Nonlinear Wave Equations and Fuchsian Equations

By

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Abstract

We consider some classes of nonlinear wave equations by reducing them to Fuchsian equations. Singular solutions are constructed near noncharacteristic surfaces.

§ 1.   The KdV Equation

Construction of singular solutions to partial differential equations (PDEs for short) is an interesting problem. In the linear case, singularities must be characteristic. In the nonlinear case, however, singularities can be *non*characteristic. A well-known example is the KdV equation:

$$(1.1) \qquad u_{ttt} - 6uu_t + u_x = 0 \quad (t, x \in \mathbb{R}).$$

For any real-analytic function $\psi(x)$, set $T = t - \psi(x)$. Then the equation (1.1) has a family of singular solutions of the form

$$u = \frac{2}{T^2} - \frac{1}{6}\psi_x + gT^2 - \frac{1}{36}\psi_{xx}T^3 + hT^4 - \frac{1}{24}g_xT^5 + \cdots,$$

where $g = g(x)$ and $h = h(x)$ are arbitrary real-analytic functions. The surface $T = t - \psi(x) = 0$ is *non*characteristic. A more striking fact is that ψ is arbitrary.

So a natural question arises: for a given nonlinear PDE, what kind of singular solutions can one construct? This question consists of two parts:

(Q1: Location of Singularities). Let a solution be singular along $t = \psi(x), x \in \mathbb{R}^n$. Can the analytic function ψ be arbitrary or do one have to impose certain conditions?

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(Q2: Expansion of Solutions). Can the singular solution be expressed by a Laurent series? If not, what kind of terms should be incorporated (fractional powers, logarithms, etc.)? In particular, what is the asymptotic behaviour of the solution?

We have different answers for different equations. The answers for the KdV equation have been given above. Other examples of integrable PDEs are explained from the viewpoint of the Painlevé PDE test in [1, §7.2]. In some works, singularities appear along characteristic surfaces (see [9] and its references).

In the present paper, we shall sketch some results about nonlinear wave equations with two kinds of nonlinearities, exponential and polynomial. The former was studied by Kichenassamy-Littman ([5], [6]) and the latter was by the present author ([10]). In both cases, singular solutions are constructed near noncharacteristic surfaces.

§2.   Exponential Nonlinearity

First let us study a nonlinear ordinary differential equation

$$u'' = e^u.$$

It is easy to see that $u = \log 2 - 2\log t = \log(2/t^2)$ is a solution, which is singular at $t = 0$. In [5] and [6], Kichenassamy-Littman studied

$$(2.1) \qquad\qquad\qquad \Box u = e^u \quad (\Box = \partial_t^2 - \Delta_x).$$

They constructed a family of solutions which blow up along a real-analytic hypersurface $\Sigma = \{T = 0\}$, where $T = t - \psi(x)$. Such a solution is of the form:

$$u = \log 2 - 2\log T + \text{the remainder}$$

and is defined in $T > 0$. If the surface Σ is space-like (hence noncharacteristic) and has zero scalar curvature, then the remainder is analytic. If it is only assumed to be space-like, then the remainder involves logarithms multiplied by positive powers of T. They proved these results by Fuchsian Reduction, i.e. by reducing the original equation (2.1) to a Fuchsian equation. They formulated their Fuchsian equation in a first-order system. Note that the Gérard-Tahara theory on Fuchsian equations lays emphasis on higher-order single equations.

§3.   Quadratic Nonlinearity

Let $f(t, x; \tau, \xi)$ be a polynomial of degree two in (τ, ξ). We assume that its coefficients are real-analytic functions in $(t, x) \in U \subset \mathbb{R} \times \mathbb{R}^n$. For a real-analytic function $\psi(x)$, set

$$\Sigma = \{(t, x) \in U; t = \psi(x)\}.$$

We shall construct singular solutions to the nonlinear wave equation

$$(3.1) \qquad\qquad u_{tt} - \Delta u = f(t, x; \partial_t u, \nabla u),$$

where ∇ is the gradient with respect to x.

It helps to consider an elementary ordinary differential equation $u'' = (u')^2$, which has a solution $u = -\log t$. It suggests that (3.1) has a singular solution led by a logarithmic term. We shall construct a solution which blows up along $T = 0$, where $T = t - \psi(x)$.

Theorem 3.1. *Let f_2 be the homogeneous part of degree two of f. We assume that $\psi(x)$ satisfies the "pseudo-Eikonal equation"*

$$(3.2) \qquad\qquad 0 \neq 1 - |\nabla\psi(x)|^2 = af_2\big(\psi(x), x; -1, \nabla\psi(x)\big)$$

in U for a nonzero constant a. Then for any analytic function v_0 on Σ, there exists an analytic function $v(t, x)$ in an open neighborhood Ω of Σ such that:

- *$u(t, x) = -a\log(t - \psi(x)) + v(t, x)$ is a solution to (3.1) in $\Omega \cap \{t > \psi(x)\}$,*
- *the restriction of v on Σ coincides with v_0.*

Such a function $v(t, x)$ is unique in a neighborhood of Σ.

The surface Σ is noncharacteristic because $1 - |\nabla\psi(x)|^2 \neq 0$. Although ψ is not arbitrary, there are many ψ's that satisfy the first-order equation (3.2). Note that (3.2) reduces to $1 - |\nabla\psi(x)|^2 \neq 0$ (which means Σ is noncharacteristic) if $f_2(\tau, \xi) = a^{-1}(\tau^2 - |\xi|^2)$.

Following the idea of [5] and [6], we construct singular solutions by Fuchsian Reduction. We reduce our nonlinear wave equation (3.1) to a second-order single Fuchsian equation rather than to a first-order Fuchsian system.

We introduce a new system of coordinates (T, X) by

$$T = t - \psi(x), \ X_i = x_i \quad (i = 1, 2, \ldots, n).$$

Note that $\Sigma = \{t = \psi(x)\} = \{T = 0\}$. We set

$$\partial_t = \partial/\partial t, \quad \partial_i = \partial/\partial x_i, \quad \psi_i = \partial_i \psi,$$
$$\widehat{\partial}_T = \partial/\partial T, \quad \widehat{\partial}_i = \partial/\partial X_i, \quad \widehat{\partial}_X = (\widehat{\partial}_1, \ldots, \widehat{\partial}_n).$$

Then we have $\partial_t = \widehat{\partial}_T$, $\partial_i = -\psi_i\widehat{\partial}_T + \widehat{\partial}_i$ and

$$(3.3) \qquad\qquad \Box = \partial_t^2 - \Delta = \Psi\widehat{\partial}_T^2 + 2\sum_{i=1}^{n} \psi_i\widehat{\partial}_i\widehat{\partial}_T + (\Delta\psi)\widehat{\partial}_T - \widehat{\Delta}.$$

Here $\Psi = \Psi(X) = \Psi(x) = 1 - |\nabla\psi(x)|^2 \neq 0$ and $\widehat{\Delta}$ is the Laplacian in X.

Set

$$u(t, x) = -a \log T + v,$$

where v is a new unknown function which is analytic in a neighborhood of $\Sigma = \{T = 0\}$.

By (3.3), the left-hand side of (3.1) is

$$(3.4) \qquad\qquad \Box u = \frac{a\Psi}{T^2} - \frac{a\Delta\psi}{T} + \Box v.$$

Set $f_2(\Sigma) = f_2(\psi(x), x; -1, \nabla\psi(x))$. The right-hand side of (3.1) is

$$f(t, x; \partial_t u, \nabla u) = \frac{a^2 f_2(\Sigma)}{T^2} + \text{higher order terms.}$$

Therefore in both sides of (3.1), the leading terms are of order T^{-2} and the pseudo-Eikonal equation just means that they coincide. We subtract these terms from both sides of (3.1). Then, after multiplication by $\Psi^{-1}T$, we obtain a nonlinear Fuchsian equation in v. It belongs to the following class (Caution: we denote T, X by t, x for typographical reasons):

$$
(3.5) \qquad
\begin{aligned}
t\partial_t^2 v + 2\partial_t v &= \alpha(t, x) + tP(t, x, \partial_x)\partial_t v + Q(t, x, \partial_x)v \\
&\quad + t\beta(t, x)(\partial_t v)^2 + R(t, x, \partial_x)v \cdot t\partial_t v + S(t, x; \nabla v),
\end{aligned}
$$

where we assume:

- α and β are analytic functions.
- P, Q and R are linear differential operators with analytic coefficients of orders 1, 2 and 1 respectively and $[P, t] = [Q, t] = [R, t] = 0$.
- $S(t, x; \xi)$ is a quadratic form in ξ whose coefficients are analytic functions in (t, x).

Direct application of the Gérard-Tahara theory shows that the equation (3.5) has a unique real-analytic solution $v(t, x)$ with $v(0, x) \equiv 0$ near $t = 0$. Moreover, it has a unique real-analytic solution $v(t, x)$ for an arbitrary initial value $v(0, x)$. This fact can be proved by the following observation.

Even if we replace $v(t, x)$ by its sum with a function in x, the equation (3.5) keeps the same form except for a change of the coefficients. Therefore if $v(t, x) = w(t, x) + (\text{a function in } x)$ and v is a solution to (3.5), then w is a solution to an equation of the same type. Hence the Cauchy problem can be reduced to the case of the null initial value.

§4.   Nonlinearity of Higher Degree

If $m \geq 2$, the ordinary differential equation $d^2 u/dt^2 = -(du/dt)^{m+1}$ has a solution $u(t) = C_m t^{(m-1)/m}$, $C_m = m^{(m-1)/m}/(m-1)$. We can treat a multi-variable version of this equation.

Let $f(t, x; \tau, \xi)$ be a polynomial of degree $m + 1$ $(m \geq 2)$. The homogeneous part of degree l is denoted by f_l. We shall construct singular solutions to the following equation:

$$(4.1) \qquad u_{tt} - \Delta u = f(t, x; \partial_t u, \nabla u) = \sum_{l=0}^{m+1} f_l(t, x; \partial_t u, \nabla u).$$

Theorem 4.1. *Let Σ be a hypersurface defined by $\Sigma = \{(t, x) \in U; t = \psi(x)\}$. Assume*

$$(4.2) \qquad 0 \neq 1 - |\nabla\psi(x)|^2 = \frac{(-m+1)^m a^m}{m^{m-1}} f_{m+1}\big(\psi(x), x; -1, \nabla\psi(x)\big),$$

$$(4.3) \qquad f_m\big(\psi(x), x; -1, \nabla\psi(x)\big) = 0,$$

for a nonzero constant a. Then there exists an open neighborhood Ω of Σ and an analytic solution $u(t, x)$ to (4.1) in $\Omega \cap \{t > \psi(x)\}$ such that

$$u(t, x) \sim a\big(t - \psi(x)\big)^{(m-1)/m}$$

as $t - \psi(x) \to +0$.

To prove it, we set $u = aT^{(m-1)/m} + Tv$, $s = T^{1/m}$. Then v satisfies a nonlinear Fuchsian equation in s. Note that a similar problem has been solved in [4], although no surface other than $t = 0$ is mentioned explicitly there. See also the related works by Tahara ([7], [8]).

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