RIMS Kôkyûroku Bessatsu
B52 (2014), 175–191

On b-functions and algebraic local cohomology classes attached to hypersurfaces with line singularities

Dedicated to Professor Takashi Aoki on the occasion of his sixtieth birthday

By

Shinichi Tajima*

Abstract

Algebraic local cohomology classes attached to Lê cycles of isolated line singularities are considered. A simple method that uses algebraic local cohomology classes to guess the microlocal b-functions associated with line singularities is described. The correctness of the method is shown by using the index theorem of regular singular ordinary differential equations and the notion of vertical monodromy on the stratum of singular locus of a line singularity.

§ 1. Introduction

In 1978, T. Yano [32] studied b-functions from the point of view of algebraic analysis. He considered b-functions of hypersurfaces in several cases and computed in particular explicite forms of b-functions for many cases by using the concept of algebraic local cohomology and holonomic D-modules. Algebraic local cohomology classes were used in [32] as eigenvectors of the action of local monodromy on vanishing cycles.

In 2002, I considered (micro-)local b-functions of non-isolated quasi-homogeneous functions and examined in particular (micro-)local b-functions of hypersurfaces with a smooth one-dimensional singular locus Σ stratified by two strata $\Sigma - \{O\}$ and $\{O\}$. Based on an observation and a guess, I obtained an elementary but conjectural method to determine microlocal b-functions for these line singularities. The main idea of the study is an use of the concept of Lê cycles that correspond to multiplicity structures of characteristic varieties or micro-supports of vanishing cycle sheaves.

Received April 08, 2014. Accepted July 31, 2014.
2010 Mathematics Subject Classification(s): 32S65, 32S05, 32C38.
Key Words: b-function, local cohomology, Lê cycle, holonomic D-module.
*Institute of Mathematics, University of Tsukuba, 1-1-1, Tennoudai, Tsukuba, Ibaraki, Japan.

The purpose of this paper is to give a correct interpretation of the observation that was obtained in 2002 and to explain the reason why the elementary method works.

In section one we consider the (micro-) local b-function of Whitney umbrella by using algebraic local cohomology classes and present the guess obtained in 2002.

In section two, we examine (micro-)local b-functions of non-isolated quasi-homogeneous hypersurfaces with isolated line singularities by using results of computation. We describe the elementary but conjectural method to determine microlocal b-functions for these singularities.

In section three, we examine microlocal b-functions of hypersurfaces with simple line singularities. We show that how the conjectural method works for these case.

In section four, we compute D-modules relevant to the vaniching cycle sheaves for two cases and compute their algebraic local cohomology solution spaces , supported on the stratum $\Sigma - \{O\}$ and on $\{O\}$ respectively, that describe local systems of vanishing cycles ([3], [5]). By using these data, we study monodromy structures of vanishing cycles and microlocal b-functions. Note that the notion of vertical monodromy, due to D. Siersma ([25], [26]), is the key to find a correct interpretation of the observation.

We use the computer algebra system Risa/Asir developed by M. Noro et al ([17]) to study (micro-)local b-functions , algebraic local cohomology classes and D-modules.

I obtained the guess and the conjectural method to find microlocal b-functions of line singularities in 2002, while I stayed at RIMS, Kyoto University as a short term visiting researcher. I am grateful to Kyoji Saito and Lê Dũng Tráng. I also would like to thank Research Institute of Mathematical Science for hospitality. An especially big "thank you" goes to Toshinori Oaku.

§ 2. Whitney umbrella

Let $f(t,x,y) = y^2 - x^3 - tx^2$ and let $S = \{(t,x,y) \in X \mid f(t,x,y) = 0\}$, where X is an open neighborhood in $\mathbb{C}^3$ of the origin. The hypersurface S is the Whitney umbrella. The singular locus Σ of S is the t-axis. For $t_0 \in \mathbb{C}$, let H_{t_0} denote the hyperplane $H_{t_0} = \{(t,x,y) \in X \mid t = t_0\}$. Then, for $t_0 \neq 0$ the curve $S \cap H_{t_0}$ has a node at $(t_0, 0.0)$ and for $t_0 = 0$ the curve $S \cap H_0$ has a cusp at $O = (0,0,0)$. The singular locus Σ is therefore stratified by two strata : $\Sigma = (\Sigma - \{O\}) \cup \{O\}$. In fact, the primary decomposition of the Jacobian ideal $< \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$ is $< x, y > \cap < t, x^2, y >$.

Set $r(x,y) = f(1,x,y)$. Since $r(x,y)$ is a Morse function of two variables, the b-function b_r of r is $b_r(s) = (s+1)^2$. The b-function $b_f(s)$ of the Whitney umbrella S is $b_f(s) = (s+1)^2(s+\frac{3}{2})$. Therefore, the factor $s + \frac{3}{2}$ in b_f comes from the origin $\{O\}$.

Let $\begin{bmatrix} 1 \\ \frac{\partial f}{\partial x} \ \frac{\partial f}{\partial y} \end{bmatrix}$ denote the algebraic local cohomology class in $\mathcal{H}^2_{[V]}(\mathcal{O}_X)$, where [ ] is the Grothendieck symbol, V is the variety

$$V = V(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = \{(t,x,y) \in X \mid 2tx + 3x^2 = y = 0\},$$

$\mathcal{O}_X$ is the sheaf on X of holomorphic functions. Since the primary decomposition of the ideal $< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$ is $< x, y > \cap < 2t + 3x, y >$, $\begin{bmatrix} 1 \\ \frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y} \end{bmatrix} |_{\Sigma - \{O\}}$ is well-defined on the stratum $\Sigma - \{O\}$ as a local cohomology class. We have

$$\begin{bmatrix} 1 \\ \frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y} \end{bmatrix} |_{\Sigma - \{O\}} = -\frac{1}{2(2t+3x)} \begin{bmatrix} 1 \\ xy \end{bmatrix} |_{\Sigma - \{O\}},$$

which is equal to $-\frac{1}{4t}\begin{bmatrix} 1 \\ xy \end{bmatrix}$ on $\Sigma - \{O\}$. Note that the local cohomology class $\frac{1}{t}\begin{bmatrix} 1 \\ xy \end{bmatrix}$ can be regarded as a section of a local system on $\Sigma - \{O\}$.

Now we consider the weighted degrees of algebraic local cohomology classes, supported on the stratum $\{O\}$, to study the microlocal b-function. Since the polar variety Γ^1 of the Whitney umbrella is $\Gamma^1 = \{(t,x,y) \mid 2t + 3x = y = 0\}$, the zero-dimensional Lê cycle (see [15]) Λ^0 , as an ideal in the ring $\mathcal{O}_{X,O}$ of germs of holomorphic functions, is defined to be

$$\Lambda^0 =< \frac{\partial f}{\partial t}, 2t + 3x, y >=< 2t + 3x, x^2, y > .$$

Let

$$H_{J_O} = \{\eta \in \mathcal{H}^3_{[O]}(\mathcal{O}_X) \mid J_O \eta = 0\} \text{ and } H_{\Lambda^0} = \{\eta \in \mathcal{H}^3_{[O]}(\mathcal{O}_X) \mid \Lambda^0 \eta = 0\},$$

where J_O stands for the primary component $< t, x^2, y >$ at the origin of the Jacobian ideal in $\mathcal{O}_{X,O}$. H_{J_O} (resp H_{Λ^0}) is the set of algebraic local cohomology classes in $\mathcal{H}^3_{[O]}(\mathcal{O}_X)$ that are annihilated by the zero-dimensional ideal J_O (resp Λ^0). We have

$$H_{J_O} = \mathrm{Span}_{\mathbb{C}}\{\begin{bmatrix} 1 \\ txy \end{bmatrix}, \begin{bmatrix} 1 \\ tx^2y \end{bmatrix}\}$$

and

$$H_{\Lambda^0} = \mathrm{Span}_{\mathbb{C}}\{\begin{bmatrix} 1 \\ txy \end{bmatrix}, \begin{bmatrix} 1 \\ tx^2y \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ t^2xy \end{bmatrix}\}.$$

The defining function f is a weighted homogeneous polynomial of weight $(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$ and thus the weighted degree of the algebraic local cohomology class $\begin{bmatrix} 1 \\ txy \end{bmatrix}$ is $-\frac{7}{6}$ and that of $\begin{bmatrix} 1 \\ tx^2y \end{bmatrix}$ and of $\begin{bmatrix} 1 \\ tx^2y \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ t^2xy \end{bmatrix}$ are both equal to $-\frac{3}{2}$.

Since the algebraic local cohomology class $\begin{bmatrix} 1 \\ tx^2y \end{bmatrix}$ (or $\begin{bmatrix} 1 \\ tx^2y \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ t^2xy \end{bmatrix}$) is an eigenvector of the Euler operator

$$E = \frac{1}{3}t\frac{\partial}{\partial t} + \frac{1}{3}x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y}$$

and its eigenvalue is equal to $-\frac{3}{2}$, one can regard the weighted degree of the algebraic local cohomology class $\begin{bmatrix} 1 \\ tx^2y \end{bmatrix}$ (or $\begin{bmatrix} 1 \\ tx^2y \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ t^2xy \end{bmatrix}$) as an eigenvalue of the monodromy action for the local system on $\{O\}$ of vanishing cycles. In contrast, even though the algebraic local cohomology class $\begin{bmatrix} 1 \\ txy \end{bmatrix}$ is an eigenvector of the Euler operator E, one can not regard it as a section on $\{O\}$ of the local system of vanishing cycles, because its eigenvalue $-\frac{7}{6}$ is not a root of b-function b_f.

Now, with an intension to seek a method to compute microlocal b-functions, we consider algebraic local cohomology classes supported on the stratum $\Sigma - \{O\}$ and on the stratum $\{O\}$.

Let i denote the open inclusion map $i : \Sigma - \{O\} \longrightarrow \Sigma$ and consider the boundary value map ([14])

$$\mathrm{bv} : i_*i^{-1}(\mathcal{H}^2_{[\Sigma]}(\mathcal{O}_X)) \longrightarrow \mathcal{H}^3_{[O]}(\mathcal{O}_X).$$

Set $\sigma = \frac{1}{t}\begin{bmatrix} 1 \\ xy \end{bmatrix} \in i_*i^{-1}(\mathcal{H}^2_{[\Sigma]}(\mathcal{O}_X))$. Then we have $\mathrm{bv}(\sigma) = \begin{bmatrix} 1 \\ txy \end{bmatrix} \in H_{\Lambda^0}$.

Therefore, it seems natural to *guess* as follows : Since the algebraic local cohomology class $\begin{bmatrix} 1 \\ txy \end{bmatrix}$ is the boundary value of the section σ of the local system on $\Sigma - \{O\}$, the class $\begin{bmatrix} 1 \\ txy \end{bmatrix}$ supported on $\{O\}$ is not a section of the local system on $\{O\}$ of vanishing cycles.

This is just *a guess*, but is in accord with a result of Siersma on Betti number of the Milnor fibre ([24]).

§ 3. Siersma's isolated line singularities

In 1983, D. Siersma ([23]) considered germs of holomorphic functions f with a smooth one-dimensional critical set Σ such that the transversal singularity of f at each point in $\Sigma - \{O\}$ is of type A_1. These singularities are called isolated line singularities. He studied in particular the homotopy type of the Milnor fibre of isolated line singularity by using *relative* Morsification ([23], [24]).

In this section, we consider b-functions of isolated line singularities by using the concept of algebraic local cohomology classes and examine in particular two cases by using the computer algebra system Risa/Asir.

Example 3.1.

Let $f(x,y) = x^5y^2 + y^3$ and let $S = \{(x,y) \in X \mid f(x,y) = 0\}$, where X is a neighborhood of the origin 0 in $\mathbb{C}^2$. The singular locus Σ of S is the x-axis : $\Sigma = \{(x,y) \mid y = 0\}$. The defining function f is a weighted homogeneous polynomial of weight $(1,5)$ and the weighted degree of f is equal to 15.

Set $r(y) = f(1,y) = y^2 + y^3$. Since r is a Morse function, the b-function b_r of the function r is $b_r(s) = (s+1)(s+\frac{1}{2})$. The factor $s+1$ comes from the non-singular part of S and thus $s+\frac{1}{2}$ is the (micro-)local b-function of f on the stratum $\Sigma - \{O\}$. We first compute the b-function b_f of f by using the algorithm constructed by T. Oaku ([18]). We have

$$b_f(s) = (s+1)(s+\frac{1}{2})(s+\frac{6}{15})(s+\frac{7}{15})(s+\frac{8}{15})(s+\frac{9}{15})$$
$$\times(s+\frac{11}{15})(s+\frac{12}{15})(s+\frac{13}{15})(s+\frac{14}{15})(s+\frac{16}{15})(s+\frac{17}{15})(s+\frac{18}{15})(s+\frac{19}{15}).$$

We see $b_r|b_f$ ([9], [32]) and $\deg(b_f) - \deg(b_r) = 12$.

The primary decomposition of the ideal $< \frac{\partial f}{\partial y} >$ is $< y > \cap < 2x^5 + 3y >$. Since the polar variety Γ^1 is $V(2x^5+3y)$, we define the zero-dimensional Lê cycle Λ^0, as an ideal in the local ring $\mathcal{O}_{X,0}$ of germs of holomorphic functions, by $< \frac{\partial f}{\partial x}, 2x^5 + 3y >$.

Let $H_{\Lambda^0} = \{\eta \in \mathcal{H}^2_{[O]}(\mathcal{O}_X) \mid \Lambda^0\eta = 0\}$. Then, it is easy to see that the following 14 algebraic local cohomology classes constitute a basis of the vector space H_{Λ^0}.

$$\begin{bmatrix} 1 \\ x^iy \end{bmatrix}, \begin{bmatrix} 1 \\ x^iy^2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ x^{5+i}y \end{bmatrix}, i = 1,2,3,4,5$$

and

$$\begin{bmatrix} 1 \\ x^iy^3 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ x^{5+i}y^2 \end{bmatrix}, i = 1,2,3,4.$$

The dimension of the vector space H_{Λ^0} is 14. Note that, since H_{Λ^0} is dual to $\mathcal{O}_{X,0}/\Lambda^0$, the Grothendieck local duality implies that

$$\dim_{\mathbb{C}}(\mathcal{O}_{X,0}/\Lambda^0) = \dim_{\mathbb{C}}(H_{\Lambda^0}) = 14.$$

The weighted degree of these algebraic local cohomology classes are

$$-\tfrac{5+i}{15}, -\tfrac{10+i}{15}, i = 1,2,3,4,5 \text{ and } -\tfrac{15+i}{15}, i = 1,2,3,4.$$

Notice that the rational number $-\frac{10}{15}$ in the list above is not a root of the b-function b_f and the rational number -1 in the list above is a root of the b-function b_r. The others are roots of b_f.

Consider the algebraic local cohomology class $\left[\begin{matrix}1\\ \frac{\partial f}{\partial y}\end{matrix}\right]|_{\Sigma-\{O\}}$ defined on the stratum $\Sigma-\{O\}$. We have the following

$$\left[\begin{matrix}1\\ \frac{\partial f}{\partial y}\end{matrix}\right]|_{\Sigma-\{O\}}=\frac{1}{2x^5+3y}\left[\begin{matrix}1\\ y\end{matrix}\right]|_{\Sigma-\{O\}}=\frac{1}{2x^5}\left[\begin{matrix}1\\ y\end{matrix}\right]|_{\Sigma-\{O\}}.$$

Set $\sigma=\frac{1}{x^5}\left[\begin{matrix}1\\ y\end{matrix}\right]\in i_*i^{-1}(\mathcal{H}^1_{[\Sigma]}(\mathcal{O}_X))$, where i is the open inclusion map

$$i:\Sigma-\{O\}\longrightarrow\Sigma.$$

Then we have $\mathrm{bv}(\sigma)=\left[\begin{matrix}1\\ x^5y\end{matrix}\right]\in H_{\Lambda^0}$, where bv is the boundary value map

$$\mathrm{bv}:i_*i^{-1}(\mathcal{H}^1_{[\Sigma]}(\mathcal{O}_{X,O}))\longrightarrow\mathcal{H}^2_{[O]}(\mathcal{O}_X).$$

The wieghted degree of the algebraic local cohomology class $\mathrm{bv}(\sigma)$ is equal to $-\frac{10}{15}$. This is consistent with the *guess* that $-\frac{10}{15}$ is not a root of the b-function b_f

Example 3.2.

Let $f(x,y)=x^2y^2+y^7$ and let $S=\{(x,y)\in X\mid f(x,y)=0\}$. The singular locus of S is $\Sigma=\{(x,y)\mid y=0\}$. The defining function f is a weighted homogeneous polynomial of weight $(5,2)$ and the weighted degree of f is equal to 14.

Let $r(y)=f(1,y)=y^2+y^7$. The b-function b_r of $r(y)$ is $b_r(s)=(s+1)(s+\frac{1}{2})$. The b-function b_f of f is given by

$$b_f(s)=(s+1)^2(s+\frac{1}{2})^2(s+\frac{9}{14})(s+\frac{11}{14})(s+\frac{13}{14})(s+\frac{15}{14})(s+\frac{17}{14})(s+\frac{19}{14}).$$

Thus, we see $b_r|b_f$ and $\deg(b_f)-\deg(b_r)=8$.

The primary decomposition of the ideal $<\frac{\partial f}{\partial y}>$ is $<y>\cap<2x^2+7y^5>$. The polar variety Γ^1 is therefore $V(2x^5+7y^5)$. We define Λ^0 by $<\frac{\partial f}{\partial x},2x^5+7y^5>$.

Let $H_{\Lambda^0}=\{\eta\in\mathcal{H}^2_{[O]}(\mathcal{O}_X)\mid\Lambda^0\eta=0\}$. The vector space H_{Λ^0} is generated by the following 9 algebraic local cohomology classes :

$$\left[\begin{matrix}1\\ xy^j\end{matrix}\right],j=1,2,3,4,5\quad\left[\begin{matrix}1\\ x^2y^k\end{matrix}\right]\text{ and }\left[\begin{matrix}1\\ x^3y^k\end{matrix}\right]-\frac{2}{7}\left[\begin{matrix}1\\ xy^{5+k}\end{matrix}\right],k=1,2.$$

The weighted degree of these algebraic local cohomology classes are

$$-\tfrac{5+2j}{14},j=1,2,3,4,5,\text{ and }-\tfrac{10+2k}{14},-\tfrac{15+2k}{14},k=1,2.$$

The rational number $-\frac{12}{14}$ in the list above is not a root of b_f and the others are roots of b_f. Let E_{Λ^0} be the set of weighted degrees of algebraic local cohomology classes in

H_{Λ^0}, $R_{\{O\}}$ the set of roots of the microlocal b-function on the conormal $T^*_{\{O\}}X$. Then we have

$$E_{\Lambda^0} = R_{\{O\}} \cup \{-\frac{12}{14}\}.$$

Now we consider the algebraic local cohomology class $\begin{bmatrix} 1 \\ \frac{\partial f}{\partial y} \end{bmatrix}|_{\Sigma-\{O\}}$ defined on the stratum $\Sigma - \{O\}$. We have

$$\begin{bmatrix} 1 \\ \frac{\partial f}{\partial y} \end{bmatrix}|_{\Sigma-\{O\}} = \frac{1}{2x^2+7y^5}\begin{bmatrix} 1 \\ y \end{bmatrix}|_{\Sigma-\{O\}} = \frac{1}{2x^2}\begin{bmatrix} 1 \\ y \end{bmatrix}|_{\Sigma-\{O\}}.$$

Set $\sigma = \frac{1}{x^2}\begin{bmatrix} 1 \\ y \end{bmatrix} \in i_*i^{-1}(\mathcal{H}^1_{[\Sigma]}(\mathcal{O}_{X,O}))$. Then we have $\mathrm{bv}(\sigma) = \begin{bmatrix} 1 \\ x^2y \end{bmatrix} \in H_{\Lambda^0}$.

The weighted degree of the $\mathrm{bv}(\sigma)$ is equal to $-\frac{12}{14}$, which is not a root of b_f. These facts are consistent with the *guess* presented in the previous section.

Based on these type of results presented above, I obtained in 2002 the following observation for isolated line singularities.

- The set of weighted degrees of the weighted homogeneous algebraic local cohomology classes in the vector space H_{Λ^0} contains the roots of the microlocal b-function on the conormal $T^*_{\{O\}}X$
- There is a possibility that the weighted degree of the boundary value of the local cohomology class $\sigma \in i_*i^{-1}(\mathcal{H}^{\mathrm{codim}\Sigma}_{[\Sigma]}(\mathcal{O}_{X,O}))$ is not a root of the microlocal b-function on the conormal $T^*_{\{O\}}X$.

Now assume that a germ of weighted homogeneous holomorphic function f with a line singularity is given. Assume further that the transversal singularities of f at each point in $\Sigma - \{O\}$ are of the same type, for instance of type A_3. For such a general case, it is natural to consider a λ^1-dimensional vector space of algebraic local cohomology classes that represents the multiplicity structure of the stratum $\Sigma - \{O\}$, where λ^1 stands for the Lê number of the stratum $\Sigma - \{O\}$ ([15], [16]). Based on this guess, I arrived at in 2002 the following elementary but conjectura method to determine the microlocal b-functions on the conormal $T^*_{\{O\}}X$ for simple line singularities.

A method to quess microlocal b-functions on the conormal $T^*_{\{O\}}X$ for simple line singularities.

- compute Lê cycles Λ^1, Λ^0 and the Lê numbers λ^1, λ^0.
- compute the vector space H_{Λ^0} and the set E_{Λ^0} of weighted degree of weighted homogeneous algebraic local cohomology classes in H_{Λ^0}.

- compute the algebraic local cohomology class $\sigma \in i_* i^{-1}(\mathcal{H}^{\mathrm{codim}\Sigma}_{[\Sigma]}(\mathcal{O}_{X,O}))$ by using partial derivative of f.
- compute a basis N of the λ^1-dimensional vector space of algebraic local cohomology classes that represent Noetherian operators of the multiplicity structure of the stratum $\Sigma - \{O\}$ (see section three).
- compute the set $B_{\{O\}}$ of weighted degree of the boundary values of the algebraic local cohomology classes in N.

Then, we may have $H_\Lambda - R_{\{O\}} \subseteq B_{\{O\}}$, probably $R_{\{O\}} = H_\Lambda - B_{\{O\}}$, where $R_{\{O\}}$ stands for the set of the roots of microlocal b-function on the conormal $T^*_{\{O\}}X$.

§ 4. de Jong's simple line singularities

In 1988, T. de Jong ([8]) studied some classes of line singularities by extending methods of D. Siersma. We examine, in this section, transverse A_2 and transverse A_3 type singularities studied by T. de Jong.

We will use an algorithm, derived by T. Oaku in [19], for computing holonomic systems of algebraic local cohomology classes to study A_3 type singularity.

Example 4.1.
Let $f(x,y,z) = xy^3 + z^2$ and let $S = \{(x,y,z) \in X \mid f(x,y,z) = 0\}$, where X is a neighborhood of the origin O in $\mathbb{C}^3$. The singular locus Σ of S is the x-axis : $\Sigma = \{(x,y,z) \mid y = z = 0\}$. The defining function f is a weighted homogeneous polynomial of weight $(1,1,2)$ and the weighted degree of f is equal to 4. The function f has a transverse singularity of type A_2 at each point on the stratum $\Sigma - \{O\}$.

Set $r(y,z) = f(1,y,z) = y^3 + z^2$. Since r is a weighted homogeneous polynomial of weight $(\frac{1}{3}, \frac{1}{2})$ and the MIlnor number is equal to 2, the b-function b_r of $r(y,z)$ is

$$b_r(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6}).$$

The b-function b_f of f is

$$b_f(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})(s+\frac{3}{2}).$$

We see $b_r | b_f$ and the factor $s + \frac{3}{2}$ comes from the origin.

The primary decomposition of the ideal $< \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$ is $< x, z > \cap < y^2, z >$. Since the polar variety Γ^1 is $V(x,z)$, we define Λ^0 to be $< \frac{\partial f}{\partial x}, x, z >$. Let

$$H_{\Lambda^0} = \{\eta \in \mathcal{H}^3_{[O]}(\mathcal{O}_X) \mid \Lambda^0 \eta = 0\}.$$

Then, algebraic local cohomology classes $\begin{bmatrix} 1 \\ xy^i z \end{bmatrix}, i = 1,2,3$ constitute a basis of the vector space H_{Λ^0}. The weighted degree of these algebraic local cohomology classes are $-1, -\frac{5}{4}, -\frac{3}{2}$. The rational numbers -1 and $-\frac{5}{3}$ are not roots of b_f.

We have the following.

$$\begin{bmatrix} 1 \\ \frac{\partial f}{\partial y}\frac{\partial f}{\partial z} \end{bmatrix} |_{\Sigma-\{O\}} = \frac{1}{6x}\begin{bmatrix} 1 \\ y^2 z \end{bmatrix} |_{\Sigma-\{O\}}.$$

Note that the pair $\{\frac{1}{x}\begin{bmatrix} 1 \\ yz \end{bmatrix}|_{\Sigma-\{O\}}, \frac{1}{x}\begin{bmatrix} 1 \\ y^2 z \end{bmatrix}|_{\Sigma-\{O\}}\}$ can be regarded as the Noether operators ([6], [21], [27], [28], [29]) on $\Sigma - \{O\}$ of the primary ideal $< y^2, z >$.

We set $\sigma = \dfrac{1}{x}\begin{bmatrix} 1 \\ y^2 z \end{bmatrix} \in i_* i^{-1}(\mathcal{H}^2_{[\Sigma]}(\mathcal{O}_{X,O}))$, where i is the open inclusion map

$$i : \Sigma - \{O\} \longrightarrow \Sigma.$$

Then we have

$$\mathrm{bv}(\sigma) = \begin{bmatrix} 1 \\ xy^2 z \end{bmatrix}, \mathrm{bv}(y\sigma) = \begin{bmatrix} 1 \\ xyz \end{bmatrix},$$

where bv is the boundary value map

$$\mathrm{bv} : i_* i^{-1}(\mathcal{H}^2_{[\Sigma]}(\mathcal{O}_{X,O})) \longrightarrow \mathcal{H}^3_{[O]}(\mathcal{O}_X).$$

The wieghted degree of the algebraic local cohomology class $\mathrm{bv}(\sigma)$ is equal to -1 and that of $\mathrm{bv}(y\sigma)$ is equal to $-\frac{5}{4}$.

Example 4.2.

Let $f(x,y,z) = xz^2 + y^3$ and let $S = \{(x,y,z) \in X \mid f(x,y,z) = 0\}$. The singular locus of S is $\Sigma = \{(x,y,z) \mid y = z = 0\}$. The defining function f is a weighted homogeneous polynomial of weight $(1,1,1)$ and the weighted degree of f is equal to 3.

Set $r(y,z) = f(1,y,z) = y^3 + z^2$. Then

$$b_r(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6}).$$

The b-function of f is

$$b_f(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})(s+\frac{4}{3})(s+\frac{5}{3}).$$

Then, $b_r|b_f$ and $\deg(b_f) - \deg(b_r) = 2$. The two factors $s+\frac{4}{3}, s+\frac{5}{3}$ come from the origin.

The primary decomposition of the ideal $< \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$ is $< x, y^2 > \cap < y^2, z >$.
Since $\Gamma^1 = V(x, y^2)$, we define Λ^0 by $< \frac{\partial f}{\partial x}, x, z >$.

Let $H_{\Lambda^0} = \{\eta \in \mathcal{H}^3_{[O]}(\mathcal{O}_X) \mid \Lambda^0\eta = 0\}$. Then, algebraic local cohomology classes

$$\left[\begin{array}{c}1\\ xy^jz^k\end{array}\right], j = 1,2, k = 1,2$$

constitute a basis of the vector space H_{Λ^0}. The weighted degree of these algebraic local cohomology classes are

$$-1, -\frac{4}{3}, -\frac{4}{3}, -\frac{5}{3}.$$

Note that the weighted degree of $\left[\begin{array}{c}1\\ xy^2z\end{array}\right]$ and that of $\left[\begin{array}{c}1\\ xyz^2\end{array}\right]$ coincide. We have

$$\left[\begin{array}{c}1\\ \frac{\partial f}{\partial y}\ \frac{\partial f}{\partial z}\end{array}\right]|_{\Sigma-\{O\}} = \frac{1}{6x}\left[\begin{array}{c}1\\ y^2z\end{array}\right]|_{\Sigma-\{O\}}.$$

Set

$$\sigma = \frac{1}{x}\left[\begin{array}{c}1\\ y^2z\end{array}\right] \in i_*i^{-1}(\mathcal{H}^2_{[\Sigma]}(\mathcal{O}_{X,O})),$$

where i is the open inclusion map $i : \Sigma - \{O\} \longrightarrow \Sigma$. Then we have

$$\mathrm{bv}(\sigma) = \left[\begin{array}{c}1\\ xy^2z\end{array}\right], \mathrm{bv}(y\sigma) = \left[\begin{array}{c}1\\ xyz\end{array}\right],$$

where bv is the boundary value map

$$\mathrm{bv} : i_*i^{-1}(\mathcal{H}^2_{[\Sigma]}(\mathcal{O}_{X,O})) \longrightarrow \mathcal{H}^3_{[O]}(\mathcal{O}_X).$$

The weighted degree of the algebraic local cohomology class $\mathrm{bv}(\sigma)$ is equal to $-\frac{4}{3}$ and that of $\mathrm{bv}(y\sigma)$ is equal to -1.

Example 4.3.

Let $f(x,y,z) = xz^2 + y^2z$. The singular locus of the hypersurface $S = \{(x,y,z) \in X \mid f(x,y,z) = 0\}$ is the x-axis : $\Sigma = \{(x,y,z) \mid y = z = 0\}$. The defining function f is a weighted homogeneous polynomial of weight $(1,1,1)$ and the weighted degree of f is equal to 3. The function f has transversal singularities of type A_3 at each point on the stratum $\Sigma - \{O\}$.

Set $r(y,z) = f(1,y,z) = y^2z + z^2$. The b-function of $r(y,z)$ is

$$b_r(s) = (s+1)^2(s+\frac{3}{4})(s+\frac{5}{4}).$$

The b-function of f is

$$b_f(s) = (s+1)^2(s+\frac{3}{4})(s+\frac{5}{4}).$$

Since $b_f = b_r$, no new factor comes from the origin.

The primary decomposition of the ideal $< \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$ is

$$< \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >=< x, y > \cap < z^2, yz, 2xz + y^2 > .$$

Note that the associated prime of the ideal $< z^2, yz, 2xz + y^2 >$ is $< y, z >$. Since $\Gamma^1 = V(x, y)$, we define Λ^0 by $< \frac{\partial f}{\partial x}, x, y >$, which is equal to $< x, y, z^2 >$.

Let $H_{\Lambda^0} = \{\eta \in \mathcal{H}^3_{[O]}(\mathcal{O}_X) \mid \Lambda^0 \eta = 0\}$. Then, algebraic local cohomology classes $\begin{bmatrix} 1 \\ xyz^i \end{bmatrix}$, $i = 1, 2$ constitute a basis of the vector space H_{Λ^0}. The weighted degree of these algebraic local cohomology classes are $-1, -\frac{4}{3}$.

Let $\begin{bmatrix} 1 \\ \frac{\partial f}{\partial y} \ \frac{\partial f}{\partial z} \end{bmatrix}$ be the algebraic local cohomology class in $\mathcal{H}^2_{[V]}(\mathcal{O}_{X,O})$, where

$$V = V(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = V(yz, 2xz + y^2).$$

In order to analyze the algebraic local cohomology class above, we first compute the holonomic system of this algebraic local cohomology class ([10]) by using the algorithm constructed by T. Oaku([19]).

We have the following set of partial differential operators as a set of generators of annihilating ideals.

$$xz^2, 2xz + y^2, yz, y\frac{\partial}{\partial x}, 2xz\frac{\partial}{\partial y} - y, -x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + 1,$$

$$y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 5, z^2\frac{\partial}{\partial z} + 2z, (z\frac{\partial}{\partial z} + 2)\frac{\partial}{\partial x}$$

Since the strutum $\Sigma - \{O\}$ is non-singular, the holonomic system above on $\Sigma - \{O\}$ is simple as D-Module ([9], [10]). The dimension of the algebraic local cohomology sotution space on the stratum $\Sigma - \{O\}$ is therefore equal to one. By solving the holonomic system, we have

$$\begin{bmatrix} 1 \\ \frac{\partial f}{\partial y} \ \frac{\partial f}{\partial z} \end{bmatrix} |_{\Sigma - \{0\}} = \mathrm{const} \cdot (\frac{1}{x}\begin{bmatrix} 1 \\ yz^2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ y^3 z \end{bmatrix}).$$

Set $\sigma = \frac{1}{x}\begin{bmatrix} 1 \\ yz^2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ y^3 z \end{bmatrix} \in i_* i^{-1}(\mathcal{H}^2_{[\Sigma]}(\mathcal{O}_{X,O}))$. The set $\{\sigma, y\sigma, z\sigma\}$ can be regarded, by residue theorem, as the Noetherian operators on the stratum $\Sigma - \{O\}$ associated with the primary ideal $< z^2, yz, 2xz + y^2 >$.

We have

$$\mathrm{bv}(\sigma) = \begin{bmatrix} 1 \\ xyz^2 \end{bmatrix}, \mathrm{bv}(y\sigma) = 0, \mathrm{bv}(z\sigma) = \begin{bmatrix} 1 \\ xyz \end{bmatrix},$$

where bv is the boundary value map

$$\mathrm{bv} : i_* i^{-1}(\mathcal{H}^2_{[\Sigma]}(\mathcal{O}_{X,O})) \longrightarrow \mathcal{H}^3_{[O]}(\mathcal{O}_X).$$

We find

$$H_{\Lambda^0} = \mathrm{Span}_{\mathbb{C}}\{\mathrm{bv}(\sigma), \mathrm{bv}(z\sigma)\},$$

which is consistent with the *guess* presented previously.

Example 4.4.

Let $f(x,y,z) = xy^4 + z^2$ and let $S = \{(x,y,z) \in X \mid f(x,y,z) = 0\}$. The singular locus of S is $\Sigma = \{(x,y,z) \mid y = z = 0\}$. The defining function f is a weighted homogeneous polynomial of weight $(2,1,3)$ and the weighted degree of f is equal to 6.

Set $r(y,z) = f(1,y,z) = y^4 + z^2$. Then

$$b_r(s) = (s+1)^2(s+\frac{3}{4})(s+\frac{5}{4}).$$

The b-function of f is

$$b_f(s) = (s+1)^2(s+\frac{3}{4})(s+\frac{5}{4})(s+\frac{3}{2}).$$

Then, $b_r | b_f$ and the factor $s + \frac{3}{2}$ comes from the origin.

The primary decomposition of the ideal $< \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$ is $< x, z > \cap < y^3, z >$.
Since $\Gamma^1 = V(x,z)$, we define Λ^0 by $< \frac{\partial f}{\partial x}, x, z >$.
Let $H_{\Lambda^0} = \{\eta \in \mathcal{H}^3_{[O]}(\mathcal{O}_X) \mid \Lambda^0 \eta = 0\}$. Then, algebraic local cohomology classes

$$\begin{bmatrix} 1 \\ xy^j z \end{bmatrix}, j = 1,2,3,4$$

constitute a basis of the vector space H_{Λ^0}. The weighted degree of these algebraic local cohomology classes are

$$-1, -\frac{7}{6}, -\frac{4}{3}, -\frac{3}{2}.$$

We have

$$\begin{bmatrix} 1 \\ \frac{\partial f}{\partial y}\ \frac{\partial f}{\partial z} \end{bmatrix} |_{\Sigma - \{O\}} = \frac{1}{8x} \begin{bmatrix} 1 \\ y^3 z \end{bmatrix} |_{\Sigma - \{O\}}.$$

Set

$$\sigma = \frac{1}{x} \begin{bmatrix} 1 \\ y^3 z \end{bmatrix} \in i_* i^{-1}(\mathcal{H}^2_{[\Sigma]}(\mathcal{O}_{X,O})),$$

where i is the open inclusion map $i : \Sigma - \{O\} \longrightarrow \Sigma$. Then we have

$$\mathrm{bv}(\sigma) = \begin{bmatrix} 1 \\ xy^3 z \end{bmatrix}, \mathrm{bv}(y\sigma) = \begin{bmatrix} 1 \\ xy^2 z \end{bmatrix}, \mathrm{bv}(y^2\sigma) = \begin{bmatrix} 1 \\ xyz \end{bmatrix}.$$

The weighted degrees of these algebraic local cohomology classes are equal to $-1, -\frac{7}{6}, -\frac{4}{3}$. This result is again consistent with our *guess* .

Remark

In [30], [31], an algorithm for computing algebraic local cohomology classes attached to zero-dimensional ideals is described. The algorithm, which is free from S-polynomials computation and standard bases, has been implemented in the computer algebra system Risa/Asir. By using the algorithm, one can compute bases of the vector space H_{Λ^0} efficiently.

§ 5. Computer experiment with T. Oaku

Let f be a weighted homogeneous polynomial with non-isolated singularities and let E be the Euler operator with respect to the weight vector of f such that $Ef = f$. Let $\mathcal{A}nn_{D_X[s]}f^s$ be the annihilation ideal of f^s in the ring $D_X[s]$ of partial differential operators.

We consider the following ideal (cf. [9], [32])

$$I = (\mathcal{A}nn_{D_X[s]}f^s + D_X(E-s) + D_X J_f) \cap D_X,$$

where J_f is the Jacobian ideal of f. Note that the Euler operator E is used to eliminate the indeterminate s from the ideal in parenthesis. We set $M = D_X/I$.

We use algorithms, derived by T. Oaku ([18]) for computing $\mathcal{A}nn_{D_X[s]}f^s$ and the multiplicities of the characteristic variety $\mathrm{Ch}(M)$ of the holonomic D_X-module M associated with hypersurface with isolated line singularities. We also compute algebraic local cohomology solutions, supported on the stratum $\Sigma - \{O\}$, of relevant holonomic systems. We present, in this section, results of these computation.

Example 5.1.

Let $f(t,x,y) = y^2 - x^3 - tx^2$. The annihilator $\mathcal{A}nn_{D_X[s]}f^s$ is generated by

$$(2t+3x)\frac{\partial}{\partial t} - x\frac{\partial}{\partial x},\ 3y\frac{\partial}{\partial t} - y\frac{\partial}{\partial x} - tx\frac{\partial}{\partial y},$$

$$2y\frac{\partial}{\partial t} - x^2\frac{\partial}{\partial y},\ x\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + 2s.$$

Note that, in the list above, the factor $2t+3x$ appears as the coefficient of $\frac{\partial}{\partial t}$ of a partial differential operator(cf. [7], [15]) . The Euler operator is

$$E = \frac{1}{3}t\frac{\partial}{\partial t} + \frac{1}{3}x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y}.$$

Set

$$I = (\mathcal{A}nn_{D_X[s]} f^s + D_X(E-s) + D_X J_f) \cap D_X \text{ and } M = D_X/I,$$

where J_f is the Jacobian ideal of f. The characteristic variety $\mathrm{Ch}(M)$ of the holonomic system M consists of two components. One is the Lagrangian variety $T^*_\Sigma X$ and the other is the conormal to the origin, $T^*_{\{O\}}X$, where Σ is the singular locus of the Whitney umbrella $S = \{(t,x,y) \mid f(t,x,y) = 0\}$. The Hilbert-Samuel polynomial of the primary ideal associated with $T^*_{\{O\}}X$ is given by

$$\frac{2}{3!}u^3 + \frac{3}{2}u^2 + \frac{13}{6}u + 1.$$

Thus, the multiplicity of the component $T^*_{\{O\}}X$ is equal to two.

Notice that the Lê numbers λ^1, λ^0 are equal to $1, 2$ respectively ([7], [15]). Thus, we see what we have computed in section two is the multiplicity of the characteristic varieties of the holonomic system.

Let

$$M_s = D[s]/(\mathcal{A}nn_{D_X[s]} f^s + D_X J_f).$$

On the stratum $\Sigma - \{O\}$, the differental system M_s has algebraic local cohomology solutions if and only if $s+1=0$ and the set of algebraic local cohomology solutions, is a one-dimensional vector space generated by $t^{-\frac{1}{2}} \begin{bmatrix} 1 \\ xy \end{bmatrix}$, that is a section on $\Sigma - \{O\}$, of the local system of vaniching cycles.

Further, on the stratum $\{O\}$, the differental system M_s has algebraic local cohomology solutions if and only if $s + \frac{3}{2} = 0$ and the set of algebraic local cohomology solution space, is a one-dimensional vector space generated by $\begin{bmatrix} 1 \\ tx^2y \end{bmatrix} - \begin{bmatrix} 1 \\ t^2xy \end{bmatrix}$. Note that the algebraic local cohomology class $\begin{bmatrix} 1 \\ txy \end{bmatrix}$ is not a solution of the differential system M_s for any s.

In section two we used the algebraic local cohomology class $\frac{1}{t}\begin{bmatrix} 1 \\ xy \end{bmatrix}$ to represent the *horizontal* monodromy structure of the vanishing cycles in the transverse directions to the stratum $\Sigma - \{O\}$. But the class $\frac{1}{t}\begin{bmatrix} 1 \\ xy \end{bmatrix}$ do not represent the monodromy structure of the local system *on the stratum* $\Sigma - \{O\}$ of the vanishing cycles ([1], [25], [26]), which is correctly described by the algebraic local cohomology class $t^{-\frac{1}{2}}\begin{bmatrix} 1 \\ xy \end{bmatrix}$.

In terms of ordinary differential equations, the algebraic local cohomology class $\frac{1}{t}\begin{bmatrix} 1 \\ xy \end{bmatrix}$ satisfies $(t\frac{\partial}{\partial t} + 1)u = 0$ and the class $\begin{bmatrix} 1 \\ txy \end{bmatrix}$ supported on $\{O\}$ is an algebraic local cohomology sotution of this equation.

In contrast, the algebraic local cohomology class $t^{-\frac{1}{2}}\begin{bmatrix} 1 \\ xy \end{bmatrix}$ satisfies the ordinary differential equation $(t\dfrac{\partial}{\partial t}+\dfrac{1}{2})u=0$ that has no algebraic local cohomology sotution supported on $\{O\}$.

Therefore, the weighted degree of the algebraic local cohomology class $\begin{bmatrix} 1 \\ txy \end{bmatrix}$ has to be discarded from the list of roots of the (micro-)local b-function.

These fact implies that what we have observed is an effect of a kind of index theorem of differential equations with regular singularities ([13]).

Example 5.2. Let $f(x,y)=x^5y^2+y^3$. The annihilator $\mathcal{A}nn_{D_X[s]}f^s$ is generated by

$$(2x^5+3y)\tfrac{\partial}{\partial x}-5x^4y\tfrac{\partial}{\partial y} \text{ and } x\tfrac{\partial}{\partial x}+5y\tfrac{\partial}{\partial y}-15s.$$

Note also that, in the list above, $2x^5+3y$ appears as the coefficient of $\frac{\partial}{\partial x}$ of a partial differential operator. The Euler operator is

$$E=\frac{1}{15}x\frac{\partial}{\partial x}+\frac{1}{3}y\frac{\partial}{\partial y}.$$

Set

$$I=(\mathcal{A}nn_{D_X[s]}f^s+D_X(E-s)+D_XJ_f)\cap D_X \text{ and } M=D_X/I.$$

The characteristic variety $\mathrm{Ch}(M)$ of the holonomic system M consists of two components. One is the Lagrangian variety $T^*_\Sigma X$ and the other is the conormal to the origin, $T^*_{\{O\}}X$, where Σ is the singular locus of S. The multiplicity of $T^*_\Sigma X$ is equal to one. The Hilbert-Samuel polynomial of the primary ideal associated with $T^*_{\{O\}}X$ is given by

$$\frac{14}{2!}u^2-70u+287.$$

Thus, the multiplicity of the component $T^*_{\{O\}}X$ is equal to 14. Notice that the Lê numbers λ^1,λ^0 are equal to $1, 14$ respectively ([7], [15]). Thus, in section 3, the multiplicities of the characteristic varieties are correctly computed. The algebraic local cohomology solution space on the stratum $\Sigma-\{O\}$ of the holonomic system is a one-dimensional vector space generated by $x^{-\frac{5}{2}}\begin{bmatrix} 1 \\ y \end{bmatrix}$.

Note that the algebraic local cohomology class $\frac{1}{x^5}\begin{bmatrix} 1 \\ y \end{bmatrix}$ that satisfies the ordinary differential equation $(x\dfrac{\partial}{\partial x}+5)u=0$ is used in section three to represent the *horizontal* monodromy structure of the vanishing cycles in the transverse directions to the stratum

$\Sigma - \{O\}$, whereas this class do not describe the monodromy structure of the local system *on the stratum* $\Sigma - \{O\}$ of the vanishing cycles.

Note also that the algebraic local cohomology class $x^{-\frac{5}{2}} \begin{bmatrix} 1 \\ y \end{bmatrix}$ satisfies $(x\dfrac{\partial}{\partial x} + \dfrac{5}{2})u = 0$ that has no algebraic local cohomology sotution supported on $\{O\}$.

The set of algebraic local cohomology solution space, on the stratum $\{O\}$ is a 13 dimensional vector space generated by

$$\begin{bmatrix} 1 \\ x^i y \end{bmatrix}, \begin{bmatrix} 1 \\ x^i y^3 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ x^{5+i} y^2 \end{bmatrix}, i = 1, 2, 3, 4.$$

and

$$\begin{bmatrix} 1 \\ x^i y^2 \end{bmatrix} - \frac{3i}{5+2i} \begin{bmatrix} 1 \\ x^{5+i} y \end{bmatrix}, i = 1, 2, 3, 4, 5$$

The algebraic local cohomology class $\begin{bmatrix} 1 \\ x^5 y \end{bmatrix}$, which is an algebraic local cohomology solution of the differential equation $(x\dfrac{\partial}{\partial x} + 5)u = 0$ is not a solution of the holonomic system. These facts are all consistent with the interpretation.

To conclude the paper, we give the following observation.
Structures of the *vertical* monodromy of vanishing cycles are encoded in relevant holonomic D-modules.

References

[1] Barlet, D., Interaction de strates consécutives por les cycles évanescents, *Ann. Sci. Ecole Norm. Sup.* **24** 4 (1991), 401–505.

[2] Briançon, J. Maisonobe, P. et Merle, M., Localisation de sytèmes différentiels, stratifications de Whitney et condition de Thom, *Invent. Math* **117** (1994), 531–550.

[3] Deligne, P., Le formalisme des cycles evanescents, *Lecture Notes in Math.* **340** (1973), 82–115.

[4] Dimca, A., On the Milnor fibrations of weighted homogeneous polynomials, *Compositio Math.* **76** (1990), 19–47.

[5] Dimca, A., *Sheaves in Topology* Springer (2004).

[6] Ehrenpreis, L., *Fourier Analysis in Several Complex Variables*, Wiley Interscience (1970).

[7] Gaffney, T. and Massey, D., *Trends in equisingularity theory*, London Math. Soc. Lecture Notes Series **26** (1999), 207–248.

[8] Jong, T. de, Some classes of line singularities, *Math. Z.* **198** (1988), 493–517.

[9] Kashiwara, M., B-functions and hypersurface singularities (in Japanese), *RIMS Kokyuroku* **225** (1975), 16–53.

[10] Kashiwara, M., On the maximally overdetermined system of linear differential equations, I. *Pub. Res. Inst. Math. Sci.* **10** (1975), 563–579.

[11] Kashiwara, M., B-functions and holonomic systems – Rationality of roots of b-functions, *Invent. Math.* **38** (1976), 33–53.

[12] Kashiwara, M., On the holonomic systems of linear differential equations, II., *Invent. Math.* **49** (1978), 121–135.

[13] Komatsu, H., On the index of ordinary differential operators, *J. Fac. Sci. Univ. Tokyo* IA, **18** (1971–72), 379–398.

[14] Komatsu, H., Relative cohomology of sheaves of solutions of differential equations, *Lecture Notes in Math.* **287** (1973), 192–261.

[15] Massey, D., Lê Cycles and Hypersurface Singularities, *Lecture Notes in Math.*, Springer, **1615** (1995)

[16] Massey, D. and Siersma, D., Deformation of polar methods, *Ann. Inst. Fourier, Grenoble*, **42**, 4 (1992), 737–778.

[17] M. Noro and Takeshima, T., Risa/Asir - a computer algebra system, *Proc. Internat. Symp. on Symbolic and Algebraic Computation* pp. 387–396, P. S. Wang ed. ACM, New York 1992

[18] Oaku, T., Algorithms for the b-function and D-modules associated with a polynomial, *J. Pure and Applied Algebra*, **117, 118** (1997), 495–518.

[19] Oaku, T., Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules, *Adv. Applied Math.*, **19** (1997), 61–105.

[20] Palamodov, V. I., A remark on the exponential representation of solutions of differential equations with constant coefficients (Russian), *Mat. Sbornik* **76 (118)** (1968), 417–434.

[21] Pellikaan, R., Deformations of hypersurfaces with a one-dimensional singular locus, *J. Pure and Applied Algebra*, **67** (1990), 49–71.

[22] Saito, K., Quasihomogene isolierte Singularitaten von Hyperflächen, *Invent. Math.*, **14** (1971), 123–142.

[23] Siersma, D., *Isolated line singularities,* Proc. Symposia in Pure Math. **49** (1983), 485–496.

[24] Siersma, D., Hypersurfaces with singular locus a plane curve and transversal type A_1 *Banach Center Pub.*, **20** (1988), 397–410.

[25] Siersma, D., Variation mappings on singularities with a 1-dimensional critical locus, *Topology* **30** No. 3 (1991), 445–468.

[26] Siersma, D., *The vanishing topology on non isolated singularities,* New developments in Singularity Theory, NATO Sci. Ser. II Math. Phys. Chem., **21**, Kluwer Acad. Publ., Dordrecht, (2001), 447–472.

[27] Tajima. S., Laurent expansions of algebraic local cohomology classes and Ehrenpreis's Noether operators (in Japanese), *RIMS Kokyuroku* **1138** (2000), 87–95.

[28] Tajima. S., Zero-dimensional primary ideals and algorithms for computing Noether operators (in Japanese), *RIMS Kokyuroku* **1395** (2004), 57–63.

[29] Tajima. S., Noether operators and algorithms for computing Grothendieck local residues (in Japanese), *RIMS Kokyuroku* **1431** (2005), 123–136.

[30] Tajima, S and Nakamura, Y., Annihilating ideals for an algebraic local cohomology class, *J. Symbolic Computation*, **44** (2009), 435–448.

[31] Tajima, S., Nakamura, Y. and Nabeshima, K., *Standard bases and algebraic local cohomology for zero dimensional ideals*, Adv. Studies in Pure Math., **56** (2009), 341–361.

[32] Yano, T., On the theory of b-functions, *Pub. Res. Inst. Math. Sci.* **14** (1978), 111–202.