Then it is easy to show

$$\sigma(Y)\Gamma(m \otimes X_1 \wedge \cdots \wedge X_p) = \Gamma(\sigma(Y)(m \otimes X_1 \wedge \cdots \wedge X_p)) \tag{2.4}$$

for any $m \in M$ and $Y, X_i \in \mathfrak{g}$. From this formula and (1.4) follows

$$\sigma(Y)\partial(m \otimes \xi) = \partial(\sigma(Y)(m \otimes \xi)) \tag{2.5}$$

for any $m \in X$, $Y \in \mathfrak{g}$ and $\xi \in \Lambda^p\mathfrak{g}$. This proves $\partial\partial = 0 : M \otimes \Lambda^p\mathfrak{g} \to M \otimes \Lambda^{p-2}\mathfrak{g}$ by induction on $p \geq 2$. In the case $p = 2$, $\partial\partial = 0$ is equivalent to the condition (2.3). Assume $\partial\partial = 0 : M \otimes \Lambda^p\mathfrak{g} \to M \otimes \Lambda^{p-2}\mathfrak{g}$ for $p \geq 2$. Then, using (2.2) and (2.5) for $m \in M$, $\xi \in \Lambda^p\mathfrak{g}$ and $Y \in \mathfrak{g}$, we compute

$$\begin{aligned}
&\partial\partial(m \otimes \xi \wedge Y) = \partial(\partial(m \otimes \xi) \wedge Y + (-1)^{p+1}\sigma(Y)(m \otimes \xi)) \\
&= \partial\partial(m \otimes \xi) \wedge Y + (-1)^p\sigma(Y)\partial(m \otimes \xi) + (-1)^{p+1}\partial(\sigma(Y)(m \otimes \xi)) \\
&= \partial\partial(m \otimes \xi) \wedge Y = 0
\end{aligned}$$

by the inductive assumption. This proves the lemma. □

The pair (M, σ) is called a **left $\mathfrak{g}$-module** if the map σ satisfies the condition (2.3). Then the p-th homology group of the chain complex $M \otimes \Lambda^*\mathfrak{g} = \{M \otimes \Lambda^p\mathfrak{g}, \partial\}_{p \geq 0}$ is denoted by

$$H_p(\mathfrak{g}; M) = H_p(M \otimes \Lambda^*\mathfrak{g})$$

and called the p-th homology group of the Lie algebra $\mathfrak{g}$ with values in M. See, for example, [1].

If we define $\overline{\sigma} : M \otimes \mathfrak{g} \to M$ by $\overline{\sigma}(m \otimes X) = -Xm$ and the condition (2.3) holds for σ, then the pair $(M, \overline{\sigma})$ is called a **right $\mathfrak{g}$-module**. By the identification (0.1) we have

$$\Gamma_\sigma(m \otimes Y_1 \wedge Y_2) = (\overline{\sigma} \otimes 1_{\mathfrak{g}})(m \otimes Y_1 \wedge Y_2) \tag{2.6}$$

for any $m \in M$ and $Y_1, Y_2 \in \mathfrak{g}$.

§ 2.2. Comodules

Next let $(\mathfrak{g}, \delta)$ be a Lie coalgebra, and M a $\mathbb{Q}$-linear space equipped with a $\mathbb{Q}$-linear map $\mu : M \to M \otimes \mathfrak{g}$. We define a $\mathbb{Q}$-linear map $d^M = d : M \otimes \Lambda^p\mathfrak{g} \to M \otimes \Lambda^{p+1}\mathfrak{g}$, $p \geq 0$, by

$$d(m \otimes \xi) := \mu(m) \wedge \xi + (-1)^p m \otimes d\xi$$

for $m \in M$ and $\xi \in \Lambda^p\mathfrak{g}$. Here $d : \Lambda^p\mathfrak{g} \to \Lambda^{p+1}\mathfrak{g}$ is the operator introduced in §1.2. If $p = 0$, then $d = \mu : M \to M \otimes \mathfrak{g}$. From the definition and the formula (1.6) follows

$$d(m \otimes \xi \wedge \eta) = d(m \otimes \xi) \wedge \eta + (-1)^p(m \otimes \xi) \wedge (d\eta) \tag{2.7}$$

for any $m \in M$, $\xi \in \Lambda^p\mathfrak{g}$ and $\eta \in \Lambda^q\mathfrak{g}$.

Lemma 2.2. *We have $d^M d^M = 0 : M \otimes \Lambda^*\mathfrak{g} \to M \otimes \Lambda^*\mathfrak{g}$, if and only if the following diagram commutes*

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "\mu"'] & M \otimes \mathfrak{g} \arrow[d, "1_M \otimes \delta"] \\ M \otimes \mathfrak{g} \arrow[r, "(1_M \otimes (1-T))(\mu \otimes 1_{\mathfrak{g}})"] & M \otimes \mathfrak{g} \otimes \mathfrak{g} \end{tikzcd} \tag{2.8}$$

Proof. By (0.1) we have

$$d^M = (1_M \otimes (1 - T))(\mu \otimes 1_{\mathfrak{g}}) - 1_M \otimes \delta.$$

Here it should be remarked $d = -\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$. Hence the commutativity of the diagram (2.8) is equivalent to $d^M d^M = 0$ on $M = M \otimes \Lambda^0\mathfrak{g}$. In particular, $d^M d^M = 0$ implies the commutativity of the diagram (2.8).

Assume the diagram (2.8) commutes. We prove $dd = 0 : M \otimes \Lambda^p\mathfrak{g} \to M \otimes \Lambda^{p+2}\mathfrak{g}$ by induction on $p \geq 0$. In the case $p = 0$, $dd = 0$ is equivalent to the commutativity of the diagram (2.8). Assume $dd = 0 : M \otimes \Lambda^p\mathfrak{g} \to M \otimes \Lambda^{p+2}\mathfrak{g}$ for $p \geq 0$. Then, for $m \in M$, $\xi \in \Lambda^p\mathfrak{g}$ and $Y \in \mathfrak{g}$, we have $dd(m\otimes\xi\wedge Y) = d(d(m\otimes\xi)\wedge Y + (-1)^p m\otimes\xi\wedge dY) = dd(m\otimes\xi)\wedge Y + (-1)^{p+1}d(m\otimes\xi)\wedge dY + (-1)^p d(m\otimes\xi)\wedge dY + m\otimes\xi\wedge ddY = dd(m\otimes\xi)\wedge Y = 0$ by the inductive assumption. This proves the lemma. $\square$

The pair (M, μ) is called a **right $\mathfrak{g}$-comodule** if the diagram (2.8) commutes. Then the p-th cohomology group of the cochain complex $M \otimes \Lambda^*\mathfrak{g} = \{M \otimes \Lambda^p\mathfrak{g}, d\}_{p\geq 0}$ is denoted by

$$H^p(\mathfrak{g}; M) = H^p(M \otimes \Lambda^*\mathfrak{g})$$

and called the p-th cohomology group of the Lie coalgebra $\mathfrak{g}$ with values in M. In view of the formula (2.7), $H^*(\mathfrak{g}; M)$ is a graded right $H^*(\mathfrak{g})$-module.

§ 2.3. Involutive bimodules

Let $\mathfrak{g}$ be a Lie bialgebra, $(M, \overline{\sigma})$ a right $\mathfrak{g}$-module, and (M, μ) a right $\mathfrak{g}$-comodule with the same underlying vector space M. As in §1.3, we look at the operator $d^M\partial^M + \partial^M d^M : M \otimes \Lambda^p\mathfrak{g} \to M \otimes \Lambda^p\mathfrak{g}$ for $p \geq 0$. In [11] Kuno and the author introduced **the compatibility condition**

$$\forall m \in M, \forall Y \in \mathfrak{g}, \quad \sigma(Y)(dm) - d(Ym) = -\Gamma_\sigma(m \otimes dY), \tag{2.9}$$

(or equivalently

$$\forall m \in M, \forall Y \in \mathfrak{g}, \quad \sigma(Y)(\mu(m)) - \mu(Ym) - (\overline{\sigma} \otimes 1_{\mathfrak{g}})(1_M \otimes \delta)(m \otimes Y) = 0, \tag{2.10}$$

) and **the involutivity**

$$\overline{\sigma}\mu = 0 : M \to M. \tag{2.11}$$

Lemma 2.3. *Let $\mathfrak{g}$ be an involutive Lie bialgebra. Then we have $d^M\partial^M + \partial^M d^M = 0 : M \otimes \Lambda^p\mathfrak{g} \to M \otimes \Lambda^p\mathfrak{g}$ for $p = 0$ and 1, if and only if $\overline{\sigma}$ and μ satisfy the compatibility condition and the involutivity.*

Proof. From the definition, the involutivity is equivalent to $d\partial + \partial d = 0$ for $p = 0$. Assume the involutivity. Then, for $m \in \mathfrak{g}$ and $Y \in \mathfrak{g}$, we have $(d\partial + \partial d)(m \otimes Y) = -d(Ym) + \partial((dm) \wedge Y + m \otimes dY) = -d(Ym) + (\partial dm) \wedge Y + \sigma(Y)(dm) + \Gamma(m \otimes dY) + m \otimes \partial dY = -d(Ym) + \sigma(Y)(dm) + \Gamma(m \otimes dY)$. Hence $d\partial + \partial d = 0$ for $p = 1$ is equivalent to the compatibility condition. This proves the lemma. □

For a Lie bialgebra $\mathfrak{g}$, M is called a **right $\mathfrak{g}$-bimodule** if the compatibility condition holds. A right $\mathfrak{g}$-bimodule M is called **involutive**, if it satisfies the involutivity.

Proposition 2.4. *If $\mathfrak{g}$ is a Lie bialgebra, and M a right $\mathfrak{g}$-bimodule, then we have*

$$\begin{aligned}
&(d\partial + \partial d)(m \otimes X_1 \wedge \cdots \wedge X_p)\\
&= (\partial dm) \otimes X_1 \wedge \cdots \wedge X_p + m \otimes \sum_{i=1}^{p} X_1 \wedge X_{i-1} \wedge (\partial dX_i) \wedge X_{i+1} \wedge \cdots \wedge X_p
\end{aligned}$$

for $m \in M$ and $X_i \in \mathfrak{g}$.

Proof. It is clear for $p = 0$. Assume it holds for $p \geq 0$. Denote $\xi = X_1 \wedge \cdots \wedge X_p$ and $Y = X_{p+1}$. We have $\sigma(Y)d(m \otimes \xi) - (dm) \wedge \sigma(Y)\xi - (Ym) \otimes d\xi = (\sigma(Y)dm) \wedge \xi + m \otimes \sigma(Y)d\xi$. So, by (2.2), (2.1) and (2.7), we compute

$$\begin{aligned}
&(d\partial + \partial d)(m \otimes \xi \wedge Y)\\
&= (d\partial + \partial d)(m \otimes \xi) \wedge Y\\
&\quad + (-1)^p m \otimes (-(\partial\xi) \wedge dY - d\sigma(Y)(\xi) + \sigma(Y)d\xi + \partial(\xi \wedge dY))\\
&\quad + (-1)^p \left(-(dYm) + \sigma(Y)dm + \Gamma(m \otimes dY)\right) \wedge \xi
\end{aligned}$$

Hence, by Lemma 1.5 and (2.9), we obtain

$$(d\partial + \partial d)(m \otimes \xi \wedge Y) = (d\partial + \partial d)(m \otimes \xi) \wedge Y + m \otimes \xi \wedge \partial dY.$$

This proceeds the induction. □

Corollary 2.5. *Let $\mathfrak{g}$ be an involutive Lie bialgebra, and M a right $\mathfrak{g}$-bimodule. Then we have $d^M\partial^M + \partial^M d^M = 0 : M \otimes \Lambda^p\mathfrak{g} \to M \otimes \Lambda^p\mathfrak{g}$ for any $p \geq 0$, if and only if M is involutive.*

This completes the proof of Proposition 0.3.

If $\mathfrak{g}$ is an involutive Lie bialgebra and M an involutive right $\mathfrak{g}$-bimodule, then the operator d^M induces the cobounday operator

$$d = d(\delta, \mu) : H_p(\mathfrak{g}; M) \to H_{p+1}(\mathfrak{g}; M), \quad [u] \mapsto [d^M u]$$

on the homology group $H_*(\mathfrak{g}; M)$. Hence one can define the cohomology of the homology $H^*(H_*(\mathfrak{g}; M))$.

§ 3. Drinfel'd's deformation

Let $\mathfrak{g}$ be a Lie algebra equipped with a Lie cobracket $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$. As was pointed out by Drinfel'd [5], the compatibility is equivalent to that δ is a 1-cocycle of the Lie algebra $\mathfrak{g}$ with values in $\Lambda^2\mathfrak{g}$, and so one can deform the cobracket δ by a 1-coboundary of $\mathfrak{g}$ with values in $\Lambda^2\mathfrak{g}$ satisfying some condition which assures the new cobracket the coJacobi identity. Here $\mathfrak{g}$ acts on $\Lambda^2\mathfrak{g}$ by the map $\sigma : \mathfrak{g} \otimes \Lambda^2\mathfrak{g} \to \Lambda^2\mathfrak{g}$. The subspace $\mathcal{N}(\mathfrak{g}) := \mathrm{Ker}(\nabla : \Lambda^2\mathfrak{g} \to \mathfrak{g})$ is a $\mathfrak{g}$-submodule. The involutivity means $\delta(\mathfrak{g}) \subset \mathcal{N}(\mathfrak{g})$. Hence we may regard the set of involutive Lie bialgebra structures on the underlying Lie algebra $\mathfrak{g}$ as a subset of $Z^1(\mathfrak{g}; \mathcal{N}(\mathfrak{g}))$, the set of 1-cocycles of $\mathfrak{g}$ with values in $\mathcal{N}(\mathfrak{g})$. In particular, we can say two cobrackets δ and δ', which define involutive Lie bialgebra structures on $\mathfrak{g}$, are **cohomologous** to each other if and only if $[\delta] = [\delta'] \in H^1(\mathfrak{g}; \mathcal{N}(\mathfrak{g}))$. Similar observations hold for a involutive bimodule structure on a $\mathfrak{g}$-module M.

We introduced the coboundary operators $d(\delta)$ and $d(\delta, \mu)$ on the homology group $H_*(\mathfrak{g})$ and $H_*(\mathfrak{g}; M)$ in the previous sections. In this section, we prove that these operators stay invariant under Drinfel'd's deformation.

§ 3.1. Deformation of a cobracket

Let $\mathfrak{g}$ be a Lie algebra.

Lemma 3.1. *If δ and $\delta' \in Z^1(\mathfrak{g}; \mathcal{N}(\mathfrak{g}))$ are involutive Lie bialgebra structures on $\mathfrak{g}$, and cohomologous to each other, then the induced coboundary operators $d(\delta)$ and $d(\delta')$ on the homology $H_*(\mathfrak{g})$ coincide with each other*

$$d(\delta) = d(\delta') : H_*(\mathfrak{g}) \to H_{*+1}(\mathfrak{g}).$$

Proof. For $A \in \Lambda^*\mathfrak{g}$, we denote by $E_A : \Lambda^*\mathfrak{g} \to \Lambda^*\mathfrak{g}$ the multiplication by A, $u \mapsto A \wedge u$. If $A \in \Lambda^2\mathfrak{g}$, then, by some straight-forward computation, we have

$$(\partial E_A - E_A\partial + E_{\nabla A})(X_1 \wedge \cdots \wedge X_p) = \sum_{i=1}^{p}(-1)^i\sigma(X_i)(A) \wedge X_1 \wedge \overset{\hat{i}}{\cdots} \wedge X_p \tag{3.1}$$

for any $X_i \in \mathfrak{g}$.

We denote $d = d(\delta)$ and $d' = d(\delta')$. Suppose δ and δ' are cohomologous to each other. Then there exists some $A \in \mathcal{N}(\mathfrak{g})$ such that $(d-d')(X) = (\delta' - \delta)(X) = \sigma(X)(A)$ for any $X \in \mathfrak{g}$. From (3.1) follows $(d' - d)(X_1 \wedge \cdots \wedge X_p) = \sum_{i=1}^{p}(-1)^i\sigma(X_i)(A) \wedge X_1 \wedge \overset{\hat{i}}{\cdots} \wedge X_p = (\partial E_A - E_A\partial + E_{\nabla A})(X_1 \wedge \cdots \wedge X_p)$. Since $A \in \mathcal{N}(\mathfrak{g})$, we obtain $d' - d = \partial E_A - E_A\partial : \Lambda^*\mathfrak{g} \to \Lambda^{*+1}\mathfrak{g}$. This proves the lemma. □

As was pointed out by Drinfel'd [5], we have $H^1(\mathfrak{g};\mathcal{N}(\mathfrak{g})) = 0$ in the case $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra. Hence, in this case, $d(\delta) = 0$ on $H_*(\mathfrak{g})$ for any involutive Lie bialgebra structure on $\mathfrak{g}$.

Let U be an automorphism of a topological Lie algebra $\mathfrak{g}$, and $\delta \in Z^1(\mathfrak{g};\mathcal{N}(\mathfrak{g}))$ an involutive Lie bialgebra structure on $\mathfrak{g}$. Then the conjugate $U\delta := (U \otimes U)\delta U^{-1}$ is also an involutive Lie bialgebra strucuture on $\mathfrak{g}$.

Lemma 3.2. *Let $X \in \mathfrak{g}$, and suppose $e^{\mathrm{ad}X} = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{ad}(X))^k$ converges as an automorphism of the topological Lie algebra $\mathfrak{g}$. Then we have $d(\delta) = d(e^{\mathrm{ad}X}\delta)$ on $H_*(\mathfrak{g})$.*

Proof. The Lie algebra $\mathfrak{g}$ acts on $Z^1(\mathfrak{g};\mathcal{N}(\mathfrak{g}))$ in an obvious way. We have

$$(Yc)(Z) := \sigma(Y)(c(Z)) - c([Y,Z]) = \sigma(Z)(c(Y))$$

for any $c \in Z^1(\mathfrak{g};\mathcal{N}(\mathfrak{g}))$ and $Y, Z \in \mathfrak{g}$. Now we have

$$(Y^kc)(Z) = \sigma(Z)\sigma(Y)^{k-1}(c(Y)) \tag{3.2}$$

for any $k \geq 1$. If $k = 1$, (3.2) was already shown. Assume (3.2) holds for $k \geq 1$. Then $(Y^{k+1}c)(Z) = \sigma(Z)\sigma(Y)^{k-1}((Yc)(Y)) = \sigma(Z)\sigma(Y)^k(c(Y))$. This proceeds the induction.

Hence we have

$$(e^{\mathrm{ad}X}\delta - \delta)(Z) = \sum_{k=1}^{\infty}\frac{1}{k!}(X^k\delta)(Z) = \sum_{k=1}^{\infty}\frac{1}{k!}\sigma(Z)\sigma(X)^{k-1}(\delta X)$$
$$= \sigma(Z)\left(\sum_{k=1}^{\infty}\frac{1}{k!}\sigma(X)^{k-1}\right)(\delta X).$$

This means $e^{\mathrm{ad}X}\delta - \delta$ is the 1-coboundary induced by $\left(\sum_{k=1}^{\infty} \frac{1}{k!}\sigma(X)^{k-1}\right)(\delta X)$. The lemma follows from Lemma 3.1. $\square$

§ 3.2. Deformation of a cobracket and a comodule structure map

A similar results to Lemma 3.1 holds for a deformation of cobrackets and comodules.

Lemma 3.3. *Let $\mathfrak{g}$ be a Lie algebra, M a $\mathfrak{g}$-module, δ and $\delta' \in Z^1(\mathfrak{g};\mathcal{N}(\mathfrak{g}))$ involutive Lie bialgebra structures on $\mathfrak{g}$, and let μ and $\mu' : M \to M \otimes \mathfrak{g}$ make M an involutive right $(\mathfrak{g}, \delta)$-bimodule and an involutive right $(\mathfrak{g}, \delta')$-bimodule, respectively. Suppose there exist $A \in \mathcal{N}(\mathfrak{g})$ and $B \in \Lambda^2\mathfrak{g}$ such that*

(i) *$\forall X \in \mathfrak{g}$, $(\delta' - \delta)(X) = \sigma(X)(A)$,*

(ii) *$\forall m \in M$, $(\mu' - \mu)(m) = \partial(m \otimes B)$, and*

(iii) *$\forall X \in \mathfrak{g}$, $\sigma(X)(A) = \sigma(X)(B)$.*

Then we have

$$d(\delta, \mu) = d(\delta', \mu') : H_*(\mathfrak{g}; M) \to H_{*+1}(\mathfrak{g}; M).$$

Proof. We define $E_B : M \otimes \Lambda^p\mathfrak{g} \to M \otimes \Lambda^{p+2}\mathfrak{g}$ by $E_B(m \otimes \xi) := m \otimes \xi \wedge B$ for $m \in M$ and $\xi \in \Lambda^p\mathfrak{g}$. By (3.1) and (2.1), we have

$$(\partial E_B - E_B\partial)(m \otimes \xi) = \partial(m \otimes B) \wedge \xi + m \otimes \sum_{i=1}^{p}(-1)^i\sigma(X_i)(B) \wedge X_1 \wedge \overset{\hat{i}}{\cdots} \wedge X_p.$$

Using the conditions (ii) (iii) and (3.1), we compute $(\partial E_B - E_B\partial)(m \otimes \xi) = (\mu' - \mu)(m) \wedge \xi + m \otimes (\partial E_A - E_A\partial)\xi = (d' - d)(m \otimes \xi)$. Here we write simply $d = d(\delta, \mu)$ and $d' = d(\delta', \mu')$. This proves the lemma. $\square$

Let $(\mathfrak{g}, \delta)$ be a topological involutive Lie bialgebra, (M, μ) a topological involutive right $\mathfrak{g}$-bimodule, U an automorphism of the topological Lie algebra $\mathfrak{g}$, and U^M an automorphism of the topological vector space M compatible with U. We define $U\mu := (U^M \otimes U)\delta(U^M)^{-1}$. Then $(M, U\mu)$ is an involutive right $(\mathfrak{g}, U\delta)$-bimodule.

Lemma 3.4. *Let $X \in \mathfrak{g}$ and suppose $e^{\mathrm{ad}X} = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{ad}(X))^k$ and $e^{\sigma(X)} = \sum_{k=0}^{\infty} \frac{1}{k!}(\sigma(X))^k$ converge as automorphisms of the topological Lie algebra $\mathfrak{g}$ and the topological vector space M, respectively. Then we have $d(\delta, \mu) = d(e^{\mathrm{ad}X}\delta, e^{\sigma(X)}\mu)$ on $H_*(\mathfrak{g}; M)$.*

Proof. We write $A = \left(\sum_{k=1}^{\infty} \frac{1}{k!}\sigma(X)^{k-1}\right)(\delta X)$. As was shown in Lemma 3.2, $(e^{\mathrm{ad}X}\delta - \delta)(Z) = \sigma(Z)(A)$ for any $Z \in \mathfrak{g}$. From 2.9 follows $(X\mu)(m) = \Gamma(m \otimes \delta X)$.

Let $\Phi \in \Lambda^2\mathfrak{g}$. If we define $\varphi : M \to M \otimes \mathfrak{g}$ by $\varphi(m) := \Gamma(m \otimes \Phi)$, then we have $(X\varphi)(m) = \sigma(X)\varphi(m) - \varphi(Xm) = \sigma(X)\Gamma(m \otimes \Phi) - \Gamma(Xm \otimes \Phi) = \Gamma(m \otimes \sigma(X)\Phi)$. Hence, by $A \in \mathcal{N}(\mathfrak{g})$,

$$
\begin{aligned}
&(e^{\sigma(X)}\mu - \mu)(m) = \sum_{k=1}^{\infty} \frac{1}{k!}(X^k\mu)(m) = \sum_{k=1}^{\infty} \frac{1}{k!}\Gamma(m \otimes \sigma(X)^{k-1}\delta X) \\
&= \Gamma(m \otimes A) = \partial(m \otimes A).
\end{aligned}
$$

Consequently the lemma follows from Lemma 3.3. □

§ 4. Surface Topology

We discuss some relations among these homological facts and surface topology, in particular, a tensorial description of the Turaev cobracket and Kontsevich's non-commutative symplectic geometry.

§ 4.1. Symplectic derivations

It is the Lie algebra of symplectic derivations of the completed tensor algebra of a symplectic vector space that plays a central role throughout this section. Let H be a symplectic $\mathbb{Q}$-vector space of dimension $2g$, $g \geq 1$, and $\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$ the completed tensor algebra over H. $\widehat{T}$ is filtered by the two-sided ideals $\widehat{T}_p := \prod_{m=p}^{\infty} H^{\otimes m}$, $p \geq 1$, and constitutes a complete Hopf algebra whose coproduct $\Delta : \widehat{T} \to \widehat{T}\widehat{\otimes}\widehat{T}$ is given by $\Delta(X) = X\widehat{\otimes}1 + 1\widehat{\otimes}X$ for any $X \in H$. The symplectic form $\omega \in H^{\otimes 2}$ is given by $\omega = \sum_{i=1}^{g} A_iB_i - B_iA_i \in H^{\otimes 2}$ for any symplectic basis $\{A_i, B_i\}_{i=1}^{g}$ of H. We study the Lie algebra of continuous derivations on $\widehat{T}$ annihilating the form ω, which we denote by $\mathrm{Der}_\omega(\widehat{T}) = \mathfrak{a}_g^-$. We regard $\mathrm{Der}_\omega(\widehat{T})$ as a subspace of $H^* \otimes \widehat{T}$ by the restriction map to H. The symplectic vector space H is naturally isomorphic to its dual H^* by the map $X \in H \mapsto (Y \mapsto X \cdot Y) \in H^*$, so that we identify $H^* \otimes \widehat{T} = H \otimes \widehat{T} = \widehat{T}_1$. Then the image of $\mathrm{Der}_\omega(\widehat{T})$ in $\widehat{T}_1$ coincides with the cyclic invariants in $\widehat{T}_1 = \prod_{m=1}^{\infty} H^{\otimes m}$. In other words, we identify $\mathrm{Der}_\omega(\widehat{T})$ with $N(\widehat{T}_1) \subset \widehat{T}_1$, where $N : \widehat{T} \to \widehat{T}$ is the *cyclic symmetrizer* or the *cyclicizer* defined by $N|_{H^{\otimes 0}} := 0$ and $N(X_1 \cdots X_m) := \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1}$ for $X_i \in H$. See [9] for details. The subspace $N(H^{\otimes 2})$ is a Lie subalgebra naturally isomorphic to $\mathfrak{sp}_{2g}(\mathbb{Q})$.

Schedler [20] constructed a cobracket on the necklace Lie algebra associated to a quiver. The Lie algebra $\mathfrak{a}_g^-$ can be regarded as such a Lie algebra. Schedler's cobracket for $\mathfrak{a}_g^-$, which we denote by $\delta^{\mathrm{alg}} : \mathfrak{a}_g^- \to \mathfrak{a}_g^-\widehat{\otimes}\mathfrak{a}_g^-$, is given by

$$
\begin{aligned}
\delta^{\mathrm{alg}}(N(X_1X_2\cdots X_m)) = \sum_{i<j}(X_i \cdot X_j)\{&N(X_{i+1}\cdots X_{j-1})\widehat{\otimes}N(X_{j+1}\cdots X_mX_1\cdots X_{i-1}) \\
&-N(X_{j+1}\cdots X_mX_1\cdots X_{i-1})\widehat{\otimes}N(X_{i+1}\cdots X_{j-1})\}
\end{aligned}
$$

for any $X_i \in H$ and $m \geq 1$.

The cyclic symmetry suggests us a close relation between symplectic derivations and fatgraphs, which was exhausted in Kontsevich's formal symplectic geometry [13]. He studied a Lie subalgebra $a_g := \bigoplus_{m=2}^{\infty} N(H^{\otimes m})$ of $\mathfrak{a}_g^-$, which he called "associative", and proved that the primitive part of the limit of the relative homology $\lim_{g\to\infty} H_k(a_g, \mathfrak{sp}_{2g}(\mathbb{Q}))$ is isomorphic to $\bigoplus_{s>0, 2-2g-s<0} H^{4g-4+2s-k}(\mathbb{M}_g^s/\mathfrak{S}_s; \mathbb{Q})$. Here $\mathbb{M}_g^s$ is the moduli space of Riemann surfaces of genus g with s punctures, and the s-th symmetric group $\mathfrak{S}_s$ acts on it by permutation of punctures.

Schedler's cobracket δ^{alg} does not preserve the subalgebra a_g, so that $d(\delta^{\mathrm{alg}})$ does not act on the homology group $H_k(a_g)$. On the other hand, Schedler's cobracket δ^{alg} preserves the subalgebra $a_g^- := \bigoplus_{m=1}^{\infty} N(H^{\otimes m})$, whose degree completion is just the Lie algebra $\mathfrak{a}_g^-$.

Problem 4.1. Find a fatgraph interpretation of the primitive part of the limit of the relative homology $\lim_{g\to\infty} H_k(a_g^-, \mathfrak{sp}_{2g}(\mathbb{Q}))$.

The difference between a_g and a_g^- is just H, the derivations of degree -1, which seem to correspond to tails in fatgraphs. The homology group $H_*(a_g^-, \mathfrak{sp}_{2g}(\mathbb{Q}))$ seems to be related to the moduli space of Riemann surfaces with boundary and marked points studied in [4]. See [19] for details on fatgraphs. The coboundary operator $d(\delta^{\mathrm{alg}})$ is defined on $H_*(a_g^-, \mathfrak{sp}_{2g}(\mathbb{Q}))$, since δ^{alg} is $\mathfrak{sp}_{2g}(\mathbb{Q})$-invariant, and vanishes on $N(H^{\otimes 2}) = \mathfrak{sp}_{2g}(\mathbb{Q})$.

Problem 4.2. If Problem 4.1 is solved in an affirmative way, then find a fatgraph interpretation of the coboundary operator $d(\delta^{\mathrm{alg}})$.

As will be explained in the next subsection, Schedler's cobracket is closely related to the Turaev cobracket. So the operator $d(\delta^{\mathrm{alg}})$ seem to be related to degeneration of Riemann surfaces.

§ 4.2. Turaev cobracket

In this section, for simplicity, we confine ourselves to a compact connected oriented surface with connected boundary. See Appendix for the definitions of the Goldman bracket, the Turaev cobracket and the operations σ and μ stated below. We begin by recalling some results of Kuno and the author on a completion of the Goldman Lie algebra [9] [10]. Let $g \geq 1$ be a positive integer. We denote by $\Sigma = \Sigma_{g,1}$ a compact connected oriented surface of genus g with 1 boundary component, and by $\hat{\pi} = \hat{\pi}(\Sigma) = [S^1, \Sigma]$ the homotopy set of free loops on the surface Σ. Goldman [6] defines a natural Lie algebra structure on the $\mathbb{Q}$-free vector space $\mathbb{Q}\hat{\pi}$, which we call the Goldman Lie algebra. Choose a basepoint $*$ on the boundary $\partial\Sigma$, and consider the fundamental group $\pi := \pi_1(\Sigma, *)$. The group ring $\mathbb{Q}\pi$ admits a decreasing filtration given by the power of the augmentation ideal $I\pi$. Since π is a free group of rank $2g$, the completion map $\mathbb{Q}\pi \to \widehat{\mathbb{Q}\pi} := \varprojlim_{n\to\infty} \mathbb{Q}\pi/(I\pi)^n$ is injective. We can consider a similar completion of the Goldman Lie algebra $\mathbb{Q}\hat{\pi}$ as follows. The forgetful map of basepoints $|\ | : \mathbb{Q}\pi \to \mathbb{Q}\hat{\pi}$ is surjective, since Σ is connected. We define a filtration $\{\mathbb{Q}\hat{\pi}(n)\}_{n\geq 1}$ of $\mathbb{Q}\hat{\pi}$ by $\mathbb{Q}\hat{\pi}(n) := |\mathbb{Q}1 + (I\pi)^n|$, where $1 \in \pi$ is the constant loop. In [10] it is proved that $[\mathbb{Q}\hat{\pi}(n), \mathbb{Q}\hat{\pi}(n')] \subset \mathbb{Q}\hat{\pi}(n + n' - 2)$. Hence we can consider the completed Goldman Lie algebra $\widehat{\mathbb{Q}\hat{\pi}} = \widehat{\mathbb{Q}\hat{\pi}}(\Sigma)$ defined by $\widehat{\mathbb{Q}\hat{\pi}} := \varprojlim_{n\to\infty} \mathbb{Q}\hat{\pi}/\mathbb{Q}\hat{\pi}(n)$. In [9] Kuno and the author defined a natural operation $\sigma : \mathbb{Q}\hat{\pi} \otimes \mathbb{Q}\pi \to \mathbb{Q}\pi$ to introduce a natural nontrivial $\mathbb{Q}\hat{\pi}$-module structure on the group ring $\mathbb{Q}\pi$, which the completed group ring $\widehat{\mathbb{Q}\pi}$ inherits as a nontrivial $\widehat{\mathbb{Q}\hat{\pi}}$-module structure [10]. These Lie algebras act on the algebras by (continuous) derivations, respectively.

As is classically known, the group ring $\mathbb{Q}\pi$ is embedded into the completed tensor algebra $\widehat{T}$ over the first rational homology group $H := H_1(\Sigma; \mathbb{Q})$ of the surface Σ as (complete) Hopf algebras. Here we consider H a symplectic $\mathbb{Q}$-vector space by the intersection number on the surface Σ. To study the embedding in detail, Massuyeau [16] introduced the notion of a symplectic expansion of the fundamental group π. A map $\theta : \pi \to \widehat{T}$ is a **symplectic expansion** if it satisfies the following four conditions.

1. We have $\theta(xy) = \theta(x)\theta(y)$ for any x and $y \in \pi$.
2. For any $x \in \pi$ we have $\theta(x) \equiv 1+[x] \pmod{\widehat{T}_2}$, where $[x] \in H \subset \widehat{T}$ is the homology class of x.
3. For any $x \in \pi$, $\theta(x)$ is group-like, namely, $\Delta\theta(x) = \theta(x)\widehat{\otimes}\theta(x)$.
4. Let $\zeta \in \pi$ be the boundary loop in the negative direction, and $\omega \in H^{\otimes 2} \subset \widehat{T}$ the symplectic form. Then we have $\theta(\zeta) = e^\omega \in \widehat{T}$.

Symplectic expansions do exist [8] [16] [14]. A symplectic expansion θ induces an isomorphism $\theta : \widehat{\mathbb{Q}\pi} \overset{\cong}{\to} \widehat{T}$ of complete Hopf algebras. For any two symplectic expansions θ and θ', there exists an element of $u \in \mathrm{Der}_\omega(\widehat{T}) = \mathfrak{a}_g^-$ such that $(u\widehat{\otimes}u)\Delta = \Delta u$, $u(H) \subset \widehat{T}_2$ and $\theta' = e^u \circ \theta : \pi \to \widehat{T}$. See [9] for details.

In [9] and [10], Kuno and the author proved

Theorem 4.3. *Any symplectic expansion $\theta : \pi \to \widehat{T}$ induces*

1. an isomorphism of Lie algebras

$$-N\theta : \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} N(\widehat{T}_1) = \mathrm{Der}_\omega(\widehat{T}) = \mathfrak{a}_g^-$$

given by $-(N\theta)(|x|) := -N(\theta(x))$ for any $x \in \pi$, and

2. a commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{Q}\hat{\pi}} \otimes \widehat{\mathbb{Q}\pi} & \longrightarrow & \widehat{\mathbb{Q}\pi} \\ {\scriptstyle -N\theta\otimes\theta}\downarrow & & {\scriptstyle \theta}\downarrow \\ \mathrm{Der}_\omega(\widehat{T}) \otimes \widehat{T} & \longrightarrow & \widehat{T}, \end{array}$$

where the horizontal arrows mean the actions as derivations.

Let $\mathbb{Q}\hat{\pi}' = \mathbb{Q}\hat{\pi}'(\Sigma)$ be the quotient of $\mathbb{Q}\hat{\pi}$ by the linear span of the constant loop $1 \in \hat{\pi}$. Since 1 is in the center of $\mathbb{Q}\hat{\pi}$, it has a natural Lie algebra structure. In [22] Turaev introduced a cobracket δ on the Lie algebra $\mathbb{Q}\hat{\pi}'$ and proved that the pair $(\mathbb{Q}\hat{\pi}', \delta)$ is a Lie bialgebra. Later Chas [2] proved that it is involutive. Kuno and the author [11] proved the completed Goldman Lie algebra $\widehat{\mathbb{Q}\hat{\pi}}$ inherits the Turaev cobracket, so we call it the completed Goldman-Turaev Lie bialgebra. Inspired by Turaev's μ in [21], they [11] introduced a natural nontrivial comodule structure map $\mu : \mathbb{Q}\pi \to \mathbb{Q}\pi \otimes \mathbb{Q}\hat{\pi}'$, and proved that $(\mathbb{Q}\pi, \mu)$ is an involutive $\mathbb{Q}\hat{\pi}'$-bimodule. The comodule structure map μ defines a complete involutive $\widehat{\mathbb{Q}\hat{\pi}}$-bimodule structure on the completed group ring $\widehat{\mathbb{Q}\pi}$ [11].

Let $\theta : \pi \to \widehat{T}$ be a symplectic expansion. Then the Turaev cobracket δ and the isomorphisms in Theorem 4.3 defines a cobracket $\delta^\theta := ((-N\theta)\widehat{\otimes}(-N\theta)) \circ \delta \circ (-N\theta) : \mathfrak{a}_g^- \to \mathfrak{a}_g^- \widehat{\otimes} \mathfrak{a}_g^-$. Similarly the comodule structure map $\mu^\theta : (\theta\widehat{\otimes}(-N\theta))\circ\mu\circ\theta : \widehat{T} \to \widehat{T}\widehat{\otimes}\mathfrak{a}_g^-$ can be defined so that $(\widehat{T}, \mu^\theta)$ is an involutive $\mathfrak{a}_g^-$-bialgebra.

The grading on $\mathfrak{a}_g^-$ defines the Laurent expansion of the cobracket δ^θ

$$\delta^\theta(N(X_1X_2\cdots X_m)) = \sum_{p=-\infty}^{\infty} \delta^\theta_{(p)}(N(X_1X_2\cdots X_m)),$$

$$\delta^\theta_{(p)}(N(X_1X_2\cdots X_m)) \in (\mathfrak{a}_g^-\widehat{\otimes}\mathfrak{a}_g^-)_{(m+p)} := \bigoplus_{k+l=m+p} N(H^{\otimes k}) \otimes N(H^{\otimes l})$$

for $X_i \in H$. Massuyeau and Turaev [18] and Kuno and the author [11] independently proved

Theorem 4.4. *For any symplectic expansion θ we have*

1. $\delta^\theta_{(p)} = 0$ *for* $p = 0, -1,$ *and* $p \leq -3$.
2. $\delta^\theta_{(-2)}$ *is the same as Schedler's cobracket [20], i.e.,* $\delta^\theta_{(-2)} = \delta^{\mathrm{alg}}$.

Theorem 4.4 follows from some computation based on a tensorial description of the homotopy intersection form by Massuyeau and Turaev [17]. In the computation we introduce the Laurent expansion of the comodule structure map μ^θ in a similar way. The principal term is $\mu^{\mathrm{alg}} : \widehat{T} \to \widehat{T}\widehat{\otimes}\mathfrak{a}_g^-$ defined by

$$\mu^{\mathrm{alg}}(X_1\cdots X_m) := \sum_{1\leq i<j\leq m} (X_i\cdot X_j)X_1\cdots X_{i-1}X_{j+1}\cdots X_m\widehat{\otimes}N(X_{i+1}\cdots X_{j-1})$$

for $X_i \in H$. The pair $(\widehat{T}, \mu^{\mathrm{alg}})$ is a complete involutive $(\mathfrak{a}_g^-, \delta^{\mathrm{alg}})$-bimodule. So we present the following problem.

Problem 4.5. Find a fatgraph interpretation of the limit of the relative twisted homology $\lim_{g\to\infty} H_k(a_g^-, \mathfrak{sp}_{2g}(\mathbb{Q}); \widehat{T})$ and the coboundary operator $d(\delta^{\mathrm{alg}}, \mu^{\mathrm{alg}})$ on it.

As for the first term $\delta^\theta_{(1)}$ of the Laurent expansion of δ^θ, the following holds.

Proposition 4.6 ([12]). *There exist symplectic expansions θ and θ' such that $\delta^\theta_{(1)} = 0$ and $\delta^{\theta'}_{(1)} \neq 0$.*

In particular, δ^θ and μ^θ do depend on the choice of a symplectic expansion θ, and the cobracket δ^θ for some θ does not coincide with Schedler's cobracket δ^{alg}. But the cohomology classes of δ^θ and μ^θ do not depend on the choice of symplectic expansions from the following proposition.

Proposition 4.7. *Let θ' be another symplectic expansion. Then we have*

$$\begin{aligned} &d(\delta^{\theta'}) = d(\delta^\theta) \quad \text{on } H_*(\mathfrak{a}_g^-), \text{ and} \\ &d(\delta^{\theta'}, \mu^{\theta'}) = d(\delta^\theta, \mu^\theta) \quad \text{on } H_*(\mathfrak{a}_g^-; \widehat{T}). \end{aligned}$$

Proof. There exists an element of $u \in \mathrm{Der}_\omega(\widehat{T}) = \mathfrak{a}_g^-$ such that $(u\widehat{\otimes}u)\Delta = \Delta u$, $u(H) \subset \widehat{T}_2$ and $\theta' = e^u \circ \theta : \pi \to \widehat{T}$. From some straight-forward computation in [9] Lemma 4.3.1, we have $Ne^u = e^{\mathrm{ad}u}N : \widehat{T} \to \mathfrak{a}_g^-$. Therefore $\delta^{\theta'} = (e^{\mathrm{ad}u}\widehat{\otimes}e^{\mathrm{ad}u})\delta^\theta e^{-\mathrm{ad}u} = e^{\mathrm{ad}u}\delta$ and $\mu^{\theta'} = (e^u\widehat{\otimes}e^{\mathrm{ad}u})\mu^\theta e^{-u} = e^{\sigma(u)}\mu^\theta$ in the sense of Lemma 3.4. In view of Lemmas 3.2 and 3.4, this shows the proposition. □

This proposition makes us to present the following problems.

Problem 4.8. Determine whether δ^θ and μ^θ are cohomologous to Schedler's δ^{alg} and μ^{alg}, respectively, or not.

Problem 4.9. If the answer to Problem 4.8 is affirmative, determine whether there exists a symplectic expansion θ such that δ^θ and μ^θ coincide with Schedler's δ^{alg} and μ^{alg}, respectively, or not.

§ Appendix A. Operations of loops on a surface

In the appendix we briefly review some operations of loops on a surface introduced in [6] [22] [9] and [11].

Appendix A.1. Goldman bracket

Let S be an oriented surface. We denote by $\hat{\pi}(S)$ the homotopy set of free loops on the surface S. For any $p \in S$ we denote by $|\ | : \pi_1(S,p) \to \hat{\pi}(S)$ the forgetful map of the basepoint p. Let α and β be elements of $\hat{\pi}(S)$. We choose their representatives in general position, and denote them by the same symbols. Then the set of intersection points $\alpha \cap \beta$ is finite, and α and β intersect transversely at each point in $\alpha \cap \beta$. The Goldman bracket is defined to be the formal sum

$$[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta) |\alpha_p \beta_p|$$

in $\mathbb{Z}\hat{\pi}(S)$, the $\mathbb{Z}$-free module over the set $\hat{\pi}(S) = [S^1, S]$. Here $\varepsilon_p(\alpha, \beta) \in \{\pm 1\}$ is the local intersection number at p, and α_p (resp. β_p) $\in \pi_1(S,p)$ is the based loop along α (resp. β) with basepoint p. Goldman [6] proved that the bracket is well-defined, namely, homotopy invariant, and that the pair $(\mathbb{Z}\hat{\pi}(S), [\ ,\ ])$ is a Lie algebra, which we call the Goldman Lie algebra of the surface S.

Assume that the boundary ∂S is non-empty, and let $*$ be a point on the boundary ∂S. We denote by $\Pi S(p_0, p_1)$ the homotopy set of paths on S from p_0 to $p_1 \in S$. Choose representatives of $\alpha \in \hat{\pi}(S)$ and $\gamma \in \pi_1(S, *)$ in general position. The formal sum

$$\sigma(\alpha)(\gamma) := \sum_{p \in \alpha \cap \gamma} \epsilon_p(\alpha, \gamma) \gamma_{*p} \alpha_p \gamma_{p*} \in \mathbb{Z}\pi_1(S, *)$$

is well-defined, namely, homotopy invariant [9]. Here $\gamma_{*p} \in \Pi S(*, p)$ (resp. $\gamma_{p*} \in \Pi S(p, *)$) is (the homotopy class of) the restriction of γ to the segment from $*$ to p (resp. from p to $*$). Moreover σ defines a Lie algebra homomorphism $\sigma : \mathbb{Z}\hat{\pi}(S) \to \mathrm{Der}(\mathbb{Z}\pi_1(S, *))$ [9]. If $*_0$ and $*_1$ are two distinct points on ∂S, then $\mathbb{Z}\Pi S(*_0, *_1)$, the $\mathbb{Z}$-free module over the set $\Pi S(*_0, *_1)$, has a similar $\mathbb{Z}\hat{\pi}(S)$-module structure [10].

Appendix A.2. Turaev cobracket

Let S be a connected oriented surface. The constant loop $1 \in \hat{\pi}(S)$ on the surface is in the center of the Goldman Lie algebra $\mathbb{Z}\hat{\pi}(S)$, so that the quotient $\mathbb{Z}\hat{\pi}'(S) := \mathbb{Z}\hat{\pi}(S)/\mathbb{Z}1$ has a natural Lie algebra structure. We denote by $|\ |' : \mathbb{Z}\pi_1(S,p) \to \mathbb{Z}\hat{\pi}'(S)$ the composite of the forgetful map of the base point $p \in S$ and the quotient map $\mathbb{Z}\hat{\pi}(S) \to \mathbb{Z}\hat{\pi}'(S)$. Choose a representative of $\alpha \in \hat{\pi}(S)$ in general position, and denote it by the same symbol. Then the set $D_\alpha := \{(t_1, t_2) \in S^1 \times S^1;\, t_1 \neq t_2,\, \alpha(t_1) = \alpha(t_2)\}$ is finite and α intersects itself transversely at each $\alpha(t_1) = \alpha(t_2)$. The Turaev cobracket is defined to be the formal sum

$$\delta(\alpha) := \sum_{(t_1, t_2) \in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) |\alpha_{t_1 t_2}|' \otimes |\alpha_{t_2 t_1}|'$$

in $\mathbb{Z}\hat{\pi}'(S) \otimes \mathbb{Z}\hat{\pi}'(S)$. Here $\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \in \{\pm 1\}$ is the local intersection number of the velocity vectors $\dot{\alpha}(t_1)$ and $\dot{\alpha}(t_2) \in T_{\alpha(t_1)}S$, and $\alpha_{t_1 t_2}$ (resp. $\alpha_{t_2 t_1}$) $\in \pi_1(S, \alpha(t_1))$ is (the homotopy class) of the restriction of α to the interval $[t_1, t_2]$ (resp. $[t_2, t_1]$). Turaev [22] proved that the cobracket δ is well-defined, namely, homotopy invariant, and that the pair $(\mathbb{Z}\hat{\pi}'(S), \delta)$ is a Lie bialgebra. Later Chas [2] proved that it satisfies the involutivity.

Assume that the boundary ∂S is non-empty, and let $*$ be a point on the boundary ∂S. The homomorphism σ stated above factors through the quotient $\mathbb{Z}\hat{\pi}'(S)$. Choose a representative of $\gamma \in \pi_1(S, *)$ such that it is a smooth immersion whose singularities are at most ordinary double points, the image of the interior $]0, 1[$ is included in the interior of S, and the velocity vectors at the endpoints 0 and 1 are linearly independent on the tangent space T_*S. We denote it by the same symbol γ. Then the set Γ_γ of self-intersection points of γ except $*$ is finite. For $p \in \Gamma_p$, we denote $\gamma^{-1}(p) = \{t_1^p, t_2^p\}$ so that $t_1^p < t_2^p$. Inspired by Turaev [21], Kuno and the author [11] introduced the formal sum

$$\mu(\gamma) := \begin{cases} -\sum_{p\in\Gamma_\gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))(\gamma_{0t_1^p}\gamma_{t_2^p 1}) \otimes |\gamma_{t_1^p t_2^p}|', & \text{if } \varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = +1, \\ 1 \otimes |\gamma|' - \sum_{p\in\Gamma_\gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))(\gamma_{0t_1^p}\gamma_{t_2^p 1}) \otimes |\gamma_{t_1^p t_2^p}|', & \text{if } \varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = -1 \end{cases}$$

in $\mathbb{Z}\pi_1(S, *) \otimes \mathbb{Z}\hat{\pi}'(S)$. Here $\gamma_{\tau_0\tau_1} \in \Pi S(\gamma(\tau_0), \gamma(\tau_1))$ is (the homotopy class of) the restriction of γ to the interval $[\tau_0, \tau_1] \subset [0, 1]$ for $0 \le \tau_0 \le \tau_1 \le 1$. They proved that the map μ is well-defined, namely, homotopy invariant, and that the pair $(\mathbb{Z}\pi_1(S, *), \mu)$ is an involutive $\mathbb{Z}\hat{\pi}'(S)$-bimodule [11]. If $*_0$ and $*_1$ are two distinct points on ∂S, then $\mathbb{Z}\Pi S(*_0, *_1)$ has a similar involutive $\mathbb{Z}\hat{\pi}'(S)$-bimodule structure [11].

References

[1] H. Cartan and S. Eilenberg, Homological algebra, Princeton University Press, Princeton, NJ (1956)
[2] M. Chas, Combinatorial Lie bialgebras of curves on surfaces, Topology **43**, 543-568 (2004)
[3] J. Conant and K. Vogtmann, On a theorem of Kontsevich, Algebr. Geom. Topol. **3**, 1167-1224 (2003)
[4] K. Costello, A dual version of the ribbon graph decomposition of moduli space, Geometry and Topology **11**, 1637-1652 (2007)
[5] V. G. Drinfel'd, Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations, Soviet Math. Dokl. **27**, 68-71 (1983)
[6] W. M. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface groups representations, Invent. Math. **85**, 263-302 (1986)
[7] A. Hamilton, Noncommutative geometry and compactifications of the moduli space of curves, J. Noncommut. Geom. **4**, 157-188 (2010)
[8] N. Kawazumi, Harmonic Magnus expansion on the universal family of Riemann surfaces, preprint, math.GT/0603158 (2006)

[9] N. Kawazumi and Y. Kuno, The logarithms of Dehn twists, to appear in Quantum Topology, available at arXiv:1008.5017 (2010)
[10] N. Kawazumi and Y. Kuno, Groupoid-theoretical methods in the mapping class groups of surfaces, preprint, arXiv:1109.6479 (2011)
[11] N. Kawazumi and Y. Kuno, Intersections of curves on surfaces and their applications to mapping class groups, preprint, arXiv:1112.3841 (2011)
[12] N. Kawazumi and Y. Kuno, in preparation.
[13] M. Kontsevich, Formal (non)-commutative symplectic geometry, in: "The Gel'fand Mathematical Seminars, 1990-1992, Birkhäuser, Boston, 173-187 (1993)
[14] Y. Kuno, A combinatorial construction of symplectic expansions, Proc. Amer. Math. Soc. **140** 1075-1083 (2012)
[15] Y. Kuno, The generalized Dehn twist along a figure eight, J. of Topology and Analysis **5** 271-295 (2013)
[16] G. Massuyeau, Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant, Bull. Soc. Math. France **140** 101-161 (2012)
[17] G. Massuyeau and V. G. Turaev, Fox pairings and generalized Dehn twists, to appear in Ann. Inst. Fourier, available at arXiv:1109.5248 (2011)
[18] G. Massuyeau and V. G. Turaev, in preparation.
[19] R. C. Penner, Decorated Teichmüller theory, European Mathematical Society, Zürich (2012)
[20] T. Schedler, A Hopf algebra quantizing a necklace Lie algebra canonically associated to a quiver, Intern. Math. Res. Notices **2005**, 725-760 (2005)
[21] V. G. Turaev, Intersections of loops in two-dimensional manifolds, Math. USSR Sbornik **35**, 229-250 (1979)
[22] V. G. Turaev, Skein quantization of Poisson algebras of loops on surfaces, Ann. sci. École Norm. Sup. (4) **24**, 635-704 (1991)