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Kernel functions and symbols of pseudodifferential operators of infinite order with an apparent parameter

By

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Abstract

We introduce a new representation of pseudodifferential operators of infinite order (or holomorphic microlocal operators) and symbol class.

Introduction

In this report, we shall announce our recent results about a complete symbol theory for the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators in the complex analytic category. The foundation of the symbol theory of $\mathscr{E}_X^{\mathbb{R}}$ at the present stage (see [1], [4]) is quite unsatisfactory. There are the following two issues:

(i) Kamimoto and Kataoka have pointed out in their work [5] that the space of the kernel functions which comes from standard Čech representation of cohomology groups is not closed under composition of kernel functions defined by naive integration employed in [1], [4].

(ii) the relation between the action of operators by integration of kernel functions and canonical action through cohomological definition was not clarified (see [7], [8]).

For (i), Kamimoto and Kataoka [5], [6] give a possible solution by introducing the notion of formal kernels. On the other hand, we shall start our study from other point of view; we use a new isomorphism of cohomology groups (see Proposition 1.2), we establish a

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new symbol theory, and solve (i) and (ii) above. Details in this article will be published in our forthcoming paper [2].

§ 1. Local Cohomology Groups on a Vector Space

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integers, of real numbers and of complex numbers respectively. Further set $\mathbb{N} := \{m \in \mathbb{Z};\, m \geqslant 1\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{C}^{\times} := \{c \in \mathbb{C};\, c \neq 0\}$. Let X be a finite dimensional $\mathbb{R}$-vector space. Let a, b, r be real numbers such that $0 < b - a < \pi$ and $r > 0$. We define an open proper sector $S \subset \mathbb{C}$ by

$$S := \{\eta \in \mathbb{C};\, a < \arg \eta < b,\, 0 < |\eta| < r\},$$

Let (x, η) be coordinates of $X \times \mathbb{C}$, and $\pi_\eta \colon X \times \mathbb{C} \ni (x, \eta) \mapsto x \in X$ the canonical projection. Let $G \subset X$ be a closed subset (not necessarily convex) and $U \subset X$ an open neighborhood of the origin. In this section we give another representation of local cohomology groups $H^k_{G \cap U}(U; \mathcal{F})$ for a sheaf $\mathcal{F}$ on X. For this purpose, we need some preparations. Let Z be a closed subset in X and $\varphi \colon X \times [0, 1] \to X$ a continuous deformation mapping which satisfies the following conditions:

(i) $\varphi(x, 1) = x$ for any $x \in X$ and $\varphi(z, s) = z$ for any $z \in Z$ and $s \in [0, 1]$.

(ii) $\varphi(\varphi(x, s), 0) = \varphi(x, 0)$ for any $s \in [0, 1]$ and $x \in X$.

(iii) We set
$$\rho_\varphi(x, s) := |\varphi(x, s) - \varphi(x, 0)|.$$
Then $\rho_\varphi(x, s)$ is a strictly increasing function of s outside Z, i.e. if $s_1 < s_2$, we have $\rho_\varphi(x, s_1) < \rho_\varphi(x, s_2)$ for any $x \in X \setminus Z$.

We set, for short

$$\rho_\varphi(x) := \rho_\varphi(x, 1) = |\varphi(x, 1) - \varphi(x, 0)| = |x - \varphi(x, 0)|.$$

Here we remark

$$\rho_\varphi(\varphi(x, s)) = |\varphi(x, s) - \varphi(\varphi(x, s), 0)| = |\varphi(x, s) - \varphi(x, 0)| = \rho_\varphi(x, s).$$

Let $\varrho > 0$ be a positive constant. We define the subsets in $X \times \mathbb{C}$ by

$$\begin{aligned} \widehat{U} &:= \{(x, \eta) \in U \times S;\, \rho_\varphi(x) < \varrho|\eta|\}, \\ \widehat{G} &:= \{(\varphi(x, s), \eta) \in X \times \mathbb{C};\, x \in G,\, \rho_\varphi(x) \leqslant \varrho|\eta|,\, 0 \leqslant s \leqslant 1\}. \end{aligned}$$

Note that $\widehat{G} \cap \widehat{U}$ is a closed subset in $\widehat{U}$.

Example 1.1. Let $X = \mathbb{C}^n \supset Z := \{z \in X;\, z_1 = 0\}$, and we set $\varphi(z,s) := (sz_1, z_2, \ldots, z_n)$. Then $\rho_\varphi(z,s) = |sz_1|$ and $\rho_\varphi(z) = |z_1|$. Let $\boldsymbol{\kappa} := (r, r', \varrho, \theta) \in \mathbb{R}^4$ be a 4-tuple of positive constants with

$$0 < \theta < \frac{\pi}{2}, \quad 0 < \varrho < 1, \quad 0 < r < \varrho r'. \tag{1.1}$$

We set

$$\begin{aligned} S_{\boldsymbol{\kappa}} &:= S_{r,\theta/4} = \{\eta \in \mathbb{C};\, |\arg \eta| < \frac{\theta}{4},\, 0 < |\eta| < r\}, \\ U_{\boldsymbol{\kappa}} &:= \bigcap_{i=2}^{n} \{z \in X;\, |z_1| < \varrho r,\, |z_i| < r'\}, \\ G_{\boldsymbol{\kappa}} &:= \bigcap_{i=2}^{d} \{z \in X;\, |\arg z_1| \leqslant \frac{\pi}{2} - \theta,\, \varrho^2 |z_i| \leqslant |z_1|\}. \end{aligned} \tag{1.2}$$

Then we have

$$\begin{aligned} \widehat{U}_{\boldsymbol{\kappa}} &= \{(z,\eta) \in U_{\boldsymbol{\kappa}} \times S_{\boldsymbol{\kappa}};\, |z_1| < \varrho|\eta|\}, \\ \widehat{G}_{\boldsymbol{\kappa}} &= \{(\varphi(z,s),\eta) \in X \times \mathbb{C};\, z \in G_{\boldsymbol{\kappa}},\, \rho_\varphi(z) \leqslant \varrho|\eta|,\, 0 \leqslant s \leqslant 1\} \\ &= \bigcap_{i=2}^{n} \{(sz_1, z_2, \ldots, z_n, \eta) \in X \times \mathbb{C};\, |\arg z_1| \leqslant \frac{\pi}{2} - \theta,\, \varrho^2|z_i| \leqslant |z_1|, \\ &\qquad\qquad |z_1| \leqslant \varrho|\eta|,\, 0 \leqslant s \leqslant 1\} \\ &= \bigcap_{i=2}^{n} \{(z,\eta) \in X \times \mathbb{C};\; |z_1| \leqslant \varrho|\eta|,\, |\arg z_1| \leqslant \frac{\pi}{2} - \theta,\, \varrho|z_i| \leqslant |\eta|\}. \end{aligned} \tag{1.3}$$

We can have the following:

Proposition 1.2. *Let $\mathcal{F}$ be a complex of Abelian sheaves on X. Assume that U satisfies that $\sup_{x \in U} \rho_\varphi(x) < \varrho r$. Then there exists the following quasi-isomorphism*:

$$\boldsymbol{R}\mathit{\Gamma}_{G \cap U}(U; \mathcal{F}) \simeq \boldsymbol{R}\mathit{\Gamma}_{\widehat{G} \cap \widehat{U}}(\widehat{U}; \pi_\eta^{-1}\mathcal{F}).$$

§ 2. Holomorphic Microfunctions with an Apparent Parameter

Let X be an n-dimensional $\mathbb{C}$-vector space with the coordinates $z = (z_1, \ldots, z_n)$, and Y the closed complex submanifold of X defined by $\{z' = 0\}$ where $z = (z', z'')$ with $z' := (z_1, \ldots, z_d)$ for some $1 \leqslant d \leqslant n$. In what follows, we denote an object defined on the space $X \times \mathbb{C}$ by a symbol with $\widehat{\ }$ like $\widehat{U}_{\boldsymbol{\kappa}}$ etc. For any $z \in \mathbb{C}^n$, we set $\|z\| := \max_{1 \leqslant i \leqslant n}\{|z_i|\}$. Let $\mathscr{O}_X$ be the sheaf of holomorphic functions on X, and $\mathscr{C}^{\mathbb{R}}_{Y|X}$ the sheaf of real holomorphic microfunctions along Y on the conormal bundle $T^*_Y X$ to Y

(see [9], [11]). Let $z_0^* = (0; 1, 0, \dots, 0) \in T_Y^*X$. Let $\boldsymbol{\kappa} = (r, r', \varrho, \theta) \in \mathbb{R}^4$ be as in (1.1), and recall the sets of (1.2) and (1.3) in Example 1.1. By the definition of $\mathscr{C}_{Y|X}^{\mathbb{R}}$, we have

$$\mathscr{C}_{Y|X,z_0^*}^{\mathbb{R}} = \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} H^d_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X).$$

Here $\mathbf{0} := (0, 0, 0, 0)$. We can apply the result in § 1 to this case. Assume that $U_{\boldsymbol{\kappa}}$ is sufficiently small so that the assumption of Proportion 1.2 is satisfied for $\varphi(z, s) = (sz_1, z_2, \dots, z_n)$. Set $\pi_\eta\colon X \times \mathbb{C} \ni (z, \eta) \mapsto z \in X$. Thus, from the exact sequence

$$0 \to \pi_\eta^{-1}\mathscr{O}_X \to \mathscr{O}_{X\times\mathbb{C}} \xrightarrow{\partial_\eta} \mathscr{O}_{X\times\mathbb{C}} \to 0,$$

we obtain the following distinguished triangle:

$$\begin{array}{l}\boldsymbol{R}\varGamma_{\widehat{G}_{\boldsymbol{\kappa}}\cap\widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \pi_\eta^{-1}\mathscr{O}_X) \longrightarrow \boldsymbol{R}\varGamma_{\widehat{G}_{\boldsymbol{\kappa}}\cap\widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{X\times\mathbb{C}}) \xrightarrow{\partial_\eta} \boldsymbol{R}\varGamma_{\widehat{G}_{\boldsymbol{\kappa}}\cap\widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{X\times\mathbb{C}}) \xrightarrow{+1} \\ \qquad\quad \uparrow\wr \\ \boldsymbol{R}\varGamma_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X).\end{array}$$

Further we can prove:

Proposition 2.1. *If $k \neq d$, then*

$$H^k_{\widehat{G}_{\boldsymbol{\kappa}}\cap\widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{X\times\mathbb{C}}) = H^k_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) = 0.$$

Definition 2.2. By Proposition 2.1, we define

$$\begin{aligned}\widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) &:= H^d_{\widehat{G}_{\boldsymbol{\kappa}}\cap\widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{X\times\mathbb{C}}),\\ C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) &:= \operatorname{Ker}(\partial_\eta\colon \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) \to \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})).\end{aligned}$$

Summing up, we obtain:

Theorem 2.3. *There exist isomorphisms*

$$\begin{array}{ccc} H^d_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) & \xrightarrow{\sim} & C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) \\ \downarrow & & \downarrow \\ \mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} & \xrightarrow{\sim} & \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}). \end{array}$$

We now consider a Čech representation of $C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})$. We set

$$\begin{aligned} V^{(1)}_{\boldsymbol{\kappa}} &:= \{z \in U_{\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg z_1 < \frac{3\pi}{2} + \theta\},\\ V^{(i)}_{\boldsymbol{\kappa}} &:= \{z \in U_{\boldsymbol{\kappa}};\ \varrho^2|z_i| > |z_1|\} \qquad (2 \leqslant i \leqslant d),\end{aligned}$$

$$\widehat{V}^{(1)}_{\boldsymbol{\kappa}} := \{(z,\eta) \in \widehat{U}_{\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg z_1 < \frac{3\pi}{2} + \theta\},$$
$$\widehat{V}^{(i)}_{\boldsymbol{\kappa}} := \{(z,\eta) \in \widehat{U}_{\boldsymbol{\kappa}};\ \varrho|z_i| > |\eta|\} \qquad (2 \leqslant i \leqslant d).$$

Let $\mathcal{P}_d$ be the set of all the subset of $\{1,\dots,d\}$ and $\mathcal{P}_d^\vee \subset \mathcal{P}_d$ consisting of $\alpha \in \mathcal{P}_d$ with $\#\alpha = d-1$ ($\#\alpha$ denotes the number of elements in α). For $\alpha \in \mathcal{P}_d$, we define

$$\widehat{V}^{(\alpha)}_{\boldsymbol{\kappa}} := \bigcap_{i\in\alpha} \widehat{V}^{(i)}_{\boldsymbol{\kappa}}, \qquad V^{(\alpha)}_{\boldsymbol{\kappa}} := \bigcap_{i\in\alpha} V^{(i)}_{\boldsymbol{\kappa}}, \qquad \widehat{V}^{(*)}_{\boldsymbol{\kappa}} := \widehat{V}^{(\{1,\dots,d\})}_{\boldsymbol{\kappa}} = \bigcap_{i=1}^{d} \widehat{V}^{(i)}_{\boldsymbol{\kappa}}.$$

As each $\widehat{V}^{(\alpha)}_{\boldsymbol{\kappa}}$ (resp. $V^{(\alpha)}_{\boldsymbol{\kappa}}$) is pseudoconvex, we have

$$H^d_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}};\mathscr{O}_X) = \frac{\varGamma(V^{(*)}_{\boldsymbol{\kappa}};\mathscr{O}_X)}{\sum\limits_{\alpha\in\mathcal{P}_d^\vee}\varGamma(V^{(\alpha)}_{\boldsymbol{\kappa}};\mathscr{O}_X)} \xrightarrow{\sim} \{u \in \frac{\varGamma(\widehat{V}^{(*)}_{\boldsymbol{\kappa}};\mathscr{O}_{X\times\mathbb{C}})}{\sum\limits_{\alpha\in\mathcal{P}_d^\vee}\varGamma(\widehat{V}^{(\alpha)}_{\boldsymbol{\kappa}};\mathscr{O}_{X\times\mathbb{C}})};\ \partial_\eta u = 0\} = C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}).$$

§ 3. Cohomological Representation of $\mathscr{E}^{\mathbb{R}}_X$ with an Apparent Parameter

We inherit the same notation from the previous section. Let (z,w) be coordinates of $X \times X$. Let $\varDelta \subset X \times X$ be the diagonal set. We identify X with $\varDelta$, and

$$T^*X = \{(z;\zeta)\} \simeq \{(z,z;\zeta,-\zeta)\} = T^*_\varDelta(X\times X).$$

Let $\mathscr{E}^{\mathbb{R}}_X$ denote the sheaf of pseudodifferential operators on the cotangent bundle T^*X of X. Let $\varOmega_X$ be the sheaf of holomorphic n-forms on X, and set $p_2\colon T^*(X\times X) \ni (z,w;\zeta,\tilde{\zeta}) \mapsto w \in X$. Then by the definition, $\mathscr{E}^{\mathbb{R}}_X = \mathscr{C}^{\mathbb{R}}_{X|X\times X} \underset{p_2^{-1}\mathscr{O}_X}{\otimes} p_2^{-1}\varOmega_X$. We fix $z_0^* = (z_0;\zeta_0) := (0;1,0,\dots,0) \in \dot{T}^*X$. Let $\boldsymbol{\kappa} = (r,r',\varrho,\theta) \in \mathbb{R}^4$ be as in (1.1). We set

$$\widehat{U}_{\varDelta,\boldsymbol{\kappa}} := \bigcap_{i=2}^{n}\{(z,w,\eta) \in X\times X\times\mathbb{C};\ \|z\| < r',\ \eta \in S_{\boldsymbol{\kappa}},\ |z_1 - w_1| < \varrho|\eta|,\ |z_i - w_i| < r'\},$$
$$\widehat{V}^{(1)}_{\varDelta,\boldsymbol{\kappa}} := \{(z,w,\eta) \in \widehat{U}_{\varDelta,\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg(z_1 - w_1) < \frac{3\pi}{2} + \theta\},$$
$$\widehat{V}^{(i)}_{\varDelta,\boldsymbol{\kappa}} := \{(z,w,\eta) \in \widehat{U}_{\varDelta,\boldsymbol{\kappa}};\ \varrho|z_i - w_i| > |\eta|\} \quad (2 \leqslant i \leqslant n),$$
$$U_{\varDelta,\boldsymbol{\kappa}} := \bigcap_{i=2}^{n}\{(z,w) \in X\times X;\ \|z\| < r',\ |z_1 - w_1| < \varrho r,\ |z_i - w_i| < r'\},$$
$$G_{\varDelta,\boldsymbol{\kappa}} := \bigcap_{i=2}^{n}\{(z,w) \in X\times X;\ |\arg(z_1 - w_1)| \leqslant \frac{\pi}{2} - \theta,\ \varrho^2|z_i - w_i| \leqslant |z_1 - w_1|\}.$$
$$V^{(1)}_{\varDelta,\boldsymbol{\kappa}} := \{(z,w) \in U_{\varDelta,\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg(z_1 - w_1) < \frac{3\pi}{2} + \theta\},$$
$$V^{(i)}_{\varDelta,\boldsymbol{\kappa}} := \{(z,\ w) \in U_{\varDelta,\boldsymbol{\kappa}};\ \varrho^2|z_i - w_i| > |z_1 - w_1|\} \quad (2 \leqslant i \leqslant n).$$

We also set

$$E^{\mathbb{R}}_X(\boldsymbol{\kappa}) := C^{\mathbb{R}}_{X|X\times X}(\boldsymbol{\kappa}) \underset{p_2^{-1}\mathscr{O}_X}{\otimes} p_2^{-1}\Omega_X \,.$$

Further set for short:

$$\begin{aligned}
N^{\mathbb{R}}_X(\boldsymbol{\kappa}) &:= \sum_{\alpha\in\mathcal{P}_n^\vee} \Gamma(V^{(\alpha)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X\times X}) \subset K^{\mathbb{R}}_X(\boldsymbol{\kappa}) := \Gamma(V^{(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X\times X}),\\
\widehat{N}^{\mathbb{R}}_X(\boldsymbol{\kappa}) &:= \sum_{\alpha\in\mathcal{P}_n^\vee} \Gamma(\widehat{V}^{(\alpha)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{X\times X\times\mathbb{C}}) \subset \widehat{K}^{\mathbb{R}}_X(\boldsymbol{\kappa}) := \{\psi \in \Gamma(\widehat{V}^{(*)}_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n,0)}_{X\times X\times\mathbb{C}});\ \partial_\eta\psi \in \widehat{N}^{\mathbb{R}}_X(\boldsymbol{\kappa})\}.
\end{aligned}$$

Here $\mathscr{O}^{(0,n,0)}_{X\times X\times\mathbb{C}} := \mathscr{O}_{X\times X\times\mathbb{C}} \underset{q_2^{-1}\mathscr{O}_X}{\otimes} q_2^{-1}\Omega_X$ with $q_2\colon X\times X\times\mathbb{C} \ni (z,w,\eta) \mapsto w \in X$, and $\mathscr{O}^{(0,n)}_{X\times X}$ is defined in a similar way. Then by Theorem 2.3, we obtain

$$\begin{tikzcd}
H^n_{G_{\Delta,\boldsymbol{\kappa}}\cap U_{\Delta,\boldsymbol{\kappa}}}(U_{\Delta,\boldsymbol{\kappa}}; \mathscr{O}^{(0,n)}_{X\times X}) \arrow[r, "\sim"] \arrow[d, equal] & E^{\mathbb{R}}_X(\boldsymbol{\kappa}) \arrow[d, equal] \\
K^{\mathbb{R}}_X(\boldsymbol{\kappa})/N^{\mathbb{R}}_X(\boldsymbol{\kappa}) \arrow[r, "\sim"] \arrow[d] & \widehat{K}^{\mathbb{R}}_X(\boldsymbol{\kappa})/\widehat{N}^{\mathbb{R}}_X(\boldsymbol{\kappa}) \arrow[d] \\
\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \arrow[r, "\sim"] & \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} E^{\mathbb{R}}_X(\boldsymbol{\kappa}).
\end{tikzcd}$$

Let $(z,\eta)\in X\times\mathbb{C}$. Set $\beta_0 := \dfrac{\varrho}{2}e^{-\sqrt{-1}(\pi+\theta)/2}$ and $\beta_1 := \dfrac{\varrho}{2}e^{\sqrt{-1}(\pi+\theta)/2}$, and we define, for a sufficiently small $\varepsilon>0$, the path $\gamma_i(z,\eta;\varrho,\theta)$ in $\mathbb{C}_{w_i}$ by

$$\begin{aligned}
\gamma_1(z,\eta;\varrho,\theta) &:= \{w_1 = z_1 + t\beta_0\eta;\ 1\geqslant t\geqslant\varepsilon\} \vee \{w_1 = z_1 + \varepsilon\beta_0\eta\, e^{\sqrt{-1}(\pi+\theta)t};\ 0\leqslant t\leqslant 1\}\\
&\qquad \vee \{w_1 = z_1 + t\beta_1\eta;\ \varepsilon\leqslant t\leqslant 1\},\\
\gamma_i(z,\eta;\varrho) &:= \{w_i = z_i + \Bigl(\frac{|\eta|}{\varrho}+\varepsilon\Bigr)e^{2\pi\sqrt{-1}t};\ 0\leqslant t\leqslant 1\} \quad (2\leqslant i\leqslant n).
\end{aligned}$$

Note that $\gamma_1(z,\eta;\varrho,\theta)$ joins the two points $z_1+\beta_0\eta$ and $z_1+\beta_1\eta$, which depend on the variables z_1 and η holomorphically. Define the real n-dimensional chain in X made from these paths by

$$\gamma(z,\eta;\varrho,\theta) := \gamma_1(z,\eta;\varrho,\theta)\times\gamma_2(z,\eta;\varrho)\times\cdots\times\gamma_n(z,\eta;\varrho) \subset X.$$

Theorem 3.1. *The bi-linear morphism*

$$\begin{array}{ccc}
\mu\colon E^{\mathbb{R}}_X(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) & \longrightarrow & C^{\mathbb{R}}_{Y|X}(\widetilde{\boldsymbol{\kappa}})\\
\cup & & \cup\\
[\psi(z,w,\eta)\,dw]\otimes[u(z,\eta)] & \longmapsto & [\displaystyle\int_{\gamma(z,\eta;\varrho,\theta)} \psi(z,w,\eta)\,u(w,\eta)\,dw]
\end{array}$$

is well defined. Here $\widetilde{\boldsymbol{\kappa}} = (\tilde{r}, \tilde{r}', \tilde{\varrho}, \tilde{\theta})$ *is a 4-tuple of positive constants satisfying*

$$0 < \tilde{r} < r, \quad 0 < \tilde{r}' < \frac{r'}{2}, \quad 0 < \tilde{\theta} < \frac{\theta}{4}, \quad 0 < \tilde{\varrho} < \frac{\varrho}{2} \sin\frac{\theta}{4}\,.$$

Moreover,

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \underset{\mathbb{C}}{\otimes} \mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} (E^{\mathbb{R}}_X(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})) \xrightarrow{\mu} \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) = \mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}$$

coincides with the cohomological action of $\mathscr{E}^{\mathbb{R}}_{X,z_0^*}$ *on* $\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*}$*.*

As a corollary of the theorem, we have the result on the composition on $E^{\mathbb{R}}_X(\boldsymbol{\kappa})$:

Corollary 3.2. *Let* $\widetilde{\boldsymbol{\kappa}} = (\tilde{r}, \tilde{r}', \tilde{\varrho}, \tilde{\theta}) \in \mathbb{R}^4$ *satisfying*

$$0 < \tilde{r} < r, \quad 0 < \tilde{r}' < \frac{r'}{8}, \quad 0 < \tilde{\theta} < \frac{\theta}{4}, \quad 0 < \tilde{\varrho} < \frac{\varrho}{2} \sin\frac{\theta}{4},$$

and the corresponding conditions to (1.1)*. Then there exists the bi-linear morphism*

$$\begin{array}{ccc} \mu\colon E^{\mathbb{R}}_X(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} E^{\mathbb{R}}_X(\boldsymbol{\kappa}) & \longrightarrow & E^{\mathbb{R}}_X(\widetilde{\boldsymbol{\kappa}}) \\ \uplus & & \uplus \\ [\psi_1(z,w,\eta)\,dw] \otimes [\psi_2(z,w,\eta)\,dw] & \longmapsto & \Big[\Big(\displaystyle\int_{\gamma(z,\eta;\varrho,\theta)} \psi_1(z,\widetilde{w},\eta)\,\psi_2(\widetilde{w},w,\eta)\,d\widetilde{w}\Big)dw\Big]. \end{array}$$

Moreover the multiplication of the ring $\mathscr{E}^{\mathbb{R}}_{X,z_0^*}$ *coincides with*

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \underset{\mathbb{C}}{\otimes} \mathscr{E}^{\mathbb{R}}_{X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} (E^{\mathbb{R}}_X(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} E^{\mathbb{R}}_X(\boldsymbol{\kappa})) \xrightarrow{\mu} \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} E^{\mathbb{R}}_X(\boldsymbol{\kappa}) = \mathscr{E}^{\mathbb{R}}_{X,z_0^*}.$$

Remark 3.3. For any $\psi_1(z,w,\eta)$, $\psi_2(z,w,\eta) \in \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} \widehat{K}^{\mathbb{R}}_X(\boldsymbol{\kappa})$, we can show

$$\int_{\gamma(z,\eta;\varrho,\theta)} \psi_1(z,\widetilde{w},\eta)\,\psi_2(\widetilde{w},w,\eta)\,d\widetilde{w} \in \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} \widehat{K}^{\mathbb{R}}_X(\boldsymbol{\kappa}).$$

Further if either ψ_1 or ψ_2 belongs to $\varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} \widehat{N}^{\mathbb{R}}_X(\boldsymbol{\kappa})$, then

$$\int_{\gamma(z,\eta;\varrho,\theta)} \psi_1(z,\widetilde{w},\eta)\,\psi_2(\widetilde{w},w,\eta)\,d\widetilde{w} \in \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} \widehat{N}^{\mathbb{R}}_X(\boldsymbol{\kappa}).$$

However in general, for $\psi_1(z,w)$, $\psi_2(z,w) \in \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} K^{\mathbb{R}}_X(\boldsymbol{\kappa})$,

$$\int_{\gamma(z,\eta_0;\varrho,\theta)} \psi_1(z,\widetilde{w})\,\psi_2(\widetilde{w},w)\,d\widetilde{w} \notin \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} K^{\mathbb{R}}_X(\boldsymbol{\kappa}),$$

for any fixed η_0 (see Kamimoto-Kataoka [5], [6]). In order to solve this problem, Kamimoto-Kataoka [5], [6] introduce a new class of kernel functions. On the other hand, our starting point is a new representation of cohomology as in § 1.

Our parameter η controls the convergent radius with respect to z_1 variable. Note that

$$\psi'(z,w,\eta) := \int_{\gamma(z,\eta;\varrho,\theta)} \psi_1(z,\widetilde{w})\,\psi_2(\widetilde{w},w)\,d\widetilde{w} \in \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} \widehat{K}^{\mathbb{R}}_X(\boldsymbol{\kappa}).$$

§ 4. Symbols and Formal Symbols

For an explicit calculus of operators in $\mathscr{E}^{\mathbb{R}}_X$, it is very useful to introduce the notions of symbols and formal symbols. In this section, we recall definitions of symbols and formal symbols of $\mathscr{E}^{\mathbb{R}}_X$ (see [1], [3], [4]).

Let $X := \mathbb{C}^n$ and consider $T^*X \simeq X \times \mathbb{C}^n = \{(z;\zeta)\}$. Let $\pi\colon T^*X \to X$ be a canonical projection. If $V \subset T^*X$ is a conic set and $d > 0$, we set

$$V[d] := \{(z,\zeta) \in V;\ \|\zeta\| \geqslant d\}.$$

For any open conic subset $\Omega \subset T^*X$ and $\rho \geqslant 0$, we set

$$\Omega_\rho := \mathrm{Cl}\Bigl[\bigcup_{(z,\zeta)\in\Omega}\{(z+z';\zeta+\zeta') \in \mathbb{C}^{2n};\ \|z'\| \leqslant \rho,\ \|\zeta'\| \leqslant \rho\|\zeta\|\}\Bigr].$$

Here Cl means the closure. In particular, $\Omega_0 = \mathrm{Cl}\,\Omega$. For any $d > 0$ and $\rho \in \left[0,1\right[$, we set $d_\rho := d(1-\rho)$ for short. Let U, V be conic subsets of T^*X. Then we write $V \underset{\text{conic}}{\Subset} U$ if V is generated by a compact subset of U.

Definition 4.1. Let $\Omega \underset{\text{conic}}{\Subset} T^*X$ be an open conic subset.

(1) We call $P(z,\zeta)$ a *symbol* on Ω if there exist $d > 0$ and $\rho \in \left]0,1\right[$ such that $P(z,\zeta) \in \Gamma(\Omega_\rho[d_\rho];\mathscr{O}_{T^*X})$, and for any $h > 0$ there exists $C_h > 0$ such that

$$|P(z,\zeta)| \leqslant C_h e^{h\|\zeta\|} \quad ((z;\zeta) \in \Omega_\rho[d_\rho]).$$

We denote by $\mathscr{S}(\Omega)$ the set of symbols on Ω.

(2) We call $P(z,\zeta)$ a *null-symbol* on Ω if there exist $d > 0$ and $\rho \in \left]0,1\right[$ such that $P(z,\zeta) \in \Gamma(\Omega_\rho[d_\rho];\mathscr{O}_{T^*X})$, and there exist C, $\delta > 0$ such that

$$|P(z,\zeta)| \leqslant Ce^{-\delta\|\zeta\|} \quad ((z;\zeta) \in \Omega_\rho[d_\rho]).$$

We denote by $\mathscr{N}(\Omega)$ the set of null-symbols on Ω.

(3) For any $z^*_0 \in \dot{T}^*X$, we set

$$\mathscr{S}_{z^*_0} := \varinjlim_{\Omega\ni z^*_0} \mathscr{S}(\Omega) \supset \mathscr{N}_{z^*_0} := \varinjlim_{\Omega\ni z^*_0} \mathscr{N}(\Omega)$$

where $\Omega \underset{\text{conic}}{\Subset} T^*X$ ranges though open conic neighborhoods of z^*_0.

Definition 4.2. Let t be an indeterminate.

(1) $P(t;z,\zeta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z,\zeta) \in \widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ if $P(t;z,\zeta) \in \Gamma(\Omega_{\rho}[d_{\rho}];\mathscr{O}_{T^*X})[\![t]\!]$ for some $d>0$ and $\rho \in \left]0,1\right[$, and there exists a constant $A>0$ satisfying the following: for any $h>0$ there exists a constant $C_h>0$ such that

$$|P_{\nu}(z,\zeta)| \leqslant \frac{C_h A^{\nu} \nu!\, e^{h\|\zeta\|}}{\|\zeta\|^{\nu}} \quad (\nu \in \mathbb{N}_0,\ (z;\zeta) \in \Omega_{\rho}[d_{\rho}]).$$

(2) $P(t;z,\zeta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z,\zeta,\eta) \in \widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ is an element of $\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)$ if there exists a constant $A>0$ satisfying the following: for any $h>0$ there exists a constant $C_h>0$ such that

$$\Big|\sum_{\nu=0}^{m-1} P_{\nu}(z,\zeta)\Big| \leqslant \frac{C_h\, A^m m!\, e^{h\|\zeta\|}}{\|\zeta\|^m} \quad (m \in \mathbb{N},\ (z;\zeta) \in \Omega_{\rho}[d_{\rho}]).$$

(3) We set

$$\widehat{\mathscr{S}}_{\mathrm{cl},z_0^*} := \varinjlim_{\Omega} \widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega) \supset \widehat{\mathscr{N}}_{\mathrm{cl},z_0^*} := \varinjlim_{\Omega} \widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega).$$

We call each element of $\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ (resp. $\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)$) a *classical formal symbol* (resp. *classical formal null-symbol*) *on* Ω.

Definition 4.3. Let t be an indeterminate.

(1) $P(t;z,\zeta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z,\zeta) \in \widehat{\mathscr{S}}(\Omega)$ if $P_{\nu}(z,\zeta) \in \Gamma(\Omega_{\rho}[(\nu+1)d_{\rho}];\mathscr{O}_{T^*X})$ for some $d>0$ and $\rho \in \left]0,1\right[$, and there exists a constant $A \in \left]0,1\right[$ satisfying the following: for any $h>0$ there exists a constant $C_h>0$ such that

$$|P_{\nu}(z,\zeta)| \leqslant C_h A^{\nu} e^{h\|\zeta\|} \quad (\nu \in \mathbb{N}_0,\ (z;\zeta) \in \Omega_{\rho}[(\nu+1)d_{\rho}]).$$

(2) Let $P(t;z,\zeta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z,\zeta) \in \widehat{\mathscr{S}}(\Omega)$. Then $P(t;z,\zeta)$ is an element of $\widehat{\mathscr{N}}(\Omega)$ if there exists a constant $A \in \left]0,1\right[$ satisfying the following: for any $h>0$ there exists a constant $C_h>0$ such that

$$\Big|\sum_{\nu=0}^{m-1} P_{\nu}(z,\zeta)\Big| \leqslant C_h A^m e^{h\|\zeta\|} \quad (m \in \mathbb{N},\ (z;\zeta) \in \Omega_{\rho}[md_{\rho}]).$$

(3) For $z_0^* \in \dot{T}^*X$, we set

$$\widehat{\mathscr{S}}_{z_0^*} := \varinjlim_{\Omega} \widehat{\mathscr{S}}(\Omega) \supset \widehat{\mathscr{N}}_{z_0^*} := \varinjlim_{\Omega} \widehat{\mathscr{N}}(\Omega).$$

We call each element of $\widehat{\mathscr{S}}(\Omega)$ (resp. $\widehat{\mathscr{N}}(\Omega)$) a *formal symbol* (resp. *formal null-symbol*) *on* Ω. Then we can prove:

Theorem 4.4 (see [4]). (1) $\mathscr{S}(\Omega) \subset \widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega) \subset \widehat{\mathscr{S}}(\Omega)$ *and* $\mathscr{N}(\Omega) \subset \widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega) \subset \widehat{\mathscr{N}}(\Omega)$.

(2) *For any* $z_0^* \in \dot{T}^*X$, *the inclusions* $\mathscr{S}_{z_0^*} \subset \widehat{\mathscr{S}}_{\mathrm{cl},z_0^*} \subset \widehat{\mathscr{S}}_{z_0^*}$ *and* $\mathscr{N}_{z_0^*} \subset \widehat{\mathscr{N}}_{\mathrm{cl},z_0^*} \subset \widehat{\mathscr{N}}_{z_0^*}$ *induce*

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}} (K^{\mathbb{R}}_X(\boldsymbol{\kappa})/N^{\mathbb{R}}_X(\boldsymbol{\kappa})) \simeq \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*} \xrightarrow{\sim} \widehat{\mathscr{S}}_{\mathrm{cl},z_0^*}/\widehat{\mathscr{N}}_{\mathrm{cl},z_0^*} \xrightarrow{\sim} \widehat{\mathscr{S}}_{z_0^*}/\widehat{\mathscr{N}}_{z_0^*}.$$

Take any $K(z,w)\,dw = [\psi(z,w)\,dw] \in \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} (K^{\mathbb{R}}_X(\boldsymbol{\kappa})/N^{\mathbb{R}}_X(\boldsymbol{\kappa}))$. Then we set

$$\sigma(\psi)(z,\zeta) := \int_{\gamma(0,\eta_0;\varrho,\theta)} \psi(z,z+w)\,e^{\langle w,\zeta\rangle} dw.$$

Here $0 < \eta_0 \ll 1$ is fixed. Then σ induces an isomorphism (see [1], [4]):

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} (K^{\mathbb{R}}_X(\boldsymbol{\kappa})/N^{\mathbb{R}}_X(\boldsymbol{\kappa})) \xrightarrow{\sim} \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*}.$$

§ 5. Symbols with an Apparent Parameter

For our new cohomological definition of $\mathscr{E}^{\mathbb{R}}_X$ in § 3, we shall develop the symbol theory as in § 4. First, we give the definition of symbols with an apparent parameter. We inherit the notation from the preceding section. Set for short

$$S := S_{\boldsymbol{\kappa}} \tag{5.1}$$

for some $r,\ \theta \in \left]0, \frac{1}{2}\right[$ (recall (1.2)). In particular we always assume that $|\eta| < \frac{1}{2}$ for any $\eta \in S$. For $Z \Subset S$, we set $m_Z := \min_{\eta \in Z} |\eta| > 0$.

Definition 5.1. We define a set $\mathfrak{N}(\Omega; S)$ as follows: $P(z,\zeta,\eta) \in \mathfrak{N}(\Omega; S)$ if

(i) $P(z,\zeta,\eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d > 0$ and $\rho \in \left]0,1\right[$,

(ii) there exists $\delta > 0$ so that for any $Z \Subset S$, there exists a constant $C_Z > 0$ satisfying

$$|P(z,\zeta,\eta)| \leqslant C_Z e^{-\delta\|\eta\zeta\|} \quad ((z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z).$$

Lemma 5.2. *If* $P(z,\zeta,\eta) \in \mathfrak{N}(\Omega; S)$, *it follows that* $\partial_\eta P(z,\zeta,\eta) \in \mathfrak{N}(\Omega; S)$.

Proposition 5.3. *Assume that* $P(z,\zeta,\eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$ *satisfies that* $\partial_\eta P(z,\zeta,\eta) \in \mathfrak{N}(\Omega; S)$.

(1) *The following conditions are equivalent*:

(i) *there exists a constant* $\upsilon > 0$ *satisfying the following*: *for any* $Z \Subset S$ *there exists a constant* $C_Z > 0$ *such that*

$$|P(z,\zeta,\eta)| \leqslant C_Z e^{\upsilon\|\eta\zeta\|} \quad ((z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z).$$

(ii) *for any* $h > 0$ *and* $Z \Subset S$ *there exists constant* $C_{h,Z} > 0$ *such that*

$$|P(z,\zeta,\eta)| \leqslant C_{h,Z}\, e^{h\|\zeta\|} \quad ((z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z).$$

(2) *Further assume that* $P(z,\zeta,\eta)$ *satisfies the equivalent conditions of* (1) (*resp.* $P(z,\zeta,\eta) \in \mathfrak{N}(\Omega;S)$). *Then for any* $\eta_0 \in S$, *it follows that* $P(z,\zeta,\eta_0) \in \mathscr{S}(\Omega)$ (*resp.* $P(z,\zeta,\eta_0) \in \mathscr{N}(\Omega)$) *and further* $P(z,\zeta,\eta) - P(z,\zeta,\eta_0) \in \mathfrak{N}(\Omega;S)$.

Definition 5.4. (1) We define a set $\mathfrak{S}(\Omega;S)$ as follows: $P(z,\zeta,\eta) \in \mathfrak{S}(\Omega;S)$ if

(i) $P(z,\zeta,\eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d > 0$ and $\rho \in \,]0,1[$,

(ii) $\partial_\eta P(z,\zeta,\eta) \in \mathfrak{N}(\Omega;S)$,

(iii) $P(z,\zeta,\eta)$ satisfies the equivalent conditions of Proposition 5.3 (1).

Note that $\mathfrak{N}(\Omega;S) \subset \mathfrak{S}(\Omega;S)$ holds by Lemma 5.2.

(2) For $z_0^* \in \dot{T}^*X$, we set

$$\mathfrak{S}_{z_0^*} := \varinjlim_{\Omega,S} \mathfrak{S}(\Omega;S) \supset \mathfrak{N}_{z_0^*} := \varinjlim_{\Omega,S} \mathfrak{N}(\Omega;S).$$

Here $\Omega \underset{\text{conic}}{\Subset} T^*X$ ranges though open conic neighborhoods of z_0^*, and the inductive limits with respect to S are taken by r, $\theta \to 0$ in (5.1).

We call each element of $\mathfrak{S}(\Omega;S)$ (resp. $\mathfrak{N}(\Omega;S)$) a *symbol* (resp. *null-symbol*) *on* Ω *with an apparent parameter in* S. It is easy to see that $\mathfrak{S}(\Omega;S)$ is a $\mathbb{C}$-algebra under the ordinary operations of functions, and $\mathfrak{N}(\Omega;S)$ is a subalgebra. By definition, we can regard that

$$\mathscr{S}(\Omega) = \{P(z,\zeta,\eta) \in \mathfrak{S}(\Omega;S);\ \partial_\eta P(z,\zeta,\eta) = 0\} \subset \mathfrak{S}(\Omega;S),$$
$$\mathscr{N}(\Omega) = \mathscr{S}(\Omega) \cap \mathfrak{N}(\Omega;S) \subset \mathfrak{N}(\Omega;S).$$

Hence we have an injective mapping $\mathscr{S}(\Omega)/\mathscr{N}(\Omega) \hookrightarrow \mathfrak{S}(\Omega;S)/\mathfrak{N}(\Omega;S)$. Moreover

Proposition 5.5. *There exists the following isomorphism*:

$$\mathscr{S}(\Omega)/\mathscr{N}(\Omega) \xrightarrow{\sim} \mathfrak{S}(\Omega;S)/\mathfrak{N}(\Omega;S).$$

Take any $K(z,w,\eta)\,dw = [\psi(z,w,\eta)\,dw] \in \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} \widehat{E}^{\mathbb{R}}_X(\boldsymbol{\kappa})$. We set

$$\sigma(\psi)(z,\zeta,\eta) := \int_{\gamma(0,\eta;\varrho,\theta)} \psi(z,z+w,\eta)\,e^{\langle w,\zeta\rangle} dw.$$

Then we can prove the following theorem:

Theorem 5.6. *The mapping σ induces an isomorphism*

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} E^{\mathbb{R}}_X(\boldsymbol{\kappa}) \xrightarrow{\sim} \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}\,.$$

The inverse ϖ of σ is defined as follows: Let $z_0^* = (0;1,0,\dots,0) \in \dot{T}^*X$, and $P(z,\zeta,\eta) \in \mathfrak{S}_{z_0^*}$. By Proposition 5.3, for any sufficiently small $\eta_0 > 0$ we have $P(z,\zeta,\eta) - P(z,\zeta,\eta_0) \in \mathfrak{N}_{z_0^*}$. We develop $P(z,\zeta,\eta_0)$ into the Taylor series with respect to $\zeta'/\zeta_1 = (\zeta_2/\zeta_1,\dots,\zeta_n/\zeta_1)$:

$$P(z,\zeta,\eta_0) = \sum_{\alpha\in\mathbb{N}_0^{n-1}} P_\alpha(z,\zeta_1,\eta_0)\Bigl(\frac{\zeta'}{\zeta_1}\Bigr)^{\alpha}.$$

Definition 5.7. Under the preceding notation, we set

$$\varpi_\alpha(P)(z,w_1,\eta) := \int_d^\infty P_\alpha(z,\zeta_1,\eta_0)\,\mathit{\Gamma}_{|\alpha|}(\zeta_1,\eta)\,e^{-w_1\zeta_1}\,d\zeta_1,$$

where we set (see [10]):

$$\mathit{\Gamma}_\nu(\tau,\eta) := \begin{cases} 1 & (\nu = 0), \\ \dfrac{1}{(\nu-1)!}\displaystyle\int_0^\eta e^{-s\tau}s^{\nu-1}ds & (\nu\in\mathbb{N}). \end{cases}$$

Further we define

$$\varpi(P)(z,w,\eta) := \sum_{\alpha\in\mathbb{N}_0^{n-1}} \frac{\alpha!\,\varpi_\alpha(P)(z,w_1-z_1,\eta)}{(2\pi\sqrt{-1}\,)^n\,(w'-z')^{\alpha+\mathbf{1}_{n-1}}}\,.$$

Here we set $w' := (w_2,\dots,w_n)$ and $\mathbf{1}_{n-1} := (1,\dots,1)$.

We can show that ϖ induces a mapping $\mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \to \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$ and

$$\varpi\circ\sigma = \mathbb{1}\colon \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \to \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}, \quad \sigma\circ\varpi = \mathbb{1}\colon \mathscr{E}^{\mathbb{R}}_{X,z_0^*} \to \mathscr{E}^{\mathbb{R}}_{X,z_0^*}\,.$$

§ 6. Classical Formal Symbols with an Apparent Parameter

In order to describe several operations of $\mathscr{E}^{\mathbb{R}}_X$ (coordinate transformations, products and formal adjoints) in terms of symbols, we need a notion of classical formal symbols as in non-parameter case: In this section, we give the definition of classical formal symbols with an apparent parameter.

Definition 6.1. Let t be an indeterminate. Then we define the set $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$ as follows: $P(t;z,\zeta,\eta)=\sum\limits_{\nu=0}^{\infty}t^{\nu}P_{\nu}(z,\zeta,\eta)\in\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$ if

(i) $P(t;z,\zeta,\eta)\in\Gamma(\Omega_{\rho}[d_{\rho}]\times S;\mathscr{O}_{T^*X\times\mathbb{C}})[\![t]\!]$ for some $d>0$ and $\rho\in\left]0,1\right[$,

(ii) there exists a constant $A>0$, and for any $Z\Subset S$, $h>0$, there exists $C_{h,Z}>0$ such that

$$\Big|\sum_{\nu=0}^{m-1}P_{\nu}(z,\zeta,\eta)\Big|\leqslant\frac{C_{h,Z}\,A^{m}\,m!\,e^{h\|\zeta\|}}{\|\eta\zeta\|^{m}}\quad(m\in\mathbb{N},\ (z;\zeta,\eta)\in\Omega_{\rho}[d_{\rho}]\times Z).$$

Definition 6.2. We say that $P(t;z,\zeta,\eta)=\sum\limits_{\nu=0}^{\infty}t^{\nu}P_{\nu}(z,\zeta,\eta)\in\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$ if

(i) $P(t;z,\zeta,\eta)\in\Gamma(\Omega_{\rho}[d_{\rho}]\times S;\mathscr{O}_{T^*X\times\mathbb{C}})[\![t]\!]$ for some $d>0$ and $\rho\in\left]0,1\right[$,

(ii) there exists a constant $A>0$, and for any $Z\Subset S$, $h>0$ there exists $C_{h,Z}>0$ such that

$$|P_{\nu}(z,\zeta,\eta)|\leqslant\frac{C_{h,Z}\,A^{\nu}\nu!\,e^{h\|\zeta\|}}{\|\eta\zeta\|^{\nu}}\quad(\nu\in\mathbb{N}_0,\ (z;\zeta,\eta)\in\Omega_{\rho}[d_{\rho}]\times Z).$$

(iii) $\partial_{\eta}P(t;z,\zeta,\eta)\in\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$.

We call each element of $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$ (resp. $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$) a *classical formal symbol* (resp. *classical formal null-symbol*) *on Ω with an apparent parameter in S*. We remark that $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)\subset\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$. For any $z_0^*\in\dot{T}^*X$, we set

$$\widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*}:=\varinjlim_{\Omega,S}\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)\supset\widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*}:=\varinjlim_{\Omega,S}\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S).$$

Proposition 6.3. *Let $P(t;z,\zeta,\eta)\in\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$. Then for any $\eta_0\in S$, it follows that $P(t;z,\zeta,\eta_0)\in\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ and $P(t;z,\zeta,\eta)-P(t;z,\zeta,\eta_0)\in\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$.*

We can regard that

$$\begin{aligned}\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)&=\{P(t;z,\zeta,\eta)\in\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S);\ \partial_{\eta}P(t;z,\zeta,\eta)=0\}\subset\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S),\\ \widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)&=\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)\cap\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)\subset\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S).\end{aligned}$$

Hence we have an injective mapping $\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)/\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)\hookrightarrow\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$. Moreover

Proposition 6.4. $\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)/\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)\simeq\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$.

Theorem 6.5. *Let $\Omega\underset{\mathrm{conic}}{\Subset}T^*X$ be any sufficiently small neighborhood of $z_0^*\in\dot{T}^*X$. Then $\mathfrak{S}(\Omega;S)/\mathfrak{N}(\Omega;S)\simeq\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$.*

Sketch of Proof. We may assume $z_0^* = (z_0; 1, 0, \dots, 0)$. For any $P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$, we set

$$P(z, \zeta) := \sum_{\nu=0}^{\infty} P_\nu(z, \zeta, \eta_0)\, \zeta_1^\nu \Gamma_\nu(\zeta_1, a).$$

Here we fix any $\eta_0 \in S$ and choose a sufficiently small $0 < a < 1$. Then we can show that $P(z, \zeta) \in \mathscr{S}(\Omega) \subset \mathfrak{S}(\Omega; S)$ and $P(t; z, \zeta, \eta) \equiv P(z, \zeta) \bmod \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$. $\square$

Definition 6.6. As in the case of $\mathfrak{S}(\Omega; S)$, for any $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$ we set

$$:\!P(t; z, \zeta, \eta)\!: := P(t; z, \zeta, \eta) \bmod \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S).$$

Take $\Omega \underset{\text{conic}}{\Subset} T^*\mathbb{C}^n$. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be local coordinates on a neighborhood of $\mathrm{Cl}\,\pi(\Omega) \subset X$, and $(z; \zeta)$, $(w; \lambda)$ corresponding local coordinates on a neighborhood of $\mathrm{Cl}\,\Omega$. Let $z = \Phi(w)$ be the coordinate transformation. We define $J_\Phi^*(z', z)$ by the relation $\Phi^{-1}(z') - \Phi^{-1}(z) = J_\Phi^*(z', z)(z' - z)$. Then ${}^{\mathrm{t}}J_\Phi^*(z, z)\lambda = {}^{\mathrm{t}}\big[\dfrac{\partial w}{\partial z}(z)\big]\lambda = \zeta$. Let $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$ with respect to $(z; \zeta)$. Then we set $\Phi^* P(t; w, \lambda, \eta) = \sum_{\nu=0}^{\infty} t^\nu \Phi^* P_\nu(w, \lambda, \eta)$ by

$$\Phi^* P(t; w, \lambda, \eta) := e^{t\langle \partial_{\zeta'}, \partial_{z'}\rangle} P(t; \Phi(w), \zeta' + {}^{\mathrm{t}}J_\Phi^*(\Phi(w) + z', \Phi(w))\lambda, \eta)\big|_{\substack{z'=0\\ \zeta'=0}},$$

i.e.

$$\Phi^* P_\nu(w, \lambda, \eta) = \sum_{k+|\alpha|=\nu} \frac{1}{\alpha!} \partial_{\zeta'}^\alpha \partial_{z'}^\alpha P_k(\Phi(w), \zeta' + {}^{\mathrm{t}}J_\Phi^*(\Phi(w) + z', \Phi(w))\lambda, \eta)\big|_{\substack{z'=0\\ \zeta'=0}}.$$

Theorem 6.7. (1) *$\Phi^* P(t; w, \lambda, \eta)$ defines an element of $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$ with respect to $(w; \lambda)$. Moreover if $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, then $\Phi^* P(t; w, \lambda, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$.*

(2) *$\mathbb{1}^*$ is the identity, and if $z = \Phi(w)$ and $w = \Psi(v)$ are complex coordinate transformations, $\Psi^*\Phi^* = (\Phi\Psi)^*$ holds.*

Definition 6.8. Under the notation above, we define a coordinate transformation Φ^* associated with Φ by

$$\Phi^*(:\!P\!:)(t; w, \lambda, \eta) := :\!\Phi^* P(t; w, \lambda, \eta)\!:.$$

Theorem 6.9. *For any $P(t; z, \zeta, \eta)$, $Q(t; z, \zeta\eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$, set*

$$\begin{aligned} Q \circ P(t; z, \zeta, \eta) &:= e^{t\langle \partial_{\zeta'}, \partial_{z'}\rangle} Q(t; z, \zeta', \eta)\, P(t; z', \zeta, \eta)\big|_{\substack{z'=z\\ \zeta'=\zeta}} \\ &= e^{t\langle \partial_{\zeta'}, \partial_{z'}\rangle} Q(t; z, \zeta + \zeta', \eta)\, P(t; z + z', \lambda, \eta)\big|_{\substack{z'=0\\ \zeta'=0}}. \end{aligned}$$

(1) $Q \circ P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$. *Moreover if either* $P(t; z, \zeta, \eta)$ *or* $Q(t; z, \zeta, \eta)$ *is an element of* $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, *it follows that* $Q \circ P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$.

(2) $R \circ (Q \circ P) = (R \circ Q) \circ P$ *holds.*

(3) *Let* $\Phi(w) = z$ *be a holomorphic coordinate transformation. Then*

$$\Phi^* Q \circ \Phi^* P(t; w, \lambda, \eta) = \Phi^*(Q \circ P)(t; w, \lambda, \eta).$$

Definition 6.10. For any $:\!P(t; z, \zeta, \eta)\!:$, $:\!Q(t; z, \zeta, \eta)\!: \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, we define the product by:

$$:\!Q(t; z, \zeta, \eta)\!:\, :\!P(t; z, \zeta, \eta)\!: := \,:\!Q \circ P(t; z, \zeta, \eta)\!:.$$

Theorem 6.11. *For any* $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$, *set*

$${}^{\mathsf{t}}P(t; z, \zeta, \eta) := e^{t\langle \partial_\zeta, \partial_z \rangle} P(t; z, -\zeta, \eta).$$

(1) ${}^{\mathsf{t}}P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega^a; S)$, *where* $\Omega^a := \{(z; \zeta);\ (z; -\zeta) \in \Omega\}$, *and* ${}^{\mathsf{tt}}P = P$. *Moreover if* $P(t; z, \zeta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, *it follows that* ${}^{\mathsf{t}}P(t; z, \zeta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega^a; S)$.

(2)

$${}^{\mathsf{t}}(Q \circ P)(t; z, \zeta, \eta) = {}^{\mathsf{t}}P(t; z, \zeta) \circ {}^{\mathsf{t}}Q(t; z, \zeta, \eta).$$

(3) *For any holomorphic coordinate transformation* $\Phi(w) = z$, *on* $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega^a; S) \underset{\mathscr{O}_X}{\otimes} \Omega_X$ *it follows that*

$$\det \frac{\partial z}{\partial w}\, \Phi^*({}^{\mathsf{t}}P) \otimes dw = {}^{\mathsf{t}}(\Phi^* P) \circ \det \frac{\partial z}{\partial w} \otimes dw.$$

Here $\det \dfrac{\partial z}{\partial w}$ *stands for the Jacobian determinant.*

Definition 6.12. For any $:\!P(t; z, \zeta, \eta)\!: \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, we define the formal adjoint by

$${}^{\mathsf{t}}(:\!P(t; z, \zeta, \eta)\!:) := \,:\!e^{t\langle \partial_\zeta, \partial_z \rangle} P(t; z, -\zeta, \eta)\!: \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega^a; S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega^a; S).$$

Theorem 6.13. *Let* $[\psi(z, w, \eta)\, dw]$, $[\varphi(z, w, \eta)\, dw] \in \mathscr{E}^{\mathbb{R}}_{X, z_0^*}$. *Then the following hold*:

(1) $\sum_\alpha \dfrac{1}{\alpha!} \partial_\zeta^\alpha \sigma(\psi)(z, \zeta, \eta)\, \partial_z^\alpha \sigma(\varphi)(z, \zeta, \eta) \in \mathfrak{S}_{z_0^*}$.

(2) $\sum_\alpha \dfrac{1}{\alpha!} \partial_\zeta^\alpha \sigma(\psi)(z, \zeta, \eta)\, \partial_z^\alpha \sigma(\varphi)(z, \zeta, \eta) - \sigma(\psi) \circ \sigma(\varphi)(z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}, z_0}$.

(3) $\sigma(\mu(\psi \otimes \varphi))(z, \zeta, \eta) - \sum_\alpha \dfrac{1}{\alpha!} \partial_\zeta^\alpha \sigma(\psi)(z, \zeta, \eta)\, \partial_z^\alpha \sigma(\varphi)(z, \zeta, \eta) \in \mathfrak{N}_{z_0^*}$.

This theorem justifies the following formal calculations:

$$\sigma(\mu(\psi \otimes \varphi))(z, \zeta, \eta) = \int \mu(\psi \otimes \varphi)(z, z + \widetilde{z}, \eta)\, e^{\langle \widetilde{z}, \zeta \rangle} d\widetilde{z}$$

$$
\begin{aligned}
&= \int d\widetilde{z}\, e^{\langle \widetilde{z}-z,\zeta\rangle} \int \psi(z,w,\eta)\,\varphi(w,\widetilde{z},\eta)\,dw \\
&= \int d\widetilde{z}\, e^{\langle \widetilde{z}-z,\zeta\rangle} \int \psi(z,z+w,\eta)\,\varphi(z+w,\widetilde{z},\eta)\,dw \\
&= \iint \psi(z,z+w,\eta)\,e^{\langle w,\zeta\rangle}\,\varphi(z+w,\widetilde{z},\eta)\,e^{\langle \widetilde{z}-w,\zeta\rangle} d\widetilde{z}\,dw \\
&= \int dw\,\psi(z,z+w,\eta)\,e^{\langle w,\zeta\rangle} \int \varphi(z+w,\widetilde{z},\eta)\,e^{\langle \widetilde{z}-w,\zeta\rangle} d\widetilde{z} \\
&= \int dw\,\psi(z,z+w,\eta)\,e^{\langle w,\zeta\rangle} \int \varphi(z+w,\widetilde{z}+w,\eta)\,e^{\langle \widetilde{z},\zeta\rangle} d\widetilde{z} \\
&= \int \psi(z,z+w,\eta)\,e^{\langle w,\zeta\rangle} \sigma(\varphi)(z+w,\zeta,\eta)\,dw \\
&= \int \psi(z,z+w,\eta)\,e^{\langle w,\zeta\rangle} \sum_{\alpha} \frac{w^{\alpha}}{\alpha!}\,\partial_z^{\alpha}\sigma(\varphi)(z,\zeta,\eta)\,dw \\
&= \sum_{\alpha} \Bigl(\int \psi(z,z+w,\eta)\,e^{\langle w,\zeta\rangle} \frac{w^{\alpha}}{\alpha!}\,dw\Bigr) \partial_z^{\alpha}\sigma(\varphi)(z,\zeta,\eta) \\
&= \sum_{\alpha} \frac{1}{\alpha!}\,\partial_\zeta^{\alpha}\Bigl(\int \psi(z,z+w,\eta)\,e^{\langle w,\zeta\rangle} dw\Bigr) \partial_z^{\alpha}\sigma(\varphi)(z,\zeta,\eta) \\
&= \sum_{\alpha} \frac{1}{\alpha!}\,\partial_\zeta^{\alpha}\sigma(\psi)(z,\zeta,\eta)\,\partial_z^{\alpha}\sigma(\varphi)(z,\zeta,\eta).
\end{aligned}
$$

Remark 6.14. Let $[\psi(z,w,\eta)\,dw] \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$. Then we can also prove the following:

(1) We have

$$
{:}P^*(t;z,\zeta,\eta){:} = {:}\int_{\gamma(0,\eta;\varrho,\theta)} \psi(z-w,z,\eta)\,e^{-\langle w,\zeta\rangle}\,dw{:}.
$$

(2) Let $z = \varPhi(w)$ be a complex coordinate transformation. Then

$$
{:}\varPhi^* P(t;w,\lambda,\eta){:} = {:}\int_{\gamma(z,\eta;\varrho,\theta)} \psi(z,z',\eta)\,e^{\langle \varPhi^{-1}(z')-\varPhi^{-1}(z),\lambda\rangle} dz'{:}.
$$

§ 7. Formal Symbols with an Apparent Parameter

For the analysis of $\mathscr{E}^{\mathbb{R}}_X$ by using symbols, it is useful to introduce a wider class $\widehat{\mathfrak{S}}(\varOmega;S)$ than $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\varOmega;S)$. In this section, we introduce this class $\widehat{\mathfrak{S}}(\varOmega;S)$; that is, formal symbols with an apparent parameter.

For $U \subset S$ and $m \in \mathbb{N}$, we set

$$
(\varOmega_\rho * U)[md_\rho] := \{(z;\zeta,\eta) \in \varOmega_\rho \times U;\ \|\eta\zeta\| \geqslant md_\rho\} \subset \varOmega_\rho[md_\rho] \times S.
$$

Definition 7.1. Let t be an indeterminate. Then, we say that $P(t;z,\zeta,\eta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z,\zeta,\eta)$ is an element of $\widehat{\mathfrak{N}}(\Omega;S)$ if

(i) $P_\nu(z,\zeta,\eta) \in \Gamma((\Omega_\rho * S)[(\nu+1)d_\rho];\mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d>0$ and $\rho \in \left]0,1\right[$,

(ii) there exists a constant $A \in \left]0,1\right[$, and for any $Z \Subset S$, $h>0$ there exists $C_{h,Z}>0$ such that

$$\Big|\sum_{\nu=0}^{m-1} P_\nu(z,\zeta,\eta)\Big| \leqslant C_{h,Z} A^m e^{h\|\zeta\|} \quad (m \in \mathbb{N},\ (z;\zeta,\eta) \in (\Omega_\rho * Z)[md_\rho]).$$

Definition 7.2. (1) We say that $P(t;z,\zeta,\eta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z,\zeta,\eta)$ is an element of $\widehat{\mathfrak{S}}(\Omega;S)$ if

(i) $P_\nu(z,\zeta,\eta) \in \Gamma((\Omega_\rho * S)[(\nu+1)d_\rho];\mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d>0$ and $\rho \in \left]0,1\right[$,

(ii) there exists a constant $A \in \left]0,1\right[$, and for any $Z \Subset S$, $h>0$, there exists $C_{h,Z}>0$ such that

$$|P_\nu(z,\zeta,\eta)| \leqslant C_{h,Z} A^\nu e^{h\|\zeta\|} \quad (\nu \in \mathbb{N}_0,\ (z;\zeta,\eta) \in (\Omega_\rho * Z)[(\nu+1)d_\rho]).$$

(iii) $\partial_\eta P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S)$.

We call each element of $\widehat{\mathfrak{S}}(\Omega;S)$ (resp. $\widehat{\mathfrak{N}}(\Omega;S)$) a *formal symbol* (resp. *formal null-symbol*) *on Ω with an apparent parameter in S*.

We set

$$\widehat{\mathfrak{S}}_{z_0^*} := \varinjlim_{\Omega,S} \widehat{\mathfrak{S}}(\Omega;S) \supset \widehat{\mathfrak{N}}_{z_0^*} := \varinjlim_{\Omega,S} \widehat{\mathfrak{N}}(\Omega;S).$$

Proposition 7.3. (1) *Let $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$. Then for any $\eta_0 \in S$, it follows that $P(t;z,\zeta,\eta_0) \in \widehat{\mathscr{S}}(\Omega)$ and $P(t;z,\zeta,\eta) - P(t;z,\zeta,\eta_0) \in \widehat{\mathfrak{N}}(\Omega;S)$.*

(2) *There exists the following isomorphism*:

$$\widehat{\mathscr{S}}(\Omega)/\widehat{\mathscr{N}}(\Omega) \xrightarrow{\sim} \widehat{\mathfrak{S}}(\Omega;S)/\widehat{\mathfrak{N}}(\Omega;S).$$

Theorem 7.4. (1) $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S) \subset \widehat{\mathfrak{S}}(\Omega;S)$ *and* $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \subset \widehat{\mathfrak{N}}(\Omega;S)$.

(2) *For any $z_0^* \in \dot{T}^*X$, the inclusions $\mathfrak{S}_{z_0^*} \subset \widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*} \subset \widehat{\mathfrak{S}}_{z_0^*}$ and $\mathfrak{N}_{z_0^*} \subset \widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*} \subset \widehat{\mathfrak{N}}_{z_0^*}$ induce*

$$\begin{tikzcd}
\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*} \arrow[r, "\sim"] \arrow[d, "\wr"] & \widehat{\mathscr{S}}_{\mathrm{cl},z_0^*}/\widehat{\mathscr{N}}_{\mathrm{cl},z_0^*} \arrow[r, "\sim"] \arrow[d, "\wr"] & \widehat{\mathscr{S}}_{z_0^*}/\widehat{\mathscr{N}}_{z_0^*} \arrow[d, "\wr"] \\
\varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} E^{\mathbb{R}}_X(\boldsymbol{\kappa}) \arrow[r, "\sim"] & \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} \arrow[r, "\sim"] & \widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*}/\widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*} \arrow[r, "\sim"] & \widehat{\mathfrak{S}}_{z_0^*}/\widehat{\mathfrak{N}}_{z_0^*}\,.
\end{tikzcd}$$

Remark 7.5. (1) It is sometimes convenient to deal with formal symbols than classical formal symbols. However, in Theorem 6.5, any sufficiently small neighborhood of $z_0^* \in \dot{T}^*X$ we can obtain an isomorphism $\mathfrak{S}(\Omega;S)/\mathfrak{N}(\Omega;S) \xrightarrow{\sim} \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$, and in Theorem 7.4, we only obtain an isomorphism between stalks at this stage.

(2) We obtain an explicit description of the isomorphism $\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \simeq \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} E^{\mathbb{R}}_X(\boldsymbol{\kappa})$ as follows: We may assume that $z_0^* = (z_0;1,0,\dots,0) \in \dot{T}^*X$. Let $P \in \mathfrak{S}_{z_0^*}$, and consider $[\varpi(P)(z,w,\eta)\,dw] \in \varinjlim_{\boldsymbol{\kappa}} E^{\mathbb{R}}_X(\boldsymbol{\kappa})$. Then we define

$$\varpi_{0,\alpha}(P)(z,w_1) := \int_d^{\infty} P_\alpha(z,\zeta_1,\eta_0)\,\Gamma_{|\alpha|}(\zeta_1, c_0\varepsilon_{|\alpha|} - w_1)\,e^{-w_1\zeta_1}\,d\zeta_1\,,$$

$$\varpi_0(P)(z,z+w) := \sum_{\alpha\in\mathbb{N}_0^{n-1}} \frac{\alpha!\,\varpi_{0,\alpha}(P)(z,w_1)}{(2\pi\sqrt{-1}\,)^n\,(w')^{\alpha+\mathbf{1}_{n-1}}}\,.$$

Here $c_0 > 1 \gg \varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_\nu \underset{\nu}{\to} 0$ satisfying some conditions (cf. [3], [4]). Then we can show that $[\varpi_0(P)(z,w)\,dw] \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$ and

$$[\varpi(P)(z,w,\eta)\,dw] = [\varpi_0(P)(z,w)\,dw] \in \varinjlim_{\boldsymbol{\kappa}\to\mathbf{0}} E^{\mathbb{R}}_X(\boldsymbol{\kappa}).$$

We use the notation of Theorem 6.7. For any $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$, we also set

$$\Phi^*P(t;w,\lambda,\eta) := e^{t\langle\partial_{\zeta'},\partial_{z'}\rangle} P(t;\Phi(w),\zeta' + {}^{\mathrm{t}}J^*_{\Phi}(\Phi(w)+z',\Phi(w))\lambda,\eta)\big|_{\substack{z'=0\\ \zeta'=0}}\,.$$

Theorem 7.6. *Let $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$.*

(1) *$\Phi^*P(t;w,\lambda,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$ with respect to coordinate system $(w;\lambda)$. Further if $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S)$, it follows that $\Phi^*P(t;w,\lambda,\eta) \in \widehat{\mathfrak{N}}(\Omega;S)$.*

(2) *$\mathbb{1}^*$ is the identity, and for complex coordinate transformations $z = \Phi(w)$ and $w = \Psi(v)$, it follow that $\Psi^*\Phi^*P(t;v,\xi,\eta) - (\Phi\Psi)^*P(t;v,\xi,\eta) \in \widehat{\mathfrak{N}}_{(v;\xi)}$.*

Theorem 7.7. *For any $P(t;z,\zeta,\eta)$, $Q(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$, set*

$$\begin{aligned} Q\circ P(t;z,\zeta,\eta) &:= e^{t\langle\partial_{\zeta'},\partial_{z'}\rangle} Q(t;z,\zeta',\eta)\,P(t;z',\zeta,\eta)\big|_{\substack{z'=z\\ \zeta'=\zeta}} \\ &= e^{t\langle\partial_{\zeta'},\partial_{z'}\rangle} Q(t;z,\zeta+\zeta',\eta)\,P(t;z+z',\zeta,\eta)\big|_{\substack{z'=0\\ \zeta'=0}}\,. \end{aligned}$$

(1) *$Q\circ P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$. Moreover if either $P(t;z,\zeta,\eta)$ or $Q(t;z,\zeta,\eta)$ is an element of $\widehat{\mathfrak{N}}(\Omega;S)$, it follows that $Q\circ P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S)$.*

(2) *$R\circ(Q\circ P) = (R\circ Q)\circ P$ holds.*

(3) *Let $\Phi(w) = z$ be a holomorphic coordinate transformation. Then*

$$\Phi^*Q\circ\Phi^*P(t;w,\lambda,\eta) - \Phi^*(Q\circ P)(t;w,\lambda,\eta) \in \widehat{\mathfrak{N}}_{(w;\lambda)}.$$

Theorem 7.8. *For any $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$ set*

$$ {}^{\mathrm{t}}P(t;z,\zeta,\eta) := e^{t\langle \partial_\zeta, \partial_z\rangle} P(t;z,-\zeta,\eta). $$

(1) ${}^{\mathrm{t}}P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega^a;S)$ *and* ${}^{\mathrm{tt}}P = P$. *Moreover if* $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S)$, *it follows that* ${}^{\mathrm{t}}P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega^a;S)$.

(2) ${}^{\mathrm{t}}(Q \circ P)(t;z,\zeta,\eta) = {}^{\mathrm{t}}P(t;z,\zeta,\eta) \circ {}^{\mathrm{t}}Q(t;z,\zeta,\eta)$.

(3) *For any holomorphic coordinate transformation $\Phi(w) = z$, on $\widehat{\mathfrak{S}}(\Omega^a;S) \underset{\mathscr{O}_X}{\otimes} \Omega_X$ it follows that*

$$ \det \frac{\partial z}{\partial w}\, \Phi^*({}^{\mathrm{t}}P) \otimes dw = {}^{\mathrm{t}}(\Phi^* P) \circ \det \frac{\partial z}{\partial w} \otimes dw. $$

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