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Multilinear fractional integral operators on weighted Morrey spaces

By

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§ 1. Introduction

This article is a summary of the results of inequalities for the linear and multilinear fractional integral operators on weighted Morrey spaces. The main contents in this article are from ([21, 22, 27, 28]).

This article consists of five sections. In Section 2, we discuss the boundedness of multilinear fractional integral operators on the product of m weighted Lebesgue spaces. In this section, we characterize $A_{\vec{P},q}(\mathbb{R}^n)$ in terms of the class $A_p(\mathbb{R}^n)$.

In Section 3, we discuss the boundedness of multilinear fractional integral operators on the product of m Morrey spaces. The purpose of Section 3 is to introduce multi-Morrey spaces. Multi-Morrey space strictly contain the product of m Morrey spaces. Hence, to treat the boundedness of some multilinear operators, the multi-Morrey space is appropriate.

In Section 4, we discuss the Adams inequality and the Olsen inequality on weighted Morrey spaces for linear and multilinear fractional integral operators.

In Section 5, we state the recent results.

We shall introduce some of definitions and notation used in this article and mention the fundamental results and theory of weights. The following notation is used: Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. For a set $E \subset \mathbb{R}^n$ we denote the Lebesgue measure of E by $|E|$. We denote the characteristic function of E by χ_E. We write a ball of radius R centered at x_0 by $B = B(x_0, R) := \{x; |x - x_0| < R\}$ and $aB := B(x_0, aR)$, for any $a > 0$. On the other hand, all cubes are assumed to have their sides parallel to the coordinate axes. $Q = Q(x_0, r)$ denotes the cube centered at x_0 with side length r. $aQ := Q(x_0, ar)$, for any $a > 0$. We use $l(Q)$ to denote the side-length of Q. We call a nonnegative locally integrable function w on $\mathbb{R}^n$ a weight

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function. Also we write $w(E) = \int_E w(x)dx$. For $1 < p < \infty$, p' is called the conjugate of p index if p' satisfies $1/p + 1/p' = 1$. The letter C shall always denote a positive constant which is independent of essential parameters and not necessarily the same at each occurrence. We list three operators.

Definition 1.1 (Hardy-Littlewood maximal operators). Let f be a locally integrable function. We define the Hardy-Littlewood maximal operator M by

$$Mf(x) := \sup_{Q \ni x} \fint_Q |f(y)|\, dy.$$

Definition 1.2.
(1) For $0 < \alpha < n$, the fractional integral operator (or the Riesz potential) I_α is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

(2) For $0 \le \alpha < n$, the fractional maximal operator M_α is defined by

$$M_\alpha f(x) := \sup_{Q \ni x} l(Q)^\alpha \fint_Q |f(y)| dy.$$

If $1 < p < \infty$, then the Hardy-Littlewood maximal operator M is bounded on $L^p(\mathbb{R}^n)$ (see [8, 12, 13, 14, 35]).

Theorem 1.3 (Hardy-Littlewood maximal theorem). *If $1 < p < \infty$, then we have*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)}.$$

It is well known that the fractional integral operator I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ (cf. [8, 12, 13, 14, 35]).

Theorem 1.4 (Hardy-Littlewood-Sobolev). *Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then the following inequality holds. If $1 < p < \frac{n}{\alpha}$, then we have*

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)}.$$

Next, we list three multilinear operators. Firstly, we define the multi(sub)linear maximal function which Lerner, Ombrosi, Pérez, Torres and Trujillo-González [34] introduced. For $\vec{f} = (f_1, \dots, f_m)$, we assume that

$$(f_1, \dots, f_m) \in L^1_{loc}(\mathbb{R}^n) \times \cdots \times L^1_{loc}(\mathbb{R}^n).$$

Definition 1.5. For a vector valued function $\vec{f} := (f_1, \dots, f_m)$, we define

$$\mathcal{M}\left(\vec{f}\right)(x) := \sup_{Q \ni x} \prod_{j=1}^m \fint_Q \left|f_j(y_j)\right| dy_j.$$

Since pointwise inequality $\mathcal{M}\left(\vec{f}\right)(x) \le \prod_{j=1}^{m} Mf_j(x)$ holds, the multi(sub)linear maximal function is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ with $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

Proposition 1.6. *If* $1 < p_1, \dots, p_m < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, *then we have*

$$\left\|\mathcal{M}\left(\vec{f}\right)\right\|_{L^p(\mathbb{R}^n)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

On the other hand, we define multilinear fractional integral operators which Kenig and Stein [30] introduced.

Definition 1.7. Let $0 < \alpha < mn$. For $\vec{f} = (f_1, \dots, f_m)$, we define the multilinear fractional integral operator:

$$I_{\alpha,m}\left(\vec{f}\right)(x) := \int_{\mathbb{R}^{mn}} \frac{f_1(y_1)\cdots f_m(y_m)}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} d\vec{y},$$

where

$$|(x-y_1,\dots,x-y_m)| := \sqrt{|x-y_1|^2 + \cdots + |x-y_m|^2},$$

and $d\vec{y} := dy_1 \cdots dy_m$.

Definition 1.8. Let $0 \le \alpha < mn$. For $\vec{f} = (f_1, \dots, f_m)$, we define the multilinear fractional maximal operator:

$$\mathcal{M}_\alpha(\vec{f})(x) := \sup_{Q \ni x} l(Q)^\alpha \prod_{i=1}^{m} \fint_Q |f_i(y)|\, dy.$$

Kenig and Stein [30] proved the following proposition.

Proposition 1.9. *Let* $0 < \alpha < mn$, $1 < p_1, \dots, p_m < \infty$ *and* $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$, *then we have*

$$\left\|I_{\alpha,m}\left(\vec{f}\right)\right\|_{L^q(\mathbb{R}^n)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

We shall define Morrey spaces.

Definition 1.10 (Morrey spaces)**.** Let $0 < p \le p_0 < \infty$. We define the classical Morrey space $\mathcal{M}_p^{p_0}(\mathbb{R}^n)$ by

$$\mathcal{M}_p^{p_0}(\mathbb{R}^n) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}_p^{p_0}} < \infty\},$$

where the norm is given by

$$\|f\|_{\mathcal{M}_p^{p_0}} := \sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{p_0}} \left(\fint_Q |f(x)|^p\, dx\right)^{\frac{1}{p}}.$$

Note that $\mathcal{M}_p^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

The Hardy-Littlewood maximal function M and the fractional integral operators I_α are bounded on Morrey spaces:

Theorem 1.11 ([4]). *If* $1 < p \le p_0 < \infty$, *then we have*

$$\|Mf\|_{\mathcal{M}_p^{p_0}} \le C\, \|f\|_{\mathcal{M}_p^{p_0}}\,.$$

Theorem 1.12 ([1]). *Let* $0 < \alpha < n$, $1 < p \le p_0 < \infty$, $0 < q \le q_0 < \infty$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ *and* $\frac{q}{q_0} = \frac{p}{p_0}$. *Then, we have*

$$\|I_\alpha f\|_{\mathcal{M}_q^{q_0}} \le C\, \|f\|_{\mathcal{M}_p^{p_0}}\,.$$

The proof of Theorem 1.12 depends on the basic idea due to Hedberg [19]. We have the following theorem from Theorem 1.12 using Hölder's inequality, which was obtained by Spanne but published by Peetre [47].

Theorem 1.13. *Let* $0 < \alpha < n$, $1 < p \le p_0 < \infty$, $0 < q \le q_0 < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$. *Then we have*

$$\|I_\alpha f\|_{\mathcal{M}_q^{q_0}} \le C\, \|f\|_{\mathcal{M}_p^{p_0}}\,.$$

Remark 1. Since the pointwise inequality $M_\alpha f(x) \le I_\alpha(|f|)(x)$, we note that the fractional maximal operators M_α is bounded on Morrey spaces; $M_\alpha : \mathcal{M}_p^{p_0}(\mathbb{R}^n) \to \mathcal{M}_q^{q_0}(\mathbb{R}^n)$.

A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. Weighted Lebesgue spaces with respect to the measure $w(x)dx$ will be by $L^p(w)$ with $0 < p < \infty$. First we shall define weighted Lebesgue spaces.

Definition 1.14. Let w be weight (function) on $\mathbb{R}^n$ and $0 < p < \infty$. We define weighted Lebesgue spaces.

$$L^p(w) := \left\{ f : \|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p\, w(x) dx \right)^{\frac{1}{p}} < \infty \right\}.$$

We shall define the class A_p. The theory of the class A_p is basic of the theory of weights in harmonic analysis and real analysis.

Definition 1.15. A weight w belongs to the class A_p, $1 < p < \infty$ if there exists $C > 1$ such that for any cube Q

$$\left(\fint_Q w(x)\, dx \right) \left(\fint_Q w(x)^{\frac{1}{1-p}} dx \right)^{p-1} \le C, \tag{1.1}$$

where the infimum of C satisfying the inequality (1.1) is denoted by $[w]_{A_p}$. We define $A_\infty = \bigcup_{1<p<\infty} A_p$. When $p = 1$, $w \in A_1$ if there exists $C > 1$ such that for almost every x,

$$Mw(x) \le Cw(x), \tag{1.2}$$

and the infimum of C satisfying the inequality (1.2) is denoted by $[w]_{A_1}$.

Muckenhoupt [39] introduced the class A_p and used it to characterize the weighted inequalities for the Hardy-Littlewood maximal operator. The importance of the A_p condition is shown by the following result.

Theorem 1.16. *Suppose that $1 < p < \infty$ and w be weight on $\mathbb{R}^n$. Then $w \in A_p(\mathbb{R}^n)$ if and only if*

$$\|Mf\|_{L^p(w)} \le C \|f\|_{L^p(w)} .$$

In the case of the corresponding to the fractional maximal operators and the fractional integral operators, we may handle the following class due to Muckenhoupt and Wheeden [40].

Definition 1.17. A weight function w belongs to $A_{p,q}$ for $1 < p < \infty$ and $0 < q < \infty$ if

$$[w]_{A_{p,q}} := \sup_{Q\subset\mathbb{R}^n} \left(\fint_Q w(x)^q \, dx \right)^{\frac{1}{q}} \left(\fint_Q w(x)^{-p'} \, dx \right)^{\frac{1}{p'}} < \infty, \tag{1.3}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 1.18 ([40]). *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} > 0$. Then*

$$\|I_\alpha f\|_{L^q(w^q)} \le C \|f\|_{L^p(w^p)}$$

if and only if $w \in A_{p,q}(\mathbb{R}^n)$.

It is well known and easily shown that the class $A_{p,q}$ is characterized in terms of A_p-weights.

Remark 2 ([35]). If $w \in A_{p,q}(\mathbb{R}^n)$ with $1 \le p \le \infty$ and $0 < q \le \infty$, then the following are equivalent:

(i) $w^q \in A_{1+\frac{q}{p'}}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

(ii) $w^{-p'} \in A_{1+\frac{p'}{q}}$.

§ 2. An observation of the class of weights associated to the multilinear fractional integral operators

In this section, we summarize the results of the paper [21]. Lerner et al [34] introduced the following class:

Definition 2.1 (Multiple weights class). Let $1 < p_1, \ldots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. A vector $\vec{w} = (w_1, \ldots, w_m)$ belongs to the multiple weights class $A_{\vec{p}}(\mathbb{R}^n)$, if the following holds:

$$[\vec{w}]_{A_{\vec{p}}(\mathbb{R}^n)} := \left(\fint_Q v_{\vec{w}}(x) dx \right)^{\frac{1}{p}} \prod_{j=1}^m \left(\fint_Q w_j(y_j)^{\frac{1}{1-p_j}} dy_j \right)^{\frac{1}{p'_j}} < \infty,$$

where $v_{\vec{w}} = w_1^{\frac{p}{p_1}} \cdots w_m^{\frac{p}{p_m}}$.

Lerner et al [34] also proved that the multiple weights class $A_{\vec{P}}(\mathbb{R}^n)$ is characterized in terms of A_p-weights.

Theorem 2.2. *Let $\vec{w}$ be a vector weight, $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then $\vec{w} \in A_{\vec{P}}(\mathbb{R}^n)$ if and only if*

$$\begin{cases} v_{\vec{w}} \in A_{mp}(\mathbb{R}^n), \\ w_j^{\frac{1}{1-p_j}} \in A_{mp'_j}(\mathbb{R}^n) \qquad (j = 1, \dots, m). \end{cases}$$

Precisely, we have the following inequalities; for $j = 1, \dots, m$,

$$\left[w_j^{\frac{1}{1-p_j}} \right]_{A_{mp'_j}}^{\frac{1}{p'_j}}, [v_{\vec{w}}]_{A_{mp}}^{\frac{1}{p}} \le [\vec{w}]_{A_{\vec{P}}} \le [v_{\vec{w}}]_{A_{mp}}^{\frac{1}{p}} \prod_{j=1}^m \left[w_j^{\frac{1}{1-p_j}} \right]_{A_{mp'_j}}^{\frac{1}{p'_j}}.$$

Theorem 2.2 automatically gives us the following theorem:

Theorem 2.3. *Suppose that $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then,*

$$\begin{cases} v_{\vec{w}} \in A_{mp}(\mathbb{R}^n), \\ w_j^{\frac{1}{1-p_j}} \in A_{mp'_j}(\mathbb{R}^n) \qquad (j = 1, \dots, m), \end{cases}$$

if and only if the following inequality holds:

$$\left\| \mathcal{M}\left(\vec{f}\right) \right\|_{L^p(v_{\vec{w}})} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

For the sake of convenience, we list the basic properties the class A_p and the class $A_{\vec{P}}$. Comparing with fundamental properties of A_p-weights, we shall investigate some properties of the class $A_{\vec{P}}$ (cf. [8, 14, 35, 60]).

Proposition 2.4.
(1-a) *If $1 < p < q < \infty$ then we have*

$$A_1(\mathbb{R}^n) \subsetneq A_p(\mathbb{R}^n) \subsetneq A_q(\mathbb{R}^n) \subsetneq A_\infty(\mathbb{R}^n).$$

(1-b) ([34]). *However $A_{\vec{P}}(\mathbb{R}^n)$ is not increasing : Let us consider the partial order relation between vectors $\vec{P} = (p_1, \dots, p_m)$ and $\vec{Q} = (q_1, \dots, q_m)$ given by $\vec{P} \le \vec{Q}$ if $p_j \le q_j$ for all j. Then, for $\vec{P} \le \vec{Q}$ we have*

$$\prod_{j=1}^m A_{p_j}(\mathbb{R}^n) \subset \prod_{j=1}^m A_{q_j}(\mathbb{R}^n).$$

but $A_{\vec{P}}(\mathbb{R}^n)$ is not contained in $A_{\vec{Q}}(\mathbb{R}^n)$. Lerner et al [34] describes a counter example. The following is the counter example: Let $n = 1$, $m = 2$, $\vec{P} = (p_1, p_2) = (2, 2)$, $\vec{Q} = (2, 6)$ and

$\vec{w} = (w_1, w_2) = \left(|x|^{-\frac{5}{3}}, 1\right)$. *Then we have* $\vec{w} \in A_{\vec{P}}(\mathbb{R}^n)$ *and* $\vec{w} \notin A_{\vec{Q}}(\mathbb{R}^n)$.

(2-a) *If* $\delta^\lambda(w)(x) = w(\lambda x)$, *then we have*

$$\left[\delta^\lambda(w)\right]_{A_p(\mathbb{R}^n)} = [w]_{A_p(\mathbb{R}^n)}.$$

(2-b) *If* $\delta^\lambda(\vec{w}) = (\delta^\lambda(w_1), \ldots, \delta^\lambda(w_m))$, *then we have*

$$\left[\delta^\lambda(\vec{w})\right]_{A_{\vec{P}}(\mathbb{R}^n)} = [\vec{w}]_{A_{\vec{P}}(\mathbb{R}^n)}.$$

(3-a) *If* $\tau^z(w) = w(x - z)$, *then we have*

$$[\tau^z(w)]_{A_p(\mathbb{R}^n)} = [w]_{A_p(\mathbb{R}^n)}.$$

(3-b) *If* $\tau^z(\vec{w}) = (\tau^z(w_1), \ldots, \tau^z(w_m))$, *then we have*

$$[\tau^z(\vec{w})]_{A_{\vec{P}}(\mathbb{R}^n)} = [\vec{w}]_{A_{\vec{P}}(\mathbb{R}^n)}.$$

(4-a) *For* $\lambda > 0$, *then we have*

$$[\lambda w]_{A_p(\mathbb{R}^n)} = [w]_{A_p(\mathbb{R}^n)}.$$

(4-b) *For* $\lambda > 0$, *if* $\lambda\vec{w} = (\lambda w_1, \ldots, \lambda w_m)$, *then we have*

$$[\lambda\vec{w}]_{A_{\vec{P}}(\mathbb{R}^n)} = [\vec{w}]_{A_{\vec{P}}(\mathbb{R}^n)}.$$

(5-a) *If* $0 < \varepsilon < 1$ *and* $w \in A_p(\mathbb{R}^n)$, *then we have* $w^\varepsilon \in A_p(\mathbb{R}^n)$
(5-b) *If* $0 < \varepsilon < 1$ *and* $\vec{w} \in A_{\vec{P}}(\mathbb{R}^n)$, *then we have* $\left(w_1^\varepsilon, \ldots, w_m^\varepsilon\right) \in A_{\vec{P}}(\mathbb{R}^n)$.

Reverse Hölder's inequality gives us the following properties (cf. [8, 14, 35, 60]).

Proposition 2.5.
(6-a) *If* $1 < p < \infty$ *and* $w \in A_p(\mathbb{R}^n)$, *there exists* $\sigma > 0$ *for all* $0 < \varepsilon \le \sigma$ *such that* $w \in A_{p-\varepsilon}(\mathbb{R}^n)$.
(6-b) *Let* $1 < p_j < \infty$ $(j = 1, \ldots, m)$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $\vec{P} := (p_1, \ldots, p_m)$. *If* $\vec{w} \in A_{\vec{P}}(\mathbb{R}^n)$ *then there exists* $\sigma > 0$, *for all* $0 < \varepsilon \le \sigma$ *such that* $\vec{w} \in A_{\vec{P}-\frac{\varepsilon}{p}\vec{P}}(\mathbb{R}^n)$.

Moen [36] introduced the following class:

Definition 2.6. Let $1 \le p_1, \ldots, p_m \le \infty$ and $0 < q \le \infty$. A vector $\vec{w} = (w_1, \ldots, w_m)$ belongs to the multiple weights class $A_{\vec{P},q}(\mathbb{R}^n)$, if the following holds:

$$[\vec{w}]_{A_{\vec{P},q}} := \sup_{Q \subset \mathbb{R}^n} \left(\fint_Q (w_1 \cdots w_m)(x)^q dx\right)^{\frac{1}{q}} \prod_{j=1}^m \left(\fint_Q w_j(y_j)^{-p_j'} dy_j\right)^{\frac{1}{p_j'}} < \infty.$$

On the other hand, it seems that the class $A_{\vec{P},q}$ is not completely characterized in terms of the class A_p. So we will characterize the class $A_{\vec{P},q}$ in terms of the class A_p.

In 2010, Chen and Xue [3] proved the following theorem:

Theorem 2.7. *Let $0 < \alpha < mn$, $1 \le p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that $\vec{w} \in A_{\vec{P},q}(\mathbb{R}^n)$ and $\frac{\alpha}{n} < m - 2 + \frac{1}{p_i} + \frac{1}{p_j}$ for any $1 \le i, j \le m$, then*

$$\begin{cases} (w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbb{R}^n), \\ w_i^{-p_i'} \in A_{1+p_i'\left(\frac{1}{q}+m-\frac{1}{p}-\frac{1}{p_i'}\right)}(\mathbb{R}^n) \qquad (i = 1, \dots, m). \end{cases}$$

When $p_i = 1$, we regard the condition $w_i^{-p_i'} \in A_{1+p_i'\cdot(\frac{1}{q}+m-\frac{1}{p}-\frac{1}{p_i'})}(\mathbb{R}^n)$ as $w_i^{(\frac{1}{q}+m-\frac{1}{p})^{-1}} \in A_1(\mathbb{R}^n)$.

Remark 3.
1. In [3] the authors assumed a superfluous assumption. In fact, the restriction $\frac{\alpha}{n} < m-2+\frac{1}{p_i}+\frac{1}{p_j}$ for all $i, j = 1, \dots, m$ was unnecessary.
2. When $1 < p < \infty$, we know $w \in A_p(\mathbb{R}^n)$ if and only if $w^{\frac{1}{1-p}} \in A_{p'}(\mathbb{R}^n)$ (cf. [8, 11, 35]). Hence if $1 < p_j < \infty$, then we have $w_j^{-p_j'} \in A_{1+p_j' s_j}(\mathbb{R}^n)$ if and only if $w_j^{s_j^{-1}} \in A_{1+\frac{1}{p_j' s_j}}(\mathbb{R}^n)$,

$$s_j := \frac{1}{q} + m - \frac{1}{p} - \frac{1}{p_j'} \quad (j = 1, \dots, m)$$

Therefore, it is natural to regard the case of $p_i = 1$ such as stated above.

We will completely characterize the class $A_{\vec{P},q}$ in terms of the class A_p.

Theorem 2.8 ([21])**.** *Let $1 \le p_1, \dots, p_m \le \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $0 < q \le \infty$. A vector $\vec{w}$ of weights satisfies $\vec{w} \in A_{\vec{P},q}(\mathbb{R}^n)$ if and only if*

$$\begin{cases} (w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbb{R}^n), \\ w_j^{-p_j'} \in A_{1+p_j'\left(\frac{1}{q}+m-\frac{1}{p}-\frac{1}{p_j'}\right)}(\mathbb{R}^n) \qquad (j = 1, \dots, m). \end{cases}$$

An analogy is available for $q = \infty$, if we regard the condition $(w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbb{R}^n)$ as the condition $(w_1 \cdots w_m)^{-\frac{1}{m-\frac{1}{p}}} \in A_1(\mathbb{R}^n)$. Also a tacit understanding which was the same as Theorem 2.7 is made when $p_i = 1$.

Remark 4.
1. When $0 < q < \infty$, we have $(w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbb{R}^n)$ if and only if $(w_1 \cdots w_m)^{-\frac{1}{m-\frac{1}{p}}} \in A_{1+\frac{1}{q}\left(m-\frac{1}{p}\right)^{-1}}(\mathbb{R}^n)$, which justifies the above understanding that $(w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbb{R}^n)$.
2. In [36, 37], the restriction $\frac{1}{m} < q < \infty$ is imposed. However, we are still interested in the case $0 < q \le \frac{1}{m}$ and $q = \infty$. Thus, we incorporate this case.

Reexamining the proof of Theorem 2.8, we obtain the following precise inequalities:

Corollary 2.9 ([21]). *Under the condition of Theorem 2.8, one has the following inequalities: for $0 < q < \infty$ and $1 < p_1, \ldots, p_m < \infty$, one has*

$$[(w_1 \cdots w_m)^q]_{A_{1+q\left(m-\frac{1}{p}\right)}}^{\frac{1}{q}}, \left[w_j^{-p'_j}\right]_{A_{1+p'_j\left(\frac{1}{q}+m-\frac{1}{p}-\frac{1}{p'_j}\right)}}^{\frac{1}{p'_j}} \leq [\vec{w}]_{A_{\vec{P},q}}, \tag{2.1}$$

for $j = 1, \ldots, m$. Conversely, one has

$$[\vec{w}]_{A_{\vec{P},q}} \leq [(w_1 \cdots w_m)^q]_{A_{1+q\left(m-\frac{1}{p}\right)}}^{\frac{1}{q}} \cdot \prod_{j=1}^{m} \left[w_j^{-p'_j}\right]_{A_{1+p'_j\left(\frac{1}{q}+m-\frac{1}{p}-\frac{1}{p'_j}\right)}}^{\frac{1}{p'_j}}. \tag{2.2}$$

In the remaining cases in Theorem 2.8, modify inequalities (2.1) *and* (2.2) *following Remarks 3 and 4.*

By Hölder's inequality, we obtain the following :

Proposition 2.10. *Let $1 \leq p_1, \ldots, p_m \leq \infty$ and $\vec{P} = (p_1, \ldots, p_m)$. If $0 < q_1 \leq q_2$, then we have*

$$A_{\vec{P},\infty}(\mathbb{R}^n) \subset A_{\vec{P},q_2}(\mathbb{R}^n) \subset A_{\vec{P},q_1}(\mathbb{R}^n).$$

Recall that the prototypical A_p-weights are the power weights: for $\theta \in \mathbb{R}$ and $p > 1$, $|x|^\theta \in A_p(\mathbb{R}^n)$ if and only if $-n < \theta < n(p-1)$. Theorem 2.2, Theorem 2.8 and the example of A_p-weights give us typical examples of the multiple weights classes $A_{\vec{P}}(\mathbb{R}^n)$ and $A_{\vec{P},q}(\mathbb{R}^n)$.

Example 2.11 (A_p-weights, $A_{p,q}$-weights, $A_{\vec{P}}(\mathbb{R}^n)$, $A_{\vec{P},q}(\mathbb{R}^n)$)**.**
(A) Let $\theta \in \mathbb{R}$, $1 < p < \infty$ and $0 < q < \infty$.
(1) $|x|^\theta \in A_p(\mathbb{R}^n)$ if and only if $-n < \theta < n(p-1)$.

(2) $|x|^\theta \in A_{p,q}(\mathbb{R}^n)$ if and only if $-\frac{n}{q} < \theta < \frac{n}{p'}$.
(B) Let $\theta_1, \ldots, \theta_m \in \mathbb{R}$, $1 < p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $0 < q < \infty$.
(3) $\left(|x|^{\theta_1}, \ldots, |x|^{\theta_m}\right) \in A_{\vec{P}}(\mathbb{R}^n)$ if and only if $\begin{cases} n\left(\frac{1}{p'_j} - m\right) < \frac{\theta_j}{p_j} < \frac{n}{p'_j} \\ -\frac{n}{p} < \frac{\theta_1}{p_1} + \cdots + \frac{\theta_m}{p_m}. \end{cases}$

(4) $\left(|x|^{\theta_1}, \ldots, |x|^{\theta_m}\right) \in A_{\vec{P},q}(\mathbb{R}^n)$ if and only if $\begin{cases} -\frac{n}{q} < \theta_1 + \cdots + \theta_m, \\ -n\left(\frac{1}{q} + m - \frac{1}{p} - \frac{1}{p'_j}\right) < \theta_j < \frac{n}{p'_j}. \end{cases}$
(5) Let $n = 1$, $m = 2$, $p_1 = 2$ and $p_2 = 2$. If $w_1(x) = \sqrt{|x|}$ and $w_2(x) = \frac{1}{\sqrt{|x|^3}}$, then we have

$$\begin{cases} w_1 \in A_2(\mathbb{R}), \\ w_2 \notin A_2(\mathbb{R}) \end{cases}$$

and

$$(w_1, w_2) \in A_{(2,2)}(\mathbb{R}).$$

Proof of Example 2.11. Since Example 2.11 (1) is a well known result, we omit the proof (see [8, 11, 35]). Next we prove Example 2.11 (2). By Remark 2 and Example 2.11 (1), we obtain the following:

$$\begin{aligned} w(x) = |x|^{\theta} \in A_{p,q}(\mathbb{R}^n) &\iff |x|^{-p'\theta} \in A_{1+\frac{p'}{q}}(\mathbb{R}^n) \\ &\iff -n < -p'\theta < n\left(1 + \frac{p'}{q} - 1\right) \\ &\iff -\frac{n}{q} < \theta < \frac{n}{p'}. \end{aligned}$$

Theorem 2.2 and Example 2.11 (1) give the proof of Example 2.11 (3). Theorem 2.8 and Example 2.11 (1) give the proof of Example 2.11 (4). Next, we prove Example 2.11 (5).

1. Since $-1 < \frac{1}{2} < 1\,(2-1) = 1$, by Example 2.11 (1), we obtain
$$w_1(x) = |x|^{\frac{1}{2}} \in A_2(\mathbb{R}).$$
2. Since $-\frac{3}{2} < -1$, by Example 2.11 (1), we obtain
$$w_2(x) = |x|^{-\frac{3}{2}} \notin A_2(\mathbb{R}).$$
3. Since $-1 < -\frac{1}{2} < 1(4-1) = 3$, by Example 2.11 (1), we obtain
$$w_1(x)^{\frac{1}{1-2}} = |x|^{-\frac{1}{2}} \in A_{2\cdot 2}(\mathbb{R}) = A_4(\mathbb{R}).$$
4. Since $-1 < \frac{3}{2} < 3$, by Example 2.11 (1), we obtain
$$w_2(x)^{\frac{1}{1-2}} = |x|^{\frac{3}{2}} \in A_4(\mathbb{R}).$$
5. Since $-1 < -\frac{1}{2} < 1$, by Example 2.11 (1), we obtain
$$v_{\vec{w}}(x) = w_1(x)^{\frac{p}{p_1}} w_2(x)^{\frac{p}{p_2}} = |x|^{\frac{1}{4}-\frac{3}{4}} = |x|^{-\frac{1}{2}} \in A_{2\cdot 1}(\mathbb{R}) = A_2(\mathbb{R}).$$

Therefore by Theorem 2.2, we obtain $\vec{w} \in A_{(2,2)}(\mathbb{R})$ and $(w_1, w_2) \notin A_2(\mathbb{R}) \times A_2(\mathbb{R})$. □

Remark 5. Example 2.11 (5) gives us

$$A_{p_1}(\mathbb{R}^n) \times \cdots \times A_{p_m}(\mathbb{R}^n) \subsetneq A_{\vec{p}}(\mathbb{R}^n).$$

§ 3. Multi-Morrey spaces and multillinear fractional integral operators

In this section, we summarize the results of the paper [27]. We define the multi-Morrey spaces(see [27]).

Definition 3.1. Let $\vec{P} = (p_1, \dots, p_m)$ and $0 < p \le p_0 < \infty$, where $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. For $\vec{f} = (f_1, \dots, f_m)$, we define

$$\left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{p_0}} \prod_{j=1}^{m} \left(\fint_Q |f_j(y_j)|^{p_j} dy_j \right)^{\frac{1}{p_j}}.$$

We define the multi-Morrey spaces $\mathcal{M}^{p_0}_{\vec{P}}(\mathbb{R}^n)$ by

$$\mathcal{M}^{p_0}_{\vec{P}}(\mathbb{R}^n) := \left\{ \vec{f} = (f_1, \dots, f_m) \in L^{p_1}_{loc}(\mathbb{R}^n) \times \cdots \times L^{p_m}_{loc}(\mathbb{R}^n) : \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}(\mathbb{R}^n)} < \infty \right\}.$$

We have the following theorem.

Theorem 3.2 ([27]). *Let* $0 < \alpha < mn$, $1 < p_1, \dots, p_m \le \infty$, $\vec{P} = (p_1, \dots, p_m)$, $0 < p \le p_0 < \infty$, $0 < q \le q_0 < \infty$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ *and* $\frac{q}{q_0} = \frac{p}{p_0}$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. *Then, for* $\vec{f} = (f_1, \dots, f_m)$ *we have*

$$\|I_{\alpha,m}\left(\vec{f}\right)\|_{\mathcal{M}^{q_0}_q} \le C \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Remark 6 ([27]). We let $\delta(x)$ denote the Dirac measure on the real line which concentrates unit point mass at the origin. An example of the case when $m = 2$, $n = 1$, $f_1(x) = \delta(x)$ and $f_2(x) = \delta(x-1)$ shows that, for the exponents $1 < p_1, p_2 < \infty$,

$$\|f_1\|_{\mathcal{M}^{p_1}_1} \|f_2\|_{\mathcal{M}^{p_2}_1} = \infty \times \infty = \infty,$$

meanwhile

$$\sup_{a<b} (b-a)^{\frac{1}{p_1}+\frac{1}{p_2}} \left(\fint_{(a,b)} f_1(x)\,dx \right) \left(\fint_{(a,b)} f_2(x)\,dx \right) = 1.$$

We consider another example with respect to the multi-Morrey spaces (see [22]).

Example 3.3 ([22]). Let $n = 1$, $m = 2$, $0 < p_0 < \infty$, $1 < p_1, p_2 < \infty$ and $\frac{1}{p_0} = \frac{1}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)$. Moreover we take $-\frac{1}{p_1} < \theta_1 \le 0$ and $-\frac{1}{p_2} < \theta_2 \le 0$. $\theta_1 + \theta_2 = -\frac{1}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)$ and $\theta_1 \ne -\frac{1}{2p_1}$ or $\theta_2 \ne -\frac{1}{2p_2}$ Then, we have

$$\left(|x|^{\theta_1}, |x|^{\theta_2}\right) \in \mathcal{M}^{p_0}_{(p_1,p_2)}(\mathbb{R}) \backslash \left(\mathcal{M}^{2p_1}_{p_1}(\mathbb{R}) \times \mathcal{M}^{2p_2}_{p_2}(\mathbb{R})\right).$$

Example 3.3 tells us that, in general, the multi-Morrey norm is strictly smaller than the m-fold product of the Morrey norms.

§ 4. The Adams inequality and the Olsen inequality

In this section, we summarize the results of the paper [28]. The purpose of this section is to develop a theory of weights for multilinear fractional integral operators and multilinear fractional maximal operators in the framework of Morrey spaces. We give natural sufficient conditions for the weighted inequalities of these operators. In [28], we showed the following inequality:

Theorem 4.1 ([28]). *Let $1 < p \le p_0 < \infty$ and w be a weight. Then, for every cube $Q \subset \mathbb{R}^n$, the weighted inequality*

$$|Q|^{\frac{1}{p_0}} \left(\fint_Q Mf(x)^p w(x)^p dx \right)^{\frac{1}{p}} \le C \sup_{Q' \supset Q} |Q'|^{\frac{1}{p_0}} \left(\fint_{Q'} |f(x)|^p w(x)^p dx \right)^{\frac{1}{p}} \tag{4.1}$$

holds if and only if

$$[w]_{p_0,p,p} := \sup_{Q \subset Q'} \left(\frac{|Q|}{|Q'|} \right)^{\frac{1}{p_0}} \left(\fint_Q w(x)^p dx \right)^{\frac{1}{p}} \left(\fint_{Q'} w(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty, \tag{4.2}$$

for any measurable function f.

Two clarifying remarks may be in order.

Remark 7. If $1 < p = p_0 < \infty$ (the case of Lebesgue spaces), then

(4.3)
$$\sup_{Q \subset Q'} \left(\frac{|Q|}{|Q'|} \right)^{\frac{1}{p_0}} \left(\fint_Q w(x)^p dx \right)^{\frac{1}{p}} \left(\fint_{Q'} w(x)^{-p'} dx \right)^{\frac{1}{p'}} = \sup_{Q \subset \mathbb{R}^n} \left(\fint_Q w(x)^p dx \right)^{\frac{1}{p}} \left(\fint_Q w(x)^{-p'} dx \right)^{\frac{1}{p'}}.$$

From (4.3) we see that (4.2) extends the class A_p to Morrey spaces in a certain sense.

Remark 8. The condition (4.2) holds if a weight w satisfies $[w]_{A_{p,p_0}} < \infty$.

We consider a theory of multiple weights related to the Adams inequality and the Olsen inequality. Main results are stated in Theorems 4.4. The following is an Adams type weighted inequality for the fractional integral operators.

Theorem 4.2 ([28]). *Let $0 < \alpha < n$, $1 < p \le p_0 < \infty$, $1 < q \le q_0 < \infty$ and w be a weight. Suppose that $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{q}{q_0} = \frac{p}{p_0}$. Then the weighted inequality*

$$\|(I_\alpha f)w\|_{\mathcal{M}^{q_0}_q} \le C \|fw\|_{\mathcal{M}^{p_0}_p} \tag{4.4}$$

holds if there exists a small number $a > 1$ such that $[w]_{aq_0,q,p} < \infty$.

Remark 9. The inequality $[w]_{aq_0,q,p} < \infty$ holds if a weight w satisfies $w \in A_{p,q_0}(\mathbb{R}^n)$. Thus, when $q = q_0$ and $p = p_0$ (the case of Lebesgue spaces), Theorem 4.2 recovers the result due to Muckenhoupt and Wheeden [40]. If $w \in A_{p,q_0}(\mathbb{R}^n)$, then $w^{q_0} \in A_{1+q_0/p'}(\mathbb{R}^n)$. By Hölder's inequality and the reverse Hölder inequality we have, for any pair of cubes $Q \subset Q'$,

$$\left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{aq_0}} \left(\fint_Q w(x)^q dx\right)^{\frac{1}{q}} \left(\fint_{Q'} w(x)^{-p'} dx\right)^{\frac{1}{p'}} \le C[w]_{A_{p,q_0}} < \infty.$$

Remark 10 ([28]). As the example $w(x) = |x|^{-n/q_0}$, $f(x) = |x|^{-\alpha}$ shows, we cannot choose $a = 1$ in Theorem 4.2. Professor Yasuo Furuya-Komori suggested us the following example: In fact, we can check that the example satisfies $I_\alpha f \equiv \infty$ but $\|fw\|_{\mathcal{M}^{p_0}_p} \le C$. This means the bound (4.4) is false. Meanwhile, a simple calculation shows that

$$\sup_{0<b<c<\infty} \left(\frac{b}{c}\right)^{\frac{n}{aq_0}} \left(\fint_{|x|\le b} |x|^{-n\frac{q}{q_0}} dx\right)^{\frac{1}{q}} \left(\fint_{|x|\le c} |x|^{n\frac{p'}{q_0}} dx\right)^{\frac{1}{p'}} \approx \sup_{0<b<c<\infty} b^{\frac{n}{q_0}\left(\frac{1}{a}-1\right)} \cdot c^{\frac{n}{q_0}\left(1-\frac{1}{a}\right)},$$

which is finite if and only if $a = 1$.

The following inequality is the Olsen inequality on Morrey spaces.

Theorem 4.3 ([43]). *Let $0 < \alpha < n$, $1 < p \le p_0 < \infty$, $1 < q \le q_0 < r_0 < \infty$ and $r_0 \ge \frac{n}{\alpha}$. Assume that*

$$\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n} \text{ and } \frac{q}{q_0} = \frac{p}{p_0}.$$

Suppose that $\|g\|_{\mathcal{M}^{r_0}_q} < \infty$, then we have

$$\|gI_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C \|g\|_{\mathcal{M}^{r_0}_q} \|f\|_{\mathcal{M}^{p_0}_p}.$$

In [28], we investigated the weighted inequality which includes Theorem 4.3 as corollary. The following result is multilinear weighted inequality of Theorem 4.3.

Theorem 4.4 ([28]). *Let v be a weight and $\vec{w} = (w_1, \ldots, w_m)$ be a multiple weight. Let $0 \le \alpha < mn$, $\vec{P} = (p_1, \ldots, p_m)$, $1 < p_1, \ldots, p_m < \infty$, $0 < p \le p_0 < \infty$, $0 < q \le q_0 < r_0 \le \infty$, $\frac{n}{\alpha} < r_0$ and small number $a > 1$, where $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Suppose that $\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n}$ and $\frac{q}{q_0} = \frac{p}{p_0}$.*

(a) *Suppose that $\alpha > 0$, $q > 1$ and*

(4.5)

$$[v, \vec{w}]_{aq_0,r_0,aq,\frac{\vec{P}}{a}} = \sup_{Q\subset Q'} \left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{aq_0}} |Q'|^{\frac{1}{r_0}} \left(\fint_Q v(x)^{aq} dx\right)^{\frac{1}{aq}} \prod_{i=1}^m \left(\fint_{Q'} w_i(x)^{-\left(\frac{p_i}{a}\right)'} dx\right)^{\frac{1}{\left(\frac{p_i}{a}\right)'}} < \infty.$$

Then there exist the constants $C > 0$ for vector valued function $\vec{f}$,

$$\left\| I_{\alpha,m}(\vec{f})v \right\|_{\mathcal{M}^{q_0}_q} \le C[v, \vec{w}]_{aq_0,r_0,aq,\frac{\vec{P}}{a}} \left\| \vec{f}_w \right\|_{\mathcal{M}^{p_0}_{\vec{P}}},$$

where $\vec{f}_w = (f_1 w_1, \ldots, f_m w_m)$.

(b) *Suppose that* $\alpha > 0$, $0 < q \leq 1$ *and*

(4.6)

$$[v, \vec{w}]_{aq_0, r_0, q, \frac{\vec{P}}{a}} = \sup_{Q \subset Q'} \left(\frac{|Q|}{|Q'|} \right)^{\frac{1}{aq_0}} |Q'|^{\frac{1}{r_0}} \left(\fint_Q v(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^m \left(\fint_{Q'} w_i(y_i)^{-(\frac{p_i}{a})'} dy_i \right)^{\frac{1}{(\frac{p_i}{a})'}} < \infty.$$

Then there exist the constants $C > 0$ *for vector valued function* $\vec{f}$,

$$\|I_{\alpha,m}(\vec{f})v\|_{\mathcal{M}^{q_0}_q} \leq C[v, \vec{w}]_{aq_0, r_0, q, \frac{\vec{P}}{a}} \left\| \vec{f}_w \right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

In Theorem 4.4, taking $w_1 = \cdots = w_m = 1$, we obtain Theorem 4.3 for multilinear fractional integral operators.

Corollary 4.5. *Let* v *be a weight. Moreover let* $0 \leq \alpha < mn$, $1 < p_1, \ldots, p_m < \infty$, $0 < p \leq p_0 < \infty$, $0 < q \leq q_0 < r_0 \leq \infty$, $\frac{n}{\alpha} < r_0$, $a > 1$ *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. *Suppose that* $\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n}$ *and* $\frac{q}{q_0} = \frac{p}{p_0}$.

(a) *If* $0 < \alpha < mn$ *and* $q > 1$, *then*

$$\left\| I_{\alpha,m}(\vec{f})v \right\|_{\mathcal{M}^{q_0}_q} \leq C\|v\|_{\mathcal{M}^{r_0}_{aq}} \left\| \vec{f} \right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

(b) *If* $0 < \alpha < mn$ *and* $0 < q \leq 1$, *then*

$$\left\| I_{\alpha,m}(\vec{f})v \right\|_{\mathcal{M}^{q_0}_q} \leq C \|v\|_{\mathcal{M}^{r_0}_q} \left\| \vec{f} \right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

The following is the Fefferman-Stein type inequality for multilinear fractional operators in the framework of Morrey spaces.

Corollary 4.6 ([28]). *Suppose that* $0 < q_i < r_i \leq \infty$, $i = 1, \ldots, m$, *satisfy* $\frac{1}{aq_0} = \sum_{i=1}^m \frac{1}{q_i}$ *and* $\frac{1}{r_0} = \sum_{i=1}^m \frac{1}{r_i}$. *Then, for any collection of* m *weights* $w_1, \ldots, w_m$, *we have*

$$\left\| I_{\alpha,m}(\vec{f})w_1 \cdots w_m \right\|_{\mathcal{M}^{q_0}_q} \leq C \|(f_1 W_1, \ldots, f_m W_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}$$

where

$$W_i(x) = \sup_{Q \ni x} |Q|^{\frac{1}{r_i}} \left(\frac{1}{|Q|} \int_Q w_i(y_i)^{q_i} dy_i \right)^{\frac{1}{q_i}} \quad \textit{for } i = 1, \ldots, m.$$

§ 5. Recent results

We consider extend the result in §4 to homogeneous kernels. The result depends on [22]. We shall define linear and multilinear fractional integral operators with homogeneous kernels (see [7, 35] for onelinear version).

Definition 5.1. Let f be a locally integrable function on $\mathbb{R}^n$.
(1) Given $0 < \alpha < n$ and a measurable function Ω on $\mathbb{R}^n \backslash \{0\}$, define

$$I_{\Omega,\alpha} f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y)f(y)}{|x-y|^{n-\alpha}} dy.$$

Let $\vec{f} = (f_1, \ldots, f_m)$ be a collection of m locally integrable functions on $\mathbb{R}^n$.
(2) Given $0 < \alpha < mn$ and a measurable function Ω_* on $\mathbb{R}^{mn} \backslash \{\vec{0}\}$, define

$$I_{\Omega_*,\alpha,m}\left(\vec{f}\right)(x) := \int_{\mathbb{R}^{mn}} \frac{\Omega_*(x-y_1, \ldots, x-y_m) \prod_{j=1}^m f_j(y_j)}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha}} d\vec{y}.$$

Remark 11. Let $0 < \alpha < n$ and Ω be a kernel as above. Then,

$$\left|I_{\Omega,\alpha}(f)(x)\right| \leq I_{|\Omega|,\alpha}(|f|)(x).$$

In the actual proof, $I_{|\Omega|,\alpha}(|f|)(x)$ will be controlled and as a consequence $I_{\Omega,\alpha}$ is proven to be bounded. In view of this pointwise inequality, there is no need to take care of the problem of the absolute convergence of the integral defining $I_{\Omega,\alpha}(f)(x)$.

Suppose that $\Omega \in L^s(\mathbb{S}^{n-1})$ is homogeneous of order 0: For any $\lambda > 0$, $\Omega(\lambda x) = \Omega(x)$.

Theorem 5.2 ([22]). *Suppose that $0 < \alpha < n$, $1 < s \leq \infty$, $(1 \leq) s' < p \leq p_0 < \infty$ and $1 < q \leq q_0 < \infty$. Suppose that $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{q}{q_0} = \frac{p}{p_0}$. If there exists $a > 1$ such that $[w^{s'}]_{\frac{aq_0}{s'}, \frac{q}{s'}, \frac{p}{s'}} < \infty$, then we have*

$$\left\|I_{\Omega,\alpha}(f)w\right\|_{\mathcal{M}^{q_0}_q} \leq C[w^{s'}]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{q}{s'}, \frac{p}{s'}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|fw\|_{\mathcal{M}^{p_0}_p}.$$

We pass to the multilinear case. The next theorem is the Olsen inequality on weighted Morrey spaces for multilinear fractional integral operators with homogeneous kernels. Firstly, let $\mathbb{S}_{m,n} := \mathbb{S}^{n-1} \times \cdots \times \mathbb{S}^{n-1}$. Moreover assume that $\Omega_* \in L^s(\mathbb{S}_{m,n})$ satisfies the following homogeneity: For any $\lambda_1, \ldots, \lambda_m > 0$, $\Omega_*(\lambda_1 x_1, \ldots, \lambda_m x_m) = \Omega_*(x_1, \ldots, x_m)$.

We obtain the following inequalities.

Theorem 5.3 ([22]). *Let $1 < s \leq \infty$, $0 < \alpha < mn$, $1 \leq s' < p_j < \infty$, $0 < p \leq p_0 < \infty$, $0 < q \leq q_0 < r_0 \leq \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n}$, $\frac{q}{q_0} = \frac{p}{p_0}$ and $r_0 \geq \frac{n}{\alpha}$.*
Case 1. *Let $q > 1$. Suppose that for small number $a > 1$, $\left[v^{s'}, \vec{w}^{s'}\right]_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{aq}{s'}, \frac{\vec{p}}{s'a}} < \infty$. Then, we have*

$$\left\|I_{\Omega_*,\alpha,m}\left(\vec{f}\right)v\right\|_{\mathcal{M}^{q_0}_q} \leq C\left[v^{s'}, \vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{aq}{s'}, \frac{\vec{p}}{s'a}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left\|\vec{f_w}\right\|_{\mathcal{M}^{p_0}_{\vec{p}}},$$

where $\vec{f_w} = (f_1 w_1, \ldots, f_m w_m)$.
Case 2. *Let $0 < q \leq 1$. Suppose that for small number $a > 1$, $\left[v^{s'}, \vec{w}^{s'}\right]_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{p}}{s'a}} < \infty$. Then, we have*

$$\left\|I_{\Omega_*,\alpha,m}\left(\vec{f}\right)v\right\|_{\mathcal{M}^{q_0}_q} \leq C\left[v^{s'}, \vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{p}}{s'a}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left\|\vec{f_w}\right\|_{\mathcal{M}^{p_0}_{\vec{p}}}.$$

Remark 12. In Theorem 5.3, we can replace the kernel Ω_* with the following kernels Ω_{**}: $\Omega_{**} \in L^s\left(\mathbb{S}^{mn-1}\right)$ and for any $\lambda > 0$, $\Omega_{**}(\lambda x_1, \ldots, \lambda x_m) = \Omega_{**}(x_1, \ldots, x_m)$. However, this case does not cover results due to Chen and Xue [3]. Hence we use the kernel Ω_*.

Theorem 5.3 has two corollaries. In Theorem 5.3, if we take $v = g$ and $\vec{w} = (1, \ldots, 1)$, then we have the Olsen inequality for multilinear fractional integral operators with homogeneous kernels. For the linear case, we refer to [17, 18, 43, 49, 50, 51, 52, 57, 58].

Corollary 5.4 ([22]). *Maintain that the conditions of Theorem 5.3.*
Case 1. *Let* $q > 1$. *For* $g \in \mathcal{M}^{r_0}_{aq}(\mathbb{R}^n)$, *we have*

$$\left\|g \cdot I_{\Omega_*,\alpha,m}\left(\vec{f}\right)\right\|_{\mathcal{M}^{q_0}_q} \leq C \|g\|_{\mathcal{M}^{r_0}_{aq}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Case 2. *Let* $0 < q \leq 1$. *For* $g \in \mathcal{M}^{r_0}_q(\mathbb{R}^n)$, *we have*

$$\left\|g \cdot I_{\Omega_*,\alpha,m}\left(\vec{f}\right)\right\|_{\mathcal{M}^{q_0}_q} \leq C \|g\|_{\mathcal{M}^{r_0}_q} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Finally, we introduce the recent result for the boundedness of the fractional integral operator on weighted Morrey spaces. The following theorem is proved by Iida, Komori-Furuya and Sato proved (see [24]).

Theorem 5.5 ([24]). *Let* $0 < \alpha < n$, $0 < \lambda < 1 - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}(1-\lambda)$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}\frac{1}{1-\lambda}$. *Suppose that*

$$\sup_{\substack{B\subset\mathbb{R}^n;\\ B;ball}} \left(⨍_B w(x)dx\right)^{\lambda} \left(⨍_B w(x)^{\frac{\lambda}{1-p}} dx\right)^{p-1} < \infty.$$

Then we have

$$\sup_{\substack{B\subset\mathbb{R}^n;\\ B;ball}} \left(\frac{1}{w(B)^{\lambda}} \int_B |I_\alpha f(x)|^q w(x)^{\lambda} dx\right)^{\frac{1}{q}} \leq C \sup_{\substack{B\subset\mathbb{R}^n;\\ B;ball}} \left(\frac{1}{w(B)^{\lambda}} \int_B |f(x)|^p w(x)^{\lambda} dx\right)^{\frac{1}{p}}.$$

The following theorem is proved by Izumi, Komori-Furuya and Sato proved (see [29]).

Theorem 5.6 ([29]). *Let* $0 < \alpha < n$, $0 < \lambda < 1 - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}(1 - \lambda)$,

$$\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n} \text{ and } \frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n}\frac{1}{1-\lambda}.$$

Suppose that $w \in A_{p,q_1}(\mathbb{R}^n)$, *then we have*

$$\sup_{\substack{B\subset\mathbb{R}^n;\\ B;ball}} \left(\frac{1}{w^{q_1}(B)^{\lambda}} \int_B |I_\alpha f(x)|^{q_2} w(x)^{q_1} dx\right)^{\frac{1}{q_2}} \leq C \sup_{\substack{B\subset\mathbb{R}^n;\\ B;ball}} \left(\frac{1}{w^{q_1}(B)^{\lambda}} \int_B |f(x)|^p w(x)^p dx\right)^{\frac{1}{p}}.$$

We know three kinds of the Adams inequalities on weighted Morrey spaces:

- The result 1 (Theorem 5.5) is in Iida, Komori-Furuya and Sato [24] .
- The result 2 (Theorem 4.2) is in Iida, Sawano, Sato and Tanaka [28].
- The result 3 (Theorem 5.6) is in Izumi, Komori-Furuya and Sato [29].

The relationship among these theorems is unknown. We shall aim to unify these results in the future.

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