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Resurgent functions and linear differential operators of infinite order — Their happy marriage in exact WKB analysis

By

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Abstract

We study the singularity structure of Borel transformed WKB solutions of simple-pole type equations. In the description of their singularities, differential operators of infinite order naturally appear.

§ 0. Introduction

Linear (micro)differential operators of infinite order have played decisive roles in showing the structure theorem of general systems of microdifferential equations ([SKK]), clarifying the role of regular holonomic systems in the theory of general holonomic systems ([KK]), and so on. Since a differential operator of infinite order acts on the sheaf $\mathscr{O}$ of holomorphic functions as a sheaf homomorphism, it is natural to expect that it should fit in with the framework of resurgent function theory ([E], [P], [S1], [S2] and references cited therein). The purpose of this article is to validate such an expectation by showing

(i) in Section 1.1 that the employment of differential operators of infinite order enables us to resolve an open problem raised in [KKoT1], i.e., finding out the connection formula for WKB solutions of a Shrödinger equation whose potential belongs to the class $(\tilde{C})$ (in the notation of [KKoT1]),

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(ii) in Section 1.2 that a similar problem for the ghost equation ([Ko2], [KKKoT]) can be neatly analyzed with the aid of differential operators of infinite order,

and

(iii) in Section 2 we perform the alien calculus of the Voros coefficient and its exponentiation for the boosted Whittaker equation to be defined later by making use of linear differential operators of infinite order together with the results of [KoT] concerning the Voros coefficient for the Whittaker equation given by (2.1).

The most symbolic relation in this article is probably (2.31), that is, the alien derivative of the Voros coefficient $\tilde{V}$ of the boosted Whittaker equation at $y = 2m\pi i\alpha$ ($m = \pm 1, \pm 2, \cdots$) is obtained by applying a linear differential operator of infinite order

$$\frac{(-1)^m}{m} \cos\left(m\pi\sqrt{1 + 4\tilde{\beta}\partial/\partial y}\right) \tag{0.1}$$

to

$$\delta = \mathrm{sing}_{y=2m\pi i\alpha}\left(\frac{1}{2\pi i z}\right) \qquad \text{with } z = y - 2m\pi i\alpha \tag{0.2}$$

(in the notation of [S1]). We note that $\cos\left(m\pi\sqrt{1 + 4\tilde{\beta}\eta}\right)$ is an exponential type entire function of η with order 1/2; hence (0.1) defines a linear differential operator of infinite order, that is, an operator acting on the sheaf $\mathscr{O}$ of holomorphic functions as a sheaf homomorphism. An important observation here is that the alien derivative of $\tilde{V}$ at $y = 2m\pi i\alpha$ has an essential singularity there.

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§ 1. How and why are linear differential operators of infinite order useful in exact WKB analysis?

Since linear differential operators of infinite order preserve local properties of analytic functions, it is natural to expect that they should be useful in exact WKB analysis, where the main target is analytic functions on the Borel plane. Actually [AKT] shows that the micro-differential operator $\mathscr{X}$ that relates a Borel transformed Schrödinger equation and its canonical form has the following representation:

$$\mathscr{X}\psi_B = \int_{y_0}^{y} K(x, t, y - y', d/dx)\psi_B(x, t, y')dy', \tag{1.0.1}$$

where K is a differential operator of infinite order depending holomorphically on t and $(y - y')$. (See [AKT, Theorem 2.7] for the precise statement.) Similar integral representations are effectively used also in [KKoT2], [KKKoT], [KKT1] and [KKT2].

The role of differential operators of infinite order in this article is of somewhat different flavour from that of the role they played in the above mentioned papers. This time we use differential operators of infinite order to obtain new relations for WKB solutions of the "boosted" equation by "applying" such operators to the known relations for the original (non-boosted) equation. Here, a "boosted" equation means, by definition, the equation obtained by the replacement of a constant β by $\tilde{\beta}\eta$, where β is a constant in the coefficient of $\eta^{-2}x^{-2}$ in the potential of the original equation. For example, the main target equation of Section 1.1, i.e., the equation (1.0.3) below is the "boosted" equation of the equation (1.0.2) below:

$$\Big[\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{x} + \eta^{-2}\frac{\beta}{x^2}\Big)\Big]\psi = 0, \tag{1.0.2}$$

$$\Big[\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{x} + \eta^{-1}\frac{\tilde{\beta}}{x^2}\Big)\Big]\psi = 0. \tag{1.0.3}$$

We note that the potential

$$\tilde{Q} = \frac{1}{x} + \eta^{-1}\frac{\tilde{\beta}}{x^2} \tag{1.0.4}$$

is a typical potential in class $(\tilde{C})$ in the notation of [KKoT1]. In what follows we call (1.0.3) as the boosted simple-pole type equation.

Remark. Although it is not a main subject of this article, we note that (1.0.3) is the WKB-theoretic canonical form for the Schrödinger equation of the form

$$\Big[\frac{d^2}{dx^2} - \eta^2\Big(\frac{Q_0(x)}{x} + \frac{Q_1(x)}{x^2}\eta^{-1}\Big)\Big]\varphi = 0 \tag{1.0.5}$$

with

$$Q_0(0) \neq 0,\ Q_1(0) = \tilde{\beta}. \tag{1.0.6}$$

The proof is essentially the same as the proof of Proposition 1.1 in [Ko1]. Actually, by using the reasoning there, we can confirm that the equation

$$\Big[\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{x} + \frac{\tilde{\beta}}{x^2}\,\eta^{-1} + \frac{\beta}{x^2}\,\eta^{-2}\Big)\Big]\psi = 0 \tag{1.0.7}$$

gives the WKB-theoretic canonical form (in the sense of e.g. [Ko1]) for equations of the form

$$\Big[\frac{d^2}{dx^2} - \eta^2\Big(\frac{Q_0(x)}{x} + \frac{Q_1(x)}{x^2}\eta^{-1} + \frac{Q_2(x)}{x^2}\eta^{-2}\Big)\Big]\varphi = 0 \tag{1.0.8}$$

with

$$Q_0(0) \neq 0,\ Q_1(0) = \tilde{\beta},\ Q_2(0) = \beta. \tag{1.0.9}$$

§ 1.1. Connection formula for the Borel transformed WKB solutions of the boosted simple-pole type equation

As [Ko1] shows, the WKB solution $\psi_\pm$ of (1.0.2) has the form

$$\psi_\pm = \exp\Big(\pm\eta\int_0^x \frac{dx}{\sqrt{x}}\Big)\Big(\sum_{n=0}^{\infty}\psi_{\pm,n}(\beta)\sqrt{x}^{\,-n+1/2}\eta^{-n-1/2}\Big), \tag{1.1.1}$$

where

$$\psi_{\pm,0} = 1, \tag{1.1.2}$$

$$\psi_{+,n} = \psi_{+,n}(\beta) = \frac{(-1)^n}{n!}\prod_{j=0}^{n-1}\Big(\beta - \frac{1}{4}\big(j-\frac{1}{2}\big)\big(j+\frac{3}{2}\big)\Big) \quad (n \geq 1) \tag{1.1.3}$$

and

$$\psi_{-,n} = \psi_{-,n}(\beta) = (-1)^n\psi_{+,n}(\beta). \tag{1.1.4}$$

Hence their Borel transform $\psi_{\pm,B}(x,y,\beta)$ has the form

$$\psi_{\pm,B} = \sum_{n=0}^{\infty}\frac{\psi_{\pm,n}(\beta)}{\Gamma(n+1/2)}\Big(\frac{y}{\sqrt{x}} \pm 2\Big)^{n-1/2}. \tag{1.1.5}$$

Thus $\psi_{+,B}(x,y,\beta)$ has the form

$$f_+(s,\beta), \tag{1.1.6}$$

where

$$s = \frac{1}{4}\big(\frac{y}{\sqrt{x}} + 2\big). \tag{1.1.7}$$

Considering the Borel transform of (1.0.2), we obtain

$$\begin{aligned} 0 &= \Big[\frac{\partial^2}{\partial x^2} - \Big(\frac{1}{x}\frac{\partial^2}{\partial y^2} + \frac{\beta}{x^2}\Big)\Big]\psi_{+,B}(x,y,\beta) \\ &= -\frac{1}{4x^2}\Big[\Big(s(1-s)\frac{d^2}{ds^2} + \Big(\frac{3}{2} - 3s\Big)\frac{d}{ds} + 4\beta\Big)f_+(s,\beta)\Big]\Big|_{s=\frac{1}{4}\left(\frac{y}{\sqrt{x}}+2\right)}. \end{aligned} \tag{1.1.8}$$

Hence, for $x \neq 0$, we find

$$\psi_{+,B}(x,y,\beta) = \frac{1}{\sqrt{4\pi}} s^{-1/2} F\Big(a - \frac{1}{2}, a' - \frac{1}{2}, \frac{1}{2}; s\Big)\Big|_{s=\frac{1}{4}\left(\frac{y}{\sqrt{x}}+2\right)}, \tag{1.1.9}$$

for Gauss' hypergeometric function F with

$$a + a' = 2, \quad aa' = -4\beta. \tag{1.1.10}$$

Similarly we find

$$\begin{aligned} &\psi_{-,B}(x,y,\beta) \\ &= \frac{1}{\sqrt{-4\pi}}(1-s)^{-1/2} F\Big(\frac{3}{2} - a, \frac{3}{2} - a', \frac{1}{2}; 1-s\Big)\Big|_{1-s=\frac{1}{4}\left(-\frac{y}{\sqrt{x}}+2\right)}. \end{aligned} \tag{1.1.11}$$

On the other hand, Gauss' connection formula tells us (e.g. [Er, p.108])

$$\begin{aligned} &s^{-1/2} F\big(a - \frac{1}{2}, a' - \frac{1}{2}, \frac{1}{2}; s\big) \\ &= \frac{\Gamma\big(\frac{1}{2}\big)\Gamma\big(-\frac{1}{2}\big)}{\Gamma(1-a)\Gamma(1-a')} F\big(a, a', \frac{3}{2}; 1-s\big) \\ &\quad + \frac{\big(\Gamma\big(\frac{1}{2}\big)\big)^2}{\Gamma\big(a-\frac{1}{2}\big)\Gamma\big(a'-\frac{1}{2}\big)}(1-s)^{-\frac{1}{2}} F\big(\frac{3}{2} - a, \frac{3}{2} - a', \frac{1}{2}; 1-s\big). \end{aligned} \tag{1.1.12}$$

Then, by comparing the singular part in each side of (1.1.12) near $s = 1$, we find that the discontinuity function $\operatorname{disc} \psi_{+,B}(x,y,\beta)$ along the cut

$$\big\{(x,y) \in \mathbb{C}^2; \operatorname{Im} y = \operatorname{Im}(2\sqrt{x}), \operatorname{Re} y \geq \operatorname{Re}(2\sqrt{x})\big\} \tag{1.1.13}$$

coincides with

$$2i \cos\big(\pi\sqrt{1+4\beta}\big)\psi_{-,B}(x,y,\beta). \tag{1.1.14}$$

We have so far dealt with β just as an arbitrary constant; we now try to endow it with a particular meaning as the symbol of a differential operator $\partial/\partial y$. We first observe, in view of the integral representation of the hypergeometric function, that each term in (1.1.12) is an entire function of β that is of exponential type of order strictly smaller than 1, that is, infra-exponential entire function. (We also note hypergeometric functions involved here are non-singular with respect to β even at $1 + 4\beta = 0$, as

$a(a+1)\cdots(a+n-1)\ a'(a'+1)\cdots(a'+n-1)$ is non-singular there.) Hence on their domain of holomorphy both

$$\psi_{+,B}(x,y,\beta) \tag{1.1.15}$$

$$= \frac{1}{\sqrt{4\pi}}\, s^{-1/2} F\big(a-\frac{1}{2}, a'-\frac{1}{2}, \frac{1}{2}; s\big)\Big|_{s=\frac{1}{4}\left(\frac{y}{\sqrt{x}}+2\right)},$$

and

$$\psi_{-,B}(x,y,\beta) \tag{1.1.16}$$

$$= \frac{1}{\sqrt{-4\pi}}(1-s)^{-1/2} F\big(\frac{3}{2}-a, \frac{3}{2}-a', \frac{1}{2}; 1-s\big)\Big|_{1-s=\frac{1}{4}\left(-\frac{y}{\sqrt{x}}+2\right)}$$

may be regarded as symbols of linear differential operators ([SKK], [B], [A]) if we identify β with $b\sigma_1(\partial/\partial y)$ for some fixed non-zero constant b. Here, and in what follows, $\sigma_1(\partial/\partial y)$ denotes the symbol of differential operator $\partial/\partial y$. Therefore we can obtain operators from symbols $\psi_{\pm,B}(x,y,b\sigma_1(\partial/\partial y))$ by assigning the ordering of the operators $\partial/\partial y$ and $y\cdot$ (the multiplication operator by y). Here we choose the anti-Wick product $\overset{\circ}{\circ}\psi_{\pm,B}\overset{\circ}{\circ}$ as the assignment, that is, we define the operators so that every differential operator stands to the left of any multiplication operator. Then we find

$$\overset{\circ}{\circ}\cos\big(\pi\sqrt{1+4\beta}\big)\psi_{-,B}(x,y,\beta)\overset{\circ}{\circ} \tag{1.1.17}$$

$$= \overset{\circ}{\circ}\cos\big(\pi\sqrt{1+4\beta}\big)\overset{\circ}{\circ}\ \overset{\circ}{\circ}\psi_{-,B}(x,y,\beta)\overset{\circ}{\circ}.$$

Hence, by the comparison of singular parts of (1.1.12), we find that the singular part (near $y=2\sqrt{x}$) of $\overset{\circ}{\circ}\psi_{+,B}(x,y,\beta)\overset{\circ}{\circ}$ is given by

$$i\,\overset{\circ}{\circ}\cos\big(\pi\sqrt{1+4\beta}\big)\overset{\circ}{\circ}\ \overset{\circ}{\circ}\psi_{-,B}(x,y,\beta)\overset{\circ}{\circ}. \tag{1.1.18}$$

Then, the next step in our reasoning is to extract an analytic function of (x,y) from the obtained differential operator that contains both $y\cdot$ and $\partial/\partial y$. In order to make our procedure be compatible with the action of differential operators with constant coefficients in y such as the Borel transformed Schrödinger operators and $\overset{\circ}{\circ}\cos\big(\pi\sqrt{1+4\beta}\big)\overset{\circ}{\circ}$, we choose the way of extracting functions from differential operators by first re-ordering the operator by the Wick-ordering $\mathscr{W}$ and then applying the following quotient mapping $\mathfrak{F}$ given below to the operator:

$$\mathscr{D}^\infty \xrightarrow{\mathfrak{F}} \mathscr{D}^\infty/\big(\mathscr{D}^\infty(\partial/\partial x)+\mathscr{D}^\infty(\partial/\partial y)\big). \tag{1.1.19}$$

Here $\mathscr{D}^\infty$ stands for the sheaf of linear differential operators of infinite order (in (x,y)). Using these notations we now define

$$\varphi_\pm(x,y,b) = \mathfrak{F}\big(\mathscr{W} \circ\!\circ \psi_{\pm,B}(x,y,\beta) \circ\!\circ\big). \tag{1.1.20}$$

We first confirm

$$\Big(\frac{\partial^2}{\partial x^2} - \Big(\frac{1}{x}\frac{\partial^2}{\partial y^2} + \frac{b}{x^2}\frac{\partial}{\partial y}\Big)\Big)\varphi_\pm(x,y,b) = 0. \tag{1.1.21}$$

In fact, $\psi_\pm(x,y,\beta)$ is an infra-exponential type entire function in β in its domain of holomorphy, and hence we can expand it there as follows:

$$\psi_{\pm,B}(x,y,\beta) = \sum_{p\geq 0} f_{\pm,p}(x,y)\beta^p. \tag{1.1.22}$$

Since

$$\Big(\frac{\partial^2}{\partial x^2} - \Big(\frac{1}{x}\frac{\partial^2}{\partial y^2} + \frac{\beta}{x^2}\Big)\Big)\psi_{\pm,B}(x,y,\beta) = 0, \tag{1.1.23}$$

the substitution of (1.1.22) into (1.1.23) and the comparison of the coefficients of terms of like powers of β entail the following:

$$\Big(\frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial^2}{\partial y^2}\Big)f_{\pm,0} = 0, \tag{1.1.24}$$

$$\Big(\frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial^2}{\partial y^2}\Big)f_{\pm,p} = \frac{1}{x^2}f_{\pm,p-1} \qquad (p\geq 1). \tag{1.1.25}$$

On the other hand, (1.1.20) implies

$$\varphi_\pm(x,y,b) = \sum_{p\geq 0} b^p \frac{\partial^p}{\partial y^p} f_{\pm,p}(x,y). \tag{1.1.26}$$

Hence we find

$$\begin{aligned}
&\Big(\frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial^2}{\partial y^2}\Big)\varphi_\pm(x,y,b) \\
&= \Big(\frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial^2}{\partial y^2}\Big)f_{\pm,0} + \sum_{p\geq 1}\Big(\frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial^2}{\partial y^2}\Big)b^p\frac{\partial^p}{\partial y^p}f_{\pm,p}(x,y) \\
&= \sum_{p\geq 1} b^p \frac{\partial^p}{\partial y^p}\frac{1}{x^2}f_{\pm,p-1}(x,y)
\end{aligned} \tag{1.1.27}$$

$$= \frac{b}{x^2}\frac{\partial}{\partial y}\Big(\sum_{q\geq 0} b^q \frac{\partial^q}{\partial y^q} f_{\pm,q}(x,y)\Big)$$

$$= \frac{b}{x^2}\frac{\partial}{\partial y}\,\varphi_{\pm}(x,y,b).$$

This proves (1.1.21).

Furthermore, by using (1.1.18) and (1.1.20), we also find

$$\operatorname{disc}_{y=2\sqrt{x}}\varphi_+(x,y,b) = 2i\,\cos\big(\pi\sqrt{1+4b\partial/\partial y}\big)\varphi_-(x,y,b). \tag{1.1.28}$$

Let us finally confirm that $\varphi_+(x,y,b)$ (resp., $\varphi_-(x,y,b)$) is the Borel transform $\mathscr{B}(\tilde{\psi}_+(x,\eta,b))$ (resp., $\mathscr{B}(\tilde{\psi}_-(x,\eta,b))$ of WKB solutions $\tilde{\psi}_\pm$ of the boosted simple-pole type equation (1.0.3) with $\tilde{\beta}=b$. For this purpose let us define

$$\tilde{\chi}_\pm(x,\eta,b) = \Big[\sum_{n=0}^{\infty}\psi_{\pm,n}(\beta)\sqrt{x}^{\,-n+1/2}\eta^{-n-1/2}\Big]\Big|_{\beta=b\eta}, \tag{1.1.29}$$

using $\psi_{\pm,n}(\beta)$ given in (1.1.3) and (1.1.4). Then it is clear that

$$\tilde{\psi}_\pm(x,\eta,b) \underset{\text{def}}{=} \exp\Big(\pm\eta\int_0^x \frac{dx}{\sqrt{x}}\Big)\tilde{\chi}_\pm(x,\eta,b) \tag{1.1.30}$$

are WKB solutions of the boosted simple-pole type equation (1.0.3) with $\tilde{\beta}=b$. On the other hand, it follows from the definition of operations $\mathfrak{F}, \mathscr{W}$ and $\overset{\circ}{\circ}\ \overset{\circ}{\circ}$ that we find the following relation (1.1.31) for a holomorphic function $a(x)$ and any non-negative integers k and n:

$$\mathfrak{F}\Big(\mathscr{W}\overset{\circ}{\circ}\beta^k\mathscr{B}\big(\exp(a(x)\eta)\eta^{-n-1/2}\big)\overset{\circ}{\circ}\Big) \tag{1.1.31}$$

$$= \mathfrak{F}\Big(\mathscr{W}\overset{\circ}{\circ}\beta^k\big(y+a(x)\big)^{n-1/2}/\Gamma(n+1/2)\overset{\circ}{\circ}\Big)$$

$$= \mathfrak{F}\Big(\mathscr{W}\Big(b^k\frac{\partial^k}{\partial y^k}\Big(\big(y+a(x)\big)^{n-1/2}/\Gamma(n+1/2)\Big)\Big)\Big)$$

$$= b^k\mathfrak{F}\Big(\sum_{0\leq l\leq k}\binom{k}{l}\Big(\big(y+a(x)\big)^{n-l-1/2}/\Gamma(n-l+1/2)\Big)\frac{\partial^{k-l}}{\partial y^{k-l}}\Big)$$

$$= b^k\mathfrak{F}\Big(\big(y+a(x)\big)^{n-k-1/2}/\Gamma(n-k+1/2)$$

$$+\sum_{0\le l\le k-1}\binom{k}{l}\Big(\big(y+a(x)\big)^{n-l-1/2}/\Gamma(n-l+1/2)\Big)\frac{\partial^{k-l}}{\partial y^{k-l}}\Big)$$

$$=b^k\big(y+a(x)\big)^{n-k-1/2}/\Gamma(n-k+1/2)$$

$$=\mathscr{B}\Big(b^k\exp\big(a(x)\eta\big)\eta^k\eta^{-n-1/2}\Big).$$

Thus we have

$$\varphi_\pm(x,y,b)=\mathscr{B}(\tilde{\psi}_\pm(x,\eta,b)). \tag{1.1.32}$$

Summing up, we have thus obtained Borel transformed WKB solutions $\varphi_\pm(x,y,b) = \mathscr{B}(\tilde{\psi}_\pm(x,\eta,b))$ of the boosted simple-pole type equation which satisfy the Borel transformed connection formula (1.1.28). We note that the confirmation of the Borel summability of $\tilde{\psi}_\pm(x,\eta,b)$ entails the connection formula in (x,η)-space with the Stokes multiplier

$$2i\cos\big(\pi\sqrt{1+4b\eta}\,\big). \tag{1.1.33}$$

(See [Ko3] for the details.)

§ 1.2. Connection formula for the Borel transformed WKB solutions of the boosted ghost equation

A ghost equation has no turning points by its definition; still its WKB solutions present singularity structures similar to those near a turning point, which originate from the singularities contained in the coefficients of η^{-k} $(k\ge 1)$ in the potential. Such an equation was first studied in [Ko2] (with the tentative wording such as "new" turning points instead of ghost (points) etc.) and its importance in WKB analysis was fully recognized in [KKKoT]. Since we are interested in the boosted ghost equation we restrict our study to equations of the form

$$\Big[\frac{d^2}{dx^2}-\eta^2\Big(Q_0(x)+\eta^{-2}\,\frac{Q_2(x)}{x^2}\Big)\Big]\varphi=0 \tag{1.2.1}$$

with

$$Q_0(0)\ne 0. \tag{1.2.2}$$

Then the WKB-theoretic canonical form of such equations is, as [Ko2] shows,

$$\Big[\frac{d^2}{dx^2}-\eta^2\Big(\frac{1}{4}+\eta^{-2}\,\frac{\gamma}{x^2}\Big)\Big]\psi=0 \tag{1.2.3}$$

with

$$\gamma = Q_2(0). \tag{1.2.4}$$

In this article we employ a simplified naming, that is, we call (1.2.3) the ghost equation, and we call the equation (1.2.5) below as the boosted ghost equation:

$$\Big[\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{4} + \eta^{-1}\,\frac{\tilde{\gamma}}{x^2}\Big)\Big]\psi = 0. \tag{1.2.5}$$

Following [Ko2], we consider the odd part S_{odd} of the solution S of the Riccati equation associated with (1.2.3) and define the WKB solutions $\psi_\pm$ of (1.2.3) that have the form

$$\psi_\pm = \frac{1}{\sqrt{2}}(S_{\mathrm{odd}})^{-1/2}\exp(\pm\eta x/2)\exp\Big(\pm\int_\infty^x \big(S_{\mathrm{odd}} - \frac{1}{2}\eta\big)dx\Big), \tag{1.2.6}$$

which are normalized at infinity. Then we can readily confirm that their Borel transform $\psi_{\pm,B}(x,y,\gamma)$ has the form

$$x^{-1/2}h_\pm(t,\gamma)\big|_{t=y/x}, \tag{1.2.7}$$

where $h_\pm(t,\gamma)$ satisfies

$$\Big[\big(\frac{1}{4} - t^2\big)\frac{d^2}{dt^2} - 3t\frac{d}{dt} + \big(\gamma - \frac{3}{4}\big)\Big]h_\pm(t,\gamma) = 0, \tag{1.2.8}$$

or, for $s = t + 1/2$,

$$\Big[s(1-s)\frac{d^2}{ds^2} + \big(\frac{3}{2} - 3s\big)\frac{d}{ds} + \big(\gamma - \frac{3}{4}\big)\Big]h_\pm(s,\gamma) = 0. \tag{1.2.9}$$

Then it follows from (1.2.6) that

$$h_+(s,\gamma) = s^{-1/2}F\big(a - \frac{1}{2}, a' - \frac{1}{2}, \frac{1}{2}; s\big) \tag{1.2.10}$$

and

$$h_-(s,\gamma) = (1-s)^{-1/2}F\big(\frac{3}{2} - a, \frac{3}{2} - a', \frac{1}{2}; s-1\big), \tag{1.2.11}$$

where

$$a + a' = 2,\ aa' = \frac{3}{4} - \gamma. \tag{1.2.12}$$

Then Gauss' connection formula ([Er, p.108]) implies

$$\begin{aligned} &h_+(s,\gamma) \\ &= \frac{\Gamma\big(\frac{1}{2}\big)\Gamma\big(-\frac{1}{2}\big)}{\Gamma(1-a)\Gamma(1-a')}F\big(a, a', \frac{3}{2}; 1-s\big) + \frac{\Gamma\big(\frac{1}{2}\big)^2}{\Gamma\big(a - \frac{1}{2}\big)\Gamma\big(a' - \frac{1}{2}\big)}h_-(s,\gamma). \end{aligned} \tag{1.2.13}$$

Comparing the singular part near $s = 1$ in each side of (1.2.13) and then setting

$$s = \frac{y}{x} + \frac{1}{2}, \tag{1.2.14}$$

we find

$$\operatorname{disc}_{y=x/2} \psi_{+,B}(x, y, \gamma) = 2i \cos(\pi\sqrt{\gamma + 1/4})\psi_{-,B}(x, y, \gamma). \tag{1.2.15}$$

Then, by defining $\varphi_{\pm}(x, y, c)$ respectively by

$$\mathfrak{F}\big(\mathscr{W} \mathop{\circ}_{\circ} \psi_{\pm,B}(x, y, c\sigma_1(\partial/\partial y)) \mathop{\circ}_{\circ}\big) \tag{1.2.16}$$

we find

$$\Big(\frac{\partial^2}{\partial x^2} - \frac{1}{4}\frac{\partial^2}{\partial y^2} + \frac{c}{x^2}\frac{\partial}{\partial y}\Big)\varphi_{\pm}(x, y, c) = 0 \tag{1.2.17}$$

and

$$\operatorname{disc}_{y=x/2} \varphi_{+}(x, y, c) = 2i \cos(\pi\sqrt{c\partial/\partial y + 1/4})\varphi_{-}(x, y, c), \tag{1.2.18}$$

where $\operatorname{disc}_{y=x/2}$ means the discontinuity function along $\{(x, y) \in \mathbb{C}^2;\ \operatorname{Im} y = \operatorname{Im}(x/2),\ \operatorname{Re} y \geq \operatorname{Re}(x/2)\}$. Furthermore, the same reasoning as in Section 1.1 shows that $\varphi_{+}(x, y, c)$ and $\varphi_{-}(x, y, c)$ are respectively the Borel transform of a WKB solution of the boosted ghost equation (1.2.5) with $\tilde{\gamma} = c$, which contains an exponential factor $\exp(+\eta\, x/2)$ and $\exp(-\eta\, x/2)$. Thus we have found WKB solutions of the boosted ghost equation which satisfy the Borel transformed connection formula (1.2.18).

§ 2. Alien calculus of the Voros coefficient and its exponentiation for the boosted Whittaker equation

In Section 1 we have derived the discontinuity formula for WKB solutions of the boosted simple-pole type equation (resp., the boosted ghost equation) by regarding the discontinuity formula for WKB solutions of a simple-pole type equation (resp., a ghost equation) as relations among symbols of linear differential operators of infinite order through the identification of the constant β (resp., γ) with the symbol $\tilde{\beta}\sigma_1(\partial/\partial y)$ (resp., $\tilde{\gamma}\sigma_1(\partial/\partial y)$).

The discontinuity formulas studied in Section 1 are concerned with the so-called movable singular points of WKB solutions. Hence it is reasonable to try to seek for their counterparts at the so-called fixed singular points of WKB solutions. The core part of the analysis of WKB solutions at fixed singular points is, in general, the analysis of the Voros coefficient V and its exponentiation $\exp V$, as is explained, for example, in [KKKoT] for the Whittaker equation with a large parameter given by (2.1) below.

Hence, in this article we concentrate our attention on the study of the Voros coefficient $\tilde{V}$ and its exponentiation $\exp \tilde{V}$ for the boosted Whittaker equation with a large parameter given by (2.3) below. The analysis of WKB solutions of the boosted Whittaker equation with a large parameter will be given in [KKKo]. Fortunately, as [KoT] has concretely described the Voros coefficient for the Whittaker equation with a large parameter, we can use the same idea as is used in Section 1 to obtain the required results for the structure of $\tilde{V}$ and its exponentiation by making use of the results in [KoT].

Let us first fix some notations to be used below; the Whittaker equation with a large parameter is, by definition,

$$\Big[\frac{d^2}{dx^2} - \eta^2 Q(x,\eta;\alpha,\beta)\Big]\psi(x,\eta;\alpha,\beta) = 0, \tag{2.1}$$

where

$$Q(x,\eta;\alpha,\beta) = \frac{1}{4} - \frac{\alpha}{x} + \eta^{-2}\frac{\beta}{x^2} \tag{2.2}$$

with a non-zero constant α and a constant β, and the boosted Whittaker equation with a large parameter is, by definition,

$$\Big[\frac{d^2}{dx^2} - \eta^2 \tilde{Q}(x,\eta;\alpha,\tilde{\beta})\Big]\psi(x,\eta;\alpha,\tilde{\beta}) = 0, \tag{2.3}$$

where

$$\tilde{Q}(x,\eta;\alpha,\tilde{\beta}) = \frac{1}{4} - \frac{\alpha}{x} + \eta^{-1}\frac{\tilde{\beta}}{x^2} \tag{2.4}$$

with a non-zero constant α and a constant $\tilde{\beta}$. In what follows we normally omit "with a large parameter" for the sake of simplicity. It follows from their definition that both the Whittaker equation and the boosted Whittaker equation have a simple turning point

$$x = 4\alpha \tag{2.5}$$

together with a simple pole at the origin in the top part of their potential, i.e., the degree 0 in η part of the potential. The pair of the simple turning point and the simple pole generates fixed singular points of WKB solutions, as is described in [KoT].

Next we briefly recall the definition of the Voros coefficient and its explicit form for the Whittaker equation. (See [KoT] for the details.) The Voros coefficient $V \underset{\text{def}}{=} V(\eta;\alpha,\beta)$ for the Whittaker equation is, by definition,

$$V = \int_{4\alpha}^{\infty} \big(S_{\text{odd}}(x,\eta) - \eta S_{-1}(x)\big)dx, \tag{2.6}$$

where S_{odd} designates the odd part of solution S of the Riccati equation associated with (2.1), that is,

$$S^2 + \frac{\partial S}{\partial x} = \eta^2 Q(x,\eta;\alpha,\beta). \tag{2.7}$$

Here we note that S_{odd} is, in general, defined by

$$\frac{1}{2}\big(S^{(+)} - S^{(-)}\big) \tag{2.8}$$

for solutions $S^{(\pm)}$ of the Riccati equation whose top order parts $S_{-1}^{(\pm)}$ satisfy

$$S_{-1}^{(\pm)}(x) = \pm\sqrt{Q_0(x)}, \tag{2.9}$$

where $Q_0(x)$ denotes the top part of the potential $Q(x,\eta)$. In the case of the Whittaker equation S_{odd} has the form

$$\sum_{l\geq 0} \eta^{-2l+1} S_{2l-1} \tag{2.10}$$

with $S_{-1} = \sqrt{Q_0}$.

Now, an important result of [KoT] is:

$$V = \sum_{n\geq 1} (\alpha\eta)^{1-2n} \frac{B_{2n}(-\gamma)}{(2n)(2n-1)}\,, \tag{2.11}$$

where $B_{2n}(z)$ denotes the Bernoulli polynomial of degree $2n$ and γ satisfies $\gamma(\gamma+1) = \beta$. Since $B_{2n}(z)$ satisfies ([Er, p.37])

$$B_{2n}(1-z) = B_{2n}(z), \tag{2.12}$$

we can readily confirm that $B_{2n}(-\gamma)$ is actually a polynomial of β of degree n, regardless of the choice of γ; for example,

$$B_2(-\gamma) = \beta + \frac{1}{6},\ \ B_4(-\gamma) = \beta^2 - \frac{1}{30}\,, \tag{2.13}$$

$$B_6(-\gamma) = \beta^3 - \frac{1}{2}\,\beta^2 + \frac{1}{4}\,, \cdots .$$

The Borel transform $V_B \underset{\mathrm{def}}{=} V_B(y;\alpha,\beta)$ of the Voros coefficient V is also given explicitly in Theorem 3.1 of [KoT]:

$$\begin{aligned} V_B &= \frac{e^{-\gamma y/\alpha} + e^{(\gamma+1)y/\alpha}}{2y(e^{y/\alpha}-1)} - \frac{\alpha}{y^2} \\ &= \frac{\cosh\big(\frac{1}{2\alpha}\sqrt{1+4\beta}\,y\big)}{2y\sinh(y/(2\alpha))} - \frac{\alpha}{y^2}\,. \end{aligned} \tag{2.14}$$

We note that [KoT] first proves (2.14) and then confirms (2.11) by using it. We also note that V_B has no singularity at $y=0$ despite the existence of the term α/y^2.

It is evident from (2.14) (after the above observation of the regularity of V_B at the origin) that V_B is singular only at $y = 2m\pi i\alpha\,(m = \pm 1, \pm 2, \cdots)$ and that it has a simple-pole there with its residue

$$\operatorname*{Res}_{y=2m\pi i\alpha} V_B = \frac{\cos(2m\pi\gamma)}{2m\pi i} = \frac{(-1)^m \cos(m\pi\sqrt{1+4\beta}\,)}{2m\pi i}\,. \tag{2.15}$$

In what follows we let y_m denote $2m\pi i\alpha$; without mentioning so, we always assume $m \neq 0$ for y_m.

In parallel with (2.6) we define the Voros coefficient $\tilde{V} = \tilde{V}(\eta; \alpha, \tilde{\beta})$ of the boosted Whittaker equation by

$$\tilde{V} = \int_{4\alpha}^{\infty} (\tilde{S}_{\mathrm{odd}} - \eta \tilde{S}_{-1}) dx, \tag{2.16}$$

where $\tilde{S}_{\mathrm{odd}}$ designates the odd part of a solution $\tilde{S}$ of the Riccati equation associated with the boosted Whittaker equation, i.e.,

$$\tilde{S}^2 + \frac{\partial \tilde{S}}{\partial x} = \eta^2 \tilde{Q}. \tag{2.17}$$

We note that $\tilde{S}_{\mathrm{odd}}$ contains, unlike S_{odd}, terms of even degree in η^{-1}; still we find

$$\tilde{S}_{\mathrm{odd}}(x, \eta; \alpha, \tilde{\beta}) = S_{\mathrm{odd}}(x, \eta; \alpha, \beta)\big|_{\beta=\tilde{\beta}\eta}\,. \tag{2.18}$$

Thus we obtain

$$\tilde{V}(\eta; \alpha, \tilde{\beta}) = V(\eta; \alpha, \beta)\big|_{\beta=\tilde{\beta}\eta}\,. \tag{2.19}$$

One important fact in defining $\tilde{V}(\eta)$ as in (2.16) is that $\tilde{S}_{\mathrm{odd},j}$ $(j \geq 0)$ is integrable near $x = \infty$; for example, we have

$$\tilde{S}_{\mathrm{odd},0} = \frac{2\tilde{\beta}}{x^{3/2}(x-4\alpha)^{1/2}}\,. \tag{2.20}$$

In order to study the structure of the Borel transform of $\tilde{V}(\eta; \alpha, \tilde{\beta})$ by making use of the analytic structure of the Borel transform of $V(\eta; \alpha, \beta)$, we introduce the auxiliary series $W(\eta; \alpha, \beta)$ and $\tilde{W}(\eta; \alpha, \tilde{\beta})$ by the following:

$$W(\eta; \alpha, \beta) = V(\eta; \alpha, \beta) - \frac{\beta}{2\alpha}\,\eta^{-1}, \tag{2.21}$$

$$\tilde{W}(\eta; \alpha, \tilde{\beta}) = \tilde{V}(\eta; \alpha, \tilde{\beta}) - \frac{\tilde{\beta}}{2\alpha}\,. \tag{2.22}$$

It is then clear from (2.19) that

$$\tilde{W}(\eta;\alpha,\tilde{\beta}) = W(\eta;\alpha,\beta)\big|_{\beta=\tilde{\beta}\eta} \tag{2.23}$$

holds. The purpose of this introduction of auxiliary series W and $\tilde{W}$ is to set aside the constant term $-\tilde{\beta}/(2\alpha)$ in $\tilde{V}$ so that a relation similar to (1.1.31) may be readily used in relating the Borel transforms of $W(\eta;\alpha,\beta)$ and $\tilde{W}(\eta;\alpha,\tilde{\beta})$, i.e., $W_B(y;\alpha,\beta)$ and $\tilde{W}_B(y;\alpha,\tilde{\beta})$.

In developing the alien calculus of $\tilde{V}$ making use of the concrete form of V given by [KoT], we use the same notions and notations in [KKKo]; they are basically the same as those given in [S1] and [S2], but for the sake of clarity of notations we use the symbol $\mathscr{B}^\dagger$ to denote the extension of the formal (= ordinary in our wording) Borel transformation whose range is in the quotient space $\mathrm{SING} \underset{\mathrm{def}}{=} \mathrm{ANA}/\mathbb{C}\{y\}$, which is introduced by [S1, Section 3.2]; that is, we denote the Borel transform of $\varphi(\eta)$ by $\mathscr{B}^\dagger(\varphi(\eta))$.

Now, it follows from the explicit form of V_B that it is a single-valued analytic function, and hence (2.14) and (2.15) may be summarized with the help of the alien derivative as follows:

$$\Delta_{y_m}(\mathscr{B}^\dagger(V)) = \frac{(-1)^m \cos\left(m\pi\sqrt{1+4\beta}\,\right)}{m}\,\delta. \tag{2.24}$$

Here we have used the notation $\mathscr{B}^\dagger(V)$ instead of V_B to emphasize that (2.24) is a relation in SING. On the other hand we know

$$\Delta_{y_m}(\mathscr{B}^\dagger(W)) = \Delta_{y_m}(\mathscr{B}^\dagger(V)), \tag{2.25}$$

as the transform of $\beta/(2\alpha\eta)$ is a genuine constant $\beta/(2\alpha)$. Hence (2.24) implies that the Borel transform W_B of W has the following form (2.26) near $y = y_m$:

$$\frac{(-1)^m \cos\left(m\pi\sqrt{1+4\beta}\,\right)}{m} \cdot \frac{1}{2\pi i(y-y_m)} + \Phi_m(y;\alpha,\beta), \tag{2.26}$$

where Φ_m is a single-valued analytic function in y which is holomorphic near $y = y_m$ and it depends holomorphically on β in $\mathbb{C}$; it is an infra-exponential type entire function in β near $y = y_m$. To make use of the concrete description (2.26) of W_B near $y = y_m$ for the study of the structure of the Borel transform $\tilde{W}_B$ of $\tilde{W}$ near $y = y_m$, we note that $\tilde{W}(\eta;\alpha,\tilde{\beta})$ is in $\eta^{-1}\mathbb{C}[[\eta^{-1}]]$. This property of $\tilde{W}$ enables us to use a relation similar to (1.1.31) with $\eta^{-n-1/2}$ being replaced by η^{-n-1} $(n \geq 0)$ (and $a(x)$ being set to be 0) so that we may find

$$\mathfrak{F}\Big(\mathscr{W} \mathbin{\overset{\circ}{\circ}} W_B\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big) \mathbin{\overset{\circ}{\circ}}\Big) = \tilde{W}_B(y;\alpha,\tilde{\beta}). \tag{2.27}$$

Hence, by combining (2.26) and (2.27) we find the following:

(2.28)

$$\begin{aligned}\tilde{W}_B(y;\alpha,\tilde{\beta}) &= \mathfrak{F}\Big(\mathscr{W} \overset{\circ}{\circ} W_B\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big) \overset{\circ}{\circ}\Big)\\ &= \mathfrak{F}\Big(\mathscr{W} \overset{\circ}{\circ} \Big(\frac{(-1)^m}{2m\pi i}\cos\big(m\pi\sqrt{1+4\tilde{\beta}\sigma_1(\partial/\partial y)}\,\big)\frac{1}{y-y_m} + \Phi_m\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big)\Big) \overset{\circ}{\circ}\Big)\\ &= \frac{(-1)^m}{2m\pi i}\cos\big(m\pi\sqrt{1+4\tilde{\beta}\partial/\partial y}\,\big)\frac{1}{y-y_m} + \varphi_m(y;\alpha,\tilde{\beta}),\end{aligned}$$

where $\varphi_m(y;\alpha,\tilde{\beta})$ is a single-valued analytic function in y that is holomorphic near $y=y_m$. Then (2.28) entails the following relation in SING:

$$\Delta_{y_m}\big(\mathscr{B}^\dagger(\tilde{W})(y;\alpha,\tilde{\beta})\big) = \frac{(-1)^m}{m}\cos\big(m\pi\sqrt{1+4\tilde{\beta}\partial/\partial y}\,\big)\delta. \tag{2.29}$$

It also follows from (2.22) that we have

$$\begin{aligned}\Delta_{y_m}\big(\mathscr{B}^\dagger(\tilde{V})\big) &= \Delta_{y_m}\big(\mathscr{B}^\dagger(\tilde{W}) + \frac{\beta}{2\alpha}\delta\big)\\ &= \Delta_{y_m}\big(\mathscr{B}^\dagger(\tilde{W})\big).\end{aligned} \tag{2.30}$$

Thus we obtain

$$\Delta_{y_m}\big(\mathscr{B}^\dagger(\tilde{V})\big) = \frac{(-1)^m}{m}\cos\big(m\pi\sqrt{1+4\tilde{\beta}\partial/\partial y}\,\big)\delta. \tag{2.31}$$

Once we obtain the concrete expression of $\Delta_{y_m}\big(\mathscr{B}^\dagger(\tilde{W})\big)$ etc., the alien calculus provides us also with the concrete expression of $\Delta_{y_m}\big(\mathscr{B}^\dagger(\exp\tilde{W}-1)\big)$ etc. In order to confirm that the alien derivation and our way of extracting an analytic function from the symbol of a differential operator are consistent procedures in general (i.e., even if unlike (2.28) etc., the Borel transform involved is not single-valued), we briefly recall the concrete expression of the alien derivative of a resurgent function in terms of the analytic continuation of its minor. (See [CNP] and [S1] for the details.) An important point is that this expression of the alien derivative gives us a kind of discontinuity formula for a multi-valued analytic function, similar to the relations we encountered in Section 1.

For the sake of simplicity we restrict our consideration here to the situation where the singularities of the minor $\hat{\varphi}$ of the resurgent function φ to be considered are confined to the set

$$S = \bigcup_{l\in\mathbb{Z},l\neq 0}\{y=y_l\}. \tag{2.32}$$

For a positive integer m, we consider a path from the origin to a small neighborhood U of y_m which runs along the purely imaginary axis detouring the points $y = y_l$ $(1 \le l \le m-1)$ by encircling y_l either clockwise or anticlockwise. (In order to fix the situation we assume $m > 0$. The case $m < 0$ is dealt with in the same manner.) Then we assign $\varepsilon_l = +1$ (resp., $\varepsilon_l = -1$) when the path encircles $y = y_l$ clockwise (resp., anticlockwise), and we designate the path by $\gamma(\varepsilon) = \gamma(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{m-1})$. We also denote by $p(\varepsilon)$ (resp., $q(\varepsilon)$) the number of l's with $\varepsilon_l = +1$ (resp., $\varepsilon_l = -1$). Then, denoting $\gamma(\varepsilon)_*(\hat{\varphi})$ the analytic continuation of $\hat{\varphi}$ to U along $\gamma(\varepsilon)$, we can describe the alien derivative of φ concretely as follows:

$$\Delta_{y_m}\varphi = \sum_{\varepsilon_1,\cdots,\varepsilon_{m-1}\in\{+1,-1\}} \frac{p(\varepsilon)!q(\varepsilon)!}{m!} \operatorname{sing}_{z=0}\big((\gamma(\varepsilon)_*(\hat{\varphi}))(y_m+z)\big). \tag{2.33}$$

In order to apply this description to the study of the exponentiated Voros coefficients, we introduce the following auxiliary series E and $\tilde{E}$:

$$E(\eta;\alpha,\beta) = \exp\big(W(\eta;\alpha,\beta)\big) - 1, \tag{2.34}$$

$$\tilde{E}(\eta;\alpha,\tilde{\beta}) = \exp\big(\tilde{W}(\eta;\alpha,\tilde{\beta})\big) - 1. \tag{2.35}$$

It then follows from (2.22) that

$$\tilde{E}(\eta;\alpha,\tilde{\beta}) = E(\eta;\alpha,\beta)\big|_{\beta=\tilde{\beta}\eta} \tag{2.36}$$

holds. Considering the alien derivative acting on SING, we find

$$\begin{aligned}\Delta_{y_m}(\mathscr{B}^\dagger(E)) &= \Delta_{y_m}(\mathscr{B}^\dagger(W)) * \mathscr{B}^\dagger(\exp W) \\ &= \Delta_{y_m}(\mathscr{B}^\dagger(W)) * (\delta + \mathscr{B}^\dagger(E)).\end{aligned} \tag{2.37}$$

Combining (2.24) and (2.25) we also obtain the following expression of $\Delta_{y_m}(\mathscr{B}^\dagger(W))$ in SING:

$$\Delta_{y_m}(\mathscr{B}^\dagger(W)) = \frac{(-1)^m \cos(m\pi\sqrt{1+4\beta}\,)}{m}\,\delta. \tag{2.38}$$

Hence (2.37) implies

$$\Delta_{y_m}(\mathscr{B}^\dagger(E)) = \frac{(-1)^m \cos(m\pi\sqrt{1+4\beta}\,)}{m}\,(\delta + \mathscr{B}^\dagger(E)). \tag{2.39}$$

Since E is in $\eta^{-1}\mathbb{C}[[\eta^{-1}]]$, $\mathscr{B}^\dagger(E)$ can be identified with the ordinary Borel transform $E_B(y)$ of E. Hence, thanks to the concrete description (2.33) of the alien derivative,

(2.39) entails

(2.40)
$$\sum_{\varepsilon} \frac{p(\varepsilon)!q(\varepsilon)!}{m!} \big(\gamma(\varepsilon)_*(E_B(y;\alpha,\beta))\big)(y)$$
$$= \frac{(-1)^m \cos(m\pi\sqrt{1+4\beta}\,)}{2m\pi i}\Big(\frac{1}{y-y_m} + E_B(y-y_m;\alpha,\beta)\log(y-y_m)\Big) + \Psi_m(y;\alpha,\beta),$$

where $\Psi_m(y;\alpha,\beta)$ is a holomorphic function near $y=y_m$. Furthermore the reasoning of [S2] together with the explicit form of V_B shows ([KKKo]) that E_B is an infra-exponential type entire function in β outside S. Hence (2.40) may be understood as a relation among symbols of linear differential operators of infinite order. Thus we obtain the following relation:

$$\sum_{\varepsilon} \frac{p(\varepsilon)!q(\varepsilon)!}{m!}\mathfrak{F}\Big(\mathscr{W} \mathbin{\overset{\circ}{\circ}} \Big(\gamma(\varepsilon)_*\big(E_B\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big)\big)\Big)(y) \mathbin{\overset{\circ}{\circ}}\Big) \tag{2.41}$$
$$= \frac{(-1)^m \mathbin{\overset{\circ}{\circ}} \cos\big(m\pi\sqrt{1+4\tilde{\beta}\sigma_1(\partial/\partial y)}\,\big) \mathbin{\overset{\circ}{\circ}}}{4m\pi i}\mathfrak{F}\Big(\mathscr{W} \mathbin{\overset{\circ}{\circ}} \Big(\frac{1}{y-y_m}$$
$$+E_B\big(y-y_m;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big)\log(y-y_m)\Big) \mathbin{\overset{\circ}{\circ}}\Big)$$
$$+\mathfrak{F}\Big(\mathscr{W} \mathbin{\overset{\circ}{\circ}} \Psi_m\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big) \mathbin{\overset{\circ}{\circ}}\Big).$$

Here we note that

$$\mathfrak{F}\Big(\mathscr{W} \mathbin{\overset{\circ}{\circ}} \Big(\gamma(\varepsilon)_*\big(E_B\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big)\big)\Big) \mathbin{\overset{\circ}{\circ}}\Big) \tag{2.42}$$
$$= \mathfrak{F}\Big(\mathscr{W}\gamma(\varepsilon)_*\Big(\mathbin{\overset{\circ}{\circ}} E_B\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big) \mathbin{\overset{\circ}{\circ}}\Big)\Big)$$
$$= \mathfrak{F}\Big(\gamma(\varepsilon)_*\Big(\mathscr{W} \mathbin{\overset{\circ}{\circ}} E_B\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big) \mathbin{\overset{\circ}{\circ}}\Big)\Big)$$
$$= \gamma(\varepsilon)_*\Big(\mathfrak{F}\Big(\mathscr{W} \mathbin{\overset{\circ}{\circ}} E_B\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big) \mathbin{\overset{\circ}{\circ}}\Big)\Big)$$

holds by the definition of $\gamma(\varepsilon)_*(E_B)$. Hence the left-hand side of (2.41) is equal to

$$\sum_{\varepsilon} \frac{p(\varepsilon)!q(\varepsilon)!}{m!}\gamma(\varepsilon)_*\Big(\mathfrak{F}\Big(\mathscr{W} \mathbin{\overset{\circ}{\circ}} E_B\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big) \mathbin{\overset{\circ}{\circ}}\Big)\Big). \tag{2.43}$$

On the other hand, since both E and $\tilde{E}$ are in $\eta^{-1}\mathbb{C}[[\eta^{-1}]]$, (2.36) entails

$$\mathfrak{F}\Big(\mathscr{W} \mathbin{\overset{\circ}{\circ}} E_B\big(y;\alpha,\tilde{\beta}\sigma_1(\partial/\partial y)\big) \mathbin{\overset{\circ}{\circ}}\Big) = \tilde{E}_B(y;\alpha,\tilde{\beta}). \tag{2.44}$$

Hence, by combining (2.41), (2.43) and (2.44), we obtain

$$\sum_{\varepsilon} \frac{p(\varepsilon)!q(\varepsilon)!}{m!} \big(\gamma(\varepsilon)_* \tilde{E}_B\big)(y_m + z) \tag{2.45}$$

$$= \frac{(-1)^m \cos\big(m\pi\sqrt{1+4\tilde{\beta}\partial/\partial z}\,\big)}{2m\pi i}\big(\frac{1}{z} + \tilde{E}_B(z;\alpha,\tilde{\beta})\log z\big) + a(z;\alpha,\tilde{\beta})$$

for some holomorphic function $a(z;\alpha,\tilde{\beta})$ defined near $z = 0$. Then (2.45) entails the following relation in SING:

$$\Delta_{y_m}\big(\mathscr{B}^\dagger(\tilde{E})\big)(z) = \frac{(-1)^m \cos\big(m\pi\sqrt{1+4\tilde{\beta}\partial/\partial z}\,\big)}{m}\big(\delta + \tilde{E}_B(z;\alpha,\tilde{\beta})\theta\big). \tag{2.46}$$

Finally, to relate the left-hand side of (2.46) with the alien derivative of $\exp\tilde{V}$ at $y = y_m$, we note that (2.22) implies the following relation (2.47) in SING:

$$\mathscr{B}^\dagger(\exp\tilde{V}) = \mathscr{B}^\dagger\big[\exp\big(\frac{\tilde{\beta}}{2\alpha}\big)(\exp\tilde{W} - 1) + \exp\big(\frac{\tilde{\beta}}{2\alpha}\big)\big], \tag{2.47}$$

that is,

$$\mathscr{B}^\dagger(\exp\tilde{V}) = \exp\big(\frac{\tilde{\beta}}{2\alpha}\big)\mathscr{B}^\dagger(\tilde{E}) + \exp\big(\frac{\tilde{\beta}}{2\alpha}\big)\delta. \tag{2.48}$$

Hence we find

$$\Delta_{y_m}\big(\mathscr{B}^\dagger(\exp\tilde{V})\big) = \exp\big(\frac{\tilde{\beta}}{2\alpha}\big)\Delta_{y_m}\big(\mathscr{B}^\dagger(\tilde{E})\big). \tag{2.49}$$

Thus the alien derivative of $\exp\tilde{V}$ at $y = y_m$ is given by

$$\frac{(-1)^m \exp\big(\frac{\tilde{\beta}}{2\alpha}\big)\cos\big(m\pi\sqrt{1+4\tilde{\beta}\partial/\partial z}\,\big)}{m}\big(\delta + \tilde{E}_B(z;\alpha,\tilde{\beta})\theta\big). \tag{2.50}$$

In conclusion, we find that the alien derivative at $y = y_m$ of $\exp\tilde{V}$, the exponentiated Voros coefficient of the boosted Whittaker equation, is represented by an analytic function with essential singularities.

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