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Boundedness of Littlewood-Paley operators

By

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Abstract

We survey some results related to L^p boundedness of Littlewood-Paley operators on homogeneous groups. Also, we give proofs of some results in the survey.

§ 1. Introduction

Let $f \in L^p(\mathbb{T})$ $(1 < p < \infty)$, where $\mathbb{T}$ is the one-dimensional torus, which is identified with $\mathbb{R}/\mathbb{Z}$ ($\mathbb{Z}$ denotes the integer group), and let

$$\sum_{k=-\infty}^{\infty} c_k e^{2\pi i k\theta}$$

be the Fourier series of f, where

$$c_k = \int_{\mathbb{T}} f(x) e^{-2\pi i k x}\, dx$$

is the Fourier coefficient.

The Littlewood-Paley function $\gamma(f)$ is defined as

$$\gamma(f)(\theta) = \left(\sum_{m=0}^{\infty} |\Delta_m(\theta)|^2\right)^{1/2},$$

where

$$\Delta_m(\theta) = \sum_{2^{m-1} \le |k| < 2^m} c_k e^{2\pi i k\theta}$$

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if m is a positive integer and $\Delta_0 = c_0$. Then Littlewood and Paley proved

$$A_p\|f\|_{L^p} \le \|\gamma(f)\|_{L^p} \le B_p\|f\|_{L^p} \tag{1.1}$$

for some positive constants A_p, B_p. This can be applied in proving the multiplier theorems of Marcinkiewicz type and in studying the lacunary convergence of the Fourier series.

A result analogous to (1.1) for the g function on $\mathbb{T}$ defined by

$$g(f)(\theta) = \left(\int_0^1 (1-t)|(\partial/\partial t)P_t * f(\theta)|^2\,dt\right)^{1/2} \tag{1.2}$$

was also shown by Littlewood and Paley, where

$$P_t(\theta) = \frac{1-t^2}{1-2t\cos(2\pi\theta)+t^2}$$

is the Poisson kernel for the unit disk. (See Littlewood and Paley [22, 23, 24]) and also Zygmund [43, Chap. XV] for the results above).

In this note we consider analogues on the Euclid spaces $\mathbb{R}^n$ and on the homogeneous groups of the Littlewood-Paley function $g(f)$ in (1.2). We survey a paper [10] and some back ground results in Sections 2–4. (See [37, 39, 43] for relevant results.) Also, in Sections 5–7, we shall give proofs of three results stated in Sections 2 and 3. Finally, in Section 8, we shall see some results related to Littlewood-Paley operators arising from the Bochner-Riesz means and the spherical means.

§ 2. Littlewood-Paley functions on $\mathbb{R}^n$

Let ψ be a function in $L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \psi(x)\,dx = 0. \tag{2.1}$$

We consider the Littlewood-Paley function on $\mathbb{R}^n$ defined by

$$S_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2\,\frac{dt}{t}\right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$.

Let $Q(x) = [(\partial/\partial t)P_t(x)]_{t=1}$, where

$$P_t(x) = c_n \frac{t}{(|x|^2+t^2)^{(n+1)/2}}$$

is the Poisson kernel on the upper half space $\mathbb{R}^n \times (0,\infty)$. Then $S_Q(f)$ is a version on $\mathbb{R}^n$ of the Littlewood-Paley function $g(f)$.

If $H(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x)$ is the Haar function on $\mathbb{R}$, then $S_H(f)$ coincides with the Marcinkiewicz integral

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $F(x) = \int_0^x f(y)\,dy$. Here χ_E denotes the characteristic function of a set E. We can easily see that S_Q and S_H are L^p $(1 < p < \infty)$ bounded on $\mathbb{R}^n$ and $\mathbb{R}$, respectively, from the following well-known result of Benedek, Calderón and Panzone [2].

Theorem A. *Suppose that ψ satisfies* (2.1) *and*

$$|\psi(x)| \le C(1+|x|)^{-n-\epsilon}, \tag{2.2}$$

$$\int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)|\,dx \le C|y|^\epsilon \tag{2.3}$$

for some positive constant ϵ. Then

(1) *S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$;*

(2) *S_ψ is of weak type $(1,1)$ on $\mathbb{R}^n$.*

It is known that for the L^p boundedness, the condition (2.3) is superfluous, which can be seen from the following result when $p = 2$.

Theorem B. *S_ψ is bounded on $L^2(\mathbb{R}^n)$ if ψ satisfies* (2.1) *and* (2.2) *with $\epsilon = 1$.*

We refer to Coifman and Meyer [8, p. 148] for this. A proof can be found in Journé [20]; see [20, pp. 81-82].

Let

$$H_\psi(x) = \sup_{|y| \ge |x|} |\psi(y)|$$

be the least non-increasing radial majorant of ψ. Also, define

$$B_\epsilon(\psi) = \int_{|x|>1} |\psi(x)|\,|x|^\epsilon\,dx \qquad \text{for} \quad \epsilon > 0,$$

$$D_u(\psi) = \left(\int_{|x|<1} |\psi(x)|^u\,dx \right)^{1/u} \qquad \text{for} \quad u > 1.$$

In [28], part (1) of Theorem A and Theorem B are improved as follows.

Theorem C. *Let $\psi \in L^1(\mathbb{R}^n)$. Suppose that ψ satisfies* (2.1) *and the conditions*

(1) $B_\epsilon(\psi) < \infty$ *for some $\epsilon > 0$;*

(2) $D_u(\psi) < \infty$ *for some* $u > 1$;

(3) $H_\psi \in L^1(\mathbb{R}^n)$.

Then

$$\|S_\psi(f)\|_{L^p_w} \le C_{p,w}\|f\|_{L^p_w}$$

for all $p \in (1, \infty)$ *and* $w \in A_p$.

As usual $L^p_w(\mathbb{R}^n)$ denotes the weighted L^p space of those functions f which satisfy $\|f\|_{L^p_w} = \|fw^{1/p}\|_p < \infty$. Also, here we recall the weight class A_p of Muckenhoupt. We say that $w \in A_p$ $(1 < p < \infty)$ if

$$\sup_B \left(|B|^{-1}\int_B w(x)\,dx\right)\left(|B|^{-1}\int_B w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty\,,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{x\in B}|B|^{-1}\int_B |f(y)|\,dy,$$

where the supremum is taken over all balls B containing x. We then say that $w \in A_1$ if there exists a constant C such that $M(w)(x) \le C\,w(x)$ for almost every x.

We now see some applications of Theorem C from [28].

Corollary 1. *Suppose that* $\psi \in L^1$ *satisfies* (2.1) *and* (2.2). *Let* $b \in BMO$ *and* $w \in A_2$. *We define the measure* ν *on the upper half space* $\mathbb{R}^n \times (0, \infty)$ *by*

$$d\nu(x,t) = |b * \psi_t(x)|^2\,\frac{dt}{t}\,w(x)\,dx.$$

Then, the measure ν *is a Carleson measure with respect to the measure* $w(x)\,dx$, *that is,*

$$\nu(S(Q)) \le C_w\|b\|^2_{BMO}\int_Q w(x)\,dx$$

for all cubes Q *in* $\mathbb{R}^n$, *where*

$$S(Q) = \{(x,t) \in \mathbb{R}^n \times (0,\infty) : x \in Q, 0 < t \le \ell(Q)\}$$

with $\ell(Q)$ *denoting sidelength of* Q.

This follows from the L^2_w-boundedness of the operator S_ψ. See [20, pp. 85–87]. From Corollary 1 we get the following (see [20, p. 87]).

Corollary 2. *Let $b \in BMO$. Suppose that φ satisfies* (2.2) *and that ψ satisfies* (2.1), (2.2). *Then*

$$\|T_b(f)\|_{L^p_w} \le C_{p,w}\|b\|_{BMO}\|f\|_{L^p_w}$$

for all $p \in (1,\infty)$ and $w \in A_p$, where

$$T_b(f)(x) = \left(\int_0^\infty |b * \psi_t(x)|^2 \, |f * \varphi_t(x)|^2 \, \frac{dt}{t}\right)^{1/2}.$$

We note that the conditions (2.1), (2.2) only are required for ψ in Corollaries 1, 2 (no additional regularity condition for ψ is needed).

By Corollary 2 and Theorem C we have the following.

Corollary 3. *We assume that ψ satisfies* (2.1), (2.2) *and that φ satisfies* (2.2). *Let $b \in BMO$. Furthermore, let η be a function in $L^1(\mathbb{R}^n)$ satisfying all the conditions of Theorem C imposed on ψ. Define a paraproduct π_b by the equation*

$$\pi_b(f)(x) = \int_0^\infty \eta_t * \left((b * \psi_t)\,(f * \varphi_t)\right)(x)\,\frac{dt}{t}.$$

Then

$$\|\pi_b(f)\|_{L^p_w} \le C_{p,w}\|b\|_{BMO}\|f\|_{L^p_w}$$

for all $p \in (1,\infty)$ and $w \in A_p$.

The class $L(\log L)^\alpha(\mathbb{R}^n)$, $\alpha > 0$, is defined to be the collection of the functions f on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x)|[\log(2 + |f(x)|)]^\alpha \, dx < \infty.$$

Similarly, let $L(\log L)^\alpha(S^{n-1})$ be the class of the functions Ω on S^{n-1} satisfying

$$\int_{S^{n-1}} |\Omega(\theta)| \left[\log(2 + |\Omega(\theta)|)\right]^\alpha \, d\sigma(\theta) < \infty,$$

where $d\sigma$ denotes the Lebesgue surface measure on $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

For the rest of this section we consider the cases where ψ is compactly supported. In [31] the following result was proved.

Theorem D. *The operator S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $2 \le p < \infty$ if ψ is a function in $L(\log L)^{1/2}(\mathbb{R}^n)$ with compact support and satisfies* (2.1).

This improves on a previous result of [17] which guarantees L^p boundedness of S_ψ for $p \in [2,\infty)$ under a more restrictive condition that $\psi \in L^q(\mathbb{R}^n)$ with some $q > 1$.

For $p < 2$, Duoandikoetxea [12] proved the following result.

Theorem E. *We assume that ψ has compact support.*

(1) *Suppose that $1 < q \le 2$ and $0 < 1/p < 1/2 + 1/q'$. Then S_ψ is bounded on $L^p(\mathbb{R}^n)$ if ψ is in $L^q(\mathbb{R}^n)$ and satisfies* (2.1).

(2) *Let $1 < q < 2$ and $1/p > 1/2 + 1/q'$. Then there exists $\psi \in L^q(\mathbb{R}^n)$ such that S_ψ is not bounded on $L^p(\mathbb{R}^n)$.*

Here q' denotes the exponent conjugate to q. See also [6] for a previous result for $p < 2$. Theorem E (1) was shown by arguments involving a theory of weights (see also [14]).

Let $\psi^{(\alpha)}$ be a function on $\mathbb{R}$ defined by

$$\psi^{(\alpha)}(x) = \begin{cases} \alpha(1-|x|)^{\alpha-1}\operatorname{sgn}(x), & x \in (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $1 < p < 2$, $1 < q < 2$ and $1/q' < \alpha \le 1/p - 1/2$. Then $\psi^{(\alpha)} \in L^q(\mathbb{R})$; also, Remark 2 of [17] implies that $S_{\psi^{(\alpha)}}$ is not bounded on L^p and $S_{\psi^{(\alpha)}}$ is of weak type (p,p) if $\alpha = 1/p - 1/2$.

The following result is a particular case of part (1) of Theorem E.

Proposition 1. *If ψ is compactly supported and belongs to $L^2(\mathbb{R}^n)$, then S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$.*

This can be proved by combining results of [28] and the weight theory of [12]. We shall give the proof in Section 5.

The Marcinkiewicz integral $\mu_\Omega(f)$ of Stein [36] (see also Hörmander [19]) is defined by $\mu_\Omega(f) = S_\psi(f)$ with

$$\psi(x) = |x|^{-n+1}\Omega(x')\chi_{(0,1]}(|x|) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\},$$

where $x' = x/|x|$, $\Omega \in L^1(S^{n-1})$, $\int_{S^{n-1}} \Omega \, d\sigma = 0$.

Al-Salman, Al-Qassem, Cheng and Pan [1] proved the following.

Theorem F. *μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$ if $\Omega \in L(\log L)^{1/2}(S^{n-1})$.*

See Walsh [42] for the case $p = 2$. In Section 3, we shall consider an analogue of Theorem F on homogeneous groups.

§ 3. Littlewood-Paley functions on homogeneous groups

We consider Littlewood-Paley functions on homogeneous groups. We also regard $\mathbb{R}^n$, $n \ge 2$, as a homogeneous group with multiplication given by a polynomial mapping. So, we have a dilation family $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ such that

$$A_t x = (t^{a_1}x_1, t^{a_2}x_2, \dots, t^{a_n}x_n), \quad x = (x_1, \dots, x_n),$$

with some real numbers $a_1, \dots, a_n$ satisfying $0 < a_1 \le a_2 \le \cdots \le a_n$ and such that each A_t is an automorphism of the group structure (see [18], [41] and [25, Section 2 of Chapter 1]). We also write $\mathbb{H} = \mathbb{R}^n$. $\mathbb{H}$ is equipped with a homogeneous nilpotent Lie group structure; the underlying manifold is $\mathbb{R}^n$ itself. We recall that Lebesgue measure is a bi-invariant Haar measure, the identity is the origin 0 and $x^{-1} = -x$. Multiplication xy, $x, y \in \mathbb{H}$, satisfies the following.

(1) $A_t(xy) = A_t x A_t y$, $x, y \in \mathbb{H}$, $t > 0$;

(2) $(ux)(vx) = ux + vx$, $x \in \mathbb{H}$, $u, v \in \mathbb{R}$;

(3) if $z = xy$, $z = (z_1, \dots, z_n)$, $z_k = P_k(x, y)$, then

$$\begin{aligned} P_1(x,y) &= x_1 + y_1, \\ P_k(x,y) &= x_k + y_k + R_k(x,y) \quad \text{for } k \ge 2, \end{aligned}$$

where $R_k(x, y)$ is a polynomial depending only on $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$.

We have a norm function $r(x)$ satisfying the following.

(1) $r(A_t x) = t r(x)$, for all $t > 0$ and $x \in \mathbb{R}^n$;

(2) r is continuous on $\mathbb{R}^n$ and smooth in $\mathbb{R}^n \setminus \{0\}$;

(3) $r(x + y) \le N_1(r(x) + r(y))$, $r(xy) \le N_2(r(x) + r(y))$ for some positive constants N_1, N_2;

(4) $r(x^{-1}) = r(x)$;

(5) if $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$, Σ coincides with S^{n-1};

(6) there exist positive constants $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$\begin{aligned} c_1|x|^{\alpha_1} \le r(x) \le c_2|x|^{\alpha_2} \quad &\text{if } r(x) \ge 1, \\ c_3|x|^{\beta_1} \le r(x) \le c_4|x|^{\beta_2} \quad &\text{if } r(x) \le 1. \end{aligned}$$

Let $\gamma = a_1 + \cdots + a_n$ (the homogeneous dimension of $\mathbb{H}$). Then $dx = t^{\gamma-1}\, dS\, dt$, that is,

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_\Sigma f(A_t\theta) t^{\gamma-1}\, dS(\theta)\, dt$$

with $dS = \omega\, d\sigma$, where ω is a strictly positive C^∞ function on Σ and $d\sigma$ is the Lebesgue surface measure on Σ as above.

The Heisenberg group $\mathbb{H}_1$ is an example of the homogeneous groups. Let

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2)$$

for $(x, y, u), (x', y', u') \in \mathbb{R}^3$. Then, with this group law, $\mathbb{R}^3$ is the Heisenberg group $\mathbb{H}_1$. A dilation is defined by

$$A_t(x, y, u) = (tx, ty, t^2 u) \quad \text{(2-step)}.$$

Also, we can adopt

$$A'_t(x, y, u) = (tx, t^2 y, t^3 u) \quad \text{(3-step)}$$

as an automorphism dilation.

For a function f on $\mathbb{H}$, let

$$f_t(x) = \delta_t f(x) = t^{-\gamma} f(A_t^{-1} x).$$

Convolution on $\mathbb{H}$ is defined as

$$f * g(x) = \int_{\mathbb{H}} f(y) g(y^{-1} x)\, dy.$$

Then $(f * g) * h = f * (g * h)$, $(f * g)\tilde{} = \tilde{g} * \tilde{f}$ if $\tilde{f}(x) = f(x^{-1})$.

We consider the Littlewood-Paley function on $\mathbb{H}$ defined by

$$S_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where ψ is in $L^1(\mathbb{H})$ and satisfies (2.1). Let Ω be locally integrable in $\mathbb{H} \setminus \{0\}$. We assume that Ω is homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, which means that $\Omega(A_t x) = \Omega(x)$ for $x \neq 0$, $t > 0$. Also, we assume that

$$\int_\Sigma \Omega(\theta)\, dS(\theta) = 0. \tag{3.1}$$

Let $\mu_\Omega = S_\Psi$ with

$$\Psi(x) = r(x)^{-\gamma + a} \Omega(x') \chi_{(0,1]}(r(x)), \quad a > 0, \tag{3.2}$$

where $x' = A_{r(x)^{-1}} x$ for $x \neq 0$. The spaces $L^p(\Sigma)$, $L(\log L)^\alpha(\Sigma)$ are defined with respect to the measure dS.

We recall a result of Ding and Wu [11].

Theorem G. *We assume in* (3.2) *that* $a = 1$ *and that* Ω *is a function in* $L \log L(\Sigma)$ *satisfying* (3.1). *Then* μ_Ω *is bounded on* $L^p(\mathbb{H})$ *for* $p \in (1, 2]$ *and is of weak type* $(1, 1)$.

The result on the L^p boundedness of Theorem G was improved by [10] as follows.

Theorem 1. μ_Ω *is bounded on* $L^p(\mathbb{H})$ *for all* $p \in (1, \infty)$ *if* Ω *is in* $L(\log L)^{1/2}(\Sigma)$ *and satisfies* (3.1).

To prove Theorem 1 we decompose $\Psi(x) = \sum_{k<0} 2^{ka}\Psi^{(k)}(x)$, $k \in \mathbb{Z}$, where

$$\Psi^{(k)}(x) = 2^{-ka} r(x)^{a-\gamma}\Omega(x')\chi_{(1,2]}(2^{-k}r(x)).$$

A change of variables and the property $\delta_s\delta_t = \delta_{st}$ of operators δ_t imply

$$S_{\Psi^{(k)}}f(x) = S_{\Psi^{(k)}_{2^{-k}}}f(x) = S_{\Psi^{(0)}}f(x).$$

Thus, by the sublinearity we have

$$S_\Psi f(x) \le \sum_{k<0} 2^{ka} S_{\Psi^{(k)}}f(x) = c_a S_{\Psi^{(0)}}f(x).$$

(See [16] for this observation.) So, we consider a function of the form

$$\Psi(x) = \ell(r(x))\frac{\Omega(x')}{r(x)^\gamma}, \tag{3.3}$$

where ℓ is in Λ^η_∞ (see [33]) for some $\eta > 0$ and supported in the interval $[1, 2]$.

Now we recall the definition of Λ^η_∞ (the definition of Λ^η_q, $1 \le q \le \infty$, can be found in [33]). Let h be a locally integrable function on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$. For $t \in (0, 1]$, define

$$\omega(h, t) = \sup_{|s|<tR/2} \int_R^{2R} |h(r - s) - h(r)| \frac{dr}{r},$$

where the supremum is taken over all s and R such that $|s| < tR/2$ (see [34]). Define Λ^η, $\eta > 0$, to be the family of the functions h such that

$$\|h\|_{\Lambda^\eta} = \sup_{t\in(0,1]} t^{-\eta}\omega(h, t) < \infty.$$

Let $\Lambda^\eta_\infty = L^\infty(\mathbb{R}_+) \cap \Lambda^\eta$ with $\|h\|_{\Lambda^\eta_\infty} = \|h\|_\infty + \|h\|_{\Lambda^\eta}$ for $h \in \Lambda^\eta_\infty$. Then $\Lambda^{\eta_1}_\infty \subset \Lambda^{\eta_2}_\infty$ if $\eta_2 \le \eta_1$.

Theorem 1 is a consequence of the following.

Theorem 2. *Let Ψ be as in* (3.3)*. Then S_Ψ is bounded on $L^p(\mathbb{H})$ for all $p \in (1, \infty)$ if Ω is in $L(\log L)^{1/2}(\Sigma)$ and satisfies* (3.1)*.*

Extrapolation arguments using the following estimates can prove Theorem 2 (see [32]).

Theorem 3. *Suppose that Ψ is as in* (3.3) *with Ω belonging to $L^s(\Sigma)$ for some $s \in (1, 2]$ and satisfying* (3.1)*. Let $1 < p < \infty$. Then*

$$\|S_\Psi f\|_p \le C_p(s - 1)^{-1/2}\|\Omega\|_s\|f\|_p,$$

where the constant C_p is independent of s and Ω.

For $F \in L(\log L)^a(\Sigma)$, $a > 0$, recall that

$$\|F\|_{L(\log L)^a} = \inf\left\{\lambda > 0 : \int_\Sigma \frac{|F|}{\lambda}\left[\log\left(2 + \frac{|F|}{\lambda}\right)\right]^a dS \le 1\right\}.$$

Then, under the assumptions of Theorem 2, we can in fact prove that

$$\|S_\Psi f\|_p \le C_p \|\Omega\|_{L(\log L)^{1/2}} \|f\|_p \tag{3.4}$$

for a constant C_p independent of Ω, which is not stated explicitly in Theorem 2. We shall give a proof of (3.4) in Section 6 by applying Theorem 3.

To prove Theorem 3 we apply certain vector valued inequalities, which will be controlled by a maximal function of the form

$$M_\psi(f)(x) = \sup_{t>0} |f * |\psi|_t(x)|.$$

Lemma 1. *Let Ψ be as in* (3.3) *and $p > 1$. Suppose that Ω is in $L^1(\Sigma)$. Then*

$$\|M_\Psi f\|_p \le C_p \|\Omega\|_1 \|f\|_p.$$

For $\theta \in \Sigma$, let

$$M_\theta f(x) = \sup_{s>0} \frac{1}{s}\int_0^s |f(x(A_t\theta)^{-1})|\, dt$$

be the maximal function on $\mathbb{H}$ along a curve homogeneous with respect to the dilation A_t. To prove Lemma 1, we apply a result of M. Christ [7].

Lemma 2. *Let $p > 1$. Then, there exists a constant C_p independent of θ such that*

$$\|M_\theta f\|_p \le C_p \|f\|_p.$$

We can easily prove Lemma 1 by applying Lemma 2.

Proof of Lemma 1. By a change of variables, we have

$$\begin{aligned} f * |\Psi|_t(x) &= \int f(xy^{-1})|\Psi|_t(y)\, dy \\ &= \int_1^2 \int_\Sigma f(x(A_{st}\theta)^{-1})|\Omega(\theta)\ell(s)|s^{-1}\, dS(\theta)\, ds. \end{aligned}$$

It follows that

$$M_\Psi f(x) \le C\|\ell\|_\infty \int_\Sigma M_\theta f(x)|\Omega(\theta)|\, dS(\theta).$$

Thus, Minkowski's inequality and Lemma 2 imply the conclusion. ∎

As indicated in [7], if we consider the Heisenberg group with 2-step dilation, then Lemma 2 can be proved by the boundedness of a maximal function along a curve in $\mathbb{R}^2$ (see (7.5)), which was studied by [40]. In Section 7, we shall give a straightforward proof of this fact.

Let $\mathcal{H} = L^2((0,\infty), dt/t)$. For each $k \in \mathbb{Z}$ and $\rho \geq 2$ we consider an operator T_k defined by

$$(T_k(f)(x))(t) = T_k(f)(x,t) = f * \Psi_t(x)\chi_{[1,\rho)}(\rho^{-k}t),$$

where Ψ is as in (3.3). The operator T_k maps functions on $\mathbb{H}$ to $\mathcal{H}$-valued functions on $\mathbb{H}$ and we see that

$$|T_k(f)(x)|_{\mathcal{H}} = \left(\int_{\rho^k}^{\rho^{k+1}} |f * \Psi_t(x)|^2 \frac{dt}{t}\right)^{1/2} = \left(\int_1^{\rho} |f * \Psi_{\rho^k t}(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

By Lemma 1, we have the following vector valued inequality, which will be useful in proving Theorem 3.

Lemma 3. *Let $1 < s < \infty$. Then*

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_s \leq C(\log\rho)^{1/2}\|\Omega\|_1 \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s.$$

We can apply the converse of Hölder's inequality and Lemma 1 to prove this (see [13]).

§ 4. Outline of the proof of Theorem 3

Let ϕ be a C^∞ function supported in $\{1/2 < r(x) < 1\}$ such that $\int \phi = 1$, $\phi(x) = \tilde{\phi}(x)$, $\phi(x) \geq 0$ for all $x \in \mathbb{H}$. For $\rho \geq 2$, we define

$$\Delta_k = \delta_{\rho^{k-1}}\phi - \delta_{\rho^k}\phi, \quad k \in \mathbb{Z}.$$

Then, $\operatorname{supp}(\Delta_k) \subset \{\rho^{k-1}/2 < r(x) < \rho^k\}$, $\Delta_k = \tilde{\Delta}_k$ and

$$\sum_k \Delta_k = \delta,$$

where δ is the delta function.

We decompose

$$f * \Psi_t(x) = \sum_{j\in\mathbb{Z}} F_j(x,t),$$

where

$$F_j(x,t) = \sum_{k\in\mathbb{Z}} f * \Delta_{j+k} * \Psi_t(x)\chi_{[\rho^k,\rho^{k+1})}(t).$$

Define

$$U_j f(x) = \left(\int_0^\infty |F_j(x,t)|^2 \, \frac{dt}{t} \right)^{1/2} = \left(\sum_{k\in\mathbb{Z}} \int_1^\rho |f * \Delta_{j+k} * \Psi_{\rho^k t}|^2 \, \frac{dt}{t} \right)^{1/2}$$

$$= \left(\sum_k |T_k(f * \Delta_{j+k})|^2_{\mathcal{H}} \right)^{1/2}.$$

Lemma 4. *Let $1 < s \le 2$ and $\rho = 2^{s'}$. Then, there exist positive constants C, ϵ independent of s and $\Omega \in L^s(\Sigma)$ such that*

$$\|U_j f\|_2 \le C(s-1)^{-1/2} 2^{-\epsilon|j|} \|\Omega\|_s \|f\|_2.$$

We choose $\psi_j \in C_0^\infty(\mathbb{R})$, $j \in \mathbb{Z}$, such that

$$\operatorname{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \le t \le \rho^{j+2}\}, \quad \psi_j \ge 0,$$
$$\log 2 \sum_{j\in\mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t > 0,$$
$$|(d/dt)^m \psi_j(t)| \le c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots,$$

where c_m is a constant independent of $\rho \ge 2$. Decompose

$$\frac{\Omega(x')}{r(x)^\gamma} = \sum_{j\in\mathbb{Z}} S_j(x),$$

where

$$S_j(x) = \int_0^\infty \psi_j(t) \delta_t K_0(x) \, \frac{dt}{t} = \frac{\Omega(x')}{r(x)^\gamma} \int_{1/2}^1 \psi_j(tr(x)) \, \frac{dt}{t}$$

with

$$K_0(x) = \frac{\Omega(x')}{r(x)^\gamma} \chi_{[1,2]}(r(x)).$$

We observe that S_j is supported in $\{\rho^j \le r(x) \le 2\rho^{j+2}\}$. Let

$$L_m^{(t)}(x) = \ell(t^{-1} r(x)) S_m(x).$$

Then by the restraint of the support of ℓ we have

$$\Psi_t(x) \chi_{[\rho^k, \rho^{k+1})}(t) = \sum_{m=k-3}^{k+3} L_m^{(t)}(x) \chi_{[\rho^k, \rho^{k+1})}(t).$$

Consequently,

$$F_j(x,t) = \sum_{k\in\mathbb{Z}} \sum_{m=k-3}^{k+3} f * \Delta_{j+k} * L_m^{(t)}(x) \chi_{[\rho^k, \rho^{k+1})}(t).$$

Using this expression of F_j and an analogue of the estimates in Lemma 1 of [33] (see also [9] for related results on product homogeneous groups), which can be proved by methods based on Tao [41], we can prove Lemma 4.

Now we are able to prove Theorem 3. First we recall the Littlewood-Paley inequality

$$\left\| \left(\sum_k |f * \Delta_k|^2 \right)^{1/2} \right\|_r \leq C_r \|f\|_r, \quad 1 < r < \infty,$$

where C_r is independent of ρ. Let $1 < p < \infty$, $\rho = 2^{s'}$, $1 < s \leq 2$. By Lemma 3 and the Littlewood-Paley inequality we have

$$\begin{aligned} \|U_j(f)\|_r &= \left\| \left(\sum_k |T_k(f * \Delta_{j+k})|_{\mathcal{H}}^2 \right)^{1/2} \right\|_r \\ &\leq C(\log \rho)^{1/2} \|\Omega\|_1 \left\| \left(\sum_k |f * \Delta_k|^2 \right)^{1/2} \right\|_r \\ &\leq C(\log \rho)^{1/2} \|\Omega\|_1 \|f\|_r \end{aligned} \tag{4.1}$$

for all $r \in (1, \infty)$. Also, by Lemma 4

$$\|U_j f\|_2 \leq C(\log \rho)^{1/2} 2^{-\epsilon|j|} \|\Omega\|_s \|f\|_2. \tag{4.2}$$

Thus, interpolating between (4.1) and (4.2), we have

$$\|U_j f\|_p \leq C(\log \rho)^{1/2} 2^{-\epsilon|j|} \|\Omega\|_s \|f\|_p$$

with some $\epsilon > 0$, which implies

$$\|S_\Psi f\|_p \leq \sum_j \|U_j f\|_p \leq C_p (s-1)^{-1/2} \|\Omega\|_s \|f\|_p.$$

This completes the proof of Theorem 3.

§ 5. A proof of Proposition 1

Let

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} \, dx$$

be the Fourier transform of f, where

$$\langle x, \xi \rangle = \sum_{j=1}^n x_j \xi_j, \quad x = (x_1, \ldots, x_n), \quad \xi = (\xi_1, \ldots, \xi_n).$$

To prove Proposition 1 we apply the following Fourier transform estimates.

Lemma 5. *Let $\psi \in L^2(\mathbb{R}^n)$. Suppose that ψ is compactly supported and satisfies* (2.1). *Then*

$$\int_1^2 |\hat{\psi}(t\xi)|^2 \, dt \le C \min\left(|\xi|^{\epsilon}, |\xi|^{-\epsilon}\right) \qquad \text{for all} \quad \xi \in \mathbb{R}^n$$

with some $\epsilon \in (0,1)$.

Also, we need the following.

Lemma 6. *Suppose that ψ is a function in $L^2(\mathbb{R}^n)$ with compact support. Let $w \in A_1$. If $v = w$ or w^{-1}, then we have*

$$\sup_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} \int_1^2 \left|f * \psi_{t2^k}(x)\right|^2 \, dt \, v(x) \, dx \le C\|f\|_{L^2_v}^2.$$

For a proof of Lemma 5 see [28].

Proof of Lemma 6. When $v = w$, Lemma 6 was proved in [28] (the author has learned from [12] that Lemma 6 is also valid for $v = w^{-1}$ and that it is useful for application). Now we recall the proof. We may assume that $\mathrm{supp}(\psi) \subset \{|x| \le 1\}$. Then, by Schwarz's inequality we see that

$$\left|f * \psi_t(x)\right|^2 \le t^{-n}\|\psi\|_2^2 \int_{|y|<t} |f(x-y)|^2 \, dy.$$

Since $w \in A_1$, integration with respect to the measure $w(x)\,dx$ gives

$$\begin{aligned} \int \left|f * \psi_t(x)\right|^2 w(x) \, dx &\le \|\psi\|_2^2 \int |f(y)|^2 t^{-n} \int_{|x-y|<t} w(x) \, dx \, dy \\ &\le C_w \|\psi\|_2^2 \int |f(y)|^2 w(y) \, dy \end{aligned} \tag{5.1}$$

uniformly in t. Also, by duality we can prove the uniform estimate

$$\int \left|f * \psi_t(x)\right|^2 w^{-1}(x) \, dx \le C_w \|\psi\|_2^2 \int |f(y)|^2 w^{-1}(y) \, dy. \tag{5.2}$$

The conclusion easily follows from the estimates (5.1) and (5.2). □

We choose $\Psi \in C^\infty$ that is supported in $\{1/2 \le |\xi| \le 2\}$ and satisfies

$$\sum_{j\in\mathbb{Z}} \Psi(2^j\xi) = 1 \qquad \text{for} \quad \xi \ne 0.$$

Define

$$\widehat{D_j(f)}(\xi) = \Psi(2^j\xi)\hat{f}(\xi) \qquad \text{for} \quad j \in \mathbb{Z},$$

and decompose

$$f * \psi_t(x) = \sum_{j \in \mathbb{Z}} F_j(x,t),$$

where

$$F_j(x,t) = \sum_{k \in \mathbb{Z}} D_{j+k}(f * \psi_t)(x) \chi_{[2^k, 2^{k+1})}(t).$$

Let

$$T_j(f)(x) = \left(\int_0^\infty |F_j(x,t)|^2 \frac{dt}{t} \right)^{1/2}.$$

We write $A_j = \{2^{-1-j} \le |\xi| \le 2^{1-j}\}$. Then, by the Plancherel theorem and Lemma 5 we see that

$$\begin{aligned} \|T_j(f)\|_2^2 &= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{2^k}^{2^{k+1}} \left| D_{j+k} \left(f * \psi_t \right) (x) \right|^2 \frac{dt}{t} \, dx \\ &\le \sum_{k \in \mathbb{Z}} C \int_{A_{j+k}} \left(\int_{2^k}^{2^{k+1}} \left| \hat{\psi}(t\xi) \right|^2 \frac{dt}{t} \right) \left| \hat{f}(\xi) \right|^2 d\xi \\ &\le \sum_{k \in \mathbb{Z}} C \int_{A_{j+k}} \min \left(|2^k \xi|^\epsilon, |2^k \xi|^{-\epsilon} \right) \left| \hat{f}(\xi) \right|^2 d\xi \\ &\le C 2^{-\epsilon |j|} \sum_{k \in \mathbb{Z}} \int_{A_{j+k}} \left| \hat{f}(\xi) \right|^2 d\xi. \end{aligned} \tag{5.3}$$

Since the sets A_j are finitely overlapping, (5.3) implies that

$$\|T_j(f)\|_2^2 \le C 2^{-\epsilon|j|} \|\hat{f}\|_2^2 = C 2^{-\epsilon|j|} \|f\|_2^2. \tag{5.4}$$

Let $w \in A_1$. If $v = w$ or w^{-1}, by Lemma 6 and the Littlewood-Paley inequality for L^2_v (note that $v \in A_2$) we see that

$$\begin{aligned} \|T_j(f)\|_{L^2_v}^2 &= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{2^k}^{2^{k+1}} \left| D_{j+k}(f) * \psi_t(x) \right|^2 \frac{dt}{t} v(x) \, dx \\ &\le \sum_{k \in \mathbb{Z}} C \int_{\mathbb{R}^n} \left| D_{j+k}(f)(x) \right|^2 v(x) \, dx \\ &\le C \|f\|_{L^2_v}^2. \end{aligned} \tag{5.5}$$

Thus, by interpolation with change of measures between (5.4) and (5.5)

$$\|T_j(f)\|_{L^2_{v^a}} \le C 2^{-\epsilon(1-a)|j|/2} \|f\|_{L^2_{v^a}} \tag{5.6}$$

for $a \in (0,1)$. Choosing a so that $w^{1/a} \in A_1$, by (5.6) we have

$$\|T_j(f)\|_{L^2_v} \le C 2^{-\epsilon(1-a)|j|/2} \|f\|_{L^2_v}.$$

From this it follows that

$$\|S_\psi(f)\|_{L^2_v} \le \sum_{j\in\mathbb{Z}} \|T_j(f)\|_{L^2_v} \le C\|f\|_{L^2_v}. \tag{5.7}$$

Let M be the Hardy-Littlewood maximal operator (see Section 2) and $M_s(f) = (M(|f|^s)(x))^{1/s}$. To prove Proposition 1, by Theorem D we may assume that $p < 2$. Now we apply the idea of [12]. If $1 < s < p/(2-p)$, then $M_s(|f|^{2-p})$ is in A_1 (we may assume that $0 < M_s(|f|^{2-p}) < \infty$) and M_s is bounded on $L^{p/(2-p)}$. Thus by Hölder's inequality and (5.7) with $v = M_s(|f|^{2-p})^{-1}$, we have

$$\begin{aligned}
&\int S_\psi(f)(x)^p\,dx = \int S_\psi(f)(x)^p M_s(|f|^{2-p})(x)^{-p/2} M_s(|f|^{2-p})(x)^{p/2}\,dx \\
&\le \left(\int S_\psi(f)(x)^2 M_s(|f|^{2-p})(x)^{-1}\,dx\right)^{p/2} \left(\int M_s(|f|^{2-p})(x)^{p/(2-p)}\,dx\right)^{1-p/2} \\
&\le C\left(\int |f(x)|^2 M_s(|f|^{2-p})(x)^{-1}\,dx\right)^{p/2} \|f\|_p^{p(1-p/2)} \\
&\le C\left(\int |f(x)|^2 |f(x)|^{p-2}\,dx\right)^{p/2} \|f\|_p^{p(1-p/2)} \\
&= C\|f\|_p^p.
\end{aligned}$$

This completes the proof of Proposition 1.

§ 6. Proof of (3.4)

We can prove Theorem 2 by extrapolation arguments using Theorem 3. More specifically, we can prove the estimate (3.4).

Let $a > 0$. We define the space $\mathcal{N}_a(\Sigma)$ to be the class of the functions $F \in L^1(\Sigma)$ for which we can find a sequence $\{F_m\}_{m=1}^\infty$ of functions on Σ and a sequence $\{b_m\}_{m=1}^\infty$ of non-negative real numbers such that

(1) $F = \sum_{m=1}^\infty b_m F_m$,

(2) $\sup_{m\ge1} \|F_m\|_{1+1/m} \le 1$,

(3) $\int_\Sigma F_m\,dS = 0$,

(4) $\sum_{m=1}^\infty m^a b_m < \infty$.

For $F \in \mathcal{N}_a(\Sigma)$, let

$$\|F\|_{\mathcal{N}_a} = \inf_{\{b_m\}} \sum_{m=1}^\infty m^a b_m,$$

where the infimum is taken over all such non-negative sequences $\{b_m\}$. We note that $\int_\Sigma F\,dS = 0$ if $F \in \mathcal{N}_a(\Sigma)$.

By well-known arguments we have the following (see [43, Chap. XII, pp. 119–120] for relevant results).

Proposition 2. *Suppose that $F \in L^1(\Sigma)$ and $a > 0$. Then, the following two statements (1), (2) are equivalent:*

(1) *$F \in L(\log L)^a(\Sigma)$ and $\int_\Sigma F\,dS = 0$;*

(2) *$F \in \mathcal{N}_a(\Sigma)$.*

Moreover,

(3) *there exist positive constants A, B such that*

$$\|F\|_{L(\log L)^a} \leq A\|F\|_{\mathcal{N}_a}, \quad \|F\|_{\mathcal{N}_a} \leq B\|F\|_{L(\log L)^a}$$

for $F \in \mathcal{N}_a(\Sigma)$.

To prove Proposition 2 we use the following two elementary results.

Lemma 7. *Let $1 < p < \infty, a > 0, x \geq 2$. Then, there exists a positive constant C_a depending only on a such that*

$$x(\log x)^a \leq C_a(p-1)^{-a}x^p.$$

This was also used in [32].

Lemma 8. *Let f be a continuous, non-negative, convex function on $[0, \infty)$ such that $f(0) = 0$. Suppose that a series $\sum_{k=1}^\infty c_k a_k$ converges, where $c_k \geq 0$, $\sum_{k=1}^\infty c_k \leq 1$, $a_k \in \mathbb{C}$. Then*

$$f\left(\left|\sum_{k=1}^\infty c_k a_k\right|\right) \leq \sum_{k=1}^\infty c_k f\left(|a_k|\right).$$

Proof of Proposition 2. We first see that part (1) follows from part (2). Let $F \in \mathcal{N}_a(\Sigma)$. We have already noted that $\int_\Sigma F\,dS = 0$. For any $\epsilon > 0$ there exist a sequence $\{b_m\}$ of non-negative real numbers and a sequence $\{F_m\}$ of functions on Σ with the properties required in the definition of $\mathcal{N}_a(\Sigma)$ such that

$$\|F\|_{\mathcal{N}_a} \leq \sum_{m=1}^\infty m^a b_m < \|F\|_{\mathcal{N}_a} + \epsilon.$$

Let $\lambda = \|F\|_{\mathcal{N}_a} + \epsilon$. By Lemma 8 with $f(x) = x[\log(2+x)]^a$ and $c_k = b_k/\lambda$, we have

$$\int_\Sigma \frac{|F|}{\lambda}\left[\log\left(2 + \frac{|F|}{\lambda}\right)\right]^a dS \leq \sum_{m=1}^\infty \lambda^{-1} b_m \int_\Sigma |F_m|\left[\log\left(2 + |F_m|\right)\right]^a dS.$$

It follows from Lemma 7 with $p = 1 + 1/m$ that

$$\begin{aligned} |F_m| \left[\log\left(2 + |F_m|\right)\right]^a &\le C_a m^a (2 + |F_m|)^{1+1/m} \\ &\le C_a m^a 2^{1/m} (2^{1+1/m} + |F_m|^{1+1/m}) \\ &\le 2C_a m^a (4 + |F_m|^{1+1/m}). \end{aligned}$$

Thus

$$\begin{aligned} \int_\Sigma \frac{|F|}{\lambda} \left[\log\left(2 + \frac{|F|}{\lambda}\right)\right]^a dS &\le \sum_{m=1}^{\infty} \lambda^{-1} b_m 2C_a m^a \int_\Sigma (4 + |F_m|^{1+1/m})\, dS \\ &= \sum_{m=1}^{\infty} \lambda^{-1} b_m 2C_a m^a (4S(\Sigma) + \|F_m\|_{1+1/m}^{1+1/m}) \\ &\le \sum_{m=1}^{\infty} \lambda^{-1} b_m 2C_a m^a (4S(\Sigma) + 1) \\ &\le 2C_a(4S(\Sigma) + 1). \end{aligned}$$

This implies that F belongs to $L(\log L)^a(\Sigma)$ and

$$\|F\|_{L(\log L)^a} \le A\lambda = A(\|F\|_{\mathcal{N}_a} + \epsilon)$$

for some $A > 0$. Letting ϵ tend to 0, we see that the first inequality of part (3) holds.

Next we prove that part (1) implies part (2). We take $\lambda > 0$ such that

$$\int_\Sigma \frac{|F|}{\lambda} \left[\log\left(2 + \frac{|F|}{\lambda}\right)\right]^a dS \le 1.$$

Let $F_\lambda = F/\lambda$. We define

$$\begin{aligned} U_m &= \{\theta \in \Sigma : 2^{m-1} < |F_\lambda(\theta)| \le 2^m\} \quad \text{for } m \ge 2, \\ U_1 &= \{\theta \in \Sigma : |F_\lambda(\theta)| \le 2\} \end{aligned}$$

and decompose $F_\lambda = \sum_{m=1}^\infty \tilde{F}_{\lambda,m}$, where

$$\tilde{F}_{\lambda,m} = F_\lambda \chi_{U_m} - S(\Sigma)^{-1} \int_{U_m} F_\lambda \, dS.$$

Note that $\int \tilde{F}_{\lambda,m}\, dS = 0$. If we put $e_m = S(U_m)$, $m \ge 1$, then

$$\|\tilde{F}_{\lambda,m}\|_{1+1/m} \le 22^m e_m^{m/(m+1)} \quad \text{for } m \ge 1. \tag{6.1}$$

Define

$$F_{\lambda,m} = \begin{cases} 2^{-m-1} e_m^{-m/(m+1)} \tilde{F}_{\lambda,m}, & \text{if } e_m \ne 0, \\ 0, & \text{if } e_m = 0. \end{cases}$$

Let $b_m = 2^{m+1} e_m^{m/(m+1)}$ for $m \geq 1$. Then

$$F_\lambda = \sum_{m=1}^{\infty} b_m F_{\lambda,m}, \qquad \int_\Sigma F_{\lambda,m}\, dS = 0.$$

Also, by (6.1) we see that $\sup_{m\geq 1} \|F_{\lambda,m}\|_{1+1/m} \leq 1$. Furthermore, applying Young's inequality, we have

$$\begin{aligned}
\sum_{m=1}^{\infty} m^a b_m &= \sum_{m=1}^{\infty} m^a 2^{m+1} e_m^{m/(m+1)} \\
&\leq 2\sum_{m=1}^{\infty} (m/(m+1)) m^a 2^{(m+1)(1+1/m)} e_m + 2\sum_{m=1}^{\infty} m^a 2^{-m-1}/(m+1) \\
&\leq C\sum_{m=1}^{\infty} m^a 2^m e_m + C \\
&\leq C\int_\Sigma |F_\lambda| \left(\log(2+|F_\lambda|)\right)^a\, dS + C \\
&\leq C.
\end{aligned} \tag{6.2}$$

Collecting results, we see that $F \in \mathcal{N}_a$ and, since $F = \sum_{m=1}^{\infty} \lambda b_m F_{\lambda,m}$,

$$\sum_{m=1}^{\infty} m^a b_m \geq \lambda^{-1} \|F\|_{\mathcal{N}_a},$$

which combined with (6.2) implies that $\|F\|_{\mathcal{N}_a} \leq B\lambda$ for some $B > 0$. So, taking the infimum over λ, we get the second inequality of part (3). $\square$

Let Ω and Ψ be as in Theorem 2. By Proposition 2 we can decompose Ω as

$$\Omega = \sum_{m=1}^{\infty} b_m \Omega_m,$$

where $\sup_{m\geq 1} \|\Omega_m\|_{1+1/m} \leq 1$ and each Ω_m satisfies (3.1), while $\{b_m\}$ is a sequence of non-negative real numbers such that $\sum_{m=1}^{\infty} m^{1/2} b_m < \infty$. Accordingly,

$$\Psi = \sum_{m=1}^{\infty} \Psi_m, \quad \Psi_m(x) = b_m \ell(r(x)) \frac{\Omega_m(x')}{r(x)^\gamma}.$$

Let $1 < p < \infty$. By Theorem 3 with $s = 1 + 1/m$ we have

$$\|S_{\Psi_m} f\|_p \leq C_p m^{1/2} b_m \|\Omega_m\|_{1+1/m} \|f\|_p \leq C_p m^{1/2} b_m \|f\|_p,$$

which implies

$$\|S_\Psi f\|_p \leq \sum_{m=1}^{\infty} \|S_{\Psi_m} f\|_p \leq C_p (\sum_{m=1}^{\infty} m^{1/2} b_m) \|f\|_p.$$

Taking the infimum over $\{b_m\}$ and applying Proposition 2, we get

$$\|S_\Psi f\|_p \le C_p \|\Omega\|_{\mathcal{N}_{1/2}} \|f\|_p \le C_p B \|\Omega\|_{L(\log L)^{1/2}} \|f\|_p.$$

This completes the proof of (3.4).

§ 7. Maximal functions on the Heisenberg group with two-step dilation

We give a proof of Lemma 2 for the maximal function M_θ on the Heisenberg group $\mathbb{H}_1$ with 2-step dilation by applying the boundedness of the maximal function $\mathfrak{M}g$ on $\mathbb{R}^2$ (see (7.5)).

Let $\theta = (\theta_1, \theta_2, \theta_3) \in S^2$ and $d_\theta = |\theta_1\theta_2\theta_3|$. We may assume that $d_\theta \ne 0$. Let

$$T_\theta x = (\theta_1^{-1} x_1, \theta_2^{-1} x_2, \theta_3^{-1} x_3).$$

It is convenient to define a group law $u \circ_\theta v$ on $\mathbb{R}^3$ so that

$$T_\theta x \circ_\theta T_\theta y = T_\theta(xy).$$

If $u = T_\theta x$, $v = T_\theta y$, this requires that

$$\begin{aligned} u \circ_\theta v &= T_\theta x \circ_\theta T_\theta y = T_\theta(xy) \\ &= T_\theta(x_1 + y_1, x_2 + y_2, x_3 + y_3 + (x_1y_2 - y_1x_2)/2) \\ &= (\theta_1^{-1}(x_1 + y_1), \theta_2^{-1}(x_2 + y_2), \theta_3^{-1}(x_3 + y_3) + \theta_3^{-1}(x_1y_2 - y_1x_2)/2) \\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3 + (2\theta_3)^{-1}\theta_1\theta_2(u_1v_2 - v_1u_2)). \end{aligned}$$

Since $A_t x = (tx_1, tx_2, t^2x_3)$, if $a(t) = (t, t, t^2)$,

$$f(x(A_t\theta)^{-1}) = f(T_\theta^{-1}((T_\theta x) \circ_\theta a(t)^{-1})) = f_\theta((T_\theta x) \circ_\theta a(t)^{-1}),$$

where $f_\theta(x) = f(T_\theta^{-1}x)$ and $a(t)^{-1} = (-t, -t, -t^2)$. Thus, by a change of variables, we have

(7.1)
$$\int_{\mathbb{H}_1} \left(\sup_{r>0} \frac{1}{r} \int_0^r |f(x(A_t\theta)^{-1})|\, dt \right)^p dx = d_\theta \int_{\mathbb{H}_1} \left(\sup_{r>0} \frac{1}{r} \int_0^r |f_\theta(y \circ_\theta a(t)^{-1})|\, dt \right)^p dy.$$

Let $c_\theta = (2\theta_3)^{-1}\theta_1\theta_2$. Then we note that

$$y = (y_1, y_2, y_3) = (0, y_2 - y_1, 0) \circ_\theta (y_1, y_1, y_3 + c_\theta y_1(y_2 - y_1)).$$

Thus

$$\begin{aligned} y \circ_\theta a(t)^{-1} &= ((0, y_2 - y_1, 0) \circ_\theta (y_1, y_1, y_3 + c_\theta y_1(y_2 - y_1))) \circ_\theta a(t)^{-1} \\ &= (0, y_2 - y_1, 0) \circ_\theta \left((y_1, y_1, y_3 + c_\theta y_1(y_2 - y_1)) \circ_\theta a(t)^{-1}\right). \end{aligned} \tag{7.2}$$

By (7.1) and (7.2), applying a change of variables, we have

(7.3)

$$
\begin{aligned}
&\int_{\mathbb{H}_1} \left(\sup_{r>0} \frac{1}{r} \int_0^r |f(x(A_t\theta)^{-1})|\, dt \right)^p dx \\
&= d_\theta \int_{\mathbb{H}_1} \left(\sup_{r>0} \frac{1}{r} \int_0^r |f_\theta((0, y_2 - y_1, 0) \circ_\theta \left((y_1, y_1, y_3 + c_\theta y_1(y_2 - y_1)) \circ_\theta a(t)^{-1}\right))\,|\, dt \right)^p dy \\
&= d_\theta \int_{\mathbb{H}_1} \left(\sup_{r>0} \frac{1}{r} \int_0^r |f_\theta((0, y_2, 0) \circ_\theta ((y_1, y_1, y_3) \circ_\theta a(t)^{-1}))|\, dt \right)^p dy.
\end{aligned}
$$

We observe that

$$(y_1, y_1, y_3) \circ_\theta a(t)^{-1} = (y_1 - t, y_1 - t, y_3 - t^2).$$

Thus (7.3) implies that

$$
\begin{aligned}
&\int_{\mathbb{H}_1} \left(\sup_{r>0} \frac{1}{r} \int_0^r |f(x(A_t\theta)^{-1})|\, dt \right)^p dx \\
&= d_\theta \int_{\mathbb{H}_1} \left(\sup_{r>0} \frac{1}{r} \int_0^r |f_\theta\left((0, y_2, 0) \circ_\theta (y_1 - t, y_1 - t, y_3 - t^2)\right)|\, dt \right)^p dy \\
&= d_\theta \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} (\mathfrak{M} f_{\theta, y_2}(y_1, y_3))^p \, dy_1\, dy_3 \right) dy_2,
\end{aligned}
\tag{7.4}
$$

where $f_{\theta,y_2}(y_1, y_3) = f_\theta((0, y_2, 0) \circ_\theta (y_1, y_1, y_3))$ and

$$\mathfrak{M} g(y_1, y_3) = \sup_{r>0} \frac{1}{r} \int_0^r |g(y_1 - t, y_3 - t^2)|\, dt. \tag{7.5}$$

It is known that

$$\|\mathfrak{M} g\|_{L^p(\mathbb{R}^2)} \le C_p \|f\|_{L^p(\mathbb{R}^2)}, \quad p > 1$$

(see [40]). Applying this and a change of variables, we see that

$$
\begin{aligned}
&d_\theta \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} (\mathfrak{M} f_{\theta, y_2}(y_1, y_3))^p \, dy_1\, dy_3 \right) dy_2 \\
&\le C_p^p d_\theta \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |f_{\theta, y_2}(y_1, y_3)|^p \, dy_1\, dy_3 \right) dy_2 \\
&= C_p^p d_\theta \int_{\mathbb{H}_1} |f_\theta(y_1, y_1 + y_2, y_3 - c_\theta y_1 y_2)|^p \, dy_1\, dy_2\, dy_3 \\
&= C_p^p d_\theta \int_{\mathbb{H}_1} |f_\theta(y)|^p \, dy \\
&= C_p^p \int_{\mathbb{H}_1} |f(y)|^p \, dy.
\end{aligned}
\tag{7.6}
$$

Combining (7.4) and (7.6), we get the conclusion.

§ 8. Littlewood-Paley operators related to Bochner-Riesz means and spherical means

Let

$$S_R^\delta(f)(x) = \int_{|\xi|<R} \widehat{f}(\xi)(1 - R^{-2}|\xi|^2)^\delta\ e^{2\pi i\langle x,\xi\rangle}\, d\xi = H_{R^{-1}}^\delta * f(x)$$

be the Bochner-Riesz mean of order δ on $\mathbb{R}^n$, $\delta > -1$, where

$$H^\delta(x) = \pi^{-\delta}\Gamma(\delta+1)|x|^{-(n/2+\delta)} J_{n/2+\delta}(2\pi|x|)$$

with J_ν denoting the Bessel function of the first kind of order ν.

For $\beta > 0$, let

$$M_t^\beta(f)(x) = c_\beta t^{-n} \int_{|y|<t} (1 - t^{-2}|y|^2)^{\beta-1} f(x-y)\, dy,$$

where

$$c_\beta = \frac{\Gamma\left(\beta + \frac{n}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(\beta)}.$$

By taking the Fourier transform, we can embed these operators in an analytic family of operators in β so that

$$M_t^0(f)(x) = c \int_{S^{n-1}} f(x - ty)\, d\sigma(y).$$

Now we define a Littlewood-Paley operator σ_δ, $\delta > 0$, from the Bochner-Riesz means as

$$\begin{aligned}\sigma_\delta(f)(x) &= \left(\int_0^\infty \left|(\partial/\partial R) S_R^\delta(f)(x)\right|^2 R\, dR\right)^{1/2} \\ &= \left(\int_0^\infty \left|-2\delta\left(S_R^\delta(f)(x) - S_R^{\delta-1}(f)(x)\right)\right|^2 \frac{dR}{R}\right)^{1/2},\end{aligned}$$

and also another Littlewood-Paley operator ν_β, $\beta + n/2 - 1 > 0$, from the spherical means as

$$\begin{aligned}\nu_\beta(f)(x) &= \left(\int_0^\infty \left|(\partial/\partial t)\ M_t^\beta(f)(x)\right|^2 t\, dt\right)^{1/2} \\ &= \left(\int_0^\infty \left|-2(\beta + n/2 - 1)\left(M_t^\beta(f)(x) - M_t^{\beta-1}(f)(x)\right)\right|^2 \frac{dt}{t}\right)^{1/2}.\end{aligned}$$

These Littlewood-Paley functions are related as follows.

Theorem H. *Suppose that $\delta = \beta + n/2 - 1 > 0$. Then, there exist positive constants A, B such that for all $x \in \mathbb{R}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space) we have*

$$\sigma_\delta(f)(x) \le A\,\nu_\beta(f)(x), \quad \nu_\beta(f)(x) \le B\,\sigma_\delta(f)(x).$$

This was proved by Kaneko and Sunouchi [21].

Also, we recall a result of Carbery, Rubio de Francia and Vega [5].

Theorem I. *If $\delta > 1/2$ and $-1 < \alpha \le 0$, then*

$$\int_{\mathbb{R}^n} |\sigma_\delta(f)(x)|^2 |x|^\alpha\, dx \le C_{\delta,\alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha\, dx.$$

See Rubio de Francia [27] for a different proof. Theorems H and I imply the following.

Proposition 3. *Suppose that $\beta > 3/2 - n/2$ and $-1 < \alpha \le 0$. Then*

$$\int_{\mathbb{R}^n} |\nu_\beta(f)(x)|^2 |x|^\alpha\, dx \le C_{\beta,\alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha\, dx.$$

Let

$$M_*^\beta(f)(x) = \sup_{t>0} \left| M_t^\beta(f)(x) \right|.$$

The following weighted L^2 estimate can be deduced from Proposition 3.

Proposition 4. *Suppose that $\mathrm{Re}(\beta) > 3/2 - n/2$ and $-1 < \alpha \le 0$. Then*

$$\int_{\mathbb{R}^n} \left| M_*^{\beta-1/2}(f)(x) \right|^2 |x|^\alpha\, dx \le C_{\beta,\alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha\, dx.$$

This is due to [38] when $\alpha = 0$.

To prove Proposition 4 we use the following relation.

Lemma 9. *If $\mathrm{Re}(\alpha) > \mathrm{Re}(\alpha') > -n/2$,*

$$M_t^\alpha(f)(x) = \frac{2\Gamma(\alpha + n/2)}{\Gamma(\alpha - \alpha')\Gamma(\alpha' + n/2)} \int_0^1 M_{st}^{\alpha'}(f)(x)(1 - s^2)^{\alpha - \alpha' - 1} s^{n + 2\alpha' - 1}\, ds.$$

See [38] and [40, p. 1270].

Proof of Proposition 4. Let k be the smallest non-negative integer such that $1 < \mathrm{Re}(\beta) + k$. Let $3/2 - n/2 < \eta < \mathrm{Re}(\beta)$. Then, by Lemma 9 and the Schwarz inequality we have

$$M_*^{\beta-1/2}(f)(x) \le C M^{\eta-1}(f)(x),$$

where

$$M^{\eta-1}(f)(x) = \sup_{t>0}\left(\frac{1}{t}\int_0^t \left|M_s^{\eta-1}(f)(x)\right|^2\,ds\right)^{1/2}.$$

Also, we easily see that

$$M^{\eta-1}(f)(x) \le C\nu_\eta(f)(x) + C\nu_{\eta+1}(f)(x) + \cdots + C\nu_{\eta+k}(f)(x) + CM^{\eta+k}(f)(x).$$

Note that $M^{\eta+k}(f)$ is bounded by the Hardy-Littlewood maximal function if η is sufficiently close to $\mathrm{Re}(\beta)$. Thus, applying Proposition 3, we get the weighted inequality as claimed. □

Define the spherical maximal operator $\mathcal{M}$ by

$$\mathcal{M}(f)(x) = \sup_{t>0}\left|\int_{S^{n-1}} f(x-ty)\,d\sigma(y)\right|.$$

We note that $\mathcal{M}(f)(x) = cM_*^0(f)(x)$. The following weighted norm inequality for $\mathcal{M}$ is due to Duoandikoetxea and Vega [15].

Theorem J. *Suppose that $n \ge 2$ and $n/(n-1) < p$. Then the inequality*

$$\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^p |x|^\alpha\,dx \le C\int_{\mathbb{R}^n} |f(x)|^p|x|^\alpha\,dx$$

holds for $1-n < \alpha < p(n-1)-n$.

This was partly proved by Rubio de Francia [26].

When $\alpha = 0$, Theorem J was proved by Stein [38] for $n \ge 3$ and by Bourgain [3] for $n = 2$. We can find in Sogge [35] a proof of the result of Bourgain which has some features in common with a proof, also given in [35], of Carbery's result [4] for the maximal Bochner-Riesz operator on $\mathbb{R}^2$.

We can give a different proof of Theorem J when $n \ge 3$, $1-n < \alpha \le 0$ and $p > n/(n-1)$ by applying Proposition 4. To see this, first we note that

$$\int_{\mathbb{R}^n} |M_*^\beta(f)(x)|^p|x|^\alpha\,dx \le C\int_{\mathbb{R}^n} |f(x)|^p|x|^\alpha\,dx \tag{8.1}$$

when $1 < p < \infty$, $-n < \alpha < n(p-1)$ and $\mathrm{Re}(\beta) \ge 1$, since $M_*^\beta(f)$ is pointwise bounded by the Hardy-Littlewood maximal function. On the other hand, by Proposition 4 we have

$$\int_{\mathbb{R}^n} |M_*^\beta(f)(x)|^2|x|^\alpha\,dx \le C\int_{\mathbb{R}^n} |f(x)|^2|x|^\alpha\,dx, \tag{8.2}$$

if $\mathrm{Re}(\beta) > (2-n)/2$ and $-1 < \alpha \le 0$. By an interpolation argument involving (8.1) and (8.2), we see that for any $p > n/(n-1)$ and $\alpha \in (1-n, 0)$, there exist $r \in (n/(n-1), p)$ and $\tau \in (1-n, \alpha)$ such that

$$\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^r |x|^\tau \, dx \le C \int_{\mathbb{R}^n} |f(x)|^r |x|^\tau \, dx.$$

Interpolating between this estimate and the unweighted L^r estimate for $\mathcal{M}$, since $\tau < \alpha < 0$, we have

$$\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^r |x|^\alpha \, dx \le C \int_{\mathbb{R}^n} |f(x)|^r |x|^\alpha \, dx.$$

Since $r < p < \infty$, interpolating between this and the obvious $L^\infty(|x|^\alpha)$ estimate for $\mathcal{M}$, we get the $L^p(|x|^\alpha)$ boundedness of $\mathcal{M}$ as claimed. (A similar argument can be found in [29]; see also [30].)

Finally, we prove Theorem J when $n \ge 2$, $0 \le \alpha < p(n-1)-n$ and $p > n/(n-1)$ by the methods of [15]. We write $w_\alpha(x) = |x|^\alpha$. It is known that the pointwise inequality $\mathcal{M}(w_\alpha) \le C w_\alpha$ holds if and only if $\alpha \in (1-n, 0]$ (see [15]). Let

$$T_\alpha(g) = w_\alpha^{-1} \mathcal{M}(w_\alpha g)$$

for $\alpha \in (1-n, 0]$. Then, T_α is bounded on L^∞, as we see that

$$\|T_\alpha(g)\|_\infty \le \|g\|_\infty \|w_\alpha^{-1} \mathcal{M}(w_\alpha)\|_\infty \le C\|g\|_\infty. \tag{8.3}$$

Let $r \in (n/(n-1), p)$. Since $\mathcal{M}$ is bounded on L^r, we have

$$\int_{\mathbb{R}^n} |T_\alpha(g)(x)|^r w_\alpha^r(x) \, dx = \int_{\mathbb{R}^n} |\mathcal{M}(w_\alpha g)(x)|^r \, dx \le C \int_{\mathbb{R}^n} |g(x)|^r w_\alpha^r(x) \, dx. \tag{8.4}$$

Interpolation between (8.3) and (8.4) will imply that

$$\int_{\mathbb{R}^n} |T_\alpha(g)(x)|^p w_\alpha^r(x) \, dx \le C \int_{\mathbb{R}^n} |g(x)|^p w_\alpha^r(x) \, dx.$$

This can be expressed as

$$\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^p w_\alpha^{r-p}(x) \, dx \le C \int_{\mathbb{R}^n} |f(x)|^p w_\alpha^{r-p}(x) \, dx$$

for any $\alpha \in (1-n, 0]$ and $r \in (n/(n-1), p)$, which implies the result as claimed.

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