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Thurston's metric on Teichmüller space and isomorphisms between Fuchsian groups

By

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Abstract

The aim of this paper is to establish relations between Thurston's metric on Teichmüller space and work of T. Sorvali on isomorphisms between Fuchsian groups. As a result, we give a new formula for Thurston's asymmetric metric for surfaces with punctures. We also update some results of Sorvali on boundary isomorphisms of Fuchsian groups.

§ 1. Introduction

Let $S = S_{g,n}$ be an oriented surface of genus g with n punctures and negative Euler characteristic. The Teichmüller space $\mathcal{T}_{g,n}$ of S is the space of equivalence classes of complete hyperbolic structures of finite area on S, where two hyperbolic structures X and Y on S are *equivalent* if there exists an isometry $h : X \to Y$ homotopic to the identity of S.

Thurston [13] defined an asymmetric metric d_L (that is, d_L is a metric except that the symmetry axiom $d_L(x, y) = d_L(y, x)$ is not satisfied) which we call, for brevity, the *Thurston metric*, on $\mathcal{T}_{g,n}$ by setting

$$d_L(X, Y) = \inf_f \log L_f(X, Y), \tag{1.1}$$

where

$$L_f(X, Y) = \sup_{x,y \in S, x \neq y} \frac{d_Y(x, y)}{d_X(x, y)}$$

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is the Lipschitz constant of a homeomorphism $f : X \to Y$ homotopic to the identity map of S. In the same paper, Thurston proved that there is a (non-necessarily unique) extremal Lipschitz homeomorphism that realizes the infimum in (1.1), and that

$$d_L(X,Y) = \log K(X,Y),$$

where

$$K(X,Y) = \sup_{\gamma\in\mathcal{S}} \frac{\ell_Y(\gamma)}{\ell_X(\gamma)},$$

where $\ell_X(\gamma)$ denotes the hyperbolic length of γ in X and $\mathcal{S}$ is the set of homotopy classes of essential simple closed curves on S. Several geometrical aspects of Thurston's metric, such as the description of a distinguished class of geodesics (called stretch lines) and the description of the structure of its Finsler norm unit ball, were studied by Thurston in [13]. Thurston's metric is also related to Thurston's compactification of Teichmüller space, see [9] and [14].

In this paper, we consider hyperbolic surfaces with at least one puncture. The existence of punctures is equivalent to the fact that the Fuchsian groups that represent the hyperbolic surfaces contain parabolic transformations. We make relations between Thurston's metric and work of Sorvali which was done more than ten years before the appearance of Thurston's preprint [13], by including Sorvali's work in a non-symmetric setting. Indeed, Sorvali's work concerns a symmetrization of Thurston's metric, namely, the so-called *length-spectrum* metric, but part of his theory may be used for a description of Thurston's metric. The length-spectrum metric d_{ls} on $\mathcal{T}_{g,n}$ is defined, for X and Y in $\mathcal{T}_{g,n}$, by

$$d_{ls}(X,Y) = \log \sup_{\gamma\in\mathcal{S}} \{\frac{\ell_Y(\gamma)}{\ell_X(\gamma)}, \frac{\ell_X(\gamma)}{\ell_Y(\gamma)}\}$$

which, by Thurston's result mentioned above, is equal to

$$\max\{d_L(X,Y), d_L(Y,X)\}.$$

For surfaces with punctures, the work of Sorvali gives a formula for $d_{ls}(X,Y)$ in terms of the *translation vector* (defined below) of the parabolic transformations. Using these ideas, we obtain a new formula for Thurston's distance between two hyperbolic structures X and Y in terms of translation vectors of parabolic transformations corresponding to punctures of X and Y.

Sorvali [11] also related the length-spectrum distance to the Hölder continuity of the *boundary mappings of Fuchsian groups* (see Section 4 for the definition). His results are also interesting for surfaces of infinite type. In a previous paper [1], we have observed that for surfaces of infinite type, the definition of the associated Teichmüller space depends on the choice of a base-point of that space and on the choice of a metric on that space which induces the topology. We used the name *quasiconformal Teichmüller space* for a Teichmüller space equipped with the Teichmüller metric, and *length-spectrum Teichmüller space* for a Teichmüller space equipped with the length-spectrum metric. The spaces are in general different (even set-theoretically), even if the base surfaces are the same. In particular, we showed that there exists a hyperbolic surface R of infinite type such that the quasiconformal Teichmüller space $\mathcal{T}_{qc}(R)$ is a proper subset of the length-spectrum Teichmüller space $\mathcal{T}_{ls}(R)$. Combining this with the result of Sorvali [11], we obtain a class of examples of homeomorphisms of $\mathbb{R} \cup \{\infty\}$ which are Hölder continuous but not quasisymmetric.

§ 2. Isomorphism between Fuchsian groups

Let $\mathbb{H}$ be the upper half-plane endowed with the Poincaré metric. The ideal boundary of $\mathbb{H}$ can be identified with $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. A *Fuchsian group* Γ is a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which acts properly discontinuously and freely on $\mathbb{H}$.

In this section, we consider Fuchsian groups that are not cyclic and which contain the parabolic transformation $g_0 : z \mapsto z + 1$. We do not assume that they are finitely generated and, consequently, the surfaces we consider might be of infinite type.

To a hyperbolic isometry g of the upper half-plane is associated a *multiplier* $\lambda(g) > 1$ whose logarithm is the *translation length* along the invariant geodesic of g. To a parabolic isometry, we can associate a *translation length along a horocycle*. The latter association is not canonical (it needs some normalization), but it will turn out to be useful. We now make this precise.

An isomorphism $j : \Gamma \to \Gamma'$ between Fuchsian groups is called *type-preserving* if it maps parabolic elements to parabolic elements and hyperbolic elements to hyperbolic elements. This is equivalent to saying that j and j^{-1} both preserve parabolic elements. Note that if there exists a homeomorphism $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that

$$f \circ g = j(g) \circ f$$

for all $g \in \Gamma$, then j is type-preserving. Indeed, in this case, $f \circ g \circ f^{-1} = j(g)$, and if $a \in \overline{\mathbb{R}}$ is a fixed point of g, then $f(a)$ is a fixed point of $j(g)$.

In this section, we shall only consider type-preserving isomorphisms $j : \Gamma \to \Gamma'$.

For a hyperbolic transformation g, we denote its attracting and repelling fixed points by $P(g)$ and $N(g)$ respectively. The element g is uniquely determined by $P(g)$, $N(g)$ and by the *multiplier* $\lambda(g) > 1$, defined by the fact that $\log \lambda(g)$ is the translation length along the invariant geodesic of g. From the definition, we see that $\lambda(g)$ is a conjugacy invariant. Up to conjugation by an element of $\mathrm{PSL}(2,\mathbb{R})$, g is represented by the transformation $z \mapsto \lambda(g)z$. Since the eigenvalues of a Möbuis transformation $z \mapsto \lambda(g)z$ are $\lambda(g)^{1/2}$ and $\lambda(g)^{-1/2}$, its trace is

$$\mathrm{tr}(g) = \lambda(g)^{1/2} + \lambda(g)^{-1/2}.$$

If g is parabolic, we set $\lambda(g) = 1$ and we denote the unique fixed point of g by $P(g)$.

Given an isomorphism $j : \Gamma \to \Gamma'$, we define $\delta_L(j) \in [1, \infty]$ by

$$\delta_L(j) = \inf\{a \geq 1 : \lambda(j(g)) \leq \lambda(g)^a, \ \forall \ g \in \Gamma\}.$$

This definition of $\delta_L(j)$ is a nonsymmetric version of a definition made by Sorvali in [11]. We have the following other formula for $\delta_L(j)$:

Lemma 2.1. *If $1 \leq s \leq \infty$ is the smallest number such that for all $g \in \Gamma$, $\mathrm{tr}(j(g)) \leq \mathrm{tr}(g)^s$, then $s = \delta_L(j)$.*

Proof. Suppose that $\mathrm{tr}(j(g)) \leq \mathrm{tr}(g)^a$ for all $g \in \Gamma$. For any $g \in \Gamma$, we let $\lambda = \lambda(g), \lambda' = \lambda(j(g))$. Since j is an isomorphism, $j(g^n) = j(g)^n$, $n = 1, 2, \cdots$. Since $\mathrm{tr}(j(g)^n) \leq \mathrm{tr}(g^n)^a$, we have

$$(\lambda')^{n/2} + (\lambda')^{-n/2} \leq (\lambda^{n/2} + \lambda^{-n/2})^a,$$

and then

$$(\lambda')^n \leq (\lambda')^n + (\lambda')^{-n} + 2 \leq (\lambda^n + \lambda^{-n} + 2)^a.$$

For some sufficiently large integer n_0, the right hand side of the above inequality is less than $(4\lambda^n)^a$ for all $n \geq n_0$. Therefore,

$$\lambda' \leq (4^{1/n}\lambda)^a, \ \forall \ n \geq n_0.$$

By letting $n \to \infty$, we get $\lambda' \leq \lambda^a$.

Conversely, suppose that $\lambda' \leq \lambda^a$. Then

$$\mathrm{tr}(j(g)) = (\lambda')^{1/2} + (\lambda')^{-1/2} \leq \lambda^{a/2} + \lambda^{-a/2} \leq (\lambda^{1/2} + \lambda^{-1/2})^a = \mathrm{tr}(g)^a.$$

□

Let g be a parabolic transformation. We shall associate with it a real number $\omega(g)$.

If $P(g) = \infty$ and $g(z) = z + c$, then we define $\omega(g) = c$. The value $\omega(g)$ is a real (positive or negative) number uniquely determined by g. The absolute value $|\omega(g)|$ is expressed (like the value $\lambda(g)$ associated to a hyperbolic element g) as a translation length, namely, the length of the horizontal segment (horocycle) joining the complex numbers i and $i+\omega(g)$. If g is parabolic with $P(g)\neq\infty$, then we define $\omega(g)$ to be the real number determined by the following equality:

$$\frac{1}{g(z)-P(g)}=\frac{1}{z-P(g)}+\omega(g).$$

There is a geometrical meaning to $w(g)$. There is a unique horocycle through $P(g)$ and $P(g)+i$ which is invariant by the action of g. For any point z on this horocycle, $|\omega(g)|$ is the non-euclidean length of the horocycle arc connecting z and $g(z)$. (We note by the way that when $P(g)=\infty$, the non-Euclidean length of the horocycle arc connecting the points i and $i+c$ is equal to c and that the Euclidean length of such a horocycle arc is also equal to c.)

Thus, when g is parabolic and whether $P(g)=\infty$ or $P(g)\neq\infty$, $|w(g)|$ is a translation length along a horocycle.

Following Sorvali [11], we call $\omega(g)$ the *translation vector* of g.

Note that unlike the value $\lambda(g)$ associated to a hyperbolic element g, the values $\omega(g)$ and $|w(g)|$ are not conjugacy invariants in the case where g is parabolic. To see this, consider the parabolic element $g(z)=\dfrac{z}{1+cz}$ where $c\neq 0$ is real and let h be a transformation $h(z)=\lambda z$ with $\lambda\in(0,\infty[$. Then a computation gives $\omega(g)=c$ while $\omega(h^{-1}\circ g\circ h)=c\lambda$. The fact that $\omega(g)$ is not a conjugacy invariant can also be seen from Lemma 2.2.

From now on and except in Theorem 4.2 below, we shall only consider type-preserving isomorphisms $j:\Gamma\to\Gamma'$ that fix g_0.

Lemma 2.2. *For any parabolic element h in* $\mathrm{PSL}(2,\mathbb{R})$ *with $P(h)\neq\infty$, we have*

$$|\omega(h^{-1}\circ g_0\circ h)|=|\omega(h)|^2.$$

Proof. The proof is contained in the proof of Theorem 3 of [11]. We reproduce it here for the convenience of the reader.

We have

$$\frac{1}{h(z)-P(h)}=\frac{1}{z-P(h)}+\omega(h)$$

or, equivalently,

$$h(z)=\frac{(1+\omega(h)P(h))\,z-\omega(h)P(h)^2}{\omega(h)z+1-\omega(h)P(h)}.$$

A computation shows that

$$(h^{-1}\circ g_0\circ h)(z)=\frac{\big(1+\omega(h)-\omega(h)^2P(h)\big)\,z-(1-\omega(h)P(h))^2}{-\omega(h)^2z+1-\omega(h)+\omega(h)^2P(h)}. \tag{2.1}$$

Since the fixed point of $h^{-1}\circ g_0\circ h$ is $h^{-1}(P(g_0))=h^{-1}(\infty)$, we have

$$P(h^{-1}\circ g_0\circ h)=\frac{\omega(h)P(h)-1}{\omega(h)}.$$

It is easy to show that (2.1) is equivalent to

$$\frac{1}{(h^{-1}\circ g_0\circ h)(z)-P(h^{-1}\circ g_0\circ h)}=\frac{1}{z-P(h^{-1}\circ g_0\circ h)}-\omega(h)^2.$$

It follows that $\omega(h^{-1}\circ g_0\circ h)=-\omega(h)^2$.

□

In what follows, we consider an isomorphism $j:\Gamma\to\Gamma'$ satisfying $\delta_L(j)<\infty$. Such a hypothesis is satisfied for example if j is induced by a homeomorphism between $\mathbb{H}/\Gamma$ and $\mathbb{H}/\Gamma'$, where $\mathbb{H}/\Gamma$ and $\mathbb{H}/\Gamma'$ are surfaces of finite type.

The following is an asymmetric version of Theorem 3 of Sorvali's paper [11].

Lemma 2.3. *Let $j:\Gamma\to\Gamma'$ be an isomorphism such that $s=\delta_L(j)<\infty$. Then $|\omega(j(g))|\leq|\omega(g)|^s$ for all parabolic transformations $g\in\Gamma$.*

Proof. The proof is the same as that of Theorem 10 of [12]. Let $g\neq g_0$ be a fixed parabolic transformation of Γ. Since Γ is discrete, $P(g)\neq\infty$.

Let $g_1=g^{-1}\circ g_0\circ g$ and inductively let $g_n=g_{n-1}^{-1}\circ g_0\circ g_{n-1}$ for every $n\geq 1$. By Lemma 2.2, we have

$$|\omega(g_n)|=|\omega(g_{n-1})|^2.$$

Therefore,

$$|\omega(g_n)|=|\omega(g)|^{2^n}. \tag{2.2}$$

Similarly, we have

$$|\omega(j(g_n))|=|\omega(j(g))|^{2^n}.$$

For any parabolic transformation $h\neq g_0$ of Γ, either $g_0\circ h^{-1}$ or $g_0\circ h$ is hyperbolic. Assume, for the proof, that $g_0\circ h$ is hyperbolic. Then,

$$tr(g_0\circ h)=|2+\omega(h)|. \tag{2.3}$$

We apply (2.3) to g_n and $j(g_n)$. Then by Lemma 2.1,

$$|2+\omega(f(g_n))|\leq|2+\omega(g_n)|^s.$$

By (2.2), we have

$$\begin{aligned}(|\omega(j(g))|^{2^n}-2)&=(|\omega(j(g_n))|-2)\\&\leq|2+\omega(j(g_n))|\\&\leq|2+\omega(g_n)|^s\\&\leq(2+|\omega(g)|^{2^n})^s.\end{aligned}$$

Hence

$$(|\omega(j(g))|^{2^n} - 2)^{1/2^n} \leq (2 + |\omega(g)|^{2^n})^{s/2^n}.$$

By letting $n \to \infty$, we obtain $|\omega(j(g))| \leq |\omega(g)|^s$. □

The following lemma is also due to Sorvali ([12] p. 3).

Lemma 2.4. *Suppose $g \in \Gamma$ is hyperbolic and $\lambda(j(g)) = \lambda(g)^a$. For each $n = 1, 2, \cdots$, let b_n be the real number such that*

$$|\omega(j(g)^n \circ g_0 \circ j(g)^{-n})| = |\omega(g^n \circ g_0 \circ g^{-n})|^{b_n}.$$

Then $\lim_{n\to\infty} b_n = a$.

Proof. The proof is due to Sorvali [11].

Note that the translation vectors of parabolic elements are not changed under conjugation by translations $z \mapsto z + b$ (where b is real). Therefore we may assume that $P(g) = P(j(g)) = 0$.

Set $\lambda = \lambda(g)$ and $N = N(g)$; then

$$g^n(z) = \frac{Nz}{(1-\lambda^n)z + \lambda^n N},$$

and it is easy to show that

$$g^n \circ g_0 \circ g^{-n}(z) = \frac{N(\lambda^n N + \lambda^n - 1)z + N^2}{-(\lambda^2 - 1)z + N(\lambda^n N + \lambda^n - 1)}.$$

Hence

$$\omega(g^n \circ g_0 \circ g^{-n}) = -\frac{(\lambda^n - 1)^2}{\lambda^n N^2} = -\frac{\lambda^n + \lambda^{-n} - 2}{N^2}. \tag{2.4}$$

By replacing N by $N' = N(j(g))$ and λ by $\lambda^a = \lambda(j(g))$, we obtain a similar expression for $\omega(j(g)^n \circ g_0 \circ j(g)^{-n})$. As a result,

$$\left(\frac{\lambda^{an} + \lambda^{-an} - 2}{(N')^2}\right)^{1/n} = \left(\frac{\lambda^n + \lambda^{-n} - 2}{(N)^2}\right)^{b_n/n}.$$

The left hand side of the above equation tends to λ^a as $n \to \infty$, and

$$\left(\frac{\lambda^n + \lambda^{-n} - 2}{(N)^2}\right)^{1/n}$$

tends to k as $n \to \infty$. It follows that $\lim_{n\to\infty} b_n = a$. □

It would be interesting to give a geometric interpretation of Lemma 2.4.

Now suppose that $j : \Gamma \to \Gamma'$ is an isomorphism with $\delta_L(j) < \infty$. We let

$$\rho_L(j) = \inf\{a \geq 1 : |\omega(j(g))| \leq |\omega(g)|^a \text{ for all parabolic } g \text{ in } \Gamma\}.$$

Theorem 2.5. *Let Γ and Γ' be two Fuchsian groups, both of which contain the parabolic element $g_0(z) = z + 1$. Suppose that $j : \Gamma \to \Gamma'$ is an isomorphism with $j(g_0) = g_0$ and $\delta_L(j) < \infty$. Then $\delta_L(j) = \rho_L(j)$.*

Proof. By Lemma 2.3, $\rho_L(j) \leq \delta_L(j)$.

Suppose that $\rho_L(j) < \delta_L(j)$. By definition of $\delta_L(j)$, there exists some hyperbolic element $g \in \Gamma$, some number $\epsilon > 0$ and $a \geq \rho_L(j) + \epsilon$ such that $\lambda(j(g)) = \lambda(g)^a$. By Lemma 2.4, the numbers b_n satisfy

$$|\omega(j(g)^n \circ g_0 \circ j(g)^{-n})| = |\omega(g^n \circ g_0 \circ g^{-n})|^{b_n}.$$

Then $\lim_{n\to\infty} b_n = a$. This means that $\rho_L(j) \geq \rho_L(j) + \epsilon/2$, which is impossible. As a result, $\rho_L(j) \geq \delta_L(j)$.

□

§ 3. Thurston's metric

Let $S = S_{g,n}$ with $n > 0$. Each hyperbolic structure X on S can be represented by $\mathbb{H}/\Gamma$ for some Fuchsian group Γ. Up to conjugation, we may assume that $g_0 : z \mapsto z+1$ belongs to Γ, and we shall do this throughout the rest of this paper.

Given two hyperbolic structures $X = \mathbb{H}/\Gamma$ and $Y = \mathbb{H}/\Gamma'$ on S, the identity map between (S, X) and (S, Y) lifts to a homeomorphism $f : \mathbb{H} \to \mathbb{H}$ which extends continuously to the ideal boundary $\overline{\mathbb{R}}$. We may also assume that f fixes $0, 1, \infty$. Using the map f, we define an isomorphism $j : \Gamma \to \Gamma'$ by

$$j(g) := f \circ g \circ f^{-1}, \forall\, g \in \Gamma.$$

Note that j satisfies the assumptions on the isomorphism j of Section 2, that is, j is type-preserving and it fixes the element g_0 (which, also by assumption, is in both groups Γ and Γ').

Each hyperbolic element g in Γ corresponds to a (not necessary simple) closed geodesic γ_g of X, with hyperbolic length $\ell_X(\gamma_g) = \log \lambda(g)$. By Theorem 2.5, we have

$$\sup_{g\in\Gamma} \frac{\ell_Y(\gamma_g)}{\ell_X(\gamma_g)} = \rho(j).$$

By a result of Thurston (Proposition 3.5, [13]),

$$\sup_{g\in\Gamma} \frac{\ell_Y(\gamma_g)}{\ell_X(\gamma_g)} = K(X, Y).$$

As a result, we get a new formula for the Thurston distance, which we state in the following:

Theorem 3.1. *With the above notation, Thurston's metric is given by*

$$d_L(X,Y) = \log \delta_L(j) = \log \rho_L(j).$$

By symmetrizing we obtain the following theorem of Sorvali.

Theorem 3.2 (Sorvali [12]). *With the above notation, the length-spectrum metric satisfies*

$$d_{ls}(X,Y) = \log \delta(j) = \log \rho(j),$$

where

$$\delta(j) = \max\{\delta_L(j), \delta_L(j^{-1})\}$$

and

$$\rho(j) = \max\{\rho_L(j), \rho_L(j^{-1})\}.$$

Note that it is interesting to have, like above, a definition of the Thurston metric in terms of translation lengths in the setting of groups, because such a definition can be generalized to other group actions. At the end of this paper, we address some questions in this direction.

§ 4. Boundary mappings

A homeomorphism $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is called a *boundary mapping* of an isomorphism $j : \Gamma \to \Gamma'$ if

$$\varphi \circ g = j(g) \circ \varphi$$

for all $g \in \Gamma$. We say that j is induced from φ.

The *limit set* of a Fuchsian group Γ, denoted by $\Lambda(\Gamma)$, is the set of accumulation points of the set

$$\Gamma(z_0) = \{g(z_0) : g \in \Gamma\}$$

on $\overline{\mathbb{H}} = \mathbb{H} \cup \overline{\mathbb{R}}$ for some $z_0 \in \mathbb{H}$. Since Γ acts properly discontinuously and freely on $\overline{\mathbb{H}}$, $\Lambda(\Gamma)$ is a subset of $\overline{\mathbb{R}}$. It is easy to see that the definition of $\Lambda(\Gamma)$ is independent of the choice of $z_0 \in \mathbb{H}$. The Fuchsian groups Γ is said to be *non-elementary* if $\Lambda(\Gamma)$ contains at least three points.

We denote by $F(\Gamma)$ the set of hyperbolic fixed points of a Fuchsian group Γ.

Lemma 4.1. *Suppose that Γ is non-elementary. Then Γ contain a hyperbolic element. Moreover, $F(\Gamma)$ is dense in $\Lambda(\Gamma)$.*

See e.g. Matsuzaki-Taniguchi [8] for a proof of Lemma 4.1.

A Fuchsian group Γ is *of the first kind* if $\Lambda(\Gamma) = \overline{\mathbb{R}}$. It is known that if Γ and Γ' are finitely generated and of the first kind, then any type-preserving isomorphism $j : \Gamma \to \Gamma'$ is realized by some homeomorphism $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. Moreover, the existence of such an isomorphism j implies that Γ and Γ' are quasiconformally conjugate, that is, there exists a quasiconformal map $\Phi : \mathbb{H} \to \mathbb{H}$ such that

$$\Phi \circ g = j(g) \circ \Phi$$

for all $g \in \Gamma$.

If an isomorphism $j : \Gamma \to \Gamma'$ is realized by some homeomorphism $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, then it follows from work of Douady-Earle [3] that there is a homeomorphism $\Phi : \mathbb{H} \to \mathbb{H}$ such that $\Phi|_{\overline{\mathbb{R}}} = \varphi$ and

$$\Phi \circ g = j(g) \circ \Phi$$

for all $g \in \Gamma$. Furthermore, φ is quasisymmetric if and only if Φ is quasiconformal. Recall that an orientation-preserving homeomorphism $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, normalized by $\varphi(\infty) = \infty$, is *quasisymmetric* if there exists a real number $k \geq 1$ such that for all $x, t \in \mathbb{R}, t \neq 0$,

$$1/k \leq \frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \leq k.$$

In general, for an isomorphism $j : \Gamma \to \Gamma'$ between two Fuchsian groups of the first kind, if there exists a boundary mapping φ of j, then φ is unique. The following necessary and sufficient condition for the existence of boundary mappings is due to Sorvali [10].

Theorem 4.2. *Let Γ and Γ' be two Fuchsian groups of the first kind and $j : \Gamma \to \Gamma'$ an isomorphism. (We do not make the assumption that j is type-preserving.) Then the following two conditions are equivalent:*

1. *The boundary mapping φ of j exists.*
2. *For all $g_1, g_2 \in \Gamma$ not equal to the identity, $A(g_1) \cap A(g_2) \neq \emptyset$ if and only if $A(j(g_1)) \cap A(j(g_2)) \neq \emptyset$. Here $A(g)$ denotes the geodesic connecting $P(g)$ and $N(g)$ if g is hyperbolic and denotes $P(g)$ if g is parabolic.*

An example of a type-preserving isomorphism between two Fuchsian groups of the first kind which does not have a boundary mapping is given by Sorvali [10].

Given $0 < \alpha \leq 1$ and a subset $F \subset \overline{\mathbb{R}}$, we say that a homeomorphism $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is *α-Hölder bi-continuous on F* if for each $x_0 \in F$, there exists a neighborhood $I \subset \overline{\mathbb{R}}$ of x_0 and a constant $C \geq 1$ such that

$$\frac{|x - x_0|^{1/\alpha}}{C} \leq |\varphi(x) - \varphi(x_0)| \leq C|x - x_0|^{\alpha}$$

for all $x \in I$. In the case where $x_0 = \infty$ or $\varphi(x_0) = \infty$, then we consider the Hölder bi-continuity of $\varphi(1/x)$ at 0 or of $1/\varphi(x)$ at x_0 respectively.

Suppose that φ is a *boundary mapping* of an isomorphism $j : \Gamma \to \Gamma'$ with $\delta(j) < \infty$. Let $B(j)$ be the set of real numbers α, $0 < \alpha \leq 1$, such that φ is α-Hölder bi-continuous on $F(\Gamma)$. Note that $B(j)$ is an interval contained in $[0, 1]$.

The following theorem is also due to Sorvali [11].

Theorem 4.3. *Suppose that φ is a* boundary mapping *of an isomorphism j : $\Gamma \to \Gamma'$ with $\delta(j) < \infty$. Then $B(j) \neq \emptyset$ and*

$$\delta(j) = \min_{\alpha \in B(j)} \{\frac{1}{\alpha}\}.$$

In the rest of this section, S is a connected orientable surface of infinite type and $R = \mathbb{H}/\Gamma_0$ is a hyperbolic structure on S. We assume that $\Lambda(\Gamma_0) = \overline{\mathbb{R}}$, i.e. Γ_0 is a Fuchsian group of the first kind. Up to conjugation, we may assume that the transformation $z \mapsto \lambda z$ for some $\lambda > 1$ belongs to Γ_0 and that 1 is a fixed point of some elements in Γ_0.

The *length-spectrum Teichmüller space* $\mathcal{T}_{ls}(R)$ is the space of homotopy classes of hyperbolic surfaces X homeomorphic to R, with

$$L(R, X) = \sup_{\gamma \in \mathcal{S}} \{\frac{\ell_X(\gamma)}{\ell_R(\gamma)}, \frac{\ell_R(\gamma)}{\ell_X(\gamma)}\} < \infty.$$

It is clear that for any two distinct elements $X, Y \in \mathcal{T}_{ls}(R)$, we have

$$1 < L(X, Y) = \sup_{\gamma \in \mathcal{S}} \{\frac{\ell_X(\gamma)}{\ell_Y(\gamma)}, \frac{\ell_Y(\gamma)}{\ell_X(\gamma)}\} < \infty.$$

The length-spectrum distance between X and Y is then given by

$$d_{ls}(X, Y) = \frac{1}{2} \log L(X, Y).$$

The fact that $d_{ls}(X, Y) = 0$ implies $X = Y$ is due to Sorvali [11]. In fact, Sorvali [11] showed that this result is also valid for Fuchsian groups of the second kind with some restriction on the isomorphism j.

Finally, we consider the *quasiconformal Teichmüller space* $\mathcal{T}_{qc}(R)$, i.e. the space of homotopy classes of hyperbolic metrics X on R such that the identity map between the topological surface equipped respectively with R and X is homotopic to a quasiconformal homoemorphism.

For any two (equivalence classes of) hyperbolic metrics $X, Y \in \mathcal{T}_{qc}(R)$, their quasiconformal distance $d_{qc}(X, Y)$ is defined as

$$d_{qc}(X, Y) = \frac{1}{2} \log \inf_f K(f)$$

where $K(f)$ is the quasiconformal dilatation of a quasiconformal homeomorphism $f : X \to Y$ which is homotopic to the identity.

The following lemma is called Wolpert's formula [15].

Lemma 4.4. *For any K-quasiconformal map $f : X \to Y$ and any $\gamma \in \mathcal{S}$, we have*

$$\frac{1}{K} \leq \frac{\ell_Y(f(\gamma))}{\ell_X(\gamma)} \leq K.$$

It follows from Wolpert's Lemma that for any two points $X, Y \in \mathcal{T}_{qc}(R)$,

$$d_{ls}(X,Y) \leq d_{qc}(X,Y). \tag{4.1}$$

We note that the inequality (4.1) was first obtained by Sorvali [11].

We proved in [1] that if there exists a sequence of simple closed curves $\{\alpha_i\}$ contained in the interior of R with $\ell_R(\alpha_i) \to 0$, then $\mathcal{T}_{qc}(R) \subsetneqq \mathcal{T}_{ls}(R)$. The idea was to construct a sequence of hyperbolic metrics by performing large twists along short curves.

Suppose that $X = \mathbb{H}/\Gamma \in \mathcal{T}_{ls}(R) \setminus \mathcal{T}_{qc}(R)$ and suppose that the surface R contains a sequence of simple closed curves α_i, in its interior, with $\ell_R(\alpha_i) \to 0$. The identity map between (S, R) and (S, X) lifts to a homeomorphism between the universal covers and induces a homeomorphism $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. Consider the homomorphism $j : \Gamma_0 \to \Gamma$ given by

$$\varphi \circ g = j(g) \circ \varphi$$

for all $g \in \Gamma_0$. Then φ is a boundary mapping of j. Since $\delta(j) < \infty$, it follows from Theorem 4.3 that there is some $0 < \alpha \leq 1$ such that φ is α-Hölder continuous on $F(\Gamma)$. However, φ is not quasisymmetric, since, otherwise, by Douady-Earle [3] there would exist an quasiconformal extension of φ to $\mathbb{H}$ satisfying

$$\varphi \circ g = j(g) \circ \varphi,$$

and this would induce a quasiconformal mapping between R and X. This is impossible, since $d_{qc}(R, X) = \infty$.

In a recent paper [2], we proved that endowed with the length-spectrum metric, $\mathcal{T}_{qc}(R)$ is nowhere dense in $\mathcal{T}_{ls}(R)$.

Denote by $A(\Gamma_0)$ the set of orientation-preserving homeomorphisms $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that

1. the conjugation
$$\varphi \circ g \circ \varphi^{-1},\ g \in \Gamma_0$$
gives a isomorphism between Γ_0 and some Fuchsian group Γ;

2. the map φ fixes $0, 1, \infty$;

3. there exists some $0 < \alpha \leq 1$ such that φ is α-Hölder bi-continuous on $F(\Gamma_0)$.

Note that if $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is quasisymmetric, then it is α-Hölder bi-continuous on $\overline{\mathbb{R}}$ for some $0 < \alpha \leq 1$ (see e. g. [6]).

Let $A_{qs}(\Gamma_0)$ be the subset of $A(\Gamma_0)$ consisting of the maps $\varphi \in A(\Gamma_0)$ which are quasisymmetric. We conclude with the following theorem.

Theorem 4.5. *$A_{qs}(\Gamma_0)$ is a proper subset of $A(\Gamma_0)$.*

§ 5. Cross-ratio norm

For any quadruple of distinct points p, q, r, s on $\overline{\mathbb{R}}$, the cross-ratio (p, q, r, s) is defined by

$$(p, q, r, s) = \frac{p-r}{p-s} \cdot \frac{q-s}{q-r}.$$

Given a hyperbolic transformation g, it is easy to show that

$$\lambda(g) = (g(s), s, N(g), P(g))$$

for any $s \neq P(g), N(g)$.

Suppose that there is an isomorphism $j : \Gamma \to \Gamma'$ with boundary mapping φ. Then for any $g \in \Gamma$, we have

$$\begin{aligned} \lambda(j(g)) &= (j(g)(t), t, N(j(g)), P(j(g))) \\ &= (\varphi \circ g(s), \varphi(s), \varphi(N(g)), \varphi(P(g))) \end{aligned}$$

for $s \neq P(g), N(g)$ and $t \neq P(j(g)), N(j(g))$. Define $\|\varphi\|_{ls} = \delta(j)$. It follows that

$$\|\varphi\|_{ls} = \sup \frac{|\log(\varphi \circ g(s), \varphi(s), \varphi(N(g)), \varphi(P(g)))}{(g(s), s, N(g), P(g))}$$

where the supremum is taken over all $g \in \Gamma$ and $s \neq P(g), N(g)$.

There is a natural norm on the set of orientation-preserving homeomorphisms of $\overline{\mathbb{R}}$, called the *cross-ratio norm*, defined by

$$\|\varphi\|_{cr} = \sup \frac{|\log(\varphi(p), \varphi(q), \varphi(r), \varphi(s))}{(p, q, r, s)}$$

where the supremum is taken over all quadruples (p, q, r, s) arranged counter-clockwise on $\overline{\mathbb{R}}$. It is clear that for a boundary mapping φ of some isomorphism j, $\|\varphi\|_{ls} \leq \|\varphi\|_{cr}$.

The cross-ratio norm was studied by Gardiner-Hu-Lakic [4] and Hu [5]. These authors proved that for an orientation-preserving homeomorphism $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, $\|\varphi\|_{cr}$

is equivalent to Thurston's norm on the transverse shearing measure induced by the earthquake map on $\mathbb{H}$ whose extension to $\overline{\mathbb{R}}$ is equal to φ. $\|\varphi\|_{cr}$ is finite if and only if φ is quasisymmetric.

We end this paper with the following questions:

1. There are some sufficient conditions on Γ such that $A_{qs}(\Gamma) = A(\Gamma)$ (see [1], [2]). Find sufficient and necessary conditions for the equality $A_{qs}(\Gamma) = A(\Gamma)$.

2. Adapt this theory to the setting of automorphism groups of free groups.

3. Adapt this theory to the setting of Kleinian groups.

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