*RIMS Kôkyûroku Bessatsu*
**B51** (2014), 33–53

# The 2-part of the non-abelian Brumer-Stark conjecture for extensions with group $D_{4p}$ and numerical examples of the conjecture

By

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**Abstract**

Andreas Nickel formulated in [12] non-abelian generalizations of the Brumer and Brumer-Stark conjectures. This paper has two main purposes. The first is to give an improvement of [15, Theorem 5.1 (2)] on the generalized conjectures. The second is to give numerical examples of the conjectures. Using the numerical examples, we explain the meaning of the conjectures. Especially, we explain why it is reasonable to conjecture that Stickelberger elements come from reduced norms, and why we need the "denominator ideals".

## § 1. Introduction

Let $K/k$ be a finite Galois extension of number fields with Galois group $G$. In the case $G$ is abelian, "the Brumer-Stark conjecture" and "the Brumer conjecture" have been studied for many years. These conjectures predict a deep relation between the special values of $L$-functions attached to $K/k$ and the Galois module structure of the ideal class group of $K$. There exists a large body of evidence of these conjectures, see for example [16], [10] and [5].

In the case $G$ is non-abelian, Andreas Nickel recently formulated non-abelian generalizations of those two conjectures in [12] (for the explicit formulations, see §4). Several results are obtained for these generalized conjectures, see for example [11], [13] and [15].

Let $p$ be an odd prime, $D_{4p}$ the dihedral group of order $4p$ and $\zeta_p$ a complex primitive $p$-th root of unity. In [12] Nickel also formulated the "weak versions" of the

Received March 31, 2013. Revised November 7, 2013, February 3, 2014 and February 4, 2014.
2010 Mathematics Subject Classification(s): 11R42, 11R29
*Key Words*: *equivariant L-values, class groups.*
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generalized conjectures (for the explicit formulations, see [12, §2]). Concerning Nickel's (weak) conjectures, the author proved the following theorem in his previous paper:

**Theorem 1.1** ([15] Theorem 5.1). *Let $K/k$ be a finite Galois CM-extension of number fields whose Galois group is isomorphic to $D_{4p}$. Then*

**(1)** *for an odd prime $l$ ($l$ can be $p$) which does not split in $\mathbb{Q}(\zeta_p)$, the $l$-part of the (non-weak) non-abelian Brumer-Stark conjecture is true;*

**(2)** *if the prime 2 does not split in $\mathbb{Q}(\zeta_p)$, the 2-part of the weak non-abelian Brumer-Stark conjecture is true.*

This result is proved by studying the relation between the weak version of the non-abelian Brumer-Stark conjecture and that of the abelian Brumer-Stark conjecture. In the case $G$ is isomorphic to $D_{12}$, the above theorem says that (a) the 2-part of the weak non-abelian Brumer-Stark conjecture is true ;(b) if $l \not\equiv 1 \mod 3$, the $l$-part of the non-abelian Brumer-Stark conjecture is true. Even in the case that a prime $l$ splits in $\mathbb{Q}(\zeta_p)$, we know the following results by Nickel for more general $G$: (i) We assume $k = \mathbb{Q}$ and the Iwasawa $\mu$-invariant of $K(\zeta_p)$ vanishes. Then if an odd prime $l$ is ramified (resp. unramified) in $K$, the $l$-part of the non-abelian Brumer-Stark conjecture is true by [13, Corollary 4.6] (resp. by [14, Corollary 0.5]). In other words, the non-abelian Brumer-Stark conjecture except the 2-part is true if $k = \mathbb{Q}$ and the Iwasawa $\mu$-invariant of $K(\zeta_p)$ vanishes. This result is proved via the non-commutative Iwasawa main conjecture; (ii) for more general $k$, if no prime above $l$ splits in $K/K^+$ or $K^{cl} \not\subset (K^{cl})^+(\zeta_l)$, the odd $l$-part of the *weak* non-abelian Brumer-Stark conjecture is true unconditionally by [12, Corollary 4.2], where superscripts $^{cl}$ and $^+$ mean the Galois closure over $\mathbb{Q}$ and the maximal real subfield, respectively. This result is proved via the strong Stark conjecture.

In this paper, we give an improvement of Theorem 1.1. More explicitly, we prove the following:

**Theorem 1.2.** *Let $K/k$ be a finite Galois CM-extension of number fields whose Galois group is isomorphic to the dihedral group of order $4p$. Then if the prime 2 does not split in $\mathbb{Q}(\zeta_p)$, the 2-part of the (non-weak) non-abelian Brumer-Stark conjecture is true.*

The key point of the proof of the above theorem is to use the fact that the group ring $\mathbb{Z}_2[D_{4p}]$ is a "nice Fitting order" (for the definition, see §2.2). Thanks to this fact, we can ignore the "denominator ideal". The notion "nice Fitting order" was first introduced by H. Johnston and Nickel in [8].

In §6 we give some numerical examples of the 3-parts of Nickel's conjectures with explicit extensions with group $D_{12}$. As far as the author knows, a numerical example for his conjectures is not described in the literature. Using the numerical examples, we see why the formulations are reasonable. Nickel's conjectures split into two parts. The first part is concerned with the "integrality" of the Stickelberger elements and the second part is concerned with the "annihilation" of the ideal class groups. If $K/k$ is an abelian extension, the first part states the modified Stickelberger elements (which are defined in §3) belong to $\mathbb{Z}[G]$, which was proved independently in [4], [1] and [2], and the second part states the modified elements themselves annihilate the ideal class groups. In the non-abelian case, however, the first part states the modified Stickelberger elements "come form" reduced norms over $\mathbb{Z}[G]$ (and do not belong to $\mathbb{Z}[G]$ in general), and the second part states that we need extra factor the "denominator ideal" to annihilate the ideal class groups. In §6.2 we see why it is reasonable to conjecture that the modified Stickelberger elements come from reduced norms, and in §6.3 we see why we need the denominator ideals.

I would like to express my sincere gratitude to Masato Kurihara for his helpful suggestions and comments. He indicated the possibility to improve [15, Theorem 5.1]. I am deeply grateful to the referee for his/her careful reading of an earlier version of this paper and many helpful comments.

## Notation

For each ring $A$, we denote by $\zeta(A)$ the center of $A$. Moreover, for each natural number $n$, we write $M_n(A)$ and $1_{n\times n}$ for the ring of $n \times n$ matrices over $A$ and the identity element in it, respectively.

For each prime number $p$, we write $\mathbb{C}_p$ for the $p$-adic completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, where $\mathbb{Q}_p$ is the $p$-adic completion of $\mathbb{Q}$. For each finite group $G$, we denote by $\operatorname{Irr} G$ the set of all the $\mathbb{C}_p$-valued irreducible characters of $G$.

For a finite Galois extension $K/k$ of number fields with Galois group $G$, an intermediate field $F$ of $K/k$, a set $S$ of places of $k$ and a prime number $p$, we fix the following notation:

| | |
|---|---|
| $S_\infty$ | the set of all infinite places of $k$ |
| $S_{ram}$ | the set of all finite places of $k$ which ramify in $K$ |
| $G'$ | the commutator subgroup of $G$ |
| $K^{ab}$ | the maximal abelian subextension of $K/k$ i.e. $K^{ab} = K^{G'}$ |
| $\mu(F)$ | the group of roots of unity in $F$ |
| $Cl(F)$ | the ideal class group of $F$ |
| $Cl(F)_p$ | the Sylow $p$-subgroup of $Cl(F)$ |
| $S_F$ | the set of places of $F$ which lie above those in $S$ |

## § 2. Algebraic preliminaries

### § 2.1. Idempotents and projectors

Let $G$ be a finite group. For each $\chi \in \operatorname{Irr} G$, we set

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma, \ \operatorname{pr}_\chi := \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma.$$

Then $e_\chi$ is a central primitive idempotent of $\mathbb{C}_p[G]$ and $\operatorname{pr}_\chi$ is the associated projector. If $\chi$ is a 1-dimensional character, for each subgroup $\Delta$ of $G$ which is contained in $\ker \chi$, we write $\chi_\Delta$ for the character of $G/\Delta$ whose inflation to $G$ is $\chi$. Then we have

$$e_\chi = e_{\chi_\Delta} \frac{1}{|\Delta|} Norm_\Delta, \ \operatorname{pr}_\chi = \operatorname{pr}_{\chi_\Delta} Norm_\Delta. \tag{2.1}$$

For each $\chi \in \operatorname{Irr} G$, we set $\mathbb{Q}_p(\chi) := \mathbb{Q}_p(\chi(g); g \in G)$. Then if $\chi$ is induced by an irreducible character of a subgroup of $G$, we have the following lemma:

**Lemma 2.1** ([15], Lemma 2.1). *Let $G$ be a finite group and let $H$ be a subgroup of $G$. If an irreducible character $\chi$ of $G$ is induced by an irreducible character of $H$, we have*

$$e_\chi = \sum_{\substack{\phi \in \operatorname{Irr} H/\sim_\chi, \\ \operatorname{Ind} \phi = \chi}} \ \sum_{h \in \operatorname{Gal}(\mathbb{Q}_p(\phi)/\mathbb{Q}_p(\chi))} e_{\phi^h} \tag{2.2}$$

*where* $\operatorname{Irr} H/\sim_\chi$ *means* $\operatorname{Irr} H$ *modulo* $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\chi))$*-action.*

For each $\phi \in \operatorname{Irr} H$ with $\operatorname{Ind} \phi = \chi$, we have $|G|/\chi(1) = |H|/\phi(1)$. Hence multiplying both sides of (2.2) by $|G|/\chi(1)$, we get the same formula for $\operatorname{pr}_\chi$.

### § 2.2. Nice Fitting order

Let $p$ be a prime and $G$ a finite group. In this section, following [8], we introduce the notion "nice Fitting order" for the group ring $\mathbb{Z}_p[G]$. In [8], this notion is defined for more general ring. We only treat group rings here. For details, see [8, §4].

**Definition 2.2** ([8], Definition 2). When $\mathbb{Z}_p[G] = \bigoplus_{i=1}^k \Lambda_i$ where each $\Lambda_i$ is either a maximal $\mathbb{Z}_p$-order or a ring of matrices over a commutative ring, we say that $\mathbb{Z}_p[G]$ is a nice Fitting order.

If $p$ does not divide the order of $G$, $\mathbb{Z}_p[G]$ is a maximal order in $\mathbb{Q}_p[G]$. Hence, by definition, $\mathbb{Z}_p[G]$ is a nice Fitting order. The following proposition enables us to find non-maximal nice Fitting orders:

**Proposition 2.3** ([8], Proposition 4.4)**.** *Let $G'$ be the commutator subgroup of $G$. Then the group ring $\mathbb{Z}_p[G]$ is a nice Fitting order if and only if $p$ does not divide the order of $G'$.*

Since the commutator subgroup of $D_{4p}$ is a cyclic group of order $p$, we see by this proposition that $\mathbb{Z}_2[D_{4p}]$ is a (non-maximal) nice Fitting order. To prove our main theorem, we use this fact in an essential way.

### § 2.3. Reduced norms and denominator ideals

Let $p$ be a prime and $G$ a finite group. We take a maximal $\mathbb{Z}_p$-order $\mathfrak{m}_p(G)$ in $\mathbb{Q}_p[G]$ which contains $\mathbb{Z}_p[G]$. We denote by $\mathrm{nr} : \mathbb{Q}_p[G] \to \zeta(\mathbb{Q}_p[G])$ the reduced norm of $\mathbb{Q}_p[G]$ (for the details of this map, see [3, §7D]). We extend this map to $M_n(\mathbb{Q}_p[G])$ for all $n \in \mathbb{N}$ in the natural way (we also denote the extended map by nr). Note that if $G$ is abelian, the reduced norm of a matrix is nothing but the usual determinant map. We set

$$\mathcal{I}_p(G) := \langle \mathrm{nr}(H) \mid H \in M_n(\mathbb{Z}_p[G]),\ n \in \mathbb{N}\rangle_{\zeta(\mathbb{Z}_p[G])} \subset \zeta(\mathbb{Q}_p[G]).$$

Then $\mathcal{I}_p(G)$ is a ring. If $G$ is abelian, this ring coincides with $\mathbb{Z}_p[G]$. However, if $G$ is not abelian, this ring is not contained in $\mathbb{Z}_p[G]$ in general, but in $\mathfrak{m}_p(G)$. For this reason, we need some "conductors". First, we define

$$\mathcal{H}_p(G) := \{x \in \zeta(\mathbb{Z}_p[G]) \mid xH^* \in M_n(\mathbb{Z}_p[G]),\ \text{for all } H \in M_n(\mathbb{Z}_p[G]) \text{ and } n \in \mathbb{N}\},$$

where $H^*$ is the matrix over $\mathfrak{m}_p(G)$ defined in [8] such that $HH^* = H^*H = \mathrm{nr}(H) \cdot 1_{n\times n}$. This matrix is a non-commutative analogue of the adjoint matrix which was first considered in [9] (H. Johnston and Nickel in [8] introduce a slightly different definition). If $G$ is abelian, $\mathcal{H}_p(G)$ coincides with $\mathbb{Z}_p[G]$. By definition, the set $\mathcal{H}_p(G)$ satisfies

$$\mathcal{H}_p(G)\mathcal{I}_p(G) \subset \zeta(\mathbb{Z}_p[G]).$$

Since $\zeta(\mathbb{Z}_p[G]) \subset \mathcal{I}_p(G)$, the set $\mathcal{H}_p(G)$ is actually an ideal of $\mathcal{I}_p(G)$. This ideal contains non-trivial elements, more precisely, contains the *central conductor* $\mathfrak{F}_p(G)$ of $\mathfrak{m}_p(G)$ over $\mathbb{Z}_p[G]$. The central conductor $\mathfrak{F}_p(G)$ is given by

$$\mathfrak{F}_p(G) := \{x \in \zeta(\mathbb{Z}_p[G]) \mid x\mathfrak{m}_p(G) \subset \mathbb{Z}_p[G]\}.$$

By Jacobinski's central conductor formula ([7, Theorem 3] also see [3, §27]), we have

$$\mathfrak{F}_p(G) \cong \bigoplus_{\chi \in \mathrm{Irr}\, G/\sim} \frac{|G|}{\chi(1)} \mathfrak{D}^{-1}(\mathbb{Q}_p(\chi)/\mathbb{Q}_p) \tag{2.3}$$

where $\mathfrak{D}^{-1}(\mathbb{Q}_p(\chi)/\mathbb{Q}_p)$ is the inverse different of $\mathbb{Q}_p(\chi)$ over $\mathbb{Q}_p$ and the direct sum runs over the irreducible characters of $G$ modulo $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-action. By definition, $\mathfrak{F}_p(G)$ is always contained in $\mathcal{H}_p(G)$. By contrast to $\mathfrak{F}_p(G)$, we do not know the explicit structure of $\mathcal{H}_p(G)$, in general. However, in some cases, we can determine the structure of $\mathcal{H}_p(G)$. If the prime $p$ does not divide the order of $G$ (that is, $\mathbb{Z}_p[G]$ is a maximal order), we have $\mathfrak{F}_p(G) = \zeta(\mathbb{Z}_p[G])$. Hence we have $\mathcal{H}_p(G) = \mathfrak{F}_p(G)$. Even in the case $p$ divides the order of $G$, we have the following:

**Proposition 2.4** ([8], Remark 6.5 and Corollary 6.20). *If $\mathcal{I}_p(G) = \zeta(\mathfrak{m}_p(G))$ and the degrees of all the irreducible characters of $G$ are prime to $p$, we have*

$$\mathcal{H}_p(G) = \mathfrak{F}_p(G).$$

In the case $\mathbb{Z}_p[G]$ is a nice Fitting order, we get the following stronger result:

**Proposition 2.5** ([8], Proposition 4.1). *If $\mathbb{Z}_p[G]$ is a nice Fitting order, we have $\mathcal{H}_p(G) = \mathcal{I}_p(G) = \zeta(\mathbb{Z}_p[G])$.*

For an odd prime $p$, we have by [8, Example 6] (also see [15, Lemma 3.22]) $\mathcal{I}_p(D_{4p}) = \zeta(\mathfrak{m}_p(G))$, and each of the irreducible characters of $D_{4p}$ is 1 or 2-dimensional. Moreover, $\mathbb{Z}_2[D_{4p}]$ is a nice Fitting order by Proposition 2.3. Hence Propositions 2.4 and 2.5 imply the following relations:

1. $\mathcal{H}_2(D_{4p}) = \mathcal{I}_2(D_{4p}) = \zeta(\mathbb{Z}_2[D_{4p}])$;
2. $\mathcal{I}_p(D_{4p}) = \zeta(\mathfrak{m}_p(D_{4p}))$ and $\mathcal{H}_p(D_{4p}) = \mathfrak{F}_p(D_{4p})$;
3. for an odd prime $l \neq p$, we have $\mathcal{H}_l(D_{4p}) = \mathfrak{F}_l(D_{4p}) = \mathcal{I}_l(D_{4p}) = \zeta(\mathbb{Z}_l[D_{4p}])$.

## § 3. Stickelberger elements

Let $K/k$ be a finite Galois extension of number fields with Galois group $G$. For each finite place $\mathfrak{p}$ of $k$ we fix a finite place $\mathfrak{P}$ of $K$ above $\mathfrak{p}$. We denote by $G_{\mathfrak{P}}$(resp. $I_{\mathfrak{P}}$) the decomposition subgroup (resp. inertia subgroup) of $G$ at $\mathfrak{P}$. We take a lift $\mathrm{Frob}_{\mathfrak{P}}$ to $G$ of the Frobenius automorphism in $G_{\mathfrak{P}}/I_{\mathfrak{P}}$.

Let $S$ be a finite set of places of $k$ which contains $S_{ram}$ and $S_\infty$. For each $\chi \in \mathrm{Irr}\, G$, we denote by $L_S(K/k, \chi, s)$ the $S$-truncated Artin $L$-function attached to $\chi$. We take another finite set $T$ of places of $k$ such that $S \cap T = \emptyset$ and set

$$\delta_T := \sum_{\chi \in \mathrm{Irr}\, G} \det(\prod_{\mathfrak{p} \in T} 1 - \mathrm{Frob}_{\mathfrak{P}}^{-1} N\mathfrak{p} | V_\chi) e_\chi = \mathrm{nr}(\prod_{\mathfrak{p} \in T} 1 - \mathrm{Frob}_{\mathfrak{P}}^{-1} N\mathfrak{p}) \in \zeta(\mathbb{Q}_p[G]),$$

where $V_\chi$ is an irreducible representation of $G$ which has character $\chi$. We define the $(S,T)$-modified Stickelberger element for $K/k$ by

$$\theta^T_{K/k,S} := \delta_T \cdot \sum_{\chi \in \mathrm{Irr}\, G} L_S(K/k, \check{\chi}, 0) e_\chi \in \zeta(\mathbb{C}_p[G]),$$

where $\check{\chi}$ is the contragredient character of $\chi$. This element is characterized by the formula

$$\chi(\theta^T_{K/k,S}) := \chi(1)\det(\prod_{\mathfrak{p}\in T} 1 - \mathrm{Frob}_{\mathfrak{P}}^{-1} N\mathfrak{p}|V_\chi)L_S(K/k,\check{\chi},0). \tag{3.1}$$

If $T$ is empty and $S = S_{ram} \cup S_\infty$, we abbreviate $\theta^T_{K/k,S}$ by $\theta_{K/k}$. In the case $k = \mathbb{Q}$, we always omit the trivial character component of $\theta^T_{K/k,S}$.

By [18, p24, Proposition 3.4], for each non-trivial character $\chi \in \mathrm{Irr}\, G$, the vanishing order at $s = 0$ of $L_S(K/k,\chi,s)$ is given by

$$\sum_{\mathfrak{p}\in S} \dim V_\chi^{G_{\mathfrak{P}}}. \tag{3.2}$$

Since $S$ always contains $S_\infty$, this formula implies that if $k$ is not totally real or $K$ is not totally imaginary, we always have $L_S(K/k,\chi,0) = 0$ for all $\chi$ in $\mathrm{Irr}\, G$. Therefore, in this paper, we always consider the case $K/k$ is a CM-extension, which means that $k$ is a totally real field, $K$ is a CM-field and the complex conjugation induces a unique automorphism $j$ which belongs to the center of $G$. For each $\chi \in \mathrm{Irr}\, G$, we call $\chi$ is odd (resp. even) if $\chi(j) = -\chi(1)$ (resp. $\chi(j) = \chi(1)$). Then we have by (3.2) $L_S(K/k,\chi,0) = 0$ for all even characters $\chi$. For odd characters $\chi$, by Stark's conjecture (proved by Siegel [17] if $G$ is abelian and the general result is given by Brauer induction [18, p70, Theorem 1.2]), we have

$$L_S(K/k,\chi,0)^\sigma = L_S(K/k,\chi^\sigma,S), \text{ for all } \sigma \in \mathrm{Aut}(\mathbb{C}).$$

This implies that $\theta^T_{K/k,S}$ actually belongs to $\zeta(\mathbb{Q}_p[G])$ (more precisely, belongs to $\zeta(\mathbb{Q}[G])$).

## § 4. Statements of the conjecture

In this section, we review the formulations of Nickel's non-abelian generalizations of the Brumer-Stark and Brumer's conjectures (for details, see the original paper [12]). In fact, Nickel formulated "global" and "local" conjectures, however, we only review "local" conjectures here.

Let $K/k$ be a finite Galois CM-extension of number fields with Galois group $G$. We denote by $j$ the unique complex conjugation in $G$. We take two finite sets $S$ and $T$ of places of $k$. We denote by $E_S(K)$ the $S_K$-units of $K$ and set $E^T_S := \{x \in E_S(K) \mid x \equiv 1 \mod \prod_{\mathfrak{P}\in T_K} \mathfrak{P}\}$. We refer to the following condition as $Hyp(S,T)$:

- $S$ contains $S_{ram}$ and $S_\infty$,
- $S \cap T = \emptyset$,
- $E^T_S(K)$ is torsion free.

For each $\alpha \in K^*$, we call $\alpha$ an *anti-unit* if $\alpha^{1+j} = 1$, and set

$$S_\alpha := \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime in } k \text{ and } \mathfrak{p} \text{ divides } N_{K/k}(\alpha)\},$$

where $N_{K/k}$ is the usual norm form $K$ to $k$. Finally, we set $\omega_K := \mathrm{nr}(|\mu(K)|)$. Now, Nickel's non-abelian generalization of the Brumer-Stark conjecture asserts

**Conjecture 4.1** ($BS(K/k, S, p)$). *Let $S$ be a finite set of places which contains $S_{ram}$ and $S_\infty$. Then $\omega_K \theta_{K/k,S}$ belongs to $\mathcal{I}_p(G)$, and for each fractional ideal $\mathfrak{A}$ of $K$ whose class in $Cl(K)$ has p-power order and for each $x \in \mathcal{H}_p(G)$, there exists an anti-unit $\alpha = \alpha(\mathfrak{A}, S, x) \in K^*$ such that*

$$\mathfrak{A}^{x\omega_K \theta_{K/k,S}} = (\alpha),$$

*and for each finite set $T$ of places of $k$ which satisfies $Hyp(S \cup S_\alpha, T)$, there exists $\alpha_T \in E_S^T(K)$ such that*

$$\alpha^{z\delta_T} = \alpha_T^{z\omega_K}$$

*for any $z \in \mathcal{H}_p(G)$.*

*Remark.* (1) If $G$ is abelian, the first claim $\omega_K\theta_{K/k,S} \in \mathcal{I}_p(G)$ is equivalent to $|\mu(K)|\theta_{K/k,S} \in \mathbb{Z}_p[G]$ and proved independently by [1] , [2] and [4]. (2) If $G$ is abelian, by [18, p83, Proposition 1.2], the second claim is equivalent to

$$\mathfrak{A}^{\omega_K \theta_{K/k,S}} = (\alpha), \text{ and } K(\alpha^{1/|\mu(K)|})/k \text{ is abelian.}$$

This is exactly the claim of the Brumer-Stark conjecture in the abelian case. Hence we can regard Conjecture 4.1 as a generalization of the Brumer-Stark conjecture.

For an intermediate field $F$ of $K/k$ and a set $T$ of places of $k$, we write $Cl(F)^{T_F}$ for the ray class group of $F$ to the ray $\prod_{\mathfrak{P}_F \in T_F} \mathfrak{P}_F$ and set $Cl(F)_p^{T_F} := Cl(F)^{T_F} \otimes \mathbb{Z}_p$. Then we can interpret Conjecture 4.1 as the annihilation of ray class groups as follows:

**Proposition 4.2.** *Let $S$ be a finite set of places of $k$ which contains $S_\infty$ and $S_{ram}$. We assume $\theta_{K/k,S}^T$ belongs to $\mathcal{I}_p(G)$ for each finite set $T$ of places which satisfies $Hyp(S, T)$. Then $BS(K/k, S, p)$ is true if and only if for each finite set $T$ of places of $k$ such that $Hyp(S, T)$ is satisfied, we have $\mathcal{H}_p(G)\theta_{K/k,S}^T \subset \mathrm{Ann}_{\mathbb{Z}_p[G]}(Cl(K)_p^{T_K})$.*

*Remark.* The following proof of the sufficiency is essentially the same as the proof of [12, Lemma 2.9].

*Proof.* Concerning the necessity, the same proof as [12, Proposition 3.8] works. Hence, we only prove the sufficiency. We take a finite set $T$ of places of $k$ such that

$Hyp(S,T)$ is satisfied. Let $\mathfrak{A}$ be a fractional ideal of $K$ coprime to the primes in $T_K$ whose class in $Cl(K)^{T_K}$ has $p$-power order. Then for each $x \in \mathcal{H}_p(G)$, we have

$$\mathfrak{A}^{x\omega_K\theta_{K/k,S}} = (\alpha) \tag{4.1}$$

for some anti-unit $\alpha \in K^*$. Since $\mathfrak{A}$ is coprime to the primes in $T_K$, we see that $Hyp(S \cup S_\alpha, T)$ is satisfied. Hence, there exists an element $\alpha_T \in E^T_{S_\alpha}(K)$ such that

$$\alpha^{z\delta_T} = \alpha_T^{z\omega_K} \tag{4.2}$$

for any $z$ in $\mathcal{H}_p(G)$. Since $\mathrm{nr}(|\mu(K)|^{-1})$ belongs to $\zeta(\mathbb{Q}[G])$, there exists a natural number $N$ such that $N\,\mathrm{nr}(|\mu(K)|^{-1}) \in \zeta(\mathbb{Z}[G])$. Then $N\,\mathrm{nr}(|\mu(K)|^{-1})\delta_T \in \zeta(\mathfrak{m}_p(G))$. Since $|G|$ is an element in $\mathfrak{F}_p(G) \subset \mathcal{H}_p(G)$, by (4.1) and (4.2) we have

$$\begin{aligned}
(\mathfrak{A}^{x\omega_K\theta_{K/k,S}})^{|G|N\,\mathrm{nr}(|\mu(K)|^{-1})\delta_T} &= \mathfrak{A}^{x\theta^T_{K/k,S}|G|N} \\
&= (\alpha)^{|G|N\,\mathrm{nr}(|\mu(K)|^{-1})\delta_T} \\
&= (\alpha^{|G|\delta_T})^{N\,\mathrm{nr}(|\mu(K)|^{-1})} \\
&= (\alpha_T^{|G|\omega_K})^{N\,\mathrm{nr}(|\mu(K)|^{-1})} \\
&= (\alpha_T)^{|G||N|}.
\end{aligned}$$

Since we assume $\theta^T_{K/k,S} \in \mathcal{I}_p(G)$ and the group of fractional ideals has no torsion, the above equation implies

$$\mathfrak{A}^{x\theta^T_{K/k,S}} = (\alpha_T).$$

This completes the proof. □

It is hard to give a numerical example of Conjecture 4.1 (especially the latter half of the second claim) even if we use Proposition 4.2. In this paper, we give numerical examples of a generalization of Brumer's conjecture. To see the formulation, first we set

$$\mathfrak{A}_S := \langle \delta_T \mid Hyp(S,T) \text{ is satisfied. } \rangle_{\zeta(\mathbb{Z}_p[G])}.$$

By [18, p82, Lemma1.1], if $G$ is abelian, this module coincides with $\mathbb{Z}_p[G]$-annihilator of $\mu(K)$. Then the following is the generalization of Brumer's conjecture by Nickel:

**Conjecture 4.3** ($B(K/k,S,p)$). *Let $S$ be a finite set of places of $k$ which contains $S_{ram}$ and $S_\infty$. Then $\mathfrak{A}_S\theta_{K/k,S} \subset \mathcal{I}_p(G)$, and we have*

$$\mathcal{H}_p(G)\mathfrak{A}_S\theta_{K/k,S} \subset \mathrm{Ann}_{\zeta(\mathbb{Z}_p[G])}(Cl(K))$$

*Remark.* (1) If $G$ is abelian, the first claim $\mathfrak{A}_S\theta_{K/k,S} \subset \mathcal{I}_p(G)$ is equivalent to $\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K))\theta_{K/k,S} \subset \mathbb{Z}_p[G]$ and proved independently in [1] , [2] and [4]. (2) If $G$ is abelian, the second claim is equivalent to

$$\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K))\theta_{K/k,S} \subset \mathrm{Ann}_{\mathbb{Z}_p[G]}(Cl(K)_p).$$

This is exactly the claim of Brumer's conjecture in the abelian case. Hence we can regard Conjecture 4.3 as a non-abelian generalization of Brumer's conjecture.

By [12, Lemma 2.9], $BS(K/k, S, p)$ always implies $B(K/k, S, p)$.

## § 5. Main Theorem

Let $p$ be an odd prime. In this section, we prove our main theorem for CM-extensions with group $D_{4p}$.

### § 5.1. Characters of $D_{4p}$

In this section, we review the character theory of $D_{4p}$. We first recall that the group $D_{4p}$ is the direct product of $\mathbb{Z}/2\mathbb{Z}$ and $D_{2p}$. We denote by $j$ the generator of $\mathbb{Z}/2\mathbb{Z}$ and use the presentation $D_{2p} = \langle \sigma,\ \tau \mid \sigma^p = \tau^2 = 1,\ \tau\sigma\tau^{-1} = \sigma^{-1}\rangle$. Then the commutator subgroup of $D_{4p}$ is $\langle\sigma\rangle$ and we have $D_{4p}/\langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. We return to the character theory. As is well known, all the irreducible characters of $D_{4p}$ are four 1-dimensional characters and $p-1$ 2-dimensional characters. Moreover, the 1-dimensional characters are determined by the following table: Since the center of $D_{4p}$ is $\{1,\ j\}$, the element

Table 1. 1-dimensional characters of $D_{4p}$

| | $\sigma$ | $\tau$ | $j$ |
|---|---|---|---|
| $\chi_0$ | 1 | 1 | 1 |
| $\chi_1$ | 1 | 1 | $-1$ |
| $\chi_2$ | 1 | $-1$ | 1 |
| $\chi_3$ | 1 | $-1$ | $-1$ |

$j$ corresponds to the unique complex conjugation in the case $D_{4p}$ is the Galois group of some CM-extension of number fields. Hence we see that the only 1-dimensional odd characters are $\chi_1$ and $\chi_3$. For $i = 1,\ 3$, we write $\chi_i^{ab}$ for the character of $\mathrm{Gal}(K^{ab}/k)$ whose inflation to $G$ is $\chi_i$. All the 2-dimensional odd characters are induced by the faithful odd characters of $\langle j\sigma\rangle$. For $m \in (\mathbb{Z}/p\mathbb{Z})^*$, let $\phi^m$ be the character of $\langle j\sigma\rangle$ which sends $\sigma$ and $j$ to $\zeta_p^m$ and $-1$, respectively. We set $\chi_{2m+3} = \mathrm{Ind}_{\langle j\sigma\rangle}^{D_{4p}} \phi^m$ (we use this numbering so that odd subscripts correspond to odd characters). Using the Frobenius reciprocity law and the fact that $\chi_{2m+3}(1) = 2$ and $\chi_{2m+3}(j) = -2$, we see that $\mathrm{Res}_{\langle j\sigma\rangle}^{D_{4p}} \chi_{2m+3} = \phi^m + \phi^{-m}$ and $\mathrm{Ind}_{\langle j\sigma\rangle}^{D_{4p}} \phi^m = \mathrm{Ind}_{\langle j\sigma\rangle}^{D_{4p}} \phi^{-m}$. Therefore, the number of 2-dimensional odd characters is $(p-1)/2$. Finally, we set $k_\phi := K^{\langle j\sigma\rangle}$

### § 5.2. Proof of the main theorem

In this section, we prove our main theorem. We begin with the following proposition.

**Proposition 5.1.** *Let $K/k$ be a finite Galois CM-extension of number fields whose Galois group $G$ is isomorphic to $D_{4p}$. We take two finite sets $S$ and $T$ of places of $k$ such that $Hyp(S,T)$ is satisfied. Then we have*

$$\theta^T_{K/k,S} = \theta^T_{K^{ab}/k,S}\frac{1}{p}Norm_{G'} + \theta^{T_{k_\phi}}_{K/k_\phi,S_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}),$$

*where $G'$ is the commutator subgroup of $G$. Moreover, $\theta^T_{K/k,S}$ belongs to $\zeta(\mathbb{Z}_2[G])$.*

*Proof.* Recalling that Artin $L$-functions do not change by the inflation of characters, we have by (2.1) and (3.1)

$$\begin{aligned}\chi_1(\theta^T_{K/k,S})e_{\chi_1} &= \prod_{\mathfrak{p}\in T}\det(1-\mathrm{Frob}_{\mathfrak{P}}^{-1}N\mathfrak{p}|V_{\chi_1})L_S(K/k,\check{\chi}_1,0)e_{\chi_1}\\ &= \prod_{\mathfrak{p}\in T}\det(1-\mathrm{Frob}_{\mathfrak{P}}^{-1}N\mathfrak{p}|V_{\chi_1^{ab}})L_S(K^{ab}/k,\check{\chi}_1^{ab},0)e_{\chi_1^{ab}}\frac{1}{p}Norm_{G'}\\ &= \chi_1^{ab}(\theta^T_{K^{ab}/k,S})e_{\chi_1^{ab}}\frac{1}{p}Norm_{G'}. \qquad (5.1)\end{aligned}$$

The same is true for $\chi_3$, that is, we have

$$\chi_3(\theta^T_{K/k,S})e_{\chi_3} = \chi_3^{ab}(\theta^T_{K^{ab}/k,S})e_{\chi_3^{ab}}\frac{1}{p}Norm_{G'}. \qquad (5.2)$$

Since $\chi_1^{ab}$ and $\chi_3^{ab}$ are the only odd characters of $\mathrm{Gal}(K^{ab}/k)$, we have by (5.1) and (5.2)

$$\begin{aligned}\chi_1(\theta^T_{K/k,S})e_{\chi_1}+\chi_3(\theta^T_{K/k,S})e_{\chi_3} &= (\chi_1^{ab}(\theta^T_{K^{ab}/k,S})e_{\chi_1^{ab}}+\chi_3^{ab}(\theta^T_{K^{ab}/k,S})e_{\chi_3^{ab}})\frac{1}{p}Norm_{G'}\\ &= \theta^T_{K^{ab}/k,S}\frac{1}{p}Norm_{G'}. \qquad (5.3)\end{aligned}$$

Next we compute $\chi_{2m+3}(\theta^T_{K/k,S})$ for $m = 1,\ 2,\ \ldots,\ (p-1)/2$. By the induction formula of Artin $L$-functions, we have

$$\begin{aligned}\frac{\chi_{2m+3}(\theta^T_{K/k,S})}{2}e_{\chi_{2m+3}} &= \prod_{\mathfrak{p}\in T}\det(1-\mathrm{Frob}_{\mathfrak{P}}^{-1}N\mathfrak{p}|V_{\chi_{2m+3}})L_S(K/k,\check{\chi}_{2m+3},0)e_{\chi_{2m+3}}\\ &= \prod_{\mathfrak{p}_\phi\in T_{k_\phi}}\det(1-\mathrm{Frob}_{\mathfrak{P}}^{-f_{\mathfrak{p}_\phi}}N\mathfrak{p}_\phi|V_{\phi^m})L_{S_{k_\phi}}(K/k_\phi,\phi^{-m},0)e_{\chi_{2m+3}}\\ &= \phi^m(\theta^{T_{K_\phi}}_{K/k_\phi})e_{\chi_{2m+3}}\end{aligned}$$

where $f_{\mathfrak{p}_\phi}$ is the residue degree of $\mathfrak{p}_\phi$. We recall that $\chi_{2m+3} = \mathrm{Ind}^G_{\langle j\sigma\rangle}\phi^m = \mathrm{Ind}^G_{\langle j\sigma\rangle}\phi^{-m}$. Then we have

$$L_{S_{k_\phi}}(K/k_\phi,\phi^m,0) = L_{S_{k_\phi}}(K/k_\phi,\phi^{-m},0)$$

$$\prod_{\mathfrak{p}_\phi\in T_{k_\phi}}\det(1-\mathrm{Frob}_{\mathfrak{P}}^{-f_{\mathfrak{p}_\phi}}N\mathfrak{p}_\phi|V_{\phi^m}) = \prod_{\mathfrak{p}_\phi\in T_{k_\phi}}\det(1-\mathrm{Frob}_{\mathfrak{P}}^{-f_{\mathfrak{p}_\phi}}N\mathfrak{p}_\phi|V_{\phi^{-m}})$$

These equations imply that

$$\phi^m(\theta_{K/k_\phi}^{T_{K_\phi}}) = \phi^{-m}(\theta_{K/k_\phi}^{T_{K_\phi}}).$$

We have by Lemma 2.1 $e_{\chi_{2m+3}} = e_{\phi^m} + e_{\phi^{-m}}$ and hence

$$\begin{aligned}\frac{\chi_{2m+3}(\theta_{K/k,S}^T)}{2} e_{\chi_{2m+3}} &= \phi^m(\theta_{K/k_\phi}^{T_{K_\phi}})(e_{\phi^m} + e_{\phi^{-m}}) \\ &= \phi^m(\theta_{K/k_\phi}^{T_{K_\phi}})e_{\phi^m} + \phi^{-m}(\theta_{K/k_\phi}^{T_{K_\phi}})e_{\phi^{-m}} \\ &= \theta_{K/k_\phi}^{T_{k_\phi}}(e_{\phi^m} + e_{\phi^{-m}}) \\ &= \theta_{K/k_\phi}^{T_{k_\phi}} e_{\chi_{2m+3}}.\end{aligned}$$

This implies that

$$\sum_{m=1}^{\frac{p-1}{2}} \frac{\chi_{2m+3}(\theta_{K/k,S}^T)}{2} e_{\chi_{2m+3}} = \sum_{m=1}^{\frac{p-1}{2}} \theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}} e_{\chi_{2m+3}} = \theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}).$$

Combining this with (5.3), we get the first claim of Proposition 5.1.

Since $K^{ab}/k$ is an abelian extension with Galois group $G/G'$, $\theta_{K^{ab}/k,S}^T$ belongs to $\mathbb{Z}[G/G']$ as we remarked just after the statement of Conjecture 4.3. Therefore, we see that $\theta_{K^{ab}/k,S}^T \frac{1}{p} Norm_{G'}$ belongs to $\mathbb{Z}_2[G]$. Next we show that $\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}})$ belongs to $\mathbb{Z}_2[\mathrm{Gal}(K/k_\phi)]$. First we write $\psi$ for the character of $\mathrm{Gal}(K/k_\phi)$ which sends $\sigma$ and $j$ to 1 and $-1$, respectively. Since $K/k_\phi$ is also an abelian extension, $\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}$ belongs to $\mathbb{Z}_2[\mathrm{Gal}(K/k_\phi)]$. Moreover, we have

$$\begin{aligned}\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}} &= \theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}} e_\psi + \theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}) \\ &= \theta_{K^{ab}/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \frac{1}{p} Norm_{G'} + \theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}).\end{aligned} \tag{5.4}$$

Since $K^{ab}/k_\phi$ is an abelian extension, $\theta_{K^{ab}/k_\phi,S_{k_\phi}}^{T_{k_\phi}}$ belongs to $\mathbb{Z}_2[\mathrm{Gal}(K^{ab}/k_\phi)]$. Hence $\theta_{K^{ab}/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \frac{1}{p} Norm_{G'}$ belongs to $\mathbb{Z}_2[\mathrm{Gal}(K/k_\phi)]$. Therefore, we see that

$$\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}) = \theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}} - \theta_{K^{ab}/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \frac{1}{p} Norm_{G'}$$

belongs to $\mathbb{Z}_2[\mathrm{Gal}(K/k_\phi)]$. The above arguments imply that $\theta_{K/k,S}^T$ belongs to $\mathbb{Z}_2[G]$, in particular, to $\zeta(\mathbb{Z}_2[G])$. □

**Theorem 5.2.** *Let $K/k$ be a finite Galois CM-extension of number fields whose Galois group $G$ is isomorphic to $D_{4p}$. Then if the prime 2 does not split in $\mathbb{Q}(\zeta_p)$, the 2-part of the non-abelian Brumer-Stark conjecture is true for $K/k$.*

*Proof.* First we take two finite sets $S$ and $T$ of places of $k$ such that $Hyp(S,T)$ is satisfied. Then it is enough to show the following two things by Proposition 4.2:

$$\theta^T_{K/k,S} \in \mathcal{I}_p(G) \text{ and } \mathcal{H}_2(G)\theta^T_{K/k,S} \subset \mathrm{Ann}_{\mathbb{Z}_2[G]}(Cl(K)_2^{T_K}).$$

Since $\mathbb{Z}_2[G]$ is a nice Fitting order, this is equivalent to

$$\theta^T_{K/k,S} \in \zeta(\mathbb{Z}_p[G]) \text{ and } \theta^T_{K/k,S} \subset \mathrm{Ann}_{\mathbb{Z}_2[G]}(Cl(K)_2^{T_K})$$

by Proposition 2.5. The claim $\theta^T_{K/k,S} \in \zeta(\mathbb{Z}_p[G])$ is true by Proposition 5.1. Hence we only have to show $\theta^T_{K/k,S}$ annihilates $Cl(K)_2^{T_K}$.

By [16, Theorem 2.1], the Brumer-Stark conjecture is true for biquadratic extensions and hence true for $K^{ab}/k$. Observing that $\frac{1}{p}Norm_{G'}(Cl(K)_2^{T_K}) \subset Cl(K^{ab})_2^{T_{K^{ab}}}$, we have

$$\theta^T_{K^{ab}/k,S}\frac{1}{p}Norm_{G'} \in \mathrm{Ann}_{\mathbb{Z}_2[G]}(Cl(K)_2^{T_K}). \tag{5.5}$$

By [6, Theorem 3.2], the 2-part of the Brumer-Stark conjecture is true for cyclic extensions of degree 6. If 2 does not split in $\mathbb{Q}(\zeta_p)$, exactly the same proof works for cyclic extensions of degree $2p$. Hence we have

$$\begin{aligned}\theta^{T_{k_\phi}}_{K/k_\phi,S_{k_\phi}} &= \theta^{T_{k_\phi}}_{K^{ab}/k_\phi,S_{k_\phi}}\frac{1}{p}Norm_{G'} \\ &\quad + \theta^{T_{k_\phi}}_{K/k_\phi,S_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}) \in \mathrm{Ann}_{\mathbb{Z}_2[\mathrm{Gal}(K/k_\phi)]}(Cl(K)_2^{T_K}).\end{aligned}$$

By [19, §3, case(c)], the Brumer-Stark conjecture is true for quadratic extensions and hence true for $K^{ab}/k_\phi$. Therefore, we have

$$\theta^{T_{k_\phi}}_{K^{ab}/k_\phi,S_{k_\phi}}\frac{1}{p}Norm_{G'} \in \mathrm{Ann}_{\mathbb{Z}_2[\mathrm{Gal}(K/k_\phi)]}(Cl(K)_2^{T_K}).$$

and hence

$$\theta^{T_{k_\phi}}_{K/k_\phi,S_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}) \in \mathrm{Ann}_{\mathbb{Z}_2[\mathrm{Gal}(K/k_\phi)]}(Cl(K)_2^{T_K}).$$

Combining this with (5.5), we conclude that

$$\theta^T_{K/k,S} \in \mathrm{Ann}_{\mathbb{Z}_2[G]}(Cl(K)_2^{T_K}).$$

This completes the proof. □

## § 6. Numerical examples

In this section, we give some numerical examples for the non-abelian Brumer conjecture. Throughout this section, we use the same notation as in §5. We note that all the computations in this section are valid under GRH.

### § 6.1. Stickelberger elements for $D_{12}$-extensions

We assume $K/k$ is a finite Galois CM-extension whose Galois group $G$ is isomorphic to $D_{12}$. As we observed in §5.1, $D_{12}$ is the direct product of $\mathbb{Z}/2\mathbb{Z}$ and $D_6 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ ($D_6$ coincides with the symmetric group $\mathfrak{S}_3$ of degree 3). As we have seen in §5.1, the only odd characters of $D_{12}$ are $\chi_1$, $\chi_3$ and $\chi_5$. Although in §5.2 we have computed the Stickelberger elements for extensions with group $D_{4p}$, in order to give numerical examples, we write down $\theta_{K/k,S}$ as its definition. By the definition of the Stickelberger elements, we have

$$\begin{aligned}\theta_{K/k,S} &= L_S(K/k, \chi_1, 0)e_{\chi_1} + L_S(K/k, \chi_3, 0)e_{\chi_3} + L_S(K/k, \chi_5, 0)e_{\chi_5} \\ &= \epsilon_{\chi_1,S} L_{S_\infty}(K/k, \chi_1, 0)e_{\chi_1} + \epsilon_{\chi_3,S} L_{S_\infty}(K/k, \chi_3, 0)e_{\chi_3} + \epsilon_{\chi_5,S} L_{S_\infty}(K/k, \chi_5, 0)e_{\chi_5}\end{aligned}$$

where we set

$$\epsilon_{\chi_i,S} = \lim_{s\to 0} \prod_{\mathfrak{p}\in S\setminus S_\infty} \det(1 - \mathrm{Frob}_{\mathfrak{P}} N\mathfrak{p}^{-s} \mid V_{\chi_i}^{I_{\mathfrak{P}}}).$$

For $i = 1,\ 3$, we set $K_i := K^{\ker \chi_i}$ and write $\chi_i'$ for the character of $\mathrm{Gal}(K_i/k)$ whose inflation to $G$ is $\chi_i$. Then

$$\begin{aligned}\theta_{K/k,S} &= \epsilon_{\chi_1,S} L_{S_\infty}(K_1/k, \chi_1', 0)e_{\chi_1} \\ &\quad + \epsilon_{\chi_3,S} L_{S_\infty}(K_3/k, \chi_3', 0)e_{\chi_3} + \epsilon_{\chi_5,S} L_{S_\infty}(K/k_\phi, \phi, 0)e_{\chi_5} \\ &= \epsilon_{\chi_1,S}\chi_1'(\theta_{K_1/k})e_{\chi_1} + \epsilon_{\chi_3,S}\chi_3'(\theta_{K_3/k})e_{\chi_3} + \epsilon_{\chi_5,S}\phi(\theta_{K/k_\phi})e_{\chi_5}\end{aligned} \tag{6.1}$$

This is a special case of [15, Lemma 3.1].

### § 6.2. Integrality of Stickelberger elements

In the case that $K/k$ is an abelian CM-extension, the first claim of Conjecture 4.3 is equivalent to

$$\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K))\theta_{K/k,S} \subset \mathbb{Z}_p[G]. \tag{6.2}$$

Hence, one may expect the same strong integrality $\mathfrak{A}_S\theta_{K/k,S} \subset \zeta(\mathbb{Z}_p[G])$ holds even if $G$ is non-abelian. However, the following example tells us that it is reasonable to conjecture that $\mathfrak{A}_S\theta_{K/k,S}$ is contained in $\mathcal{I}_p(G)$ not in $\zeta(\mathbb{Z}_p[G])$.

Let $\alpha$ be a root of the cubic equation $x^3-11x+7=0$ and set $K=\mathbb{Q}(\sqrt{-3},\sqrt{4001},\alpha)$. Then $K/\mathbb{Q}$ is a finite Galois CM-extension, $K$ contains 3rd roots of unity and its Galois group is isomorphic to

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q})\times\mathrm{Gal}(\mathbb{Q}(\sqrt{4001},\alpha)/\mathbb{Q})\cong\mathbb{Z}/2\mathbb{Z}\times\mathfrak{S}_3\cong D_{12}.$$

Using the same notations as §6.1, we see that

$$K_1=\mathbb{Q}(\sqrt{-3}),\ K_3=\mathbb{Q}(\sqrt{-12003})\text{ and }k_\phi=\mathbb{Q}(\sqrt{4001}).$$

The only primes which ramify in $K/\mathbb{Q}$ are 3 and 4001. If we suitably choose the primes $\mathfrak{P}_3$ and $\mathfrak{P}_{4001}$ of $K$ above 3 and 4001, we see that

$$\begin{aligned}G_{\mathfrak{P}_3}&=\mathrm{Gal}(K/\mathbb{Q}(\alpha))\cong\langle j\rangle\times\langle\tau\rangle,\ I_{\mathfrak{P}_3}=\mathrm{Gal}(K/\mathbb{Q}(\sqrt{4001},\alpha))\cong\langle j\rangle,\\ G_{\mathfrak{P}_{4001}}&=\mathrm{Gal}(K/\mathbb{Q}(\alpha))\cong\langle j\rangle\times\langle\tau\rangle,\ I_{\mathfrak{P}_{4001}}=\mathrm{Gal}(K/\mathbb{Q}(\sqrt{-3},\alpha))\cong\langle\tau\rangle.\end{aligned}$$

From this, we have

$$\begin{aligned}\epsilon_{\chi_1,S_{ram}}&=\lim_{s\to0}\prod_{p\in S_{ram}}\det(1-\mathrm{Frob}_{\mathfrak{P}}\,p^{-s}\mid V_{\chi_1}^{I_{\mathfrak{P}}})=\lim_{s\to0}\det(1-\mathrm{Frob}_{\mathfrak{P}_{4001}}4001^{-s}\mid V_{\chi_1}^{\langle\tau\rangle})\\&=\lim_{s\to0}\det(1-j4001^{-s}\mid V_{\chi_1})=2.\end{aligned}$$

By the same way, we also have $\epsilon_{\chi_3,S_{ram}}=1$ and $\epsilon_{\chi_5,S_{ram}}=2$. By PARI/GP, we can compute $L$-values attached to $\chi_1,\ \chi_3$ and $\chi_5$ as

$$L_{S_\infty}(K_1/\mathbb{Q},\chi_1',0)=\frac{1}{3},\ L_{S_\infty}(K_3/\mathbb{Q},\chi_3',0)=30,\ \text{and}\ L_{S_\infty}(K/k_\phi,\phi,0)=48.$$

Hence we see from (6.1) that

$$\theta_{K/\mathbb{Q}}=\frac{2}{3}e_{\chi_1}+30e_{\chi_3}+96e_{\chi_5}=\frac{1}{9}(1-j)(311-121(\sigma+\sigma^2)-22(\tau+\sigma\tau+\sigma^2\tau)).\tag{6.3}$$

Take the prime 7. This prime is completely decomposed in $K$ and $Hyp(S_{ram}\cup S_\infty,\{7\})$ is satisfied. Also we have

$$\delta_{\{7\}}=\mathrm{nr}(1-\mathrm{Frob}_{\mathfrak{P}_7}^{-1}7)=\mathrm{nr}(1-7)=\mathrm{nr}(-6).$$

Then

$$\delta_{\{7\}}\theta_{K/\mathbb{Q}}=\frac{1}{3}(1-j)(3410-1774(\sigma+\sigma^2)+44(\tau+\sigma\tau+\sigma^2\tau)).\tag{6.4}$$

Obviously this element does not belong to $\zeta(\mathbb{Z}_3[G])$ and hence we can not expect the strong inclusion $\mathfrak{A}_S\theta_{K/k,S} \subset \zeta(\mathbb{Z}_p[G])$ in general. However, we actually have

$$\delta_{\{7\}}\theta_{K/\mathbb{Q}} = \mathrm{nr}((1-j)(-\frac{71}{2}+\frac{1}{2}\sigma-11\sigma^2+19\tau+\frac{13}{2}\sigma\tau+\frac{37}{2}\sigma^2\tau)) \in \mathcal{I}_3(G). \tag{6.5}$$

As long as we see this example, it seems reasonable to conjecture $\mathfrak{A}_S\theta_{K/k,S} \subset \mathcal{I}_p(G)$. In fact, by [15, Lemma 4.1] (and [15, Lemma 3.11]), if $G$ is isomorphic to $D_{4p}$, we always have $\mathfrak{A}_S\theta_{K/k,S} \subset \mathcal{I}_p(G)$. Note that the preimage of $\delta_{\{7\}}\theta_{K/\mathbb{Q}}$ is found in an ad hoc way, and as far as the author knows, there are no theoretical approaches to find concrete preimages of Stickelberger elements.

We have seen where $\mathfrak{A}_S\theta_{K/k,S}$ should live. Then where does $\theta_{K/k,S}$ itself live? First we return to the case $G$ is abelian. Since $|\mu(K)|$ belongs to $\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K))$, we have by (6.2)

$$|\mu(K)|\theta_{K/k,S} \in \mathbb{Z}_p[G]$$

or equivalently,

$$\theta_{K/k,S} \in \frac{1}{|\mu(K)|}\mathbb{Z}_p[G].$$

This implies the denominator of $\theta_{K/k,S}$ is at most $|\mu(K)|$. In the case $G$ is non-abelian, we see by (6.3) that the denominator of $\theta_{K/k,S}$ can not be bounded by $|\mu(K)|$. However, if we believe the first claim of Conjecture 4.1, we have

$$\omega_K\theta_{K/k,S} \in \mathcal{I}_p(G)$$

and hence

$$\theta_{K/k,S} \in \langle \mathrm{nr}(\frac{1}{|\mu(K)|}H) \mid H \in M_n(\mathbb{Z}_p[G]), n \in \mathbb{N}\rangle_{\zeta(\mathbb{Z}_p[G])}. \tag{6.6}$$

Namely, the first claim of Conjecture 4.1 predicts that the denominators of preimages are at most $|\mu(K)|$ (not the denominators of $\theta_{K/k,S}$ itself). In fact, by (6.5), we see that

$$\theta_{K/\mathbb{Q}} = \mathrm{nr}(\frac{1}{6}(1-j)(\frac{71}{2}-\frac{1}{2}\sigma+11\sigma^2-19\tau-\frac{13}{2}\sigma\tau-\frac{37}{2}\sigma^2\tau)). \tag{6.7}$$

The reduced norm map is not injective, but the explicit computation of the reduced norm in Appendix tells us the preimages of $\theta_{K/\mathbb{Q}}$ does not belong to $\mathbb{Z}_3[G]$. More explicitly we see that the preimages of $\theta_{K/\mathbb{Q}}$ must belong to $(1/3)\mathbb{Z}_3[G] = (1/|\mu(K)|)\mathbb{Z}_3[G]$. If we set $L = \mathbb{Q}(\sqrt{-2}, \sqrt{33}, \beta)$ ($\beta$ satisfies $\beta^3 - 9\beta + 3 = 0$), we have $\mu(L) = \{\pm 1\}$, $\mathrm{Gal}(L/\mathbb{Q}) \cong D_{12}$ and as computed in [15, §5.1.3]

$$\theta_{L/\mathbb{Q}} = \frac{2}{3}(1-j)(1+\sigma+\sigma^2-\tau-\sigma\tau-\sigma^2\tau).$$

Since $L$ does not contain non-trivial roots of unity, we expect $\theta_{L/\mathbb{Q}}$ itself belongs to $\mathcal{I}_3(D_{12})$. In fact, we have

$$\theta_{L/\mathbb{Q}} = \mathrm{nr}(2(1-j)(-1+\sigma+\sigma^2-\tau+\sigma\tau-\sigma^2\tau)).$$

As long as we see these numerical examples, in the non-abelian cases it seems that the direct influence of the existence of the roots of unity appears not in the denominators of the Stickelberger elements themselves but in those of the preimages of the Stickelberger elements.

Finally, we introduce an example which tells us that Stickelberger elements can belong to $\zeta(\mathbb{Z}_p[G])$ even if $\mathbb{Z}_p[G]$ is not a nice Fitting order. We take a root $\gamma$ of the cubic equation $x^3-12x+13=0$ and set $M=\mathbb{Q}(\sqrt{-6},\sqrt{29},\gamma)$. By the same manner as the calculation of $\theta_{K/\mathbb{Q}}$, we see that

$$\epsilon_{\chi_1,S_{ram}} = \epsilon_{\chi_5,S_{ram}} = 0 \text{ and } \epsilon_{\chi_3,S_{ram}} = 1,$$

and by PARI/GP

$$L_{S_\infty}(M/\mathbb{Q},\chi_3,0)=12.$$

Therefore, we have

$$\theta_{M/\mathbb{Q}} = 12e_{\chi_3} = \mathrm{pr}_{\chi_3}.$$

Obviously, this element belongs to $\zeta(\mathbb{Z}_3[G])$. Moreover, $\theta_{M/\mathbb{Q}}$ comes from the reduced norm. In fact, we have

$$\theta_{M/\mathbb{Q}} = \mathrm{nr}(\mathrm{pr}_{\chi_3}).$$

Since $M$ does not contain non-trivial roots of unity, this is also an example of the inclusion (6.6).

## § 6.3. Annihilation of ideal class groups

As we have seen in the previous section, the elements $\delta_T\theta_{K/k,S}$ have denominators in general. Therefore, they can not act on the ideal class groups just as they are. This is one of the main reason why we adopt $\mathfrak{F}_p(G)$ and $\mathcal{H}_p(G)$ (in the latter half of this section, we will see that this is not the only reason). In this section, we see how Stickelberger elements annihilate ideal class groups with concrete Galois extensions appearing in the previous section.

First we study $K/\mathbb{Q}$, where we recall $K=\mathbb{Q}(\sqrt{-3},\sqrt{4001},\alpha)$ with $\alpha^3-11\alpha+7=0$. By PARI/GP, we can see the structure as an abelian group of the ideal class group of $K$ as follows:

$$Cl(K)\cong \mathbb{Z}/180\mathbb{Z}\oplus\mathbb{Z}/12\mathbb{Z},\ Cl(K)_3\cong\mathbb{Z}/9\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}.$$

We denote by $c_1$ and $c_2$ the basis of $Cl(K)_3$ which is chosen in the computation of PARI/GP. Then also using PARI/GP, we see the Galois action on $Cl(K)_3$ as follows:

$$\begin{cases} \sigma(c_1) = 4c_1 + c_2, & \tau(c_1) = -c_1, \quad j(c_1) = -c_1, \\ \sigma(c_2) = 6c_1 + c_2, & \tau(c_2) = c_2, \qquad j(c_2) = -c_2. \end{cases} \tag{6.8}$$

The above relations imply that $Cl(K)_3$ is generated by $c_1$ as a $\mathbb{Z}_3[G]$-module.

By Proposition 2.4, $\mathcal{H}_3(G)$ coincides with $\mathfrak{F}_3(G)$, and hence, by (2.3) each element $x$ in $\mathcal{H}_3(G)$ is of the form

$$x = \sum_{\chi \in \operatorname{Irr} G} x_\chi \operatorname{pr}_\chi, \ x_\chi \in \mathbb{Z}_3.$$

Then we have

$$x\delta_{\{7\}}\theta_{K/\mathbb{Q}} = -4x_{\chi_1} \operatorname{pr}_{\chi_1} - 180x_{\chi_3} \operatorname{pr}_{\chi_3} + 3456x_{\chi_5} \operatorname{pr}_{\chi_5}.$$

Obviously this element belongs to $\zeta(\mathbb{Z}_3[D_{12}])$. Since 180 and 3456 are multiples of 9, we have

$$180x_{\chi_3} \operatorname{pr}_{\chi_3} c_1 = 3456x_{\chi_5} \operatorname{pr}_{\chi_5} c_1 = 0.$$

Moreover, we see by (6.8) that

$$\operatorname{pr}_{\chi_1} c_1 = (1-j)(1+\sigma+\sigma^2)(1+\tau)c_1 = (1-j)(1+\sigma+\sigma^2)(1-1)c_1 = 0.$$

Hence

$$x\delta_{\{7\}}\theta_{K/\mathbb{Q}}c_1 = 0.$$

Thus thank to the denominator ideal $\mathcal{H}_p(G)$ (and the central conductor $\mathfrak{F}_p(G)$), $\delta_{\{7\}}\theta_{K/k,S}$ becomes an element in $\zeta(\mathbb{Z}_3[G])$ and annihilates $Cl(K)_3$. Then what will happen in the case the Stickelberger elements have no denominators? If $\mathbb{Z}_p[G]$ is a nice Fitting order, we do not need $\mathcal{H}_p(G)$. However, the following calculation tells us that we need $\mathcal{H}_p(G)$ in general.

We study $M/\mathbb{Q}$, where we recall $M = \mathbb{Q}(\sqrt{-6}, \sqrt{29}, \gamma)$ with $\gamma^3 - 12\gamma + 13 = 0$. By PARI/GP, we can see the explicit structure of the ideal class group of $M$ and the Galois action on it as follows:

$$Cl(M) \cong \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}, \ Cl(M)_3 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

We denote by $c_1$, $c_2$ and $c_3$ the basis of $Cl(M)_3$ which is chosen in the computation of PARI/GP. Then we have

$$\begin{cases} \sigma(c_1) = -c_1 - c_2, & \tau(c_1) = -c_1, & j(c_1) = -c_1, \\ \sigma(c_2) = c_1 + c_3, & \tau(c_2) = c_1 + c_2 - c_3, & j(c_2) = -c_2, \\ \sigma(c_3) = c_3, & \tau(c_1) = -c, & j(c_3) = -c_3. \end{cases} \tag{6.9}$$

By the above relations, we can see that $Cl(M)_3$ is generated by $c_1$ as a $\mathbb{Z}_3[G]$-module.

Take the prime 173. This prime is completely decomposed in $M$ and satisfies $Hyp(S_{ram} \cup S_\infty, \{173\})$. Also we have

$$\delta_{\{173\}}\theta_{M/\mathbb{Q}} = \mathrm{nr}(-172)\,\mathrm{pr}_{\chi_3} = -172 e_{\chi_3}\,\mathrm{pr}_{\chi_3} = -172\,\mathrm{pr}_{\chi_3}.$$

This element also belongs to $\zeta(\mathbb{Z}_3[G])$. However, from (6.9) we have

$$\begin{aligned}\delta_{\{173\}}\theta_{M/\mathbb{Q}}c_1 &= -172\,\mathrm{pr}_{\chi_3} c_1 = -172(1-j)(1+\sigma+\sigma^2)(1-\tau)c_1 \\ &= -172 \cdot 2 \cdot (-1) \cdot 2c_3 \neq 0.\end{aligned}$$

We take an element $x = \sum_{\chi \in \mathrm{Irr}\, G} x_\chi \mathrm{pr}_\chi \in \mathcal{H}_3(G)$. Then we have

$$x\delta_{\{173\}}\theta_{M/\mathbb{Q}}c_1 = -172 \cdot 2 \cdot (-1) \cdot 2 \cdot 12 x_{\chi_3} c_3 = 0.$$

Therefore, even in the case that Stickelberger elements do not have denominators, we need denominator ideal $\mathcal{H}_p(G)$.

Finally, we study why we need $\mathcal{H}_3(G)$. We recall that

$$e_{\chi_3} = \frac{1}{12}\mathrm{pr}_{\chi_3} \text{ and } \theta_{M/\mathbb{Q}} = L_{S_\infty}(M/\mathbb{Q}, \chi_3, 0)e_{\chi_3} = 12e_{\chi_3} = \mathrm{pr}_{\chi_3}.$$

The important thing here is that the $L$-value attached to $\chi_3$ is canceled by the denominator of $e_{\chi_3}$ and hence $\theta_{M/\mathbb{Q}}$ has no information on the $L$-value. However, if we multiply $\theta_{M/\mathbb{Q}}$ by $x$, we have

$$x\theta_{M/\mathbb{Q}} = x_{\chi_3}\mathrm{pr}_{\chi_3}\mathrm{pr}_{\chi_3} = x_{\chi_3} 12\,\mathrm{pr}_{\chi_3} = x_{\chi_3} L_{S_\infty}(M/\mathbb{Q}, \chi_3, 0)\,\mathrm{pr}_{\chi_3}.$$

In this way, thanks to the element $x$, we obtain information on the $L$-value from $\theta_{M/\mathbb{Q}}$. This is the reason why we need the denominator ideal $\mathcal{H}_3(G)$.

## § 7. Appendix: The reduced norm of $\mathbb{Q}_p[D_{12}]$

We fix a prime $p$. In this Appendix, we review the way how to compute the reduced norm of $\mathbb{Q}_p[D_{12}]$.

From Table 5.1, we see all the 1-dimensional representations of $D_{12}$. We set

$$\rho_{\chi_4}(\sigma) := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},\ \rho_{\chi_4}(\tau) := \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},\ \rho_{\chi_4}(j) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\rho_{\chi_5}(\sigma) := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},\ \rho_{\chi_5}(\tau) := \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},\ \rho_{\chi_5}(j) := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can easily see that these determine all the 2-dimensional representations (there is no deep reason we choose these forms). We also set $\rho_{\chi_i} := \chi_i$ for $i = 0,\ 1,\ 2,\ 3$. Then we have

$$\mathbb{Q}_p[D_{12}] \xrightarrow{\sim} \mathbb{Q}_p \oplus \mathbb{Q}_p \oplus \mathbb{Q}_p \oplus \mathbb{Q}_p \oplus M_2(\mathbb{Q}_p) \oplus M_2(\mathbb{Q}_p),\ \alpha \mapsto \oplus\rho_{\chi_i}(\alpha).$$

The reduced norm map is defined by the following composition map:

$$\mathbb{Q}_p[D_{12}] \xrightarrow[\sim]{\oplus\rho_{\chi_i}} \bigoplus_{i=0}^{5} M_{\chi_i(1)}(\mathbb{Q}_p) \xrightarrow{\oplus \det} \bigoplus_{i=0}^{5} \mathbb{Q}_p \xrightarrow[\sim]{(\oplus\rho_{\chi_i})^{-1}} \zeta(\mathbb{Q}_p[D_{12}]).$$

Take an element

$$\alpha = A + B\sigma + C\sigma^2 + D\tau + E\sigma\tau + F\sigma^2\tau + Gj + H\sigma j + I\sigma^2 j + J\tau j + K\sigma\tau j + L\sigma^2\tau j$$

in $\mathbb{Q}_p[D_{12}]$. Then the coefficient of the identity of $D_{12}$ is

$$\begin{aligned}\frac{1}{3}(A + 2A^2 + B - 2AB + 2B^2 + C - 2AC - 2BC + 2C^2 - 2D^2 + 2DE - 2E^2 + 2DF \\ +2EF - 2F^2 + 2G^2 - 2GH + 2H^2 - 2GI - 2HI + 2I^2 - 2J^2 + 2JK - 2K^2 \\ +2JL + 2KL - 2L^2).\end{aligned}$$

## References


[1] Barsky, D., Fonctions zêta p-adiques d'une classe de rayon des corps de nombres totalement réels, *Groupe d'Etude d'Analyse Ultrametrique* (1977/78), Exp. No. 16.

[2] Cassou-Noguès, P., Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques, *Invent. Math.* **51** (1979), 29–59.

[3] Curtis, C. W., Reiner, I., *Methods of Representation Theory with applications to finite groups and orders*, Vol. 1, John Wiley & Sons (1981).

[4] Deligne, P., Ribet, K. Values of abelian L-functions at negative integers over totally real fields, *Invent. Math.* **59** (1980), 227–286.

[5] Greither, C., Popescu, C., An equivariant main conjecture in Iwasawa theory and applications, to appear in Journal of Algebraic Geometry.

[6] Greither, C., Roblot, X.-F., Tangedal, B., The Brumer-Stark Conjecture in some families of extensions of specified degree, *Math. Comp.* **73** (2004), 297–315.

[7] Jacobinski, H., On extensions of lattices, *Michigan Math. J.* **13** (1966), 471–475.

[8] Johnston, H., Nickel, A., Noncommutative Fitting invariants and improved annihilation results, *J. Lond. Math. Soc.* **88 (1)** (2013), 137–160.

[9] Nickel, A., Non-commutative Fitting invariants and annihilation of class groups, *J. Algebra* **323 (10)** (2010), 2756–2778.

[10] Nickel, A., On the Equivariant Tamagawa Number Conjecture in tame CM-extensions, *Math. Z.* **268** (2011), 1–35.

[11] Nickel, A., On the equivariant Tamagawa number conjecture in tame CM-extensions, II, *Compos. Math* **147 (4)** (2011), 1179–1204.
[12] Nickel, A., On non-abelian Stark-type conjectures, *Ann. Inst. Fourier* **61 (6)** (2011), 2577–2608.
[13] Nickel, A., Equivariant Iwasawa theory and non-abelian Stark-type conjectures, *Proc. Lond. Math. Soc.* **106 (6)** (2013), 1223–1247.
[14] Nickel, A., Integrality of Stickelberger elements and the equivariant Tamagawa number conjecture, to appear in J. Reine Angew. Math.
[15] Nomura, J., On non-abelian Brumer and Brumer-Stark conjectures for monomial CM-extensions, *Int. J. Number Theory.* **10 (4)** (2014), 817–848.
[16] Sands, J., Galois groups of exponent two and the Brumer-Stark conjecture, *J. Reine Angew. Math.* **349** (1984), 129–135.
[17] Siegel, C., Über die Zetafunktionen indefiniter quadratischen Formen, I, II, *Math. Z.*, **43** (1938), 682-708; **44** (1939), 398-426.
[18] Tate, J., *Les Conjectures de Stark sur les Fonctions L d'Artin en s = 0*, Birkhauser, 1984.
[19] Tate, J., Brumer-Stark-Stickelberger, *Séminaire de Théorie des Nombres*, Université de Bordeaux I, U. E. R. de mathématiques et d'information, exposé no. 24.