Formal Solutions and True Solutions with Gevery Type Asymptotic Expansion for Some Nonlinear Partial Differential Equations in the Complex Domain

By

Hiroshi YAMAZAWA*

Abstract

Ôuchi [3] showed that for some linear partial differential equations in a complex domain, there exists a true solution $u_{S}(t, x)$ which is a holomorphic function in a sector $S$, and has an asymptotic expansion as $t \to 0$ in $S$. In this paper, we extend these results for nonlinear equations, and give another construction of such a solution.

§1. Introduction

Let $\mathbb{C}$ be the complex plane or the set of all complex numbers, $t$ a coordinate of $\mathbb{C}_{t}$, and $x = (x_{1}, \ldots, x_{n})$ coordinates of $\mathbb{C}_{x}^{n} = \mathbb{C}_{x_{1}} \times \cdots \times \mathbb{C}_{x_{n}}$. Set $\mathbb{N} := \{0, 1, 2, \ldots\}$. For $\alpha = (\alpha_{1}, \ldots, \alpha_{n}) \in \mathbb{N}^{n}$, we set $|\alpha| := \alpha_{1} + \cdots + \alpha_{n}$, and $(\partial / \partial x)^{\alpha} := (\partial / \partial x_{1})^{\alpha_{1}} \cdots (\partial / \partial x_{n})^{\alpha_{n}}$. Set $|x| := \max_{1 \leq i \leq n} \{|x_{i}|\}$, $D_{R} := \{x \in \mathbb{C}_{x}^{n}; |x| < R\}$ and $S_{\theta}(T) := \{t \in \mathbb{C}_{t}; 0 < |t| < T, |\arg t| < \theta\}$.

We denote by $\mathcal{O}(D_{R})$ (resp. $\mathcal{O}(S_{\theta}(T) \times D_{R})$) the set of all holomorphic functions defined on $D_{R}$ (resp. $S_{\theta}(T) \times D_{R}$).

In this paper, we consider the following equation:

\begin{equation}
D(u(t, x)) = f(t, x).
\end{equation}

Here $D(u(t, x))$ is a nonlinear partial differential operator with coefficients in holomorphic functions on a neighborhood of the origin for an unknown function $u(t, x)$, and $f(t, x)$ is a given function.

Received March 20, 2007, Accepted November 4, 2007.
2000 Mathematics Subject Classification(s): 34G20, 35A07, 35C10.
Key Words: true solutions, asymptotic expansion.

*Department of Language and Culture, Caritas Junior College, Yokohama 225-0011, Japan.

© 2008 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.
Problem 1. Suppose that \( f(t, x) \) is a holomorphic function. Can we construct a true solution?
Then the answer is “Yes” (Ōuchi [1], [2], Yamazawa [4]).

Problem 2. Suppose that \( f(t, x) \) is a function with Gevrey type asymptotic expansion. Can we construct a true solution of the same type?

In this paper we consider Problem 2; we construct a true solution to (1.1), and moreover, we prove that the solution has the same Gevrey type asymptotic expansion as \( f(t, x) \).

§2. Solvability in \( \text{Asy}_{\{ \gamma \}}^{0} \)

We define some function spaces which will be used in this paper.

**Definition 2.1.** Let \( \gamma > 0 \). Then we define a subspace \( \text{Asy}_{\{ \gamma \}}^{0}(S_{\theta}(T) \times D_{R}) \) of \( \mathcal{O}(S_{\theta}(T) \times D_{R}) \) as follows: \( f(t, x) \in \text{Asy}_{\{ \gamma \}}^{0}(S_{\theta}(T) \times D_{R}) \) if for any \( S_{0} = S_{\theta_{0}}(T_{0}) \) with \( 0 < \theta_{0} < \theta \) and \( 0 < T_{0} < T \) (which we denote by \( S_{0} \in S \)), there exist \( f_{k}(x) \in \mathcal{O}(D_{R}) \) and \( C \) and \( c_{0} > 0 \) such that

\[
|f(t, x)| \leq C \exp(-c_{0}|t|^{-\gamma})
\]

holds in \( S_{0} \).

**Definition 2.2.** Let \( \gamma > 0 \). Then we define a subspace \( \text{Asy}_{\{ \gamma \}}^{\gamma}(S_{\theta}(T) \times D_{R}) \) of \( \mathcal{O}(S_{\theta}(T) \times D_{R}) \) as follows: \( f(t, x) \in \text{Asy}_{\{ \gamma \}}^{\gamma}(S_{\theta}(T) \times D_{R}) \) if for any \( S_{0} \subset S_{\theta_{0}}(T_{0}) \), there exist \( f_{k}(x) \in \mathcal{O}(D_{R}) \) and \( A_{0}, B_{0} > 0 \) such that for any \( N \in \mathbb{N} \setminus \{0\} \)

\[
|f(t, x) - \sum_{k=0}^{N-1} f_{k}(x)t^{k}| \leq A_{0}B_{0}^{N}|t|^{N} \Gamma\left(\frac{N}{\gamma}+1\right)
\]

holds in \( S_{0} \). If the condition (2.1) is satisfied, then we write

\[
f(t, x) \sim_{\gamma} \tilde{f}(t, x) = \sum_{k \geq 0} f_{k}(x)t^{k} \quad \text{in} \quad S_{\theta}(T).
\]

We call \( \tilde{f}(t, x) \) an Gevrey type asymptotic expansion with index \( \gamma \) for \( f(t, x) \).

We consider the following operator \( D(u) \):

\[
D(u(t, x)) = F(t, x, \{\partial/\partial t\}^{j}(\partial/\partial x)^{\alpha}u(t, x)\}_{j+|\alpha| \leq m}).
\]

We assume that \( F(t, x, Z) (Z = \{Z_{j, \alpha}\}_{j+|\alpha| \leq m}) \) admits an expansion which is a convergent power series with respect to \( Z \):

\[
F(t, x, Z) = \sum_{|q| \geq 1} a_{q}(t, x) \prod_{j+|\alpha| \leq m} \{Z_{j, \alpha}\}^{q_{j, \alpha}},
\]
where \( q_{j, \alpha} \in \mathbb{N} \), and we set \( q := \{ q_{j, \alpha} \in \mathbb{N} ; j + |\alpha| \leq m \} \), \( |q| := \sum_{j+|\alpha| \leq m} q_{j, \alpha} \), and we assume that \( a_q(t, x) \in \text{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R) \). Then we consider the following equation \((E_0)\):

\[
D(u(t, x)) = f(t, x) \in \text{Asy}^0_{\{\gamma\}}(S_\theta(T) \times D_R).
\]

For the equation \((E_0)\) we introduce Newton polygon due to Ōuchi [2], [3]. We write each coefficient \( a_q(t, x) \in \text{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R) \) as

\[
a_q(t, x) = t^{\sigma_q} b_q(t, x) \quad (b_q(0, x) \neq 0, \sigma_q \in \mathbb{N} \setminus \{0\}).
\]

We set

\[
\Pi(a, b) := \{(x, y) \in \mathbb{R}^2 ; x \leq a \text{ and } y \geq b\}.
\]

Moreover, set \( l_q := \max\{j + |\alpha| ; q_{j, \alpha} \in q, q_{j, \alpha} \neq 0\} \) and \( e_q := \sigma_q - \sum_{q_{j, \alpha} \in q} j q_{j, \alpha} \). Then we define Newton polygon \( NP_1(D) \) for the linear part of the operator \( D(\cdot) \) by

\[
NP_1(D) := CH\{\bigcup_{|q|=1} \Pi(l_q, e_q) ; b_q(t, x) \neq 0\},
\]

where \( CH\{\cdot\} \) is the convex hull of a set.

The boundary of Newton polygon \( NP_1(D) \) consists of a vertical half line \( \Sigma_p^* \), a horizontal half line \( \Sigma_p^* \) and segments \( \Sigma_i^* (1 \leq i \leq p - 1) \). Let \( \gamma_i^* \) be the slope of \( \Sigma_i^* \) for \( i = 0, \ldots, p \). Then we have \( 0 = \gamma_p^* < \gamma_{p-1}^* < \cdots < \gamma_0^* = \infty \). Further Newton polygon \( NP_1(D) \) has \( p \)-point vertices which we denote by \( (l_i^*, e_i^*) \) with \( l_{p-1}^* < l_{p-2}^* < \cdots < l_0^* = m \).

Next let us define an operator \( \mathcal{L}_i \) with respect to \( \Sigma_i^* \) for \( i = 1, \cdots, p - 1 \). We set

\[
I_i := \{ q \subset \mathbb{N} ; e_i^* - e_q = \gamma_i^*(l_i^* - l_q) \text{ and } |q| = 1\}.
\]

Then we set

\[
\mathcal{L}_i u(t, x) := \sum_{q \in I_i} t^{\sigma_q} b_q(t, x) \prod_{j+|\alpha| \leq m} \left\{ \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u(t, x) \right\}^{q_{j, \alpha}}
\]

\[
= \sum_{(j, \alpha) \in J_i} t^{\sigma_{j, \alpha}} b_{j, \alpha}(t, x) \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u(t, x),
\]

where

\[
J_i := \{ j + |\alpha| \leq m ; e_i^* - (\sigma_{j, \alpha} - j) = \gamma_i^*(l_i^* - j - |\alpha|)\}.
\]

Let \( m_i^* \) be the order with respect to \( \partial/\partial x \) of \( \mathcal{L}_i \).

We assume the following conditions for the equation \((E_0)\):

\[
(C_1) \quad D(u) \text{ has a linear part of order } m,
\]
the operator $\mathcal{L}_i$ satisfies

(1) if $j + |\alpha| < l_i^*$ then $|\alpha| < m_i^*$ and

(2) \[ \sum_{j + |\alpha| = l_i^*} b_{j,\alpha}(0,0)\hat{\xi}^\alpha \neq 0 \] with $\hat{\xi} := (1, 0, \cdots, 0)$.

If we assume that the equation $(E_0)$ satisfies the condition $(C'_1)$, then it is sufficient to define Newton polygon for the only linear part (see Ōuchi [2, Proposition 1.7]).

We have an existence theorem concerning exponential decay solutions.

**Theorem 2.3.** Let $\gamma_{s+1}^* \leq \gamma < \gamma_s^*$ and $f(t, x) \in \text{Asy}_0^\gamma(S_\theta(T) \times D_R) (|\theta| < \pi/2\gamma)$. Suppose the conditions $(C_1)$ and $(C'_2)_i (i = 0, \ldots, s)$. Then for any $0 < r < R$, there exists a solution $u(t, x) \in \text{Asy}_0^\gamma(S_\theta(T) \times D_r) (|\theta'| < \pi/2\gamma_1^*)$ to the equation $(E_0)$.

In the case where $\gamma = \gamma_{s+1}^*$, Theorem 2.3 was obtained in Ōuchi [2] under the condition that $F(t, x, Z)$ is a polynomial in $Z$, and this condition was removed in Yamazawa [4]. If $\gamma > \gamma_{s+1}^*$, we can prove this theorem as same way as in the case where $\gamma = \gamma_{s+1}^*$.

§ 3. Construction of True Solution

We consider the following equation:

$$(E_1) \quad D(u(t, x)) = f(t, x),$$

where each $a_q(t, x)$ in (2.3) is holomorphic in a neighborhood of the origin, and $f(t, x)$ belongs to $\text{Asy}_0^\gamma(S_\theta(T) \times D_R)$.

For the equation $(E_1)$ we want to construct a solution that belongs to the same class as $f(t, x)$. In a linear case we shall recall Ōuchi’s result in [3].

Let us consider the following condition $(C'_2)_i$ for the operator $\mathcal{L}_i$:

$$(C'_2)_i \quad (1) \quad \sigma_{j,\alpha} = 0 \text{ for any } (j, \alpha) \text{ with } j + |\alpha| = l_i^* \text{ and } |\alpha| = m_i^*.$$

(2) \[ \sum_{j + |\alpha| = l_i^*} b_{j,\alpha}(0,0)\hat{\xi}^\alpha \neq 0. \]

Then we have the following theorem:

**Theorem 3.1 (Ōuchi [3]).** Assume that $D(u)$ satisfies $(C'_2)_i (i = 1, \ldots, s)$. Let $\gamma \geq \gamma_{s+1}^*$. Then for any $S_0 \subseteq S$ with $0 < \theta_0 < \pi/(2\gamma_1^*)$ there exists a solution $u(t, x) \in \text{Asy}_0^\gamma(S_0 \times D_r)$ to $(E_1)$. 

Remark. In [3], Ouchi constructed a solution to \((E_1)\) by using an integral kernel \(G\) and the solution is expressed as follows:

\[
\int_0 \! G(t, x; w) f(w) dw,
\]

but it is very complicated to construct \(G(t, x; w)\).

Next let us consider a nonlinear case. We shall get the same result as in a linear case, and give another construction of a true solution.

Let \(s\) be a nonnegative integer with \(\gamma_{s+1}^* \leq \gamma < \gamma_s^*\), and we set \(k_s^* = l_s^* - m_s^*\). We will give a condition for nonlinear terms.

For all nonlinear term, let us assume the following condition:

\((A_1)\) For any \((l_q, e_q)\) with \(|q| \geq 2\), there exist \(J_q^- > 0\) and \(J_q^+ \geq 0\) such that

\[
l_q = \begin{cases} 
- \frac{e^* - e_q}{\gamma_{s+1}} + l_s^* - J_q^- & \text{for } e^* \geq e_q, \\
\frac{e_q - e^*}{\gamma_s^*} + l_s^* - J_q^+ & \text{for } e_q > e^*.
\end{cases}
\]

Further we assume

\[
\left[ \frac{m_s}{\gamma_{s+1}} \right]_0 < J_q \quad \text{if } e^* > e_q \text{ and } |q| \geq 2,
\]

where \([a]_0\) is the decimal part of a number \(a\).

Then we have the following result for a formal solution.

**Theorem 3.2.** Suppose the conditions \((A_1)\) and \((C'_2)\) for \((E_1)\). Then for \(|\theta| < \pi/(2 \gamma)\), we can construct a formal power series \(\tilde{u}(t, x) = \sum_{h \leq 0} u_{(h)}(t, x)\) that formally satisfies \((E_1)\). Further \(u_{(h)}(t, x)\) satisfies the following estimate:

\[
|u_{(h)}(t, x)| \leq \tau^{-m_s} \tilde{U}_{(h)}(\theta) B^h \Gamma \left( - \frac{h}{\gamma_{s+1}} + 1 \right) \Gamma \left( \frac{k_s^*}{\gamma} + 1 \right) |t|^{k_s^*-h} \text{ in } S_0 \subset S_\theta(T),
\]

and \(\sum_{h \leq 0} \tilde{U}_{(h)} t^{-h}\) is a convergent power series in a neighborhood of the origin.

We shall give a sketch of proof of Theorem 3.2 in Section 5.

As for the formal solution \(\tilde{u}(t, x)\) we have the following fact.

**Lemma 3.3.** There exists a function \(u_{S_0}(t, x) \in \text{Asy}_{\gamma_{s}^*}(S_0 \times D_r)\) with \(0 < \theta_0 < \pi/(2 \gamma_s^*)\) such that for any \(S_1 \subset S_0\)

\[
|u_{S_0}(t, x) - \sum_{h=0}^{-N+1} u_{(h)}(t, x)| \leq \tilde{U}_{(h)} |t|^N \Gamma \left( \frac{N}{\gamma_{s}^*} + 1 \right) \text{ for } t \in S_1,
\]
where \( \sum_{n \geq 0} \tilde{U}'_{(h)} t^N \) is a convergent series.

Proof. Put
\[
\hat{u}_{(h)}(t, x; \xi) = \frac{u_{(h)}(t, x)}{t^{k_{\mathrm{s}}-h+\gamma_s^*}} \frac{\xi^{-h/\gamma_s^*}}{\Gamma(-h/\gamma_s^*+1)},
\]
\[
\hat{u}_{H}(t, x; \xi) = \sum_{h \leq -H-1} \hat{u}_{(h)}(x, t, \xi),
\]
\[
\hat{u}(t, x; \xi) = \sum_{h \leq 0} \hat{u}_{(h)}(x, t, \xi).
\]
It follows from Theorem 3.2 that \( \hat{u}_{H}(t, x; \xi) \) and \( \hat{u}(t, x; \xi) \) converge on \( D_R \times S_0 \times \{ |\xi| \leq \hat{\xi}_0 \} \) for some \( \hat{\xi}_0 > 0 \). Then there exist \( \hat{\xi} \) with \( 0 < \hat{\xi} < \hat{\xi}_0 \) and \( B_0 \) such that
\[
|\hat{u}_{H}(t, x; \xi)| \leq AB_0^{H+1} |t|^{-\gamma_s^*} |\xi|^{(H+1)/\gamma_s^*}
\]
on \( D_R \times S_0 \times \{ |\xi| \geq \hat{\xi} \} \). Define
\[
u_{S_0}(t, x) = t^{k_s^*} \int_{0}^{\hat{\xi}} \exp(-\xi t^{-\gamma_0^*}) \hat{u}(t, x; \xi) d\xi.
\]
By the estimates above we can see that \( \nu_{S_0}(t, x) \) is the desired function. \( \square \)

We can construct a true solution as \( u(t, x) = \nu_{S_0}(t, x) + v(t, x) \), where \( v(t, x) \) is an unknown function. Set \( D^{u_{S_0}}(v) := D(u_{S_0} + v) - D(u_{S_0}) \). Then we have the following theorem.

**Theorem 3.4.** Suppose that for the differential equation \( D^{u_{S_0}}(v) \) with respect to \( v(t, x) \), the conditions \( (C_2)_i \) are satisfied \( (i = 0, \ldots, s-1) \). Then for any \( S_1 \Subset S_0 \) with \( 0 < \theta_1 < \pi/(2\gamma_1^*) \), there exists a solution \( u_{S_1}(t, x) \in \text{Asy}_{\{\gamma\}}(S_1 \times D_r) \) to \( (E_1) \).

Proof. If the condition \( (C_1) \) is satisfied for the equation \( (E_1) \), then the same condition is also satisfied for the equation \( D^{u_{S_0}}(v) \). As the proof of in [2, Lemma 4.4] we can prove that \( f(t, x) - D(u_{S_0}) \in \text{Asy}_0(S_0 \times D_r) \). Therefore we can adapt Theorem 2.3 to an equation \( D^{u_{S_0}}(v) = f(t, x) - D(u_{S_0}) \), and this equation has a solution \( v_{S_1}(x) \in \text{Asy}_0(S_1 \times D_r) \). Then \( u_{S_0}(t, x) + v_{S_1}(t, x) \) is a solution to the equation \( (E_1) \). \( \square \)

**Remark.** We can prove Theorem 3.4 if we replace the condition \( (C_2)_i \) by \( (C'_2)_i \).

**§ 4. Majorant Function**

First let us introduce a majorant function in [2]. Set
\[
\theta(t) = \sum_{n \geq 0} \frac{ct^n}{(n+1)^{n+2}} \quad \text{and} \quad \theta^{(k)}(t) = \left( \frac{\partial}{\partial t} \right)^k \theta(t),
\]
where $m \in \mathbb{N}$.

**Lemma 4.1.** There exists a positive constant $c$ such that for $0 \leq k' \leq k \leq m$

$$\theta^{(k)}(t) \theta^{(k')}(t) \ll \theta^{(k)}(t).$$

We fix $c > 0$ so that (4.1) holds.

**Lemma 4.2.**

1. There exists a constant $C > 0$ such that the following holds for any $k$:

$$\theta^{(k)}(t) \ll \frac{C}{k+1} \theta^{(k+1)}(t).$$

2. Let $0 \leq k' \leq k \leq m$. Then

$$\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \theta^{(n-i+k+p)}(t) \theta^{(i+k'+p')}(t) \ll \frac{p!p'}{(p+p')!} \theta^{(n+k+p+p')}(t).$$

Set $\Psi_R(t) = \theta(t/R)$ and $\Psi_R^{(k)}(t) = (\partial/\partial t)^k \Psi_R(t)$ for $0 < R < 1$. By Lemma 4.2 we can prove the following proposition:

**Proposition 4.3.**

1. There exists a constant $C > 0$ such that the following holds for any $k$:

$$\Psi_R^{(k)}(t) \ll \frac{C}{k+1} R \Psi_R^{(k+1)}(t).$$

2. Let $0 \leq k' \leq k \leq m$. Then for $\delta \geq 1$

$$\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \Psi_R^{(\lfloor(n-i)/\delta\rfloor+k+p)}(t) \Psi_R^{(\lfloor i/\delta\rfloor+k'+p')}(t) \ll \frac{p!p'}{(p+p')!} R^{-k'} \Psi_R^{(\lfloor n/\delta\rfloor+k+p+p')}(t),$$

where $[a]$ denotes the integer part of a number $a$.

§ 5. Sketch of Proof of Theorem 3.2

First we study a solvability for an operator with respect to a vertex of Newton polygon.

We set

$$\Delta(s) := \{q; l_q = l^*_s \text{ and } e_q = e^*_s\}.$$ 

Then we define an operator by

$$Su(t, x) = \sum_{q \in \Delta(s)} a_q(t, x) \prod_{j + |\alpha| \leq m} \left\{ \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u \right\}^{q_j, \alpha}. $$
Under the condition (A1), the operator $S$ is a linear operator. So we can write:

$$S = \sum_{(j, \alpha) \in \Delta(s)} t^{\sigma_{j, \alpha}} b_{j, \alpha}(t, x) \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha,$$

where

$$\Delta(s) := \{(j, \alpha); j + |\alpha| = l_s^* \text{ and } \sigma_{j, \alpha} - j = e_s^*\}.$$ 

We fix $s$ and set $\Delta^* = \Delta(s)$, $k^* = k_s^*$ and $m^* = m_s^*$ and $(l^*, e^*) = (l_s^*, e_s^*)$ for short. Let us consider the following equation:

$$(E_S) \quad Su(t, x) = F(t, x),$$

where $F(t, x)$ satisfies the following estimate as a formal power series in $(t, x)$ for some $F > 0$:

$$F(t, x) \ll F \sum_{k \geq m_h} \frac{1}{k!} \Psi_R^{(\lfloor k/k^* \rfloor + m^* + c_h)}(\chi)(\frac{t}{\zeta})^k,$$

where $\chi = \tau x_1 + \sum_{i=2}^{n} x_i$, $\tau, \zeta > 0$.

**Lemma 5.1.** Under the conditions $(C_2)'_s$ and $(A1)$, the equation $(E_S)$ has a solution $u(t, x)$ satisfying

$$u(t, x) \ll \tau^{-m_s^*} \zeta^{-e^*} c(\tau) F \sum_{k \geq m_h + k^*} \frac{1}{k!} \Psi_R^{(\lfloor k/k^* \rfloor + c_h)}(\chi)(\frac{t}{\zeta})^k,$$

where $c(\tau)$ is a positive and bounded function of $\tau > 0$.

We will construct a formal solution $u(t, x) = \sum_{g \geq 0} u^g(t, x)$ to the equation $(E_1)$: We choose the sequence $\{u^g(t, x)\}$ satisfying the following equations:

$$(E^g) \begin{cases}
Su^0(t, x) + \sum_{e_q \leq e^*} t^{\sigma_q} b_q(t, x) \prod_{j+|\alpha| \leq m} \left\{ \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u^0(t, x) \right\}^{q_{j, \alpha}} = f(t, x), \\
Su^g(t, x) + \sum_{e_q \leq e^*} \left| g' \right| = g \left| e_q \right| \geq e^* t^{\sigma_q} b_q(t, x) \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u^{g_i}(t, x) \\
= - \sum_{e_q > e^*} \left| g' \right| = g - (e_q - e^*) t^{\sigma_q} b_q(t, x) \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u^{g_i}(t, x) \quad (g \geq 1).
\end{cases}$$

Here we set $|g'| := \sum_{j+|\alpha| \leq m} \sum_{i=1}^{q_{j, \alpha}} g_i$. 
We solve the equation \((E^q)\) inductively: First let us solve the equation \((E^0)\). We construct a solution \(u^0(t, x) = \sum_{h \geq 0} u^0_h(t, x)\) as
\[
SU^0(t, x) = f(t, x),
SU^0_h(t, x) := W^0_h(t, x)
\]

\[
= - \sum_{q, \alpha} t^{q_0} b^0_{q}(t, x) \prod_{j+|\alpha| \leq m} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u^0_{\alpha}(t, x)
\]

\[
= \sum_{q, \alpha} t^{q_0} b^0_{q}(t, x) \prod_{j+|\alpha| \leq m} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u^0_{\alpha}(t, x) \quad (h \geq 1).
\]

Here we set \(|h'| := \sum_{j+|\alpha| \leq m} h_i \).

Since the function \(f(t, x)\) belongs to \(\text{Asy}_{\{\gamma}\}}(S_\theta(T) \times D_R)\) we can assume that \(f(t, x)\) satisfies
\[
f(t, x) \ll F \sum_{k \geq 0} \frac{1}{k!} \Psi^{(\lceil k/k^* \rceil)}(\chi) \left( \frac{t}{\zeta} \right)^k,
\]
where \(\frac{1}{\delta} = 1 + \frac{1}{\gamma}\), and each \(b^0_{q}(t, x)\) satisfies for some \(B_q > 0\):
\[
b^0_{q}(t, x) \ll B_q \sum_{k \geq 0} \frac{1}{k!} \Psi^{(k)}(\chi) \left( \frac{t}{\zeta} \right)^k.
\]

Then by Lemma 5.1, we get
\[
u^0_0(t, x) \ll \tau^{-m^*} U_0^0 \sum_{k \geq k^*} \frac{1}{k!} \Psi^{(k/k^*)}(\chi) \left( \frac{t}{\zeta} \right)^k,
\]
where \(U_0^0 := \zeta^{-e^*} c(\tau) F\). By induction on \(h > 0\), we can obtain \(u^0_h(t, x)\) by Lemma 5.1. So we will inductively give an estimate for \(u^0_h(t, x)\).

**Proposition 5.2.** Under the conditions \((C_2')_s\) and \((A_1)\), for any \(h \in \mathbb{N}\) there exists \(U^0_h > 0\) such that
\[
u^0_h(t, x) \ll \tau^{-m^*} U^0_h \sum_{k \geq k^*} \frac{1}{k!} \Psi^{(k/k^*)}(\chi) \left( \frac{t}{\zeta} \right)^k,
\]
and that \(\sum_{h \geq 0} U^0_h\) converges for a sufficiently large \(\tau > 0\) and a sufficiently small \(R > 0\).
Proof. We set $\gamma_0 = \gamma_{s+1}^*$ for short. We already obtain the estimate (5.1) for $h = 0$. Let us assume the following estimate for $h_i < h$:

\[
\left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u_{h_i}^0(t, x) \ll \tau^{\alpha_1-m^*} \zeta^{-j} U_{h_i}^0 \sum_{k \geq (k^*-j)_+} \frac{1}{k!} \psi_{R}^{(\tau_{(k^*+m^*)}/\delta)_{+}+|\alpha|}}(\chi)(\frac{t}{\zeta})^k,
\]

where $(a)_+ = a$ if $a \geq 0$ and $(a)_+ = 0$ if $a < 0$ for $a \in \mathbb{R}$. By Proposition 4.3, we have

\[
t^{\sigma_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u_{h_i}^0(t, x)
\ll C^{\sigma_q} \zeta^{\epsilon_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j, \alpha}} \tau^{\alpha_1-m^*} U_{h_i}^0 \sum_{k \geq 0} \frac{1}{k!} \psi_{R}^{(\tau_{k^*}/\delta)_{+}+\epsilon_q}}(\chi)(\frac{t}{\zeta})^k.
\]

It follows $[\lfloor k^*/\gamma \rfloor_0 + 1 - J_{q-}] \leq 0$ from the assumption $(A_1)$. If $\epsilon_q < \epsilon^*$ and $|h'| = h - (\epsilon^* - \epsilon_q)$, then we have

\[
\left\lfloor \frac{-\epsilon_q}{\gamma} + l_q \right\rfloor = \left\lfloor \frac{-\epsilon_q}{\gamma} + \frac{\epsilon^* - \epsilon_q}{\gamma_0} + k^* + m^* - J_{q-} \right\rfloor
\leq \left\lfloor \frac{k^*}{\delta} + m^* - J_{q-} \right\rfloor = k^* + \left\lfloor \frac{k^*}{\gamma} \right\rfloor + m^* - 1 + \left\lfloor \frac{m^* - J_{q-}}{l_q} \right\rfloor
\leq \left\lfloor \frac{k^*}{\delta} \right\rfloor + m^* - 1,
\]

and if $\epsilon_q = \epsilon^*$, by the same manner we have

\[
\left\lfloor \frac{-\epsilon_q}{\gamma} + l_q \right\rfloor = \left\lfloor \frac{k^*}{\gamma} + k^* + m^* - 1 \right\rfloor = \left\lfloor \frac{k^*}{\delta} \right\rfloor + m^* - 1.
\]

Thus by Proposition 4.3, we have

\[
W_{h}^0(t, x)
\ll \sum_{\epsilon_q < \epsilon^*, |h'| = h - (\epsilon^* - \epsilon_q)} C^{\sigma_q} \zeta^{\epsilon_q} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j, \alpha}} \tau^{\alpha_1-m^*} U_{h_i}^0 \sum_{k \geq 0} \frac{1}{k!} \psi_{R}^{(\tau_{(k^*+m^*)}/\delta)_{+}+m^*}}(\chi)(\frac{t}{\zeta})^k
+ \sum_{\epsilon_q = \epsilon^*, |h'| = h - 1} C^{\sigma_q} \zeta^{\epsilon_q} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j, \alpha}} \tau^{\alpha_1-m^*} U_{h_i}^0 \sum_{k \geq 0} \frac{1}{k!} \psi_{R}^{(\tau_{(k^*+m^*)}/\delta)_{+}+m^*}}(\chi)(\frac{t}{\zeta})^k.
Thus by Lemma 5.1, we get the estimate (5.1) for any \( h \geq 0 \), where

\[
U_h^0 := c(\tau) \sum_{e_q < e^*} C^{e_q} \zeta^{e_q-e^*} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_{1}-m^{*}} U_{h_i}^0
\]

\[+ c(\tau) \sum_{e_q = e^*, |h'| = h-1} C^{e_q} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_{1}-m^{*}} U_{h_i}^0.\]

Next let us give an estimate of the coefficient \( U_h^0 \). Let us consider the following functional equation for \( Z(t) \):

\[
Z(t) = c(\tau) \sum_{e_q < e^*} C^{e_q} \zeta^{e_q-e^*} t^{e^*-e_q} B_q \prod_{j+|\alpha| \leq m} \{\tau^{\alpha_{1}-m^{*}} Z(t)\}^{q_{j,\alpha}} + c(\tau) \sum_{e_q = e^*} C^{e_q} t \prod_{j+|\alpha| \leq m} \{\tau^{\alpha_{1}-m^{*}} Z(t)\}^{q_{j,\alpha}} + \zeta^{-e^*} c(\tau) F.
\]

By the implicit function theorem, we have the holomorphic solution near \( t = 0 \). Moreover substituting \( Z(t) = \sum_{h \geq 0} Z_h^0 t^h \) into (5.5), for any \( T > 0 \), we can take a sufficiently large \( \tau > 0 \) and a sufficiently small \( R > 0 \) such that \( U_h^0 \leq Z_h^0 T^h \). Hence we have the desired result.

We construct a true solution to \((E^0)\). Set \( \tilde{u}^0(t, x) := \sum_{h \geq 0} u_h^0(t, x) \). Then we have:

\[
\tilde{u}^0(t, x) \ll \tau^{-m^{*}} U_h^0 \sum_{k \geq k^{*}} \frac{1}{k!} \Psi_{R}^{(\lfloor k/\delta \rfloor)}(\chi) \left( \frac{t}{\zeta} \right)^k.
\]

If we set \( \tilde{U}_0^0 := \sum_{h \geq 0} U_h^0 \), then by Proposition 5.2, we get

\[
\tilde{u}^0(t, x) \ll \tau^{-m^{*}} \tilde{U}_0^0 \sum_{k \geq k^{*}} \frac{1}{k!} \Psi_{R}^{(\lfloor k/\delta \rfloor)}(\chi) \left( \frac{t}{\zeta} \right)^k.
\]

For \(|\theta| < \pi/(2\gamma)\), as in Lemma 3.3 we can prove the existence of a holomorphic function \( u_{S}^0(t, x) \) such that

\[
u_{S}(t, x) \sim_{\gamma} \tilde{u}^0(t, x) \quad \text{in} \quad S_{\theta}(T).
\]

So we construct a true solution \( u^0(t, x) \) to \((E^0)\) such that \( u^0(t, x) = u_{S}^0(t, x) + v^0(t, x) \), where \( v^0(t, x) \) is an unknown function; we can adapt Theorem 2.3 to the equation for \( v^0(t, x) \), and we can solve \((E^0)\) with \( v^0(t, x) \in \text{Asy}^{0}_{\{\gamma\}}(S_{\theta}(T) \times D_{R}) \).

We can solve each equation \((E^g)\) for \( g \geq 0 \) by repeating the procedure of constructing a true solution \( u^0(t, x) \):
First we construct a formal solution \( u^g(t, x) = \sum_{h \geq -g} u_h^g(t, x) \) for \( g \geq 0 \) as follows:

\[
S u_{-g}^g(t, x) = - \sum_{e_q > e^*} \sum_{|g'| = g - (e_q - e^*)} a_q(t, x) \prod_{j+|\alpha| \leq m} \prod^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u_{h}^g(t, x),
\]

\[
S u_h^g(t, x) = - \sum_{e_q > e^*} \sum_{|g'| = g - (e_q - e^*)} a_q(t, x) \prod_{j+|\alpha| \leq m} \prod^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u_{h}^g(t, x)
\]

\[
- \sum_{e_q < e^*} \sum_{|g'| = g + (e_q - e^*)} a_q(t, x) \prod_{j+|\alpha| \leq m} \prod^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u_{h}^g(t, x)
\]

\[
- \sum_{e_q = e^*} \sum_{|g'| = g - (e_q - e^*)} a_q(t, x) \prod_{j+|\alpha| \leq m} \prod^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u_{h}^g(t, x).
\]

**Proposition 5.3.** Under the conditions \((C_2')_s\) and \((A_1)\), for any \( g \in \mathbb{N} \) and \( h \geq g \), there exists a \( U_h^g > 0 \) such that

\[
(5.6) \quad u_h^g(t, x) \ll \tau^{-m^*} U_h^g \sum_{k \geq k^*(h)_{+}} \frac{1}{k!} \Psi_R^{([k+(h)_{+}/\gamma-(h)_{+}/\gamma^*])}(\chi)(\frac{t}{\zeta})^k,
\]

and that \( \sum_{g \geq 0} \sum_{h \geq g} U_h^g \) converges for any sufficiently large \( \tau > 0 \) and sufficiently small \( \zeta \), \( R > 0 \). Here \( \gamma^* = \gamma_* \), and \((a)_{+} := 0 \) if \( a \geq 0 \) and \((a)_{-} := a \) if \( a < 0 \) for \( a \in \mathbb{R} \).

**Proof.** In the case where \( g = 0 \), we showed the estimate (5.6) by Lemma 5.2. Let us prove that \( u_h^g(t, x) \) satisfies (5.6) by induction on \( g > 0 \). We assume for \( g_i < g \)

\[
\left( \frac{\partial}{\partial t} \right)^{j} \left( \frac{\partial}{\partial x} \right)^{\alpha} u_{h_i}^{g_i}(t, x)
\]

\[
\ll \tau^{g_i-m^*} \zeta^{-j} \sum_{k \geq \gamma^*+(h_i)_{+}} \frac{1}{k!} \Psi_R^{[k+(h_i)_{+}/\gamma+(h_i)_{+}/\gamma^*+j+|\alpha|]}(\chi)(\frac{t}{\zeta})^k.
\]

Set \( |(h')_{-}| = \sum_{j+|\alpha| \leq m} (h_i)_{-} \). By Proposition 4.3, we have

\[
t^{\sigma_q} \prod_{j+|\alpha| \leq m} \prod^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x)
\]

\[
\ll C^{\sigma_q} \zeta^{e_q} \prod_{j+|\alpha| \leq m} \prod^{q_{j, \alpha}} \tau^{g_i-m^*} \sum_{k \geq k_h} \frac{1}{k!} \Psi_R^{([k+(h')_{-}/\gamma-(h')_{-}/\gamma^*+l_q])}(\chi)(\frac{t}{\zeta})^k.
\]
First let us estimate $u^g_{-g}(t, x)$. Under the conditions that $e_q > e^*$, $|g'| = g - (e_q - e^*)$ and $|h'| = -g + (e_q - e^*)$, we have

$$k_h \geq k^*|q| - |(h')_-| - \sum_{j+|\alpha| \leq m} q_j \alpha + \sigma_q \geq k^*|q| - |h'| + e_q \geq g,$$

and

$$\left[ \frac{|(h')_-| - e_q}{\gamma} - \frac{|(h')_-|}{\gamma^*} + l_q \right] \leq \left[ \frac{|h'| - e_q}{\gamma} - \frac{|h'|}{\gamma^*} + l_q \right] = \left[ \frac{k^*}{\delta} + \frac{-g}{\gamma} - \frac{-g}{\gamma^*} + m^* - J_q \right].$$

Thus by Proposition 4.3 we have

$$t^{\sigma_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u^q_{h_{i}}(t, x) \ll C^{\sigma_q} \zeta^{e_q} \prod_{j+|\alpha| < m} \prod_{i=1}^{q_{j, \alpha}} \tau^{\alpha_1 - m^*} U^q_{h_{i}}.$$ 

Therefore by Lemma 5.1, the estimate (5.6) holds for $u^g_{-g}(t, x)$, where

$$U^g_{-g} := c(\tau) \sum_{e_q > e^*} \sum_{\frac{|g'| = g - (e_q - e^*)}{|h'| = -g + (e_q - e^*)}} C^{\sigma_q} \zeta^{e_q - e^*} B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j, \alpha}} \tau^{|\alpha_1 - m^*} U^q_{h_{i}}.$$ 

Let us assume that a function $u^q_{h_{i}}(t, x)$ satisfies the estimate (5.6) for $h_i < h$. Here we give an estimate for the following three cases:

(i) $e_q > e^*$, $|g'| = g - (e_q - e^*)$ and $|h'| = h + (e_q - e^*)$;
(ii) $e_q < e^*$, $|g'| = g$ and $|h'| = h - (e^* - e_q)$;
(iii) $e_q = e^*$, $|g'| = g$ and $|h'| = h - 1$.

(A) Assume $h \leq 0$. Then as for (5.7), we have $k_h \geq -h$ for the all cases. First consider the case (i). Then as for the estimate (5.8) we have

$$\left[ \frac{|(h')_-| - e_q}{\gamma} - \frac{|(h')_-|}{\gamma^*} + l_q \right] \leq \left[ \frac{k^*}{\delta} + \frac{h}{\gamma} - \frac{h}{\gamma^*} + m^* \right].$$

Therefore by Proposition 4.3, we have

$$t^{\sigma_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u^q_{h_{i}}(t, x) \ll C^{\sigma_q} \zeta^{e_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j, \alpha}} \tau^{\alpha_1 - m^*} U^q_{h_{i}} \sum_{k \geq -h} \frac{1}{k!} \Psi_R^{\left(\left(\frac{k+k^*}{\delta}+\frac{h}{\gamma}-\frac{h}{\gamma^*}\right)+m^*\right)}(\chi) \left( \frac{t}{\zeta} \right)^k.$$


Next consider the cases (ii) and (iii). Then as for the estimates (5.2) and (5.3) we have
\[
\frac{|(h')_{-}| - e_q}{\gamma} - \frac{|(h')_{-}|}{\gamma^*} + l_q \leq \frac{k^*}{\delta} + \frac{h}{\gamma^*} - \frac{h}{\gamma} + m^* - 1,
\]
Therefore by the same proposition, we have
\[
\begin{aligned}
&\left| \frac{|(h')_{-}| - e_q}{\gamma} - \frac{|(h')_{-}|}{\gamma^*} + l_q \right| \\
&\leq \left| \frac{-e_q}{\gamma} + l_q \right| \\
&\leq \left| \frac{k^*}{\delta} + m^* - J_q \right| \\
&\leq \left| \frac{k^*}{\delta} \right| + m^*.
\end{aligned}
\]

Therefore we have
\[
\begin{aligned}
t^\sigma q \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u^q_{h_i}(t, x)
\ll C^\sigma q \zeta^{e_q} R \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j, \alpha}} \tau^{\alpha_1 - m^*} U^q_{h_i}
\times \sum_{k\geq -(h)_{-}} \frac{1}{k!} \Psi_R^\left(\frac{(k+k^*)/\delta+(h)_{-}/\gamma-(h)_{-}/\gamma^*)+m^*\right)(\chi)\left(\frac{t}{\zeta}\right)^k.
\end{aligned}
\]

(B) If \( h > 0 \), then we have \( \kappa_h \geq 0 \) for all cases.

In the case (i), we have
\[
\begin{aligned}
&\left| \frac{|(h')_{-}| - e_q}{\gamma} - \frac{|(h')_{-}|}{\gamma^*} + l_q \right| \\
&\leq \left| \frac{-e_q}{\gamma} + l_q \right| \\
&\leq \left| \frac{k^*}{\delta} + m^* - J_q \right| \\
&\leq \left| \frac{k^*}{\delta} \right| + m^*.
\end{aligned}
\]

Therefore we have
\[
\begin{aligned}
t^\sigma q \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j, \alpha}} \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\alpha u^q_{h_i}(t, x)
\ll C^\sigma q \zeta^{e_q} R \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j, \alpha}} \tau^{\alpha_1 - m^*} U^q_{h_i}
\times \sum_{k\geq -(h)_{-}} \frac{1}{k!} \Psi_R^\left(\frac{(k+k^*)/\delta+(h)_{-}/\gamma-(h)_{-}/\gamma^*)+m^*\right)(\chi)\left(\frac{t}{\zeta}\right)^k.
\end{aligned}
\]

In the cases (ii) and (iii), as for the estimates (5.2) and (5.3) we have
\[
\begin{aligned}
&\left| \frac{|(h')_{-}| - e_q}{\gamma} - \frac{|(h')_{-}|}{\gamma^*} + l_q \right| \\
&\leq \left| \frac{-e_q}{\gamma} + l_q \right| \\
&\leq \left| \frac{k^*}{\delta} \right| + m^*.
\end{aligned}
\]

Therefore we have the estimate (5.9). Hence by Lemma 5.1, we have the estimate (5.6), where
\[
U^g_h = c(\tau) \sum_{e_q > e^*} \sum_{\|g'\| = g - (e_q - e^*)} C^\sigma q \zeta^{e_q - e^*} B_q \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j, \alpha}} \tau^{\alpha_1 - m^*} U^g_{h_i}
\]
\[
+ c(\tau) \sum_{e_q < e^*} \sum_{\|g'\| = g} C^\sigma q \zeta^{e_q - e^*} R B_q \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j, \alpha}} \tau^{\alpha_1 - m^*} U^g_{h_i}
\]
\[
+ c(\tau) \sum_{e_q = e^*} \sum_{\|g'\| = g} C^\sigma q R B_q \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j, \alpha}} \tau^{\alpha_1 - m^*} U^g_{h_i}.
\]
Next let us give an estimate of the coefficient $U^g_h$. Let us consider the following functional equation:

$$Z(t, s) = c(\tau) \sum_{e_q > e^*} C^e B_q \prod_{j+|\alpha| \leq m} \{\tau^{-m^*} Z(t, s)\}^{q_j, \alpha}$$

$$+ c(\tau) \sum_{e_q < e^*} C^e t^{e^*-e_q} B_q \prod_{j+|\alpha| \leq m} \{\tau^{-m^*} Z(t, s)\}^{q_j, \alpha}$$

$$+ c(\tau) \sum_{e_q = e^*} C^e B_q \prod_{j+|\alpha| \leq m} \{\tau^{-m^*} Z(t, s)\}^{q_j, \alpha} + \zeta^{-e^*} c(\tau) F.$$  

(5.10)

As in the case of (5.5), we have the holomorphic solution near $(t, s) = (0, 0)$. Moreover substituting $Z(t, s) = \sum_{g \geq 0} \sum_{h \geq -g} Z_{h}^{g} t^{h}(ts)^{g}$ into (5.10), for any $T > 0$ and $S > 0$ we can take a sufficiently large $\tau > 0$ and sufficiently small $R > 0$ and $\zeta > 0$ such that $U^g_h \leq Z^h(TS)^g$. Hence we have the desired result.

Next for $g > 0$ we construct a true solution to $(E^g)$.

Set $	ilde{u}^g(t, x) := \sum_{h \geq -g} u^g_h(t, x) = \sum_{h = -g}^{-1} u^g_h(t, x) + \tilde{u}^g_0(t, x).$ Then we have

$$\tilde{u}^g(t, x) \ll \sum_{h \geq 0} \tau^{-m^*} U^g_h \sum_{k \geq k^*} \frac{1}{k!} \Psi_R^{(\lceil k/\delta \rceil)}(\chi) \left(\frac{t}{\zeta}\right)^k.$$  

By Proposition 5.3, $\sum_{h \geq 0} U^g_h$ converges. Further, if we set $\tilde{U}^g_0 := \sum_{h \geq 0} U^g_h$, then $\sum_{g \geq 0} \tilde{U}^g_0$ also converges by Proposition 5.3. For $h < 0$ set $\tilde{u}^g_h(t, x) := u^g_h(t, x)$ and $\tilde{U}^g_h := U^g_h$. Then we have

$$\tilde{u}^g(t, x) = \sum_{h = -g}^{0} \tilde{u}^g_h(t, x) \ll \sum_{h = -g}^{0} \tau^{-m^*} \tilde{U}^g_h \sum_{k \geq k^* - h} \frac{1}{k!} \Psi_R^{(\lceil k/\delta \rceil + h/\gamma - h/\gamma^*)}(\chi) \left(\frac{t}{\zeta}\right)^k.$$  

Therefore for each $g > 0$, we can construct a true solution of $(E^g)$ inductively as in the case $g = 0$. Hence we have the following result for $g \geq 0$:

**Proposition 5.4.** Under the assumptions $(C'_2)_s$ and $(A_1)$, there exists a true solution $u^g(t, x)$ to $(E^g)$ such that for $|\theta| \leq \pi/(2\gamma)$

$$u^g(t, x) \sim_\gamma \tilde{u}^g(t, x) = \sum_{-g \leq h \leq 0} \tilde{u}^g_h(t, x) = \sum_{-g \leq h \leq 0} \sum_{k \geq k^* - h} \tilde{u}^g_{h,k}(x)t^k \text{ in } S_\theta(T).$$

Here $\tilde{u}^g_{h,k}(x) \ll \tau^{-m^*} \tilde{U}^g_h \frac{1}{k!\zeta^k} \Psi_R^{(\lceil k/\delta + h/\gamma - h/\gamma^* \rceil)}(\chi)$ and $\sum_{g \geq 0} \sum_{-g \leq h \leq 0} \tilde{U}^g_h$ converges for a sufficiently large $\tau > 0$ and sufficiently small $R > 0$ and $\zeta > 0$. 
**Proof of Theorem 3.2.** Let us construct $u_{(h)}(t, x)$. We define

$$\tilde{u}_{h}^{g}(x, \xi) := \sum_{k \geq k^{*}-h} \frac{\overline{u}_{h,k}^{g}(x)}{\Gamma(k/\gamma+1)} \xi^{k/\gamma},$$

$$u_{h,S}^{g}(t, x) := t^{-\gamma} \int_{0}^{\xi} \exp(-\xi t^{-\gamma}) \tilde{u}_{h}^{g}(x, \xi) d\xi.$$ 

First let us estimate $\tilde{u}_{h}^{g}(x, \xi)$. By Proposition 5.4, there exist positive constants $A$ and $B$ such that

$$|\tilde{u}_{h}^{g}(x, \xi)| \leq \tau^{-m^{*}} \overline{U}_{h}^{g} A^{g} B^{h} \frac{\Gamma(-h/\gamma+1)}{\Gamma(-h/\gamma+1)} |\xi|^{(k^{*}-h)/\gamma} \quad \text{for} \quad g \geq -h \geq 0.$$ 

Therefore we have

$$|u_{h,S}^{g}(t, x)| \leq \tau^{-m^{*}} \overline{U}_{h}^{g} A^{g} B^{h} \Gamma\left(-\frac{h}{\gamma^{*}}+1\right) \Gamma\left(\frac{k^{*}}{\gamma}+1\right) \tau^{k^{*}-h} \quad \text{for} \quad g \geq -h \geq 0.$$ 

Setting $u_{(h)}(t, x) := \sum_{g \geq -h} u_{h,S}^{g}(t, x)$, we have

$$|u_{(h)}(t, x)| \leq \tau^{-m^{*}} \sum_{g \geq -h} \overline{U}_{h}^{g} A^{g} B^{h} \Gamma\left(-\frac{h}{\gamma^{*}}+1\right) \Gamma\left(\frac{k^{*}}{\gamma}+1\right) \tau^{k^{*}-h} \quad \text{for} \quad h \leq 0.$$ 

For a sufficiently large $\tau > 0$ and sufficiently small $R > 0$ and $\zeta > 0$, we can show that $\sum_{g \geq -h} \tilde{U}_{h}^{g} A^{g}$ converges. Set $\tilde{U}_{(h)} := \sum_{g \geq -h} \tilde{U}_{h}^{g} A^{g}$. Then $\sum_{h \leq 0} \tilde{U}_{(h)} t^{-h}$ also converges. This completes a proof of Theorem 3.2.

\[\square\]

**References**


