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Exact WKB analysis of confluent hypergeometric differential equations with a large parameter

By

Takashi AOKI*, Toshinori TAKAHASHI** and Mika TANDA**

Abstract

Voros coefficients of the confluent hypergeometric differential equations (the Kummer equation or the Whittaker equation) with a large parameter are defined and explicit forms of them are obtained.

Introduction

The aim of this article is to define and to compute explicit forms of the Voros coefficients of the Whittaker equation, or equivalently, the Kummer equation with a large parameter. In [5], the first and the third authors have investigated the Stokes phenomena for WKB solutions with respect to parameters of the Gauss equation with a large parameter. They have defined the Voros coefficients for the Gauss equation and given the explicit forms of them as well as their Borel sums. We shall show that the same definition works and a similar computation can be carried out for confluent hypergeometric equations. In this article, we restrict our discussion to the Voros coefficients and we do not mention the parametric Stokes phenomena for WKB solutions of the Whittaker equation. Those will be discussed in our forthcoming paper with the investigation of the case where the way of putting the large parameter to be more general.

The notion of the Voros coefficients was introduced by Voros [15] for the Weber equation and for the quartic oscillator. It was effectively used for the analysis of the

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*Department of Mathematics, School of Science and Engineering, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan.
**Interdisciplinary Graduate School of Science and Engineering, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan

Stokes phenomena of WKB solutions with respect to parameters. In [15], Voros computed the explicit form of the Jost function for the Weber equation and obtained a formal power series as its asymptotic expansion. Later, Delabaere, Dillinger and Pham [7] discussed such formal series for general polynomial potentials where two simple turning points are connected by a Stokes curve and called them (or, more precisely, their exponentials) Voros coefficients. They introduced the notion of Voros coefficients as a formal series independently of the Jost functions. Starting from this point of view, Shen and Silverstone [12] and Takei [13] obtained the explicit form of the Voros coefficient directly for the Weber equation. These results were extended by Koike and Takei [10] to a special case of the Whittaker equation. They defined and computed the Voros coefficient for the Whittaker equation in a degenerate case where a simple turning point and the regular singular point at the origin merge. In this article, we are interested in the Whittaker equation in more general case. We employ a similar idea used in [10], [13] to obtain systems of difference equations for the Voros coefficients. Since the number of parameters (besides the large parameter) is two, the same method as in those article, namely, the division for the Borel transforms, cannot be applicable to solve the systems. As in the case of the Gauss equation [5], the systems are solved by using formal differential operators of infinite order. Our results can be obtained by the procedure of confluence from some of the formulas obtained in [5]. This observation is given in [1], where a part of the results of [10] has been recovered by considering degeneration of parameters which induces the merging of a simple turning point and the regular singularity at the origin.

§ 1. The Whittaker equation and the Kummer equation with a large parameter.

Let α and γ be complex constants. Let us consider the following Whittaker equation with a large parameter η:

$$\Big(-\frac{d^2}{dx^2}+\eta^2 Q\Big)\psi=0, \tag{1.1}$$

where $Q=Q_0+\eta^{-2}Q_1$ with

$$Q_0=\frac{x^2+2(2\alpha-\gamma)x+\gamma^2}{4x^2},\quad Q_1=-\frac{1}{4x^2}.$$

Equation (1.1) has a regular singular point at $b_0:=0$ and an irregular singular point at $b_2:=\infty$. Here the lacking of the subscript index “1” in b’s comes from the usual procedure of confluence of regular singularities $b_1=1$ and $b_2=\infty$ in the Gauss equation ([1], [5]). Equation (1.1) is obtained from the following Kummer equation:

$$x\frac{d^2w}{dx^2}+(\gamma+\eta^{-1}-x)\eta\frac{dw}{dx}-\eta^2\left(\alpha+\frac{1}{2}\eta^{-1}\right)w=0 \tag{1.2}$$

by introducing a new unknown function ψ:

$$w = x^{-\frac{1}{2}-\frac{\eta\gamma}{2}} \exp\left(\frac{\eta x}{2}\right)\psi. \tag{1.3}$$

The parameters α and γ correspond to those in the Gauss equation investigated in [2], [5], [14]. The Whittaker equation is normally written as (cf. [11])

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right)W = 0. \tag{1.4}$$

If we set $\mu = \frac{1}{2}\gamma\eta$, $\kappa = (\frac{1}{2}\gamma - \alpha)\eta$ and $z = \eta x$ in (1.4), then we have (1.1). Our potential Q is invariant under the involution ι on the space of parameters defined by

$$\iota : (\alpha, \gamma) \mapsto (\alpha - \gamma, -\gamma). \tag{1.5}$$

We assume the following condition:

$$\alpha\gamma(\alpha - \gamma) \neq 0. \tag{1.6}$$

This implies that there are two distinct simple turning points a_0 and a_1, namely, simple zeros of $Q_0 dx^2$ and that $a_0, a_1 \neq b_0, b_2$.

Remark. If we consider the limit as γ tends to 0 in (1.1), we have a special case of the Whittaker equation treated in [10]. See [1] for details.

§ 2. The Voros coefficients of the Whittaker equation

Let

$$S = \sum_{j=-1}^{\infty} \eta^{-j} S_j \tag{2.1}$$

be a formal solution of the Riccati equation associated with (1.1)

$$\frac{dS}{dx} + S^2 = \eta^2 Q \tag{2.2}$$

with the leading term $S_{-1} = \sqrt{Q_0}$. Here we fix a branch of $\sqrt{Q_0}$ suitably by taking a segment or an arc connecting a_0 and a_1 in $\mathbb{C} - \{0\}$ as a branch cut. Then the higher-order terms S_j $(j \geq 0)$ are uniquely determined. Let S_{odd} be the odd-order part of S with respect to η. Then

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{x_0}^{x} S_{\mathrm{odd}}\, dx\right) \tag{2.3}$$

are formal solutions to (1.1), which are called WKB solutions (see, for example, [9] and the references cited therein). Here x_0 is a fixed point in $\mathbb{C} - \{0\}$. When we choose $x_0 = a_j$ ($j = 0$ or 1) (see [9] for the meaning of the integration), the solutions are said to be normalized at the turning point a_j. Let C_j denote a contour starting from b_j, going to a_0 through the first sheet of the Riemann surface of $\sqrt{Q_0}$, stopping just before arriving at a_0 and turning once around a_0 counterclockwise and finally going back to b_j through the second sheet ($j = 0, 2$). The branches of $\sqrt{Q_0}$ on C_j are chosen so that at the starting point b_j we have

$$\sqrt{Q_0} \sim \frac{\gamma}{2x} \quad \text{at } x = b_0, \tag{2.4}$$

$$\sqrt{Q_0} \sim \frac{1}{2} \quad \text{at } x = b_2. \tag{2.5}$$

Since $b_0 = 0$ is a regular singular point of (1.1), $S_{\mathrm{odd}}dx$ and $S_{-1}dx$ have a simple pole there and their residues at b_0 are the same (Cf. [9]). On the other hand, $b_2 = \infty$ is an irregular singularity of rank 1 of (1.1). As in the case of regular singularities, we can see that the principal part of $S_{\mathrm{odd}}dx$ coincides with that of $S_{-1}dx$ and that it is

$$-\eta\left(\frac{1}{2y^2} + \frac{2\alpha - \gamma}{2y}\right)dy$$

in the local coordinate $y = 1/x$ at the infinity. Hence $(S_{\mathrm{odd}} - S_{-1})dx$ is integrable on C_j for $j = 0, 2$.

Definition 2.1. Let $V_j = V_j(\alpha, \gamma, \eta)$ ($j = 0, 2$) denote the formal power series in η^{-1} defined by

$$\frac{1}{2}\int_{C_j}(S_{\mathrm{odd}} - S_{-1})dx. \tag{2.6}$$

We call V_j the Voros coefficient of (1.1) or of (1.2) with respect to b_j for $j = 0, 2$.

The explicit forms of V_j ($j = 0, 2$) are given by the following theorem which has been announced in [1].

Theorem 2.2. *The Voros coefficients V_j ($j = 0, 2$) have the following forms*:

$$V_0 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\left\{(1 - 2^{1-n})\left(\frac{1}{\alpha^{n-1}} + \frac{1}{(\gamma-\alpha)^{n-1}}\right) + \frac{2}{\gamma^{n-1}}\right\}, \tag{2.7}$$

$$V_2 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}(1 - 2^{1-n})\left(\frac{1}{\alpha^{n-1}} - \frac{1}{(\gamma-\alpha)^{n-1}}\right). \tag{2.8}$$

Here B_n denote the Bernoulli numbers defined by

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}t^n.$$

To prove the theorem, we first find systems of difference equations which characterize V_0 and V_2. As in the cases of the Weber equation ([13]), of the degenerate Whittaker equation ([10]) and of the Gauss equation ([2], [5]), the ladder operators yield the systems. Let $\mathcal{S}(\alpha,\gamma)$ be the space of all formal solutions of (1.2). As is well known, the operators

$$H_1 := x\frac{d}{dx} + \eta\alpha + \frac{1}{2} \tag{2.9}$$

and

$$H_2 := \frac{d}{dx} - \eta \tag{2.10}$$

induce linear isomorphisms

$$H_1 : \mathcal{S}(\alpha,\gamma) \to \mathcal{S}(\alpha+\eta^{-1},\gamma) \tag{2.11}$$

and

$$H_2 : \mathcal{S}(\alpha,\gamma) \to \mathcal{S}(\alpha,\gamma+\eta^{-1}), \tag{2.12}$$

respectively.

Lemma 2.3. *The formal solution* $S = S(\alpha,\gamma;x,\eta)$ *of* (2.2) *satisfies the following relations*:

$$\begin{aligned} &S(\alpha+\eta^{-1},\gamma;x,\eta) - S(\alpha,\gamma;x,\eta) \\ &\qquad = \frac{d}{dx}\log\left(-\frac{\eta\gamma+1}{2} + x\left(\frac{\eta}{2} + S(\alpha,\gamma;x,\eta)\right) + \eta\alpha + \frac{1}{2}\right), \end{aligned} \tag{2.13}$$

$$\begin{aligned} &S(\alpha,\gamma+\eta^{-1};x,\eta) - S(\alpha,\gamma;x,\eta) \\ &\qquad = \frac{d}{dx}\log\left(-\frac{\eta\gamma+1}{2x} - \frac{\eta}{2} + S(\alpha,\gamma;x,\eta)\right) + \frac{1}{2x}. \end{aligned} \tag{2.14}$$

Proof. Firstly we note that

$$\hat{w} := H_1\left(x^{-\frac{1}{2}-\frac{\eta\gamma}{2}}\exp\left(\frac{\eta x}{2}\right)\exp\left(\int^x S(\alpha,\gamma;x,\eta)dx\right)\right)$$

belongs to $\mathcal{S}(\alpha+\eta^{-1},\gamma)$. Let $\hat{S}$ denote the logarithmic derivative of $x^{\frac{1}{2}+\frac{\eta\gamma}{2}}\exp\left(-\frac{\eta x}{2}\right)\hat{w}$. Then we can see that $\hat{S}$ satisfies the equation obtained from (2.2) by replacing α by $\alpha+\eta^{-1}$. Since the leading term of $\hat{S}$ coincides with that of $S(\alpha+\eta^{-1},\gamma;x,\eta)$, we conclude that

$$\hat{S} = S(\alpha+\eta^{-1},\gamma;x,\eta). \tag{2.15}$$

Hence we have (2.13). Similarly, (2.14) can be proved. □

Integrating the above relations with the difference of S_{-1} on the contours C_j ($j = 0,2$), we have

Proposition 2.4. *The Voros coefficients V_j $(j = 0, 2)$ satisfies the following systems of difference equations :*

$$\begin{aligned}(2.16)\quad & V_0(\alpha+\eta^{-1},\gamma,\eta) - V_0(\alpha,\gamma,\eta) \\ &= \frac{1}{2}\log\frac{\gamma-\alpha-\frac{1}{2}\eta^{-1}}{\alpha+\frac{1}{2}\eta^{-1}} + \frac{1}{2}\eta((\alpha+\eta^{-1})\log(\alpha+\eta^{-1}) - \alpha\log\alpha) \\ &\quad + \frac{\eta}{2}((\gamma-\alpha-\eta^{-1})\log(\gamma-\alpha-\eta^{-1}) - (\gamma-\alpha)\log(\gamma-\alpha)),\end{aligned}$$

$$\begin{aligned}(2.17)\quad & V_0(\alpha,\gamma+\eta^{-1},\eta) - V_0(\alpha,\gamma,\eta) \\ &= \frac{1}{2}\log\frac{\gamma(\gamma+\eta^{-1})}{\gamma-\alpha+\frac{1}{2}\eta^{-1}} - \eta((\gamma+\eta^{-1})\log(\gamma+\eta^{-1}) - \gamma\log\gamma) + \frac{1}{2}((\gamma+\eta^{-1})-\gamma)) \\ &\quad + \frac{1}{2}\eta((\gamma+\eta^{-1}-\alpha)\log(\gamma+\eta^{-1}-\alpha) - (\gamma-\alpha)\log(\gamma-\alpha)),\end{aligned}$$

$$\begin{aligned}(2.18)\quad & V_2(\alpha+\eta^{-1},\gamma,\eta) - V_2(\alpha,\gamma,\eta) \\ &= \frac{1}{2}\log\left(\left(\alpha+\frac{1}{2}\eta^{-1}\right)\left(\gamma-\alpha-\frac{1}{2}\eta^{-1}\right)\right) + \eta((\alpha+\eta^{-1})-\alpha) \\ &\quad + \frac{1}{2}\eta((\gamma-\alpha-\eta^{-1})\log(\gamma-\alpha-\eta^{-1}) - (\gamma-\alpha)\log(\gamma-\alpha)) \\ &\quad - \frac{1}{2}\eta((\alpha+\eta^{-1})\log(\alpha+\eta^{-1}) - \alpha\log\alpha),\end{aligned}$$

$$\begin{aligned}(2.19)\quad & V_2(\alpha,\gamma+\eta^{-1},\eta) - V_2(\alpha,\gamma,\eta) \\ &= -\frac{1}{2}\log\left(\gamma-\alpha+\frac{1}{2}\eta^{-1}\right) - \frac{1}{2}\eta((\gamma+\eta^{-1})-\gamma) \\ &\quad + \frac{1}{2}\eta(\gamma+\eta^{-1}-\alpha)\log(\gamma+\eta^{-1}-\alpha)) - (\gamma-\alpha)\log(\gamma-\alpha)).\end{aligned}$$

Moreover, V_j are characterized by those systems as the formal power series solutions in η^{-1} which are homogeneous of degree 0 in $(\alpha,\gamma,\eta^{-1})$ and which do not have constant terms.

Proof. We only give the proof of (2.18), since the others can be shown similarly. As is shown in [13], $\mathrm{Res}_{x=a_0}S = \mathrm{Res}_{x=a_0}S_0 = -1/4$ holds also in this case. Hence we have

$$(2.20)\qquad \frac{1}{2}\int_{C_2}(S_{\mathrm{odd}} - S_{-1})dx = \frac{1}{2}\int_{C_2}(S - \eta S_{-1} - S_0)dx.$$

Let x_0 be a complex number and let C_{x_0} be a contour starting at x_0 on the first sheet of the Riemann surface of $\sqrt{Q_0}$, going to a_0 along a straight line connecting x_0 and a_0 just before arriving at a_0, turning a_0 once counterclockwise and going back to x_0. Modifying the contour slightly if necessary, we may assume that C_{x_0} does not hit another turning point. Let $\hat{S}$ and $\hat{S}_{-1}$ be formal series obtained by replacing α by $\alpha+\eta^{-1}$ in S and S_{-1}, respectively. To obtain $V_2(\alpha+\eta^{-1},\gamma,\eta)-V_2(\alpha,\gamma,\eta)$, it suffices to compute

$$\lim_{x_0\to\infty}\frac{1}{2}\left(\int_{C_{x_0}}(\hat{S}-S)dx-\eta\int_{C_{x_0}}(\hat{S}_{-1}-S_{-1})dx\right).$$

Integrating (2.13), we have

$$\text{(2.21)}\qquad \frac{1}{2}\int_{C_{x_0}}(\hat{S}-S)dx=\frac{1}{2}\log\frac{-\frac{1}{2}(\eta\gamma+1)+\eta\alpha+\frac{1}{2}+x_0\left(\frac{\eta}{2}+S(\alpha,\gamma;\tilde{x}_0,\eta)\right)}{-\frac{1}{2}(\eta\gamma+1)+\eta\alpha+\frac{1}{2}+x_0\left(\frac{\eta}{2}+S(\alpha,\gamma;x_0,\eta)\right)},$$

where $\tilde{x}_0$ denotes the point on the second sheet corresponding to x_0. It follows from (2.2) and (2.5) that

$$\text{(2.22)}\quad S(\alpha,\gamma,\tilde{x}_0,\eta)=-\frac{\eta}{2}-\eta\frac{2\alpha-\gamma}{2x_0}+\eta\left(\alpha-\gamma+\frac{\eta^{-1}}{2}\right)\left(\alpha+\frac{\eta^{-1}}{2}\right)\frac{1}{x_0^2}+O\left(\frac{1}{x_0^3}\right)$$

and

$$\text{(2.23)}\quad S(\alpha,\gamma,x_0,\eta)=\frac{\eta}{2}+\eta\frac{2\alpha-\gamma}{2x_0}+\eta\left(\alpha-\gamma-\frac{\eta^{-1}}{2}\right)\left(\alpha-\frac{\eta^{-1}}{2}\right)\frac{1}{x_0^2}+O\left(\frac{1}{x_0^3}\right)$$

hold. Thus (2.21) has the following leading terms:

$$\text{(2.24)}\qquad \frac{1}{2}\log\left(\left(\gamma-\alpha-\frac{\eta^{-1}}{2}\right)\left(\alpha+\frac{\eta^{-1}}{2}\right)\right)+\frac{\pi i}{2}-\log x_0+O\left(\frac{1}{x_0}\right).$$

Here we use the convention of the argument $\alpha-\gamma=e^{\pi i}(\gamma-\alpha)$, etc. On the other hand, S_{-1} can be integrated easily and we have

$$\begin{aligned}\text{(2.25)}\quad \frac{\eta}{2}\int_{C_{x_0}}(\hat{S}_{-1}-S_{-1})dx&=-1-\log x_0+\frac{\eta}{2}\left((\alpha+\eta^{-1})\log(\alpha+\eta^{-1})-\alpha\log\alpha\right)\\&-\frac{\eta}{2}\left((\gamma-\alpha-\eta^{-1})\log(\gamma-\alpha-\eta^{-1})-(\gamma-\alpha)\log(\gamma-\alpha)\right)+\frac{\pi i}{2}+O\left(\frac{1}{x_0}\right)\end{aligned}$$

Subtracting (2.25) from (2.24) and taking the limit x_0 to the infinity, we obtain (2.18). Uniqueness of the homogeneous formal solutions follows from a similar argument in the proof of Proposition 2.8 in [5]. □

We can solve the systems (2.16)–(2.19) in a similar manner as in [5] by using formal differential operators of infinite order ([6])

$$\text{(2.26)}\qquad \eta^{-1}\partial_\alpha(\exp(\eta^{-1}\partial_\alpha)-1)^{-1}=\sum_{n=0}^{\infty}\frac{(-1)^nB_n}{n!}\eta^{-n}\partial_\alpha^n$$

and

$$\eta^{-1}\partial_\gamma(\exp(\eta^{-1}\partial_\gamma)-1)^{-1}=\sum_{n=0}^{\infty}\frac{(-1)^n B_n}{n!}\eta^{-n}\partial_\gamma^n. \tag{2.27}$$

Here ∂_α stands for $\partial/\partial\alpha$, etc. We omit the computation of solving the systems because it is almost the same as a part of the proof of Theorem 2.3 in [5].

§ 3. Borel sums of the Voros coefficients

To consider the Borel sums of V_j $(j=0,2)$, we must specify the "Stokes regions" for the parameters, where V_j's are Borel summable. These regions are described by using the following sets:

$$\omega_1=\{(\alpha,\gamma)\in\mathbb{C}^2\,|\,0<\operatorname{Re}\alpha<\operatorname{Re}\gamma\}, \tag{3.1}$$

$$\omega_3=\{(\alpha,\gamma)\in\mathbb{C}^2\,|\,0<\operatorname{Re}\gamma<\operatorname{Re}\alpha\}, \tag{3.2}$$

$$\omega_4=\{(\alpha,\gamma)\in\mathbb{C}^2\,|\,\operatorname{Re}\alpha<0<\operatorname{Re}\gamma\}. \tag{3.3}$$

Regarding the relation between these sets and configurations of the Stokes curves of (1.1), we refer the readers to [1]. Those regions and the images of them by the involution ι (cf. (1.5)) cover almost all $\mathbb{C}^2$:

$$\bigcup_{j=1,3,4}(\omega_j\cup\iota(\omega_j))=\{(\alpha,\gamma)\in\mathbb{C}^2\,|\,\operatorname{Re}\alpha\ \operatorname{Re}\gamma\ \operatorname{Re}(\gamma-\alpha)\neq 0\}. \tag{3.4}$$

If (α,γ) belongs to the above set, there is no Stokes curves of (1.1) which connect turning point(s).

Theorem 3.1. *The Voros coefficients V_j $(j=0,2)$ are Borel summable in ω_k and in $\iota(\omega_k)$ for $k=1,3,4$. Here ι is the involution in the space of parameters defined* *by* (1.5). *The Borel sums* V_j^k *of* V_j *in* ω_k *have the following forms*:

$$V_0^1 = \frac{1}{2}\log\frac{\Gamma(\gamma\eta)^2\alpha^{\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta^{1-\gamma\eta}}{\Gamma\left(\frac{1}{2}+\alpha\eta\right)\Gamma\left(\frac{1}{2}+(\gamma-\alpha)\eta\right)\gamma^{2\gamma\eta-1}}+\frac{1}{2}\gamma\eta, \tag{3.5}$$

$$V_0^3 = \frac{1}{2}\log\frac{\Gamma(\gamma\eta)^2\Gamma\left(\frac{1}{2}+(\alpha-\gamma)\eta\right)\alpha^{\alpha\eta}\eta^{1-\gamma\eta}}{2\pi\Gamma\left(\frac{1}{2}+\alpha\eta\right)(\alpha-\gamma)^{(\alpha-\gamma)\eta}\gamma^{2\gamma\eta-1}}+\frac{1}{2}\gamma\eta, \tag{3.6}$$

$$V_0^4 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\gamma\eta)^2(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta^{1-\gamma\eta}}{2\pi\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}\gamma^{2\gamma\eta-1}}+\frac{1}{2}\gamma\eta, \tag{3.7}$$

$$V_2^1 = \frac{1}{2}\log\frac{\Gamma\left(\frac{1}{2}+(\gamma-\alpha)\eta\right)\alpha^{\alpha\eta}\eta^{(2\alpha-\gamma)\eta}}{\Gamma\left(\frac{1}{2}+\alpha\eta\right)(\gamma-\alpha)^{(\gamma-\alpha)\eta}}-\frac{1}{2}(2\alpha-\gamma)\eta, \tag{3.8}$$

$$V_2^3 = \frac{1}{2}\log\frac{2\pi\alpha^{\alpha\eta}(\alpha-\gamma)^{(\alpha-\gamma)\eta}\eta^{(2\alpha-\gamma)\eta}}{\Gamma\left(\frac{1}{2}+\alpha\eta\right)\Gamma\left(\frac{1}{2}+(\alpha-\gamma)\eta\right)}-\frac{1}{2}(2\alpha-\gamma)\eta, \tag{3.9}$$

$$V_2^4 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\eta^{(2\alpha-\gamma)\eta}}{2\pi(-\alpha)^{-\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}}-\frac{1}{2}(2\alpha-\gamma)\eta. \tag{3.10}$$

Remark. The Borel sum of V_j in $\iota(\omega_k)$ coincides with V_j^k up to signature, which comes from the choice of the branch of $\sqrt{Q_0}$.

Theorem 3.1 is proved by computing the Borel sums directly from the expressions given in Theorem 2.2. We can use a method developed in [13], where the Borel sum of the series

$$\sum_{n=2}^{\infty}\frac{(1-2^{1-n})B_n}{n(n-1)}(iE\eta)^{1-n} \tag{3.11}$$

is given. Hence we do not repeat the computation here. See the proof of Theorem 3.1 in [5] also.

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