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An inversion formula of Schapira type for topological Radon transforms of definable functions

By

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Abstract

We study topological Radon transforms of definable functions. Although our Euler integration of definable functions is not linear but homogeneous, under some assumptions we prove an inversion formula of Schapira type for our topological Radon transforms.

§ 1. Introduction

Let X be a real analytic manifold. We say that an integer-valued function $\varphi\colon X \longrightarrow \mathbb{Z}$ on X is constructible if $\{\varphi^{-1}(t)\}_{t\in\mathbb{Z}}$ is a locally finite family of subanalytic subsets of X. For constructible functions, the Euler integration, which is a theory of integration based on the Euler characteristics of subanalytic subsets, was introduced by Schapira [8]. Note that it was introduced by Viro [10] independently in the complex analytic case. Nowadays, it can be defined on appropriate definable spaces with suitable o-minimal structure [11]. In the theory of o-minimal category, it is important to study more general functions: definable functions. We say that a real-valued function $\varphi\colon X \longrightarrow \mathbb{R}$ on X is definable if the graph of φ is subanalytic in $X \times \mathbb{R}$. The Euler integration was generalized to definable functions by Baryshnikov and Ghrist [1], [2].

For such functions, we can define the topological Radon transforms as follows. Let X and Y be compact real analytic manifolds, S a locally closed subanalytic subset of

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$X \times Y$. Let us consider the diagram:

$$
\begin{tikzcd}
 & X \times Y \arrow[ld, "p_X"'] \arrow[rd, "p_Y"] & \\
 & S \arrow[u, hook] \arrow[ld, "f"] \arrow[rd, "g"'] & \\
X & & Y.
\end{tikzcd} \tag{1.1}
$$

Here f and g are restrictions of natural projections p_X and p_Y to S respectively. By composing the inverse image by f and the direct image by g, for a constructible (resp. definable) function φ on X the topological Radon transform $\mathcal{R}_S(\varphi)$ of φ, which is a constructible (resp. definable) function on Y, is defined by the formula:

$$\mathcal{R}_S(\varphi)(y) = \int_{g^{-1}(y)} \varphi(f(x,y)) \ [dx]. \tag{1.2}$$

See Section 2 for the precise definitions of the integration and some operations.

In [5], [6], [7], [9] etc., the topological Radon transforms of constructible functions have been studied. In this report, we would like to give a first attempt of study of the topological Radon transforms of definable functions. Note that the Euler integrations of definable functions are neither linear nor functorial so that our study is more difficult than that of constructible functions. As a first result of our study, we will prove an inversion formula of Schapira type for topological Radon transforms of definable functions with some good condition. Namely, we will generalize Schapira's formula in [9] to the case of definable functions.

The analytic Radon transform is one of the most important integral transforms in mathematics, which is applied to the CT-scan, partial differential equations and so on. Recently, the topological Radon transform of constructible functions is applied to class formulas for dual varieties in algebraic geometry, sensor networks in applied mathematics and so on (see [1], [6], [7] etc.). We shall study applications of topological Radon transforms of definable functions in the future work.

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§ 2. Definable functions

Let X be a real analytic manifold. Although we consider real analytic manifolds and subanalytic subsets here, we can easily generalize our results for appropriate definable spaces with suitable o-minimal structure.

First let us recall the definition of constructible functions and their Euler integration. The theory of constructible functions is related to that of derived category of constructible sheaves. See [4] and [9] for more details.

Definition 2.1.

(i) We say that an integer-valued function $\varphi\colon X \longrightarrow \mathbb{Z}$ on X is constructible if there exists a locally finite family $\{X_i\}_{i\in I}$ of compact subanalytic subsets X_i of X such that φ is expressed by

$$\varphi = \sum_{i\in I} c_i \mathbf{1}_{X_i} \qquad (c_i \in \mathbb{Z}). \tag{2.1}$$

Here $\mathbf{1}_{X_i}$ denotes the characteristic function of X_i. We denote the abelian group of constructible functions on X by $CF(X)$.

(ii) Let $\varphi = \sum_{i\in I} c_i \mathbf{1}_{X_i} \in CF(X)$ be a constructible function on X and assume that the support $\mathrm{supp}(\varphi)$ of φ is compact. Then we define a topological Euler integral $\int_X \varphi(x)dx \in \mathbb{Z}$ of φ by

$$\int_X \varphi(x)dx = \sum_{i\in I} c_i \cdot \chi(X_i), \tag{2.2}$$

where $\chi(X_i)$ is the topological Euler characteristic of X_i.

Now let us introduce more general functions: definable functions. See [11] for more details.

Definition 2.2. We say that a real-valued function $\varphi\colon X \longrightarrow \mathbb{R}$ on X is definable if the graph $\{(x,t) \in X \times \mathbb{R} \mid t = \varphi(x)\}$ of φ is definable (i.e. subanalytic) in $X \times \mathbb{R}$. We denote the abelian group of definable functions on X by $DF(X)$.

More generally, for a real analytic manifold Y we say that a map $f\colon X \longrightarrow Y$ is definable if the graph $\{(x,y) \in X \times Y \mid y = f(x)\}$ is definable (i.e. subanalytic) in $X \times Y$.

Baryshnikov and Ghrist gave some definitions of Euler integrations of definable functions by step function approximations and the Euler integration of constructible functions in [2] as follows.

Definition 2.3. For a definable function $\varphi \in DF(X)$ on X with compact support, we define topological Euler integrations of φ by

$$\int_X \varphi(x)\lfloor dx\rfloor = \lim_{n\to\infty} \frac{1}{n} \int_X \lfloor n\varphi\rfloor dx, \tag{2.3}$$

$$\int_X \varphi(x)\lceil dx\rceil = \lim_{n\to\infty} \frac{1}{n} \int_X \lceil n\varphi\rceil dx, \tag{2.4}$$

$$\int_X \varphi(x)\,[dx] = \frac{1}{2}\left(\int_X \varphi(x)\lfloor dx\rfloor + \int_X \varphi(x)\lceil dx\rceil\right). \tag{2.5}$$

Here $\lfloor \cdot \rfloor$ is the floor function and $\lceil \cdot \rceil$ is the ceiling function. For a real number $t \in \mathbb{R}$, $\lfloor t \rfloor$ is the largest integer not greater than t and $\lceil t \rceil$ is the smallest integer not less than t.

By definition, it is clear that for a constructible function $\varphi \in CF(X)$ we have

$$\int_X \varphi(x)\lfloor dx \rfloor = \int_X \varphi(x)\lceil dx \rceil = \int_X \varphi(x)\,[dx] = \int_X \varphi(x)dx, \tag{2.6}$$

so the integrations (2.3), (2.4), (2.5) are generalizations of the Euler integration (2.2) of constructible functions. In [2], it is proved that they are well-defined by using the following properties.

Lemma 2.4 ([3], [11]). *Let $\varphi \in DF(X)$ be a definable function on X with compact support. Then there exist a disjoint union of open simplices $\bigsqcup_{i \in I} \sigma_i$ in some Euclidean space and a definable homeomorphism $\Xi\colon \bigsqcup_{i \in I} \sigma_i \longrightarrow X$ (a triangulation of X) such that $(\varphi \circ \Xi)|_{\sigma_i}$ is an affine linear function for each $i \in I$.*

Lemma 2.5 ([2]).

(i) *Let σ be an open simplex on some Euclidean space. For an affine linear function φ on σ, we have*

$$\int_\sigma \varphi(x)\lfloor dx \rfloor = (-1)^{\dim \sigma} \inf_{x \in \sigma} \varphi(x), \tag{2.7}$$

$$\int_\sigma \varphi(x)\lceil dx \rceil = (-1)^{\dim \sigma} \sup_{x \in \sigma} \varphi(x), \tag{2.8}$$

$$\int_\sigma \varphi(x)\,[dx] = \frac{(-1)^{\dim \sigma}}{2} \left(\inf_{x \in \sigma} \varphi(x) + \sup_{x \in \sigma} \varphi(x) \right). \tag{2.9}$$

(ii) *The integrations of definable functions on X in Definition 2.3 are invariant under changes of variables by definable homeomorphisms of X.*

Hereafter we focus on the third integration (2.5). In general, it is neither linear nor functorial but only homogeneous. Note that the first and second integrations (2.3), (2.4) are not homogeneous but only positively homogeneous.

Lemma 2.6 ([2]). *Let $\varphi \in DF(X)$ be a definable function on X with compact support. Then we have the following:*

(i) *For $r \in \mathbb{R}$, we have*

$$r \int_X \varphi(x)\,[dx] = \int_X r\varphi(x)\,[dx]. \tag{2.10}$$

(ii) *For constructible functions $\psi_1, \psi_2 \in CF(X)$, we have*

$$\int_X (\psi_1(x) + \psi_2(x))\varphi(x)\,[dx] = \int_X \psi_1(x)\varphi(x)\,[dx] + \int_X \psi_2(x)\varphi(x)\,[dx]. \tag{2.11}$$

For definable functions $\psi_1, \psi_2 \in DF(X)$, (2.11) *does not hold in general.*

Let us define some basic operations for definable functions. Let X and Y be real analytic manifolds and $f\colon X \longrightarrow Y$ a definable map from X to Y.

Definition 2.7.

(i) For $\psi \in DF(Y)$, we define the inverse image $f^*\psi \in DF(X)$ of ψ by f by

$$f^*\psi(x) = \psi(f(x)). \tag{2.12}$$

(ii) Let $\varphi \in DF(X)$ be a definable function with compact support. Then we define the direct image $\int_f \varphi \in DF(Y)$ of φ by f by

$$\left(\int_f \varphi\right)(y) = \int_{f^{-1}(y)} \varphi(x)\,[dx] = \int_X (\varphi(x) \cdot \mathbf{1}_{f^{-1}(y)}(x))\,[dx]. \tag{2.13}$$

In the case of constructible functions, these operations are functorial. In the case of definable functions, the inverse image is functorial, but the direct image is not functorial. We need some assumption to have a formula of Fubini type for the Euler integration of definable functions as follows.

Lemma 2.8. *Let $\varphi \in DF(X)$ be a definable function with compact support. Assume that φ is constant on each fiber of f. Then we have*

$$\int_Y \left(\int_{f^{-1}(y)} \varphi(x)\,[dx]\right)[dy] = \int_X \varphi(x)\,[dx]. \tag{2.14}$$

Remark. Without the assumption of Lemma 2.8, (2.14) does not hold in general. In fact, there exists a counter example via the non-linearity of the integration. If φ is constructible, (2.14) holds without the assumption. The proof is similar to that of Lemma 2.8 below.

Proof. By the trivialization theorem (See [11] for the detail), there exist a definable partition $\{Y_\alpha\}_{\alpha\in A}$ of Y and definable sets $\{U_\alpha\}_{\alpha\in A}$ such that each inverse image $f^{-1}(Y_\alpha)$ of Y_α by f is definably homeomorphic to $Y_\alpha \times U_\alpha$ and via this identification

$f|_{Y_\alpha \times U_\alpha}$ is the first projection. Since φ is constant on each fiber of f and the integration is homogeneous (2.10) (not only positively homogeneous), we have

$$\int_Y \left(\int_{f^{-1}(y)} \varphi(x)\,[dx] \right) [dy] = \sum_{\alpha \in A} \int_{Y_\alpha} \varphi(f^{-1}(y))\chi(U_\alpha)\,[dy] \tag{2.15}$$

$$= \sum_{\alpha \in A} \int_{Y_\alpha \times U_\alpha} \varphi(f^{-1}(y))\,[d(y,u)] \tag{2.16}$$

$$= \int_X \varphi(x)\,[dx]. \tag{2.17}$$

This completes the proof. □

Example 2.9. Let X be a real analytic genus-2 compact surface in $\mathbb{R}^3$ and $V = [0,1]$ a closed interval in $Y = \mathbb{R}$ and consider the second projection $f\colon X \times Y \longrightarrow Y$ from $X \times Y$. Let us consider a definable function $\varphi(x,y) = y\mathbf{1}_V(y)$ on $X \times Y$. Since φ is constant on each fiber of f, the Fubini type formula holds:

$$\int_Y \left(\int_{f^{-1}(y)} \varphi(x,y)\,[dx] \right) [dy] = \int_{X \times Y} \varphi(x,y)[d(x,y)]. \tag{2.18}$$

Actually, we can check the equality (2.18), which is equal to -1, by computing the both sides with (2.9).

On the other hand, by replacing the integration (2.5) with (2.3), the Fubini type formula does not hold in general:

$$\int_Y \left(\int_{f^{-1}(y)} \varphi(x,y)\lfloor dx \rfloor \right) \lfloor dy \rfloor = \int_V -2y\lfloor dy \rfloor = 0, \tag{2.19}$$

$$\int_{X \times Y} \varphi(x,y)\lfloor d(x,y) \rfloor = -2. \tag{2.20}$$

By these calculations, Theorem 7 in [2] (Fubini theorem with respect to the first integration (2.3)) does not seem to hold in general. This is because the first integration (2.3) is not homogeneous but only positively homogeneous.

§ 3. Topological Radon transforms of definable functions

§ 3.1. Setting

Let X and Y be real analytic manifolds and S a locally closed subanalytic subset of $X \times Y$. For simplicity, we assume that X and Y are compact. Consider the diagram:

(3.1)
$$\begin{tikzcd} & X \times Y \arrow[ddl, "p_X"'] \arrow[ddr, "p_Y"] & \\ & S \arrow[u, hook] \arrow[dl, "f"] \arrow[dr, "g"'] & \\ X & & Y, \end{tikzcd}$$

where p_X and p_Y are natural projections from $X \times Y$ and f and g are restrictions of p_X and p_Y to S respectively. In the situation above, we define the topological Radon transform of definable functions as follows. See [5] and [9] in the case of constructible functions.

Definition 3.1. For a definable function $\varphi \in DF(X)$ on X, we define the topological Radon transform $\mathcal{R}_S(\varphi) \in DF(Y)$ of φ by

$$\mathcal{R}_S(\varphi) = \int_g f^*\varphi = \int_{p_Y} \mathbf{1}_S \cdot p_X^*\varphi. \tag{3.2}$$

Similarly, for a definable function $\psi \in DF(Y)$ on Y, we define the transposed topological Radon transform ${}^t\mathcal{R}_S(\psi) \in DF(X)$ of ψ by

$${}^t\mathcal{R}_S(\psi) = \int_f g^*\psi = \int_{p_X} \mathbf{1}_S \cdot p_Y^*\psi. \tag{3.3}$$

Note that in the definition above we consider $\mathbf{1}_S$ as the kernel function of the topological Radon transform $\mathcal{R}_S$. The aim of this report is to prove an inversion formula for $\mathcal{R}_S$ under some assumptions. In general case, even in the case of constructible functions, we do not expect that the transposed transform ${}^t\mathcal{R}_S$ is a left inverse transform of $\mathcal{R}_S$.

§ 3.2. Main results

In order to prove an inversion formula for $\mathcal{R}_S$, first we compute the composition of $\mathcal{R}_S$ and ${}^t\mathcal{R}_S$. Since a formula of Fubini type for definable functions does not hold in general, some assumption is needed to compute it.

Definition 3.2. We say that a definable function $\varphi \in DF(X)$ on X has a good triangulation associated with S if there exist disjoint unions of open simplices $\bigsqcup_{i \in I} \sigma_i$ and

$\bigsqcup_{j\in J}\tau_j$ in some Euclidean space and definable homeomorphisms $\Xi_X\colon \bigsqcup_{i\in I}\sigma_i \longrightarrow X$ and $\Xi_Y\colon \bigsqcup_{j\in J}\tau_j \longrightarrow Y$ (definable triangulations of X and Y) and a definable homeomorphism $\Xi\colon \bigsqcup_{(i,j)\in I\times J}\sigma_i\times\tau_j \longrightarrow X\times Y$ such that the restriction $((\varphi\cdot\mathbf{1}_S)\circ\Xi)|_{\sigma_i\times\tau_j}$ of $(\varphi\cdot\mathbf{1}_S)\circ\Xi$ to $\sigma_i\times\tau_j$ is an affine linear function for any $(i,j)\in I\times J$.

When we fix definable triangulations above, for simplicity we identify X, Y and $X\times Y$ with $\bigsqcup_{i\in I}\sigma_i$, $\bigsqcup_{j\in J}\tau_j$ and $\bigsqcup_{(i,j)\in I\times J}\sigma_i\times\tau_j$ respectively and omit Ξ_X, Ξ_Y and Ξ.

Remark. By the triangulation theorem (See Lemma 2.4, [3], [11] for the detail), we have the following properties.

(i) For any definable function $\varphi\in DF(X)$, there exists a definable triangulation $\bigsqcup_{k\in K}\rho_k$ of $X\times Y$ such that the restriction $(\varphi\cdot\mathbf{1}_S)|_{\rho_k}$ is an affine linear function for any $k\in K$.

(ii) If S is a finite union of product sets $S_{X,i}\times S_{Y,i}$ of two subanalytic subsets $S_{X,i}\subset X$ and $S_{Y,i}$ for $i\in I$ (I is a finite set), any definable function $\varphi\in DF(X)$ on X always has a good triangulation associated with S.

(iii) In general, if a definable function $\varphi\in DF(X)$ satisfies that $(\mathrm{supp}(\varphi)\times Y)\cap S$ is a finite union of product sets $\Phi_{X,i}\times\Phi_{Y,i}$ of two subanalytic subsets $\Phi_{X,i}\subset X$ and $\Phi_{Y,i}$ for $i\in I$ (I is a finite set), φ has a good triangulation associated with S.

Lemma 3.3. *Any constructible function $\varphi\in CF(X)$ on X always has a good triangulation associated with S.*

Proof. It is enough to prove the lemma in the case $\varphi=\mathbf{1}_Z$ for a definable subset $Z\subset X$ of X. Set $\widetilde{g}=p_Y|_{(Z\times Y)\cap S}\colon (Z\times Y)\cap S\longrightarrow Y$. By the trivialization theorem (See [11] for the detail), there exist a finite definable partition $\{Y_\alpha\}_{\alpha\in A}$ of Y and definable subsets $\{U_\alpha\}_{\alpha\in A}$ such that each inverse image $\widetilde{g}^{-1}(Y_\alpha)$ of Y_α is definably homeomorphic to $U_\alpha\times Y_\alpha$ and via this identification $\widetilde{g}|_{U_\alpha\times Y_\alpha}$ is the second projection and $\varphi\cdot\mathbf{1}_S$ is constant on $U_\alpha\times Y_\alpha$. By the triangulation theorem and taking a common subdivision, we get definable triangulations $\bigsqcup_{i\in I}\sigma_i$ and $\bigsqcup_{j\in J}\tau_j$ of X and Y, which are compatible with $\{U_\alpha\}_{\alpha\in A}$ and $\{Y_\alpha\}_{\alpha\in A}$ respectively, such that the restriction $(\varphi\cdot\mathbf{1}_S)|_{\sigma_i\times\tau_j}$ of $\varphi\cdot\mathbf{1}_S$ to $\sigma_i\times\tau_j$ is constant for any $(i,j)\in I\times J$. This completes the proof. $\square$

The proof above is similar to that of Fubini type theorem of constructible functions. See also the proof of Lemma 2.8. In the case of definable functions, since $(\varphi\cdot\mathbf{1}_S)|_{\sigma_i\times\tau_j}$

is not constant, we need to assume the affine linearity to prove the Fubini type theorem. Actually, by the condition of the existence of a good triangulation above, we have the following key lemma.

Lemma 3.4. *For any definable function $\varphi \in DF(X)$ on X with a good triangulation associated with S and $x \in X$, we have*

$$\int_Y \left(\int_X \varphi(\xi) \mathbf{1}_S(\xi, y) \, [d\xi] \right) \mathbf{1}_S(x, y) \, [dy] = \int_{X \times Y} \varphi(\xi) \mathbf{1}_{S \underset{Y}{\times} S}(\xi, y, x) \, [d(\xi, y)]. \tag{3.4}$$

Proof. By the assumption, there exist definable triangulations $\bigsqcup_{i \in I} \sigma_i$ and $\bigsqcup_{j \in J} \tau_j$ of X and Y respectively such that for any $(i, j) \in I \times J$ the restriction $(\varphi \cdot \mathbf{1}_S)|_{\sigma_i \times \tau_j}$ of $\varphi \cdot \mathbf{1}_S$ to $\sigma_i \times \tau_j$ is affine linear. By taking a subdivision, we may assume that there exists a subset $J' \subset J$ such that $p_X^{-1}(x) \cap S$ is definably homeomorphic to $\bigsqcup_{j \in J'} \tau_j$. By Lemma 2.6, we have

$$\int_Y \left(\int_X \varphi(\xi) \mathbf{1}_S(\xi, y) \, [d\xi] \right) \mathbf{1}_S(x, y) \, [dy] = \sum_{j \in J'} \int_{\tau_j} \left(\sum_{i \in I} \int_{\sigma_i} \varphi(\xi) \mathbf{1}_S(\xi, y) \, [d\xi] \right) [dy] \tag{3.5}$$

$$= \sum_{i \in I} \sum_{j \in J'} \int_{\tau_j} \left(\int_{\sigma_i} \varphi(\xi) \mathbf{1}_S(\xi, y) \, [d\xi] \right) [dy]. \tag{3.6}$$

By the affine linearity of $(\varphi \cdot \mathbf{1}_S)|_{\sigma_i \times \tau_j}$ for $i \in I, j \in J'$ and (2.9), we have

$$\int_{\tau_j} \left(\int_{\sigma_i} \varphi(\xi) \mathbf{1}_S(\xi, y) \, [d\xi] \right) [dy] = \int_{\sigma_i \times \tau_j} \varphi(\xi) \mathbf{1}_{S \underset{Y}{\times} S}(\xi, y, x) \, [d(\xi, y)]. \tag{3.7}$$

This completes the proof. $\square$

For a definable function with a good triangulation associated with S, we obtain the following composition formula of Schapira type for two transforms $\mathcal{R}_S$ and ${}^t\mathcal{R}_S$. This is a generalization of Schapira's formula in the case of constructible functions in [9]. Note that Theorem 3.5 is applicable to many situations. See [1] for examples satisfying the condition (3.8).

Theorem 3.5. *Assume that there exist integers $\mu, \lambda \in \mathbb{Z}$ such that the following condition is satisfied:*

$$\chi(g(f^{-1}(x_1)) \cap g(f^{-1}(x_2))) = \begin{cases} \mu & (x_1 = x_2), \\ \lambda & (x_1 \neq x_2). \end{cases} \tag{3.8}$$

Then for any definable function $\varphi \in DF(X)$ on X with a good triangulation associated with S we have

$$ {}^t\mathcal{R}_S \circ \mathcal{R}_S(\varphi) = (\mu - \lambda)\varphi + \lambda \left(\int_X \varphi(\xi)\,[d\xi] \right) \mathbf{1}_X. \tag{3.9} $$

Proof. Let us take $x \in X$. By Lemma 3.4, we have

$$ {}^t\mathcal{R}_S \circ \mathcal{R}_S(\varphi)(x) = \int_{X\times Y} \varphi(\xi)\mathbf{1}_{S\underset{Y}{\times}S}(\xi, y, x)\,[d(\xi, y)]. \tag{3.10} $$

Since the definable function $\varphi(\xi)\mathbf{1}_{S\underset{Y}{\times}S}(\xi, y, x)$ is a real-valued constructible function along each fiber of p_X, by Lemmas 2.6 and 2.8 we have

$$ \int_{X\times Y} \varphi(\xi)\mathbf{1}_{S\underset{Y}{\times}S}(\xi, y, x)\,[d(\xi, y)] = \int_X \varphi(\xi) \left(\int_Y \mathbf{1}_{S\underset{Y}{\times}S}(\xi, y, x)\,[dy] \right) [d\xi]. \tag{3.11} $$

By the assumption (3.8) and Lemma 2.6, we have

$$ \int_X \varphi(\xi) \left(\int_Y \mathbf{1}_{S\underset{Y}{\times}S}(x, \xi, y)\,[dy] \right) [d\xi] = \int_X \varphi(\xi) \left(\mu\mathbf{1}_{\{x\}}(\xi) + \lambda\mathbf{1}_{X\setminus\{x\}}(\xi) \right) [d\xi] \tag{3.12} $$

$$ = (\mu - \lambda)\varphi(x) + \lambda \int_X \varphi(\xi)\,[d\xi]. \tag{3.13} $$

This completes the proof. □

By modifying the kernel function of the transposed transform, we prove an inversion formula for the topological Radon transform $\mathcal{R}_S$ of definable functions with a good triangulation associated with S. In order to prove it, we need the following lemma.

Lemma 3.6. *In the situation of Theorem 3.5, for any definable function $\varphi \in DF(X)$ on X with a good triangulation associated with S we have*

$$ \int_Y \mathcal{R}_S(\varphi)(y)\,[dy] = \mu \int_X \varphi(\xi)\,[d\xi]. \tag{3.14} $$

Proof. Similar to the proof of Lemma 3.4, we have

$$ \int_Y \mathcal{R}_S(\varphi)(y)\,[dy] = \int_Y \left(\int_X \varphi(\xi)\mathbf{1}_S(\xi, y)\,[d\xi] \right) [dy] \tag{3.15} $$

$$ = \int_{X\times Y} \varphi(\xi)\mathbf{1}_S(\xi, y)\,[d(\xi, y)]. \tag{3.16} $$

By Lemmas 2.6 and 2.8, we have

$$ \int_{X\times Y} \varphi(\xi)\mathbf{1}_S(\xi, y)\,[d(\xi, y)] = \int_X \left(\int_Y \varphi(\xi)\mathbf{1}_S(\xi, y)\,[dy] \right) [d\xi] = \mu \int_X \varphi(\xi)\,[d\xi]. \tag{3.17} $$

This completes the proof. □

By Theorem 3.5, Lemmas 2.6 and 3.6, we have the following inversion formula for the topological Radon transform $\mathcal{R}_S$ of definable functions with a good triangulation associated with S. See [5] in the case of constructible functions.

Corollary 3.7. *In the situation of Theorem 3.5, moreover we assume the condition $\mu(\mu-\lambda) \neq 0$. For a definable function $\psi \in DF(Y)$ on Y, we define a modified transposed transform $\widetilde{{}^t\mathcal{R}_S}(\psi)$ of ψ by*

$$\widetilde{{}^t\mathcal{R}_S}(\psi) = \frac{1}{\mu(\mu-\lambda)} \left\{ \mu\, {}^t\mathcal{R}_S(\psi) - \lambda \left(\int_Y \psi(y)\,[dy] \right) \mathbf{1}_X \right\}. \tag{3.18}$$

Then for any definable function $\varphi \in DF(X)$ on X with a good triangulation associated with S we have

$$\widetilde{{}^t\mathcal{R}_S} \circ \mathcal{R}_S(\varphi) = \varphi. \tag{3.19}$$

Remark. By Lemma 2.6, the kernel function of the left inverse transform $\widetilde{{}^t\mathcal{R}_S}$ of $\mathcal{R}_S$ is $\dfrac{1}{\mu(\mu-\lambda)}(\mu \mathbf{1}_S - \lambda \mathbf{1}_{X\times Y})$.

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