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On the computation of Voros coefficients via middle convolutions

Dedicated to Professor T.Aoki on his 60th birthday

By

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Abstract

We study the Voros coefficients of linear ordinary differential equations with a large parameter obtained via middle convolutions. By the middle convolution, a new parameter is introduced to the equation. We determine its dependency to the Voros coefficients explicitly.

§ 1. Introduction

In exact WKB analysis, two kinds of Stokes phenomena for WKB solutions have been studied: Stokes phenomena with respect to the variable of the equation and those with respect to parameters included in the equation. The latter is called *parametric Stokes phenomena*, and substantial progress has been made in recent studies. One of the reasons of this progress is the success of proving Sato's conjecture ([20]) for the explicit form of the Voros coefficient of the Weber equation without using transcendental techniques ([25], [27] and §3 below; see also [28, §7 and Appendix A]).

A Voros coefficient, named after the important work of Voros [28], is defined as a properly regularized integral of the logarithmic derivative of WKB solutions from a turning point to a singular point (cf. (3.7)). It is, in a sense, formal connection coefficients between WKB solutions normalized at turning points and those normalized at singular points.

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Since the work of [7] and [28], Voros coefficients have been recognized as one of the most important objects in exact WKB analysis. The explicit form of the Voros coefficients of the Weber equation, i.e., Sato's conjecture, becomes important since it enables us to describe parametric Stokes phenomena explicitly. Furthermore, through the alien calculus and the transformation theory, we can also describe parametric Stokes phenomena for more general equations which can be reduced to the Weber equation (cf. [3], [27] and recent results obtained by Sasaki). Similar results are also obtained for the equations of special functions and for the equations reduced to them (cf. [1], [3], [5], [17], [18], [22], [26]). Voros coefficients are also studied even for the Painlevé equations in order to clarify the parametric Stokes phenomena ([12], [13], [14]) and to obtain the asymptotic behavior of solutions ([15]). Voros coefficients also play an essential role in a cluster algebraic structure hidden in the exact WKB analysis revealed in [16].

In this article we give an announcement of our studies on the computation of the explicit form of Voros coefficients. Our aim is to determine Voros coefficients of, mainly, higher order linear ordinary differential equations. We will show that such a computation becomes transparent and can be done in a unified manner if we consider the problem with the middle convolution (see, e.g., [23] and references cited there). By the middle convolution, one parameter is introduced to the equation. Our main theorem (and its corollary) gives its relevance to the Voros coefficients explicitly.

This paper is organized as follows. In §2 we recall some basic facts about the WKB analysis and the middle convolution. In §3 we state our main theorem. In §4, to illustrate our main theorem, we study the hypergeometric equation of type $(1, 4)$ and compute Voros coefficients explicitly. Although we have not succeeded in constructing algorithms to compute Voros coefficients yet, we believe our study becomes the first step for it. In §5 we give some remarks toward such study.

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§ 2. Preliminaries

The main result announced in this article is concerned with exact WKB analysis and the middle convolution. Here we recall some notations and basic facts in these theories which will be needed to state our results. As general references, we refer the reader to [21] for exact WKB analysis and [23] for the middle convolutions. In this article we follow their notations.

§ 2.1. WKB solution

In this article we discuss a linear ordinary differential equation of order n (≥ 1) containing a large parameter $\eta > 0$:

$$P(x, \eta^{-1}\partial_x)\psi = 0, \tag{2.1}$$

where

$$P(x, \eta^{-1}\partial_x) = \sum_{k=0}^{n} a_k(x)(\eta^{-1}\partial_x)^k \tag{2.2}$$

is a linear differential operator of polynomial coefficients. Let $\zeta_j(x)$ ($1 \leq j \leq n$) be the characteristic roots of $P(x, \eta^{-1}\partial_x)$; these are solutions of $\sigma(P)(x, \zeta) = 0$, where

$$\sigma(P)(x, \zeta) := \sum_{k=0}^{n} a_k(x)\zeta^k \tag{2.3}$$

is the total symbol of P. Throughout this article we assume

Assumption 2.1.

(i) The total symbol $\sigma(P)(x, \zeta)$ is an irreducible polynomial.

(ii) The point $x = \infty$ is an irregular singular point of (2.1). Moreover, the behavior of characteristic roots when x tends to ∞ is given by

$$\zeta_j(x) = x^{d_j}(c_j + O(x^{-1})), \tag{2.4}$$

with some constants d_j and c_j satisfying

$$\text{(i) } d_j \geq 0, \quad \text{(ii) } c_j \neq 0, \quad \text{(iii) } c_j \neq c_k \text{ if } d_j = d_k. \tag{2.5}$$

Note that, if some of d_j are not integers, characteristic roots are multivalued near $x = \infty$ and hence the above labeling becomes ambiguous. To avoid such an ambiguity, in what follows we put suitable branch cuts from ∞ and we regard (2.4) as the asymptotic behavior in the cut plane.

For the equation (2.1), we can construct a formal solution, called a *WKB solution*, of the form

$$\psi(x, \eta) = \exp\left[\int^x S(x, \eta)dx\right], \tag{2.6}$$

$$S(x, \eta) = \sum_{k=-1}^{\infty} \eta^{-k} S_k(x) = \eta S_{-1}(x) + S_0(x) + \eta^{-1}S_1(x) + \cdots \tag{2.7}$$

(cf. [2]). Here the function $S_{-1}(x)$ satisfies the algebraic equation $\sigma(P)(x, S_{-1}) = 0$, and hence the leading term $S_{-1}(x)$ of (2.7) is one of $\zeta_j(x)$ $(1 \le j \le n)$. Once the label j of the leading term is fixed, the functions $S_k(x)$ for $k \ge 0$ are determined uniquely and recursively. Thus we have n WKB solutions of (2.1) of the form

$$\psi_j(x,\eta) = \exp\left(\eta \int^x \zeta_j(x)dx\right) \sum_{n=0}^{\infty} \eta^{-n}\psi_{j,n}(x) \quad (1 \le j \le n), \tag{2.8}$$

where $\psi_{j,n}(x)$ are some functions.

For the asymptotic behavior of higher order terms $S_k(x)$ $(k \ge 0)$, we obtain

Lemma 2.2. *If we choose $\zeta_j(x)$ as the leading term $S_{-1}(x)$ of (2.7), then the function $S_k(x)$ behaves as*

$$S_k(x) = O(x^{-(d_j+1)k-1}) \quad (k \ge -1) \tag{2.9}$$

when x tends to ∞. Here d_j is defined in (2.4).

§ 2.2. Middle convolutions

The notion of *middle convolutions* is introduced by Nicholas Katz [19] in his study of rigid local systems, and reformulated as an operation on Fuchsian systems by Dettweiler-Reiter [8]. A remarkable development has been made based on middle convolutions together with additions recently. By using these operations, all of the rigid Fuchsian equations can be built up from (or reduced to) a trivial equation in an algorithmic way. We refer the reader to [23] for the recent studies. These operations are also studied for equations with irregular singular points (cf., e.g., [6], [9], [10] and [29]).

Here we consider middle convolutions with a large parameter η. Except for this, we follow the notation used in [23, Chapter 1] for scalar differential equations.

Definition 2.3 ([23, § 1.3]). For a linear differential operator $P(x, \eta^{-1}\partial_x)$ satisfying Assumption 2.1 (i), and a complex number $\mu \neq 0$, define an operator $\tilde{P}_\mu(x, \eta^{-1}\partial_x) = mc_{\mu\eta}(P)(x, \eta^{-1}\partial_x)$ by

$$\tilde{P}_\mu(x, \eta^{-1}\partial_x) = (\eta^{-1}\partial_x)^\ell \circ \mathrm{Ad}(\partial_x^{-\mu\eta})P(x, \eta^{-1}\partial_x). \tag{2.10}$$

Here ℓ is given by $\max\{j - k \mid a_{j,k} \neq 0\}$, where $a_{j,k}$ is defined by $a_k(x) = \sum_{j=1}^{\deg a_k} a_{j,k}x^j$. The operator $\tilde{P}_\mu$ is called the *middle convolution* of P.

Note that the operator $\tilde{P}_\mu = mc_{\mu\eta}(P)$ is an $(n+\ell)$-th order differential operator of the form

$$\tilde{P}_\mu(x, \eta^{-1}\partial_x) = \sum_{k=0}^{n+\ell} \tilde{a}_k(x,\eta)(\eta^{-1}\partial_x)^k, \tag{2.11}$$

where

$$\tilde{a}_k(x,\eta) = \tilde{a}_{k,0}(x) + \eta^{-1}\tilde{a}_{k,1}(x) + \eta^{-2}\tilde{a}_{k,2}(x) + \cdots \tag{2.12}$$

is a polynomial in η^{-1} whose coefficients $\tilde{a}_{k,m}(x)$ are polynomials in x. The explicit form of the total symbol of $\tilde{P}_\mu$ defined by

$$\sigma(\tilde{P}_\mu)(x,\zeta) = \sum_{k=0}^{n+\ell} \tilde{a}_k(x,\eta)\zeta^k, \tag{2.13}$$

is given in Lemma 2.5 below. At the level of solutions, the middle convolution is an integral transformation

$$\psi(x,\eta) \mapsto \tilde{\psi}(x,\mu,\eta) = \frac{1}{\Gamma(\mu\eta)} \int_{x_0}^{x} \psi(z,\eta)(x-z)^{\mu\eta-1}dz, \tag{2.14}$$

where x_0 is suitably chosen. The operation (2.14) is known to be the *fractional derivation.*

Example 2.4.

(i) Since $\mathrm{Ad}(\partial_x^{-\mu\eta})\partial_x = \partial_x$ and $\mathrm{Ad}(\partial_x^{-\mu\eta})x = x - \mu\eta\partial_x^{-1}$, we have

$$mc_{\mu\eta}(\eta^{-1}\partial_x) = \eta^{-1}\partial_x, \tag{2.15}$$

$$mc_{\mu\eta}(x) = \eta^{-1}\partial_x \circ (x - \eta\mu\partial_x^{-1}) = x(\eta^{-1}\partial_x) - \mu + \eta^{-1}. \tag{2.16}$$

(ii) For a second order operator $P = 3(\eta^{-1}\partial_x)^2 + 2c(\eta^{-1}\partial_x) + x$ ($c \in \mathbb{C}$ is a constant), we obtain

$$\tilde{P}_\mu = 3(\eta^{-1}\partial_x)^3 + 2c(\eta^{-1}\partial_x)^2 + x(\eta^{-1}\partial_x) - \mu + \eta^{-1}. \tag{2.17}$$

The equation $\tilde{P}_\mu\tilde{\psi} = 0$ will be analyzed in §4.

Since we deal with two operators P and $\tilde{P}_\mu = mc_{\mu\eta}(P)$ simultaneously, we will call the operator P in (2.1) the *original* operator to avoid confusions.

Lemma 2.5. *The total symbol of the operator $\tilde{P}_\mu$ is given by*

$$\sigma(\tilde{P}_\mu) = \sum_{j\geq 0} \frac{(-1)^j}{j!}[\mu - \eta^{-1}\ell]_j \frac{\partial^j\sigma(P)}{\partial x^j}\zeta^{\ell-j}, \tag{2.18}$$

where

$$[\lambda]_j = \begin{cases} 1 & j = 0, \\ \lambda(\lambda+\eta^{-1})\cdots(\lambda+(j-1)\eta^{-1}) & j \geq 1. \end{cases} \tag{2.19}$$

It follows from Lemma 2.5 that, if we expand $\sigma(\tilde{P}_\mu)$ with respect to η as

$$\sigma(\tilde{P}_\mu)(x,\zeta) = \sigma(\tilde{P}_\mu)_0(x,\zeta) + \eta^{-1}\sigma(\tilde{P}_\mu)_1(x,\zeta) + \eta^{-2}\sigma(\tilde{P}_\mu)_2(x,\zeta) + \cdots, \tag{2.20}$$

then the leading term is given by

$$\sigma(\tilde{P}_\mu)_0(x,\zeta) = \zeta^\ell \sigma(P)(x - \frac{\mu}{\zeta}, \zeta). \tag{2.21}$$

Here $\sigma(\tilde{P}_\mu)_0(x,\zeta)$ is a polynomial in ζ of degree $n+\ell$. The characteristic roots $\tilde{\zeta}_j(x,\mu)$ $(1 \le j \le n+\ell)$ of $\tilde{P}_\mu$ are defined by the roots of $\sigma(\tilde{P}_\mu)_0(x,\zeta) = 0$ with respect to ζ.

In this article, we consider the case where the middle convolution *strictly increases* the order of a differential operator:

Assumption 2.6. The original operator $P(x, \eta^{-1}\partial_x)$ satisfies $\ell > 1$. Here ℓ is the index defined in Definition 2.3.

Using the relation (2.21), we can prove

Proposition 2.7. *Under Assumptions 2.1 and 2.6, we can assign the label of characteristic roots $\{\tilde{\zeta}_j(x,\mu)\}_{j=1}^{n+\ell}$ of $\tilde{P}_\mu$ such that their asymptotic behavior is*

$$\tilde{\zeta}_j(x,\mu) = x^{d_j}(1 + O(x^{-1})) \quad (1 \le j \le n), \tag{2.22}$$

$$\tilde{\zeta}_\alpha(x,\mu) = \frac{\mu}{x} + O(x^{-2}) \qquad (n+1 \le \alpha \le n+\ell) \tag{2.23}$$

as x tends to ∞. Furthermore, for $1 \le j \le n$, we have

$$\tilde{\zeta}_j(x,\mu) - \zeta_j(x) = -\frac{\mu d_j}{x} + O(x^{-2}) \quad (x \to \infty). \tag{2.24}$$

§ 3. The main theorem: difference equations for Voros coefficients

§ 3.1. Voros coefficients

Voros coefficients were introduced intrinsically in [28], and studied by [7] etc. They are important objects in the exact WKB analysis since they appear in the expression of monodromy or Stokes data. Moreover, Voros coefficients play crucial roles in the analysis of *parametric Stokes phenomena* (see [27, 13, 5]).

A typical and historical example of the Voros coefficients is that of the Weber equation

$$\left(\eta^{-2}\frac{d^2}{dx^2} - \left(\frac{1}{4}x^2 - \lambda\right)\right)\psi = 0, \tag{3.1}$$

where η denotes a large parameter and λ is a complex parameter. This equation appeared in the study of the reduction problem of the one-dimensional Schrödinger equations when two simple turning points are connected by a Stokes curve ([20], in that case λ is a formal power series with respect to η), and Sato conjectured

$$\int_{2\sqrt{\lambda}}^{\infty}(S(x,\eta)-\eta S_{-1}(x)-S_0(x))dx=\frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{1-2n}-1}{2n(2n-1)}B_{2n}(\lambda\eta)^{1-2n}, \tag{3.2}$$

where $S(x,\eta)=\eta S_{-1}(x)+S_0(x)+\eta^{-1}S_1(x)+\cdots$ is the logarithmic derivative of a WKB solution, and B_n is the n-th Bernoulli number defined by

$$\frac{w}{e^w-1}=1-\frac{w}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}w^{2n}. \tag{3.3}$$

The left-hand side of (3.2) is called the Voros coefficient of the Weber equation. Although the proof of (3.2) is given in [28, Appendix A] when λ is a genuine constant (i.e., when λ does not depend on η), it was also proved by [25] and [27] without using transcendental techniques.

Here we consider the Voros coefficients for a differential equation obtained by the middle convolution of a certain differential equation. Let $P(x,\eta^{-1}\partial_x)$ be the operator of the original equation (2.1), and $\tilde{P}_\mu(x,\eta^{-1}\partial_x)$ be the middle convolution of $P(x,\eta^{-1}\partial_x)$ with a complex parameter $\mu\neq 0$ defined in Definition 2.3. For the equation

$$\tilde{P}_\mu(x,\eta^{-1}\partial_x)\tilde{\psi}=0, \tag{3.4}$$

we can construct a WKB solution

$$\tilde{\psi}(x,\mu,\eta)=\exp\left[\int^x\tilde{S}(x,\mu,\eta)dx\right], \tag{3.5}$$

$$\tilde{S}(x,\mu,\eta)=\sum_{k=-1}^{\infty}\eta^{-k}\tilde{S}_k(x,\mu)=\eta\tilde{S}_{-1}(x,\mu)+\tilde{S}_0(x,\mu)+\eta^{-1}\tilde{S}_1(x,\mu)+\cdots \tag{3.6}$$

similarly to (2.6).

Fix the label $1,\ldots,n+\ell$ of characteristic roots of (3.4) near $x=\infty$ as is specified by Proposition 2.7. For any $j,k\in\{1,\ldots,n+\ell\}$, consider a path $\gamma_{j,k}$ (cf. Figure 1) on the Riemann surface defined by $\{\sigma(\tilde{P}_\mu)_0(x,\zeta)=0\}$ which starts from $x=\infty$ on the j-th sheet (i.e., $\tilde{\zeta}(x,\mu)=\tilde{\zeta}_j(x,\mu)$ on the sheet), turn around a branch point (*turning point* of (3.4)), and returns to $x=\infty$ on the k-th sheet (i.e., $\tilde{\zeta}(x,\mu)=\tilde{\zeta}_k(x,\mu)$ on the sheet). Lemma 2.2 implies that the following integral is well-defined (as the term-wise integral for the coefficient of each power of η^{-1}):

$$\tilde{V}_{j,k}(\mu,\eta)=\frac{1}{2}\int_{\gamma_{j,k}}\left(\tilde{S}(x,\mu,\eta)-\eta\tilde{S}_{-1}(x,\mu)-\tilde{S}_0(x,\mu)\right)dx. \tag{3.7}$$

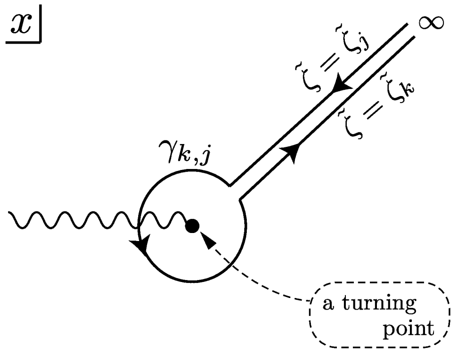


Figure 1. The path $\gamma_{j,k}$.

Definition 3.1. The formal power series $\tilde{V}_{j,k}(\mu, \eta)$ defined in (3.7) for $j, k \in \{1, \ldots, n+\ell\}$ is called the *Voros coefficient* of the type (j, k) for (3.4).

§ 3.2. Statement of the main Theorem

Our main theorem shows that the Voros coefficients satisfy certain difference equations concerning the shift of the parameter μ.

Theorem 3.2. *Suppose that the original operator $P = P(x, \eta^{-1}\partial_x)$ satisfies Assumptions 2.1 and 2.6. Let $\tilde{P}_\mu = mc_{\mu\eta}(P)$ the middle convolution of P. Then, the Voros coefficients $\tilde{V}_{k,j}(\mu, \eta)$ of (3.4) satisfy the following difference equations:*

(i) *For $1 \le j, k \le n$, we have*

$$\tilde{V}_{j,k}(\mu, \eta) - \tilde{V}_{j,k}(\mu - \eta^{-1}, \eta) = 0. \tag{3.8}$$

(ii) *For $1 \le j \le n$ and $n+1 \le \alpha \le n+\ell$, we have*

$$\tilde{V}_{j,\alpha}(\mu, \eta) - \tilde{V}_{j,\alpha}(\mu - \eta^{-1}, \eta) = \frac{1}{2}\left[1 + \left(\mu\eta - \frac{1}{2}\right)\log\left(1 - \frac{1}{\mu\eta}\right)\right]. \tag{3.9}$$

(iii) *For $n+1 \le \alpha, \beta \le n+\ell$, we have*

$$\tilde{V}_{\alpha,\beta}(\mu, \eta) - \tilde{V}_{\alpha,\beta}(\mu - \eta^{-1}, \eta) = 0. \tag{3.10}$$

Here the label $1, \ldots, n+\ell$ of characteristic roots of (3.4) near $x = \infty$ are specified by Proposition 2.7.

To be precise, the above equalities hold up to the sign because the Voros coefficient $\tilde{V}_{j,k}$ depends on the orientation of the integration path $\gamma_{j,k}$.

A similar kind of difference equations appearing in Theorem 3.2 was effectively used in the computation of the explicit form of Voros coefficients for the *Weber equation* ([27]), for the *Whitaker equation* ([22]), for the *Legendre equation* ([18]) and for the *hypergeometric equation* ([5]). The idea of the proof of Theorem 3.2 is also similar to these concrete examples listed above, i.e., Takei's method ([27]): in our case ∂_x becomes the lowering operator with respect to μ, that is to say, the derivative $\partial_x\tilde{\psi}(x,\mu,\eta)$ of a WKB solution for $\tilde{P}_\mu$ becomes the constant multiple of that for $\tilde{P}_{\mu-\eta^{-1}}$. The details will be given elsewhere.

As a corollary of Theorem 3.2, the μ-dependent part of the Voros coefficients are completely determined:

Corollary 3.3.

(i) *For* $1 \leq j,k \leq n$, $\tilde{V}_{j,k}(\mu,\eta)$ *does not depend on* μ.

(ii) *For* $1 \leq j \leq n$ *and* $n+1 \leq \alpha \leq n+\ell$, *we have*

$$\tilde{V}_{j,\alpha}(\mu,\eta) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)}(\mu\eta)^{1-2n} + (\mu\text{-independent terms})\,. \tag{3.11}$$

Here B_{2n} *is the* $2n$*-th Bernoulli number defined by* (3.3).

(iii) *For* $n+1 \leq \alpha,\beta \leq n+\ell$, $\tilde{V}_{\alpha,\beta}(\mu,\eta)$ *does not depend on* μ.

The "μ-independent terms" in (3.11) are not determined from the difference equation in Theorem 3.2. In order to find their explicit form, we have to analyze the dependence of Voros coefficients on the other parameters contained in the equation. In next section we will give an example for which we can determine the μ-independent terms completely.

§ 4. Example: The hypergeometric equation of the type $(1,4)$

In this section we will consider the following differential equation

$$\tilde{P}_\mu(x,\eta^{-1}\partial_x)\tilde{\psi} = \left\{3(\eta^{-1}\partial_x)^3 + 2c(\eta^{-1}\partial_x)^2 + x(\eta^{-1}\partial_x) - \mu + \eta^{-1}\right\}\tilde{\psi} = 0. \tag{4.1}$$

Here c and μ are complex parameters. The equation (4.1) is satisfied by a certain (confluent) hypergeometric integral which is a special solution of a degenerate Garnier system in two variables [24]. The equation (4.1) is called the *hypergeometric equation of the type* $(1,4)$ since the equation corresponds to "$1+4$" in the confluence diagram obtained from a linear differential equation with 5 regular singular points. The equation (4.1) is also studied by Hirose from a viewpoint of exact WKB analysis. See [11] for further information including Stokes geometry of (4.1).

The goal of this section is to determine the Voros coefficients of (4.1) with the aid of the results explained in Section 3. Recall that (4.1) is obtained by the middle convolution from the second order equation

$$P(x,\eta^{-1}\partial_x)\psi = \{3(\eta^{-1}\partial_x)^2 + 2c(\eta^{-1}\partial_x) + x\}\psi = 0 \tag{4.2}$$

(cf. Example 2.4 (ii)). Note that the equation (4.2) is nothing but the Airy equation (with the translated and scaled variables). By straightforward computations, we obtain asymptotic behaviors of characteristic roots:

$$\begin{cases} \tilde{\zeta}_1(x,\mu) = +\dfrac{i}{\sqrt{3}}x^{\frac{1}{2}} - \dfrac{c}{3} - \dfrac{ic^2}{6\sqrt{3}}x^{-\frac{1}{2}} - \dfrac{\mu}{2}x^{-1} + O(x^{-\frac{3}{2}}), \\ \tilde{\zeta}_2(x,\mu) = -\dfrac{i}{\sqrt{3}}x^{\frac{1}{2}} - \dfrac{c}{3} + \dfrac{ic^2}{6\sqrt{3}}x^{-\frac{1}{2}} - \dfrac{\mu}{2}x^{-1} + O(x^{-\frac{3}{2}}), \\ \tilde{\zeta}_3(x,\mu) = \mu x^{-1} + O(x^{-2}) \end{cases} \tag{4.3}$$

for $\tilde{P}_\mu$ and

$$\begin{cases} \zeta_1(x) = +\dfrac{i}{\sqrt{3}}x^{\frac{1}{2}} - \dfrac{c}{3} - \dfrac{ic^2}{6\sqrt{3}}x^{-\frac{1}{2}} + O(x^{-\frac{3}{2}}), \\ \zeta_2(x) = -\dfrac{i}{\sqrt{3}}x^{\frac{1}{2}} - \dfrac{c}{3} + \dfrac{ic^2}{6\sqrt{3}}x^{-\frac{1}{2}} + O(x^{-\frac{3}{2}}) \end{cases} \tag{4.4}$$

for P_μ. Note that the labels of characteristic roots in (4.3) and (4.4) are chosen so that they are consistent with those in Proposition 2.7. We consider the three Voros coefficients $\tilde{V}_{1,2}(\mu,\eta)$, $\tilde{V}_{1,3}(\mu,\eta)$ and $\tilde{V}_{2,3}(\mu,\eta)$ for (4.1) since other Voros coefficients of (4.1) are obtained from them; e.g., $\tilde{V}_{2,1}(\mu,\eta) = -\tilde{V}_{1,2}(\mu,\eta)$. Let $\gamma_{j,k}$ be a path defining the Voros coefficient $\tilde{V}_{j,k}(\mu,\eta)$.

The main theorem of this section is the following:

Theorem 4.1. *The Voros coefficients for* (4.1) *are independent of the parameter c, and given explicitly as follows:*

$$\tilde{V}_{1,2}(\mu,\eta) = 0. \tag{4.5}$$

$$\tilde{V}_{1,3}(\mu,\eta) = \tilde{V}_{2,3}(\mu,\eta) = \sum_{n=0}^{\infty} \frac{B_{2n}}{2n(2n-1)}(\mu\eta)^{1-2n}. \tag{4.6}$$

Here B_{2n} is the $2n$-th Bernoulli number given by (3.3).

Again we note that the above equalities hold up to the sign, depending on the orientation of the integration path $\gamma_{j,k}$ which defines the Voros coefficient $\tilde{V}_{j,k}$. The rest of this section is devoted to the proof of Theorem 4.1.

Firstly, using Theorem 3.2 (Corollary 3.3), we can decompose the Voros coefficients into μ-dependent part and μ-independent part. Let $\tilde{f}_{j,k}(c,\eta)$ the μ-independent part of $\tilde{V}_{j,k}$:

$$\tilde{f}_{1,2}(c,\eta) = \tilde{V}_{1,2}(\mu,\eta), \tag{4.7}$$

$$\tilde{f}_{1,3}(c,\eta) = \tilde{V}_{1,3}(\mu,\eta) - \sum_{n=0}^{\infty} \frac{B_{2n}}{2n(2n-1)}(\mu\eta)^{1-2n}, \tag{4.8}$$

$$\tilde{f}_{2,3}(c,\eta) = \tilde{V}_{2,3}(\mu,\eta) - \sum_{n=0}^{\infty} \frac{B_{2n}}{2n(2n-1)}(\mu\eta)^{1-2n}. \tag{4.9}$$

Here $\tilde{f}_{j,k}(c,\eta)$ is a formal power series whose coefficients may depend on c of the form $\tilde{f}_{j,k}(c,\eta) = \sum_{m=1}^{\infty} \eta^{-m} \tilde{f}_{j,k}^{(m)}(c)$. We will show that $\tilde{f}_{j,k}(c,\eta)$ vanishes.

Using a result of [11], we can show the following.

Lemma 4.2. *$\tilde{V}_{j,k}(\mu,\eta)$ (and hence, $\tilde{f}_{j,k}(c,\eta)$ also) does not depend on the parameter c.*

Proof. The equation (4.1) is a restriction of a system of linear differential equations in two variables ([11]):

$$\begin{cases} \left\{3(\eta^{-1}\partial_x)^3 + 2z(\eta^{-1}\partial_x)^2 + x(\eta^{-1}\partial_x) - \mu + \eta^{-1}\right\} \tilde{\phi} = 0, \\ \left(\eta^{-1}\partial_x^2 - \partial_z\right) \tilde{\phi} = 0. \end{cases} \tag{4.10}$$

The restriction on $\{z = c\}$ of (4.10) gives the equation (4.1). In [11] a WKB solution $\tilde{\phi}(x,z,\mu,\eta)$ of the system (4.10) was constructed. Set

$$\tilde{T}^{(1)}(x,z,\mu,\eta) = \frac{\partial_x \tilde{\phi}(x,z,\mu,\eta)}{\tilde{\phi}(x,z,\mu,\eta)} = \sum_{k=-1}^{\infty} \eta^{-k} T_k^{(1)}(x,z,\mu),$$
$$\tilde{T}^{(2)}(x,z,\mu,\eta) = \frac{\partial_z \tilde{\phi}(x,z,\mu,\eta)}{\tilde{\phi}(x,z,\mu,\eta)} = \sum_{k=-1}^{\infty} \eta^{-k} T_k^{(2)}(x,z,\mu).$$

Here we note that the restriction of $\tilde{T}^{(1)}(x,z,\mu,\eta)$ coincides with $\tilde{S}(x,c,\mu,\eta)$ defined in (3.6) for the equation (4.1): $\tilde{T}^{(1)}(x,c,\mu,\eta) = \tilde{S}(x,c,\mu,\eta)$. Moreover, it is shown in [11, Proposition 3.1] that these formal series satisfy

$$\partial_z \tilde{T}^{(1)}(x,z,\mu,\eta) = \partial_x \tilde{T}^{(2)}(x,z,\mu,\eta). \tag{4.11}$$

Therefore, we have

$$\begin{aligned} &\partial_z \int_{\gamma_{j,k}} \left(\tilde{T}^{(1)}(x,z,\mu,\eta) - \eta \tilde{T}_{-1}^{(1)}(x,z,\mu) - \tilde{T}_0^{(1)}(x,z,\mu)\right) dx \\ &= \left(\tilde{T}^{(2)}(x,z,\mu,\eta) - \eta \tilde{T}_{-1}^{(2)}(x,z,\mu) - \tilde{T}_0^{(2)}(x,z,\mu)\right)\big|_{x=\infty_j} \\ &\quad - \left(\tilde{T}^{(2)}(x,z,\mu,\eta) - \eta \tilde{T}_{-1}^{(2)}(x,z,\mu) - \tilde{T}_0^{(2)}(x,z,\mu)\right)\big|_{x=\infty_k} = 0. \end{aligned} \tag{4.12}$$

Thus the formal series

$$\int_{\gamma_{j,k}} \left(\tilde{T}^{(1)}(x,z,\mu,\eta) - \eta \tilde{T}^{(1)}_{-1}(x,z,\mu) - \tilde{T}^{(1)}_{0}(x,z,\mu) \right) dx \tag{4.13}$$

does not depend on z. Since the restriction of the formal series (4.13) on $\{z = c\}$ coincides with $\tilde{V}_{j,k}$, we have proved the claim. □

Therefore, the μ-independent part $\tilde{f}_{j,k}$ of $\tilde{V}_{j,k}$ takes the form

$$\tilde{f}_{j,k}(\eta) = \sum_{m=1}^{\infty} \eta^{-m} \tilde{f}^{(m)}_{j,k} \tag{4.14}$$

with a genuine constant $\tilde{f}^{(m)}_{j,k} \in \mathbb{C}$.

Lemma 4.3. *$\tilde{V}_{j,k}(\mu,\eta)$ is invariant under the scaling $(\mu,\eta) \mapsto (r\mu, r^{-1}\eta)$ for any complex number $r \neq 0$.*

Proof. Since the equation (4.1) is invariant under $(x,c,\eta,\mu) \mapsto (r^{\frac{2}{3}}x, r^{\frac{1}{3}}c, r\mu, r^{-1}\eta)$, the formal series $\tilde{S}$ satisfies

$$\tilde{S}(r^{\frac{2}{3}}x, r^{\frac{1}{3}}c, r\mu, r^{-1}\eta) d(r^{\frac{2}{3}}x) = \tilde{S}(x,c,\eta,\mu)dx.$$

Thus we have $\tilde{V}_{j,k}(r\mu, r^{-1}\eta) = \tilde{V}_{j,k}(\mu,\eta)$. □

Lemma 4.3 implies that $\tilde{f}_{j,k}(r^{-1}\eta) = \tilde{f}_{j,k}(\eta)$ for any $r \neq 0$. Hence, the coefficient $\tilde{f}^{(m)}_{j,k}$ in (4.14) must vanish for all m, and we have $\tilde{f}_{j,k}(\eta) = 0$. Thus we have proved Theorem 4.1.

§ 5. Toward an algorithm to compute Voros coefficients

In concluding this report, we give some remarks on algorithms to compute Voros coefficients. For an operator $\tilde{P}_{\mu}$ obtained by the middle convolution, we can determine the μ-dependent parts of Voros coefficients completely by Theorem 3.2. Since all the rigid equations can be (or are expected to be, for the equations with irregular singular points) obtained by the addition and the middle convolution (cf. [6], [9], [10], [23] and [29]), and since the addition is nothing but the gauge transformation, we may expect that we can also obtain an algorithm to compute Voros coefficients. In other words, there seems to be a possibility of determining Voros coefficients of $\tilde{P}_{\mu}$ from some information of the original equation.

In order to construct an algorithm, we need to determine μ-independent parts of the Voros coefficients. In §4, we have computed the μ-independent part in question by using some special properties which (4.1) has: its extensibility to a holonomic system, and its homogeneity. Although we expect that our method can be applied to some other equations obtained by the middle convolution of holonomic systems, it does not seem to be applicable in general situation.

In this section we show another possibility to determine μ-independent parts of Voros coefficients. The idea is to take the limit $\mu \to 0$. If we fix the independent variable x at a generic point, the characteristic roots $\tilde{\zeta}_j(x,\mu)$ of $\tilde{P}_\mu$ are holomorphic in μ on some neighborhood of $\mu = 0$. Due to the relation (2.21) between the total symbols, the limit $\tilde{\zeta}_j(x,0)$ satisfies

$$\tilde{\zeta}_j(x,0)^\ell \sigma(P)(x,\tilde{\zeta}_j(x,0)) = 0. \tag{5.1}$$

This implies that, under the label specified by Proposition 2.7, $\tilde{\zeta}_j(x,0)$ coincides with a characteristic root $\zeta_j(x)$ of the original operator P for $1 \leq j \leq n$, and $\tilde{\zeta}_\alpha(x,0) = 0$ for $n+1 \leq \alpha \leq n+\ell$. Furthermore, for $1 \leq j \leq n$, $\tilde{S}(x,\mu,\eta)$ with $\tilde{S}_{-1}(x) = \tilde{\zeta}_j(x,\mu)$ coincides with $S(x,\eta)$ at $\mu = 0$:

$$\tilde{S}(x,\mu,\eta)\big|_{\mu=0} = S(x,\eta). \tag{5.2}$$

Here the right-hand side is the formal series (2.7) defined for the original operator P whose leading term is given by $\zeta_j(x)$. The equality (5.2) follows from the uniqueness of the formal series (2.7) for a fixed leading term. Therefore, it follows from this observation and the fact that the Voros coefficient $\tilde{V}_{j,k}(\mu,\eta)$ does not depend on μ for $1 \leq j,k, \leq n$ (cf. Corollary 3.3 (i)), if a pair of turning points of $\tilde{P}_\mu$ does not "pinch" the path $\gamma_{j,k}$ in the limit $\mu \to 0$, the Voros coefficient $\tilde{V}_{j,k}$ coincides with a Voros coefficient of the original operator P:

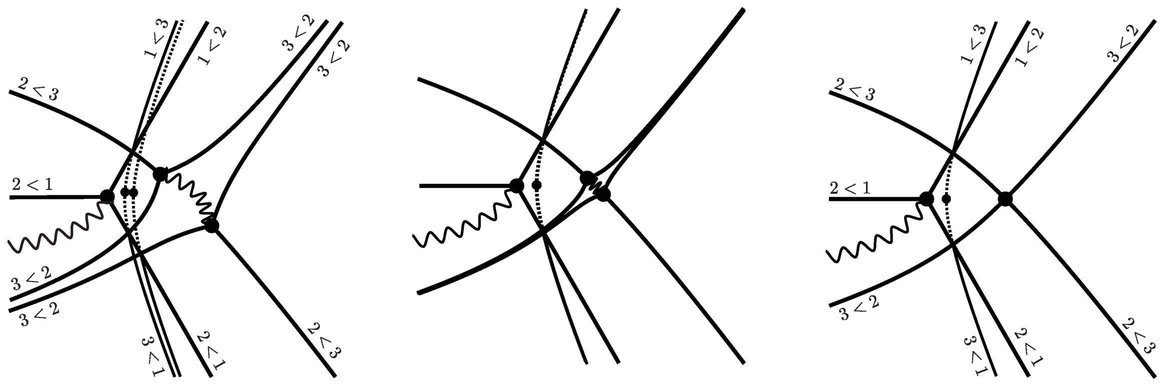
$$\tilde{V}_{j,k}(\mu,\eta) = V_{j,k}(\eta). \tag{5.3}$$

Let us examine our idea by the equation (4.1) and (4.2) discussed in §4. Turning points and Stokes curves are shown in Figure 2. A turning point is defined by a point at which two characteristic roots coincide. The equation (4.1) with generic c and μ has three turning points. If $\tilde{\zeta}_j(a,\mu) = \tilde{\zeta}_k(a,\mu)$ at a turning point a, it is said to be "of type (j,k)". From a turning point a of type (j,k), we draw Stokes curves defined by

$$\operatorname{Im} \int_a^x \left(\tilde{\zeta}_j(x,\mu) - \tilde{\zeta}_k(x,\mu)\right) dx = 0. \tag{5.4}$$

Three Stokes curves emanate from each turning point as is shown Figure 2. If a Stokes curve satisfies

$$\operatorname{Re} \int_a^x \left(\tilde{\zeta}_j(x,\mu) - \tilde{\zeta}_k(x,\mu)\right) dx > 0, \tag{5.5}$$



(a) $\mu = 0.01$ (b) $\mu = 0.001$ (c) $\mu = 0$

Figure 2. Examples of the Stokes geometry for (4.1) with $c = 1.5i$ and several μ. The dot symbols (resp., smaller dot symbols) designate turning points (resp., *virtual turning points*). The solid lines (resp., dashed lines) designate Stokes curves or “effective” parts of new Stokes curves (resp., “non-effective” parts of new Stokes curves).

we assign a label “$j > k$” to it. The labels used in Figure 2 are compatible with the labels specified in (4.3). (We refer the reader to [4] concerning the explanation of virtual turning points and new Stokes curves.)

Let us fix c to be a nonzero number. As μ tends to 0, two turning points of type (2, 3) approach to each other, and both of them converge to the origin, as is illustrated in Figure 2. As a result, the path $\gamma_{2,3}$ is pinched by two turning points as μ tends to zero. The path $\gamma_{1,2}$ is not, however, pinched by any turning points in the limit $\mu \to 0$. Hence we obtain

$$\tilde{V}_{1,2}(\mu,\eta) = \lim_{\mu\to 0} \tilde{V}_{1,2}(\mu,\eta) = V_{1,2}(\eta). \tag{5.6}$$

Now the problem is reduced to the computation of the Voros coefficients $V_{1,2}(\eta)$ of the original equation (4.2). But it is easy to compute: since the equation (4.2) has only one turning point (cf. Figure 3), $V_{1,2}(\eta)$ is identically zero. In this way we can confirm again $\tilde{V}_{1,2}(\mu,\eta) = 0$, as already proved in §4.

In a general case, we can show the following: if every zero of $a_n(x)$ (a zero-th order term of the original operator P) is not a turning of P, then the path $\gamma_{j,k}$ for $1 \le j,k \le n$ is not pinched by turning points of $\tilde{P}_\mu$ as μ tends to zero. Hence we obtain (5.3) for $1 \le j,k \le n$. (Note that this also proves Theorem 3.2 (i) since the original Voros coefficient $V_{j,k}(\eta)$ does not depend on μ.) In this way we can compute these Voros coefficients of $\tilde{P}_\mu$ if we know those of P.

It is desirable that other Voros coefficients of $\tilde{P}_\mu$ can also be expressed by those of P. We have not, however, succeeded in finding general formulas like (5.3) of μ-independent parts for other Voros coefficients. It is our future problem to find an algorithm to

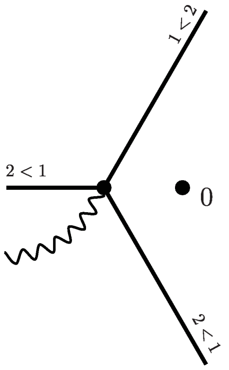


Figure 3. Stokes geometry for (4.2) with $c = 1.5i$.

compute Voros coefficients $\{\tilde{V}_{j,k}(\mu, \eta)\}$ from some data of P (e.g., from $\{V_{j,k}(\eta)\}$).

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