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Analytic extension of Birkhoff normal forms for Hamiltonian systems of one degree of freedom – Simple pendulum and free rigid body dynamics –

By

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Abstract

Birkhoff normal form is a power series expansion associated with the local behavior of a Hamiltonian system near the critical point. It is known that one can take convergent canonical transformation which puts the Hamiltonian into Birkhoff normal form for integrable systems under some non-degeneracy conditions. By means of an expression of the inverse of Birkhoff normal form by a period integral, analytic continuation of the Birkhoff normal forms is considered for two examples of Hamiltonian systems of one degree of freedom, the simple pendulum dynamics and the free rigid body dynamics on $SO(3)$. It is shown that the analytic continuation of the inverse derivative for the Birkhoff normal forms has monodromy structure, which is explicitly calculated, and that in the free rigid body case the monodromy coincides with that of an elliptic fibration which naturally arises from the dynamics.

§ 1. Introduction

Birkhoff normal form is a normal form of the Hamiltonian for a Hamiltonian system, which is at first defined locally around an equilibrium. It is first considered by G. Birkhoff [3, 4] as a formal power series in relation to the stability of Hamiltonian systems around the equilibria. Although it is known that the canonical transformation which puts the Hamiltonian into Birkhoff normal form is divergent in general from the

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result by C. L. Siegel [20], the completely integrable Hamiltonian systems (in the sense of Liouville) admit the convergence of the canonical transformation which makes the Hamiltonian into Birkhoff normal form under some non-degeneracy or, more weakly, non-resonance conditions.

In the present paper, considered is a global aspect of Birkhoff normal forms for analytic Hamiltonian systems of one degree of freedom via analytic extension, while the previous researches on Birkhoff normal forms mainly concern with their local aspects. Note that a Hamiltonian system of one degree of freedom is necessarily completely integrable and that the convergence of the canonical transformation which makes the Hamiltonian into Birkhoff normal form around elliptic and hyperbolic equilibria is known [21].

After giving preliminary facts about Birkhoff normal form in Section 2, certain formulae are given for the derivative of the inverse Birkhoff normal form in Section 3. Using those formulae, one studies the analytic extension of Birkhoff normal forms for the dynamical systems of the simple pendulum and of the free rigid body in Sections 4 and 5. As to Birkhoff normal forms for the Hamiltonian systems of the simple pendulum and of the free rigid body, detailed studies have been done in [8, 9]. In [8, 9], Birkhoff normal forms are calculated by means of the argument of relative cohomology. As an interesting result of [9], it is discovered that all the coefficients of the inverse Birkhoff normal form of the free rigid body dynamics are polynomials in one variable whose roots are on the unit circle in $\mathbb{C}$. Nevertheless, the global behavior of Birkhoff normal forms has not been considered yet. Since the system of the simple pendulum and that of the free rigid body have natural complexification, one can consider the analytic continuation of (the derivatives of the inverse for) Birkhoff normal forms as functions of the energy level viewed as a complex variable. An important feature of the derivatives of the inverse Birkhoff normal forms is their relation to a special Gauß hypergeometric differential equation. As a main result, the monodromy of the analytic continuation for the derivative of the inverse Birkhoff normal forms is found by means of the Gauß hypergeometric differential equation.

It is certainly a new viewpoint to ask the global behavior of Birkhoff normal forms, as far as the knowledge of the authors. Although this problem seems rather naive, one can observe meaningful results on the analytic continuation of the derivative of the inverse Birkhoff normal forms for the simple pendulum and the free rigid body dynamics in this article. Furthermore, the results for the free rigid body dynamics have much to do with the geometry of elliptic fibrations which naturally arise from the dynamics. In fact, the monodromy of the analytic continuation of the derivatives for the inverse Birkhoff normal forms exactly gives the monodromy of the elliptic fibrations which were studied in [16].

§ 2. Birkhoff normal forms for Hamiltonian systems

We start with the definition of Hamiltonian systems. Since we concentrate ourselves to the (real) analytic integrable systems in the present paper, all the geometric settings are considered in the (real) analytic category. Let M be a real analytic symplectic manifold of dimension $2n$, whose symplectic form is denoted by ω. (ω is a closed two-form, i.e. $\mathsf{d}\omega = 0$, on M which is non-degenerate, i.e. $\underbrace{\omega \wedge \cdots \wedge \omega}_{n} \neq 0$.) Take a real analytic function H on M as the Hamiltonian and consider the Hamiltonian system (M, ω, H), which is determined by the Hamiltonian vector field X_H given by $\iota_{X_H}\omega = -\mathsf{d}H$, where ι denotes the interior product of a tensor field with a vector field. Taking a Darboux coordinates $(p_1, \ldots, p_n; q_1, \ldots, q_n)$, for which $\omega = \sum_{i=1}^{n} \mathsf{d}p_i \wedge \mathsf{d}q_i$, the Hamiltonian vector field X_H determines the Hamilton's equation

$$\begin{cases} \dfrac{\mathsf{d}p_i}{\mathsf{d}t} = -\dfrac{\partial H}{\partial q_i}, \\ \dfrac{\mathsf{d}q_i}{\mathsf{d}t} = \dfrac{\partial H}{\partial p_i}, \end{cases} \qquad i = 1, \ldots, n.$$

Consider an isolated elliptic equilibrium $x_0 \in M$ of the Hamiltonian vector field X_H, i.e., $X_H(x_0) = 0$. We assume $H(x_0) = 0$ without loss of generality. Suppose that there exists a Darboux coordinate system $(p_1, \ldots, p_n; q_1, \ldots, q_n)$ with the origin at x_0, such that the Hamiltonian H can be written as $H = \mathcal{H}\left(\frac{p_1^2 + q_1^2}{2}, \cdots, \frac{p_n^2 + q_n^2}{2}\right)$, where $\mathcal{H}$ is a power series in n variables. Then, the power series $\mathcal{H}$ is called Birkhoff normal form. It is known that the power series $\mathcal{H}$ is uniquely determined in the category of formal power series. If the equilibrium has another type of stability than elliptic, the corresponding Birkhoff normal form is defined in a similar manner. Although the canonical transformation which puts the Hamiltonian into Birkhoff normal form is not convergent in general from the result by Siegel [20], it was shown by Vey in [22] that (real) analytic completely integrable systems in the sense of Liouville admit a convergent canonical transformation which makes the Hamiltonian into Birkhoff normal form around the isolated equilibrium x_0, if x_0 is non-degenerate in the sense that the linearization matrices of the Hamiltonian vector fields $X_{f_1}, \cdots, X_{f_n}$ at x_0 form a Cartan subalgebra in $\mathfrak{sp}\,(T_{x_0}M, \omega_{x_0})$. The case of two degrees of freedom was solved by H. Rüssmann [19]. Here, the analytic completely integrable Hamiltonian system in the sense of Liouville is a Hamiltonian system (M, ω, H) such that there exist functionally independent n analytic functions $f_1, \ldots, f_{n-1}, f_n (= H)$ on M which are in involution $\{f_i, f_j\} = 0$, $i, j = 1, \cdots, n$, with respect to the Poisson bracket $\{\cdot, \cdot\}$ defined through $\{f, g\} = \omega(X_f, X_g)$, where f, g are smooth functions on M. The convergence under the non-resonance condition was proved by Ito [12]. Here, the non-resonance, in the case of an elliptic equilibrium, means the rational independence of the eigenvalues $\lambda_1, \cdots, \lambda_n$ of the Hessian of the Hamiltonian H with which the quadratic term of H is written as $\sum_{i=1}^{n} \frac{\lambda_i}{2} \left(x_i^2 + y_i^2\right)$ for some Darboux coordinates $(x_1, \cdots, x_n; y_1 \cdots, y_n)$. The non-resonance condition is weaker than the non-degeneracy condition. In fact, around a non-degenerate isolated equilibrium of a completely integrable system, one can take a linear combination of the first integrals which is non-resonant. The case of $\mathcal{C}^\infty$ completely integrable Hamiltonian systems was proved by Eliasson [6] under the non-degeneracy condition. Later, a more geometric proof in the analytic case was given by Nguyen Tien Zung [23] in view of torus actions, which first concerns with the non-resonant case and which can in fact be extended also to resonant cases.

Since the present paper concerns with the Hamiltonian systems of one degree of freedom, which are obviously completely integrable, we mention Birkhoff normal forms around elliptic and hyperbolic equilibria for those Hamiltonian systems. Take a real analytic two-dimensional symplectic manifold (M, ω) and an analytic function H on M as the Hamiltonian.

Theorem 2.1. *If $x_0 \in M$ is an elliptic equilibrium of the (real) analytic Hamiltonian vector field X_H, where $H(x_0) = 0$, then there exists a Darboux coordinate system (p, q) with the origin at x_0 such that $\omega = \mathsf{d}p \wedge \mathsf{d}q$ and the Hamiltonian H is in Birkhoff normal form $H = \mathcal{H}\left(\frac{p^2 + q^2}{2}\right)$ with a convergent power series $\mathcal{H}$ in one variable.*

Note that the power series $\mathcal{H}$ is invertible, since the equilibrium is isolated. This theorem is known from [21].

For the hyperbolic equilibrium, we have a similar theorem.

Theorem 2.2. *If $x_0 \in M$ is a hyperbolic equilibrium of the (real) analytic Hamiltonian vector field X_H, where $H(x_0) = 0$, then there exists a Darboux coordinate system (P, Q) with the origin at x_0 such that $\omega = \mathsf{d}P \wedge \mathsf{d}Q$ and the Hamiltonian H is in Birkhoff normal form $H = \mathcal{H}(PQ)$ with a convergent power series $\mathcal{H}$ in one variable.*

§ 3. Expression of Birkhoff normal forms in terms of period integrals

On the basis of the existence of Birkhoff normal forms around elliptic and hyperbolic equilibria, we give an expression for the derivative of the inverse of Birkhoff normal form in terms of period integrals in this section. Let $x_0 \in M$ be an elliptic equilibrium of a Hamiltonian system (M, ω, H) of one degree of freedom. We can assume that $H(x_0) = 0$ without loss of generality. By Theorem 2.1, we can take a Darboux coordinate system (p, q) with the centre at x_0 such that $\omega = \mathsf{d}p \wedge \mathsf{d}q$ and the Hamiltonian

H is in Birkhoff normal form $H = \mathcal{H}\left(\frac{p^2+q^2}{2}\right)$ with an invertible analytic function $\mathcal{H}$ on a neighbourhood U of the origin $(p,q) = (0,0)$. We denote the inverse of $\mathcal{H}$ by Φ. Let η be an arbitrary one-form defined on $U \setminus \{(0,0)\}$ such that $\omega = \eta \wedge \mathsf{d}H$. The one-form η has the ambiguity of the additive factor $g\mathsf{d}H$ where g is an analytic function on $U \setminus \{(0,0)\}$.

Theorem 3.1. *The derivative of the inverse Φ of Birkhoff normal form $\mathcal{H}$ around the elliptic equilibrium where $H = 0$ can be written as*

$$\Phi'(h) = -\frac{1}{2\pi}\int_{H=h} \eta.$$

Here, the constant h denotes the level of the Hamiltonian H around the critical value $H = 0$.

For the proof of this theorem, see [10].

Similarly, we give an expression of the derivative for the inverse Ψ of Birkhoff normal form around a hyperbolic equilibrium $x_0 \in M$. By Theorem 2.2, there is a Darboux coordinate system (P,Q) with the centre at x_0 such that the Hamiltonian H is in Birkhoff normal form $H = \mathcal{H}(PQ)$, where $\mathcal{H}$ is a convergent power series in one variable and where $\omega = \mathsf{d}P \wedge \mathsf{d}Q$. Since all the setting is considered in the real analytic category, the symplectic two form ω and the Hamiltonian H can be extended to the complexification $M^{\mathbb{C}}$ of M. Note that the complexification $M^{\mathbb{C}}$ is a complex manifold of complex dimension two which, regarded as a real four-dimensional analytic manifold, contains M as a real two-dimensional submanifold, such that $T_xM^{\mathbb{C}} = T_xM \oplus \sqrt{-1}T_xM$ at any point $x \in M$. It is known that the complexification of a real-analytic manifold is unique as a complex neighbourhood and that every para-compact real analytic manifold has a complexification which is a Stein manifold. See [13, Chapter I] for a brief explanation on the complexification of real analytic manifolds.

Now, we take a real closed arc $\gamma : P = \sqrt{\epsilon}e^{\sqrt{-1}\theta}, Q = \sqrt{\epsilon}e^{-\sqrt{-1}\theta}$, with a small positive number $\epsilon > 0$ and with the parameter $\theta \in [0, 2\pi]$, in the complexification $M^{\mathbb{C}}$ of M. The arc γ is included in the complexified integral curve $H = h$ in $M^{\mathbb{C}}$, where the constant $h = \mathcal{H}(\epsilon)$ denotes the energy level. We take a one-form η' such that $\omega = \eta' \wedge \mathsf{d}H$ and the inverse Ψ of Birkhoff normal form $\mathcal{H}$ for the hyperbolic equilibrium $x_0 \in M$.

Theorem 3.2. *The derivative of the inverse Ψ of Birkhoff normal form $\mathcal{H}$ for the Hamiltonian H around the hyperbolic equilibrium $x_0 \in M$ can be written as*

$$\Psi'(h) = \frac{1}{2\pi\sqrt{-1}}\int_{\gamma} \eta'.$$

This theorem can be deduced from the previous one through the (complex) coordinate change

$$P = \frac{p + \sqrt{-1}q}{2}, Q = \frac{p - \sqrt{-1}q}{2},$$

since this coordinate change maps the hyperbolic equilibrium at the origin in the real coordinates (P, Q) into an elliptic equilibrium at the origin in the new real coordinates (p, q).

Remark 3.1. From the viewpoint of complex analytic geometry, the elliptic and the hyperbolic equilibria of a (real) analytic Hamiltonian system (M, ω, H) of one degree of freedom can be regarded as an A_1-singularity of the complex curve $H = 0$ in the complexification $M^{\mathbb{C}}$ of M. Further, the family of complex curves $H = h$ in $M^{\mathbb{C}}$, where the energy level h is seen as a complex parameter around 0, forms a deformation of the A_1-singularity on the complex curve $H = 0$. Then, we can take the integral paths in the previous two theorems as vanishing cycles in the complex curve $H = h$ in $M^{\mathbb{C}}$, which vanish into the A_1-singular point when the parameter h approaches to zero.

Remark 3.2. In [7, Theorem1], a similar formula to Theorems 3.1 and 3.2 is given by taking another kind of one-form ξ such that $\mathsf{d}\xi = \omega$. In a neighbourhood of an elliptic equilibrium, the period integral $-\frac{1}{2\pi}\int_{H=h} \xi$ coincides with the inverse function $\mathcal{H}^{-1}$ of the Birkhoff normal form, while the period integral $-\frac{1}{2\pi}\int_{H=h} \eta$ in Theorem 3.1, where $\eta \wedge \mathsf{d}H = \omega$, represents the derivative of the inverse function $\mathcal{H}^{-1}$. Around a hyperbolic equilibrium, the period integral $\frac{1}{2\pi\sqrt{-1}}\int_{\gamma} \xi$ coincides with the inverse function $\mathcal{H}^{-1}$ of the Birkhoff normal form, while the period integral $\frac{1}{2\pi\sqrt{-1}}\int_{\gamma} \eta'$ in Theorem 3.2, where $\eta' \wedge \mathsf{d}H = \omega$, represents the derivative of the inverse function $\mathcal{H}^{-1}$. Taking the latter formulae, the analytic continuation can more easily be considered than taking the former.

In the subsequent sections, we apply the expression of the derivative of the inverse Birkhoff normal form for the simple pendulum and the free rigid body dynamics, in order to consider the global property of Birkhoff normal forms.

§ 4. Simple pendulum dynamics

As applications of the results in the previous section, we start with the Hamiltonian systems of simple pendulums. We follow the notations in [8]. The phase space for a simple pendulum is the two-dimensional cylinder $\mathbb{R} \times S^1 = \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ with the coordinates (B, β) of the universal covering $\mathbb{R}^2$ where $\beta \equiv \beta + 2\pi$. Here, β denotes the angle of the pendulum measured from the vertical axis in the direction of the top and B stands for the momentum of the pendulum. See Figure 1.

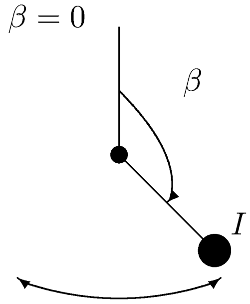


Figure 1. Simple pendulum

The symplectic form on the phase space is given as $\omega = \mathsf{d}B \wedge \mathsf{d}\beta$. We assume that the length of the string is 1, while the mass of the pendulum is $I(>0)$. The gravity constant is written as g^2, although it is normally written as g. Then, the Hamiltonian of the motion for the simple pendulum is given by

$$H(B,\beta) = \frac{B^2}{2I} - Ig^2(1-\cos\beta).$$

The motion is described by Hamilton's equation

$$\begin{cases} \dot{B} = -\dfrac{\partial H}{\partial \beta} = Ig^2 \sin\beta, \\ \dot{\beta} = \dfrac{\partial H}{\partial B} = \dfrac{B}{I}. \end{cases} \tag{4.1}$$

It is easy to observe that there are two equilibria at $(B,\beta) = (0,0)$ and $(B,\beta) = (0,\pi)$. The point $(B,\beta) = (0,0)$, which corresponds to the pendulum at the top, is a hyperbolic equilibrium, while the point $(B,\beta) = (0,\pi)$, which corresponds to the pendulum at the bottom, is an elliptic equilibrium. This is obvious, since the linearization of the Hamilton's equation (4.1) at $(B,\beta) = (0,0)$ is given by the linearization matrix

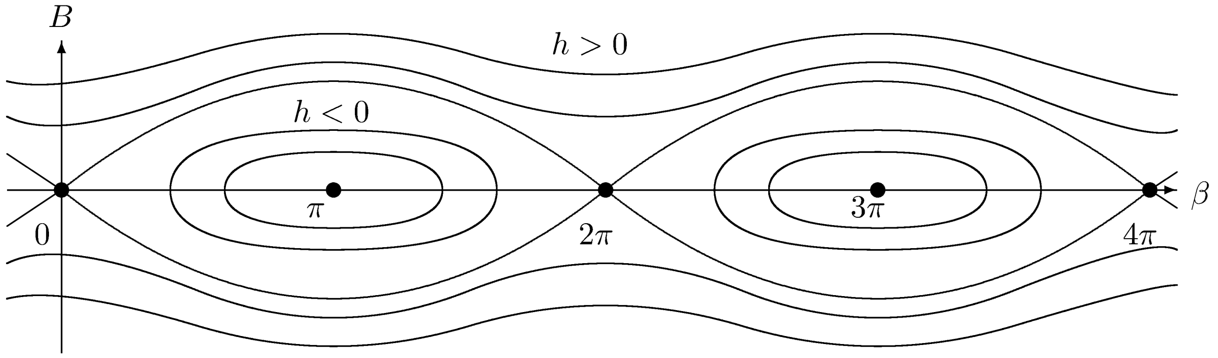
$$\begin{pmatrix} 0 & Ig^2 \\ \dfrac{1}{I} & 0 \end{pmatrix},$$

which has two real eigenvalues with opposite signs and since the linearization at $(B,\beta) = (0,\pi)$ is given by the matrix

$$\begin{pmatrix} 0 & -Ig^2 \\ \dfrac{1}{I} & 0 \end{pmatrix}$$

which has two purely imaginary eigenvalues. Note that $H(0,0) = 0, H(0,\pi) = -2Ig^2$. The phase portrait of the simple pendulum dynamics is given as in Figure 2.

Figure 2. Phase portrait for simple pendulum dynamics



Now, we calculate the derivative of the inverse of Birkhoff normal forms at the two equilibria $(B, \beta) = (0, 0)$ and $(0, \pi)$, using the formulae given in Section 3. First, we concentrate ourselves in the elliptic equilibrium $(B, \beta) = (0, \pi)$. We consider the one-form η which satisfies $\omega = \eta \wedge \mathsf{d}H$. Since $\mathsf{d}H = \dfrac{B}{I}\mathsf{d}B - Ig^2 \sin\beta \mathsf{d}\beta$, the one-form η can be written as

$$\eta = -s\frac{\mathsf{d}B}{Ig^2 \sin\beta} - (1-s)\frac{I}{B}\mathsf{d}\beta,$$

where s is an arbitrary parameter. An integral curve near the elliptic equilibrium is given by the equation $H = \dfrac{B^2}{2I} - Ig^2(1-\cos\beta) = h$, with the energy level h near $-2Ig^2$. Note that $h \geq -2Ig^2$. Since h is near $-2Ig^2$, we see that the integral curve is parameterized by $\beta \in \left[2\mathrm{Arcsin}\sqrt{\dfrac{-h}{2Ig^2}}, 2\pi - 2\mathrm{Arcsin}\sqrt{\dfrac{-h}{2Ig^2}}\right]$ with which the momentum is written as

$$B = \pm\sqrt{2I\left\{h + Ig^2(1-\cos\beta)\right\}} = \pm\sqrt{2Ih}\cdot\sqrt{1 + \frac{2Ig^2}{h}\sin^2\frac{\beta}{2}}.$$

By Theorem 3.1, the derivative of the inverse for Birkhoff normal form around the elliptic equilibrium $(B, \beta) = (0, \pi)$ can be calculated as

$$\begin{aligned}
-\frac{1}{2\pi}\int_{H=h}\eta &= \frac{I}{\pi}\int_{2\mathrm{Arcsin}\sqrt{\frac{-h}{2Ig^2}}}^{2\pi - 2\mathrm{Arcsin}\sqrt{\frac{-h}{2Ig^2}}} \frac{\mathsf{d}\beta}{\sqrt{2Ih}\cdot\sqrt{1 + \frac{2Ig^2}{h}\sin^2\frac{\beta}{2}}} \\
&\overset{\beta'=\beta/2}{=} \frac{1}{\pi}\sqrt{\frac{2I}{h}}\int_{\mathrm{Arcsin}\sqrt{\frac{-h}{2Ig^2}}}^{\pi - \mathrm{Arcsin}\sqrt{\frac{-h}{2Ig^2}}} \frac{\mathsf{d}\beta'}{\sqrt{1 + \frac{2Ig^2}{h}\sin^2\beta'}} \\
&= \frac{2}{\pi g}\sqrt{\frac{2Ig^2}{h}}\int_{\mathrm{Arcsin}\sqrt{\frac{-h}{2Ig^2}}}^{\frac{\pi}{2}} \frac{\mathsf{d}\beta'}{\sqrt{1 + \frac{2Ig^2}{h}\sin^2\beta'}} \\
&\overset{x=\sin\beta'}{=} \frac{2}{\pi g}\sqrt{\frac{2Ig^2}{h}}\int_{\sqrt{\frac{-h}{2Ig^2}}}^{1} \frac{\mathsf{d}x}{\sqrt{(1-x^2)\left(1 + \frac{2Ig^2}{h}x^2\right)}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi g}\sqrt{-\frac{2Ig^2}{h}}\int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)\left(1-\left(1+\frac{2Ig^2}{h}\right)x^2\right)}} \\
&= \frac{2}{\pi g}\sqrt{-\frac{2Ig^2}{h}}\mathcal{K}\left(1+\frac{2Ig^2}{h}\right) \\
&= \frac{2}{\pi g}\mathcal{K}\left(\frac{h+2Ig^2}{2Ig^2}\right).
\end{aligned}
$$

Here, $\mathcal{K}(\lambda) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-\lambda\sin^2\theta}} = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-\lambda x^2)}}$ is the complete elliptic integral of the first kind. We have used the formulae

$$
\int_{1/\sqrt{\lambda}}^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-\lambda x^2)}} = \sqrt{-1}\mathcal{K}(1-\lambda) = \sqrt{-1}\int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-(1-\lambda)x^2)}},
$$

which is equivalent to $\mathcal{K}\left(\frac{1}{\lambda}\right) = \sqrt{\lambda}\left(\mathcal{K}(\lambda)+\sqrt{-1}\mathcal{K}(1-\lambda)\right)$, and

$$
\mathcal{K}\left(\frac{\lambda}{\lambda-1}\right) = \sqrt{1-\lambda}\mathcal{K}(\lambda).
$$

See [11, Chapter 8, (8.128)] for these formulae of the complete elliptic integral of the first kind.

As to the hyperbolic equilibrium $(B,\beta)=(0,0)$, we choose the real closed arc

$$
\gamma : \begin{cases} B = \sqrt{2Ih}\cos\theta, \\ \beta = 2\mathrm{Arcsin}\left(\sqrt{-\frac{h}{2Ig^2}}\sin\theta\right), \end{cases}
$$

parameterized by $\theta \in [0,2\pi]$ in the complexified integral curve $H(B,\beta) = \frac{B^2}{2I} - Ig^2(1-\cos\beta) = h$, where (B,β) are regarded as the coordinates of the complexified phase space $\mathbb{C}\times\mathbb{C}^*$. Using Theorem 3.2, the derivative for the inverse of Birkhoff normal form around the hyperbolic equilibrium $(B,\beta)=(0,0)$ can be calculated as

$$
\begin{aligned}
\frac{1}{2\pi\sqrt{-1}}\int_\gamma \eta &= \frac{1}{2\pi\sqrt{-1}}\int_0^{2\pi} \frac{I\mathrm{d}\beta}{B} \\
&= \frac{I}{2\pi\sqrt{-1}}\int_0^{2\pi} \frac{1}{\sqrt{2Ih}\cos\theta}\cdot 2\frac{\sqrt{-\frac{h}{2Ig^2}}\cos\theta}{\sqrt{1+\frac{h}{2Ig^2}\sin^2\theta}}\mathrm{d}\theta \\
&= \frac{1}{2\pi g}\int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{1+\frac{h}{2Ig^2}\sin^2\theta}}
\end{aligned}
$$

$$= \frac{2}{\pi g} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 + \frac{h}{2Ig^2} \sin^2 \theta}}$$
$$= \frac{2}{\pi g} \mathcal{K} \left(-\frac{h}{2Ig^2} \right).$$

Therefore, the derivatives of the inverse Birkhoff normal forms around the elliptic and the hyperbolic equilibria, which are respectively written as $\frac{2}{\pi g}\mathcal{K}(1-\lambda)$, $\frac{2}{\pi g}\mathcal{K}(\lambda)$, are analytic functions of the variable $\lambda = -\frac{h}{2Ig^2}$. Note that the two functions depend only on this variable λ and have no other parameter than λ, since the gravity constant g^2 is a universal constant. As to the global behavior of these two functions, we consider their analytic continuation with respect to the variable $\lambda = -\frac{h}{2Ig^2}$. Since the phase space $\mathbb{R} \times S^1$ with the coordinates (B, β) can be naturally complexified to $\mathbb{C} \times \mathbb{C}^*$ and since the symplectic form ω and the Hamiltonian $H(B, \beta) = \frac{B^2}{2I} - Ig^2(1 - \cos\beta)$ can be considered as an analytic form and an analytic function on $\mathbb{C} \times \mathbb{C}^*$ with the coordinates (B, β), we concentrate ourselves into the complexified system from now on.

In view of the analytic continuation, the key observation is the relation of the derivatives, $\frac{2}{\pi g}\mathcal{K}(1-\lambda)$, $\frac{2}{\pi g}\mathcal{K}(\lambda)$, of the inverse Birkhoff normal forms to the special Gauß hypergeometric differential equation

$$(1-\lambda)\lambda \frac{\mathrm{d}^2 f}{\mathrm{d}\lambda^2} + (1-2\lambda)\frac{\mathrm{d}f}{\mathrm{d}\lambda} - \frac{1}{4} f = 0. \tag{4.2}$$

In fact, the functions $\frac{2}{\pi g}\mathcal{K}(1-\lambda)$ and $\frac{2}{\pi g}\mathcal{K}(\lambda)$ are independent solutions and hence form a basis of the solution space of the linear differential equation (4.2). By the properties of the Gauß hypergeometric differential equation (4.2), we see that the analytic continuation of the derivative of the inverse Birkhoff normal forms has the monodromy. We can compute this monodromy, by using the connection formula of the equation (4.2). See [5, 7.405-7.406, pp.167-169] for the detail. As a result, the closed contour in the Gauß plane $\mathbb{C}$ with the affine coordinate $\lambda = -\frac{h}{2Ig^2}$ which counterclockwise encloses the origin $\lambda = 0$ gives rise to the monodromy matrix

$$\begin{pmatrix} 1 & -2\sqrt{-1} \\ 0 & 1 \end{pmatrix}$$

with respect to the above basis $\frac{2}{\pi g}\mathcal{K}(1-\lambda)$, $\frac{2}{\pi g}\mathcal{K}(\lambda)$, while the closed arc in the Gauß plane $\mathbb{C}$ which counterclockwise goes around the point $\lambda = 1$ corresponds to the monodromy matrix

$$\begin{pmatrix} 1 & 0 \\ -2\sqrt{-1} & 1 \end{pmatrix}$$

with respect to the same basis.

To sum up, the analytic continuation of the derivatives of the inverse Birkhoff normal forms for the simple pendulum dynamics reveals their global behavior as the monodromy which are represented by the above monodromy matrices.

Remark 4.1. The above computation of the derivative of the inverse for Birkhoff normal form around the hyperbolic equilibrium $(B, \beta) = (0, 0)$ agrees with the result in [8], while that around the elliptic equilibrium $(B, \beta) = (0, \pi)$ does not. In fact, the argument in Appendix C of [8] uses a (complex) transformation of the parameters which does not preserve the energy level, although it is not cared there. Note that the energy at the elliptic equilibrium $(B, \beta) = (0, \pi)$ is $H = -2Ig^2$, while the energy at the hyperbolic equilibrium $(B, \beta) = (0, 0)$ is $H = 0$. Thus, the calculation in Appendix C of [8] should be understood by taking the change of the energy levels through the transformation of the parameters into account.

§ 5. Free rigid body dynamics

As the second application of the results in Section 3, we briefly mention the free rigid body dynamics. The detailed discussion can be found in [10]. A free rigid body means a rigid body under no external force. Taking the coordinates with the origin at the centre of mass of the body, we can omit the translation of the body and the rotational motion can mathematically be formulated as a Hamiltonian system on the cotangent bundle $T^*SO(3)$ to the rotation group $SO(3)$. Since this Hamiltonian system on $T^*SO(3)$ has the symmetry with respect to the left-translation by $SO(3)$, the system can be reduced to the Hamiltonian system on (the dual to) the Lie algebra $\mathfrak{so}(3)$, which is isomorphic to $(\mathbb{R}^3, \times)$, where $\times$ denote the exterior product with respect to the standard inner product $\cdot$ of $\mathbb{R}^3$. This procedure is called the Lie-Poisson reduction. See [18] for the details on the reduction of Hamiltonian systems. The reduced system on $\mathbb{R}^3$ for the free rigid body dynamics can be described by Euler equation

$$\frac{\mathsf{d}P}{\mathsf{d}t} = P \times \left(\mathcal{I}^{-1}(P)\right), \qquad P = (p_1, p_2, p_3) \in \mathbb{R}^3, \tag{5.1}$$

where $\mathcal{I}$ is the inertia tensor of the rigid body, which is determined by its mass distribution. The inertia tensor $\mathcal{I}$ is a linear operator $\mathbb{R}^3 \to \mathbb{R}^3$ given by the diagonal matrix $\operatorname{diag}(I_1, I_2, I_3)$ where $I_1, I_2, I_3 > 0$. The vector $P = (p_1, p_2, p_3) \in \mathbb{R}^3$ is called the angular momentum of the rigid body. Note that the equation (5.1) is the Hamilton's equation for the Hamiltonian $H(P) = \dfrac{1}{2}P \cdot \left(\mathcal{I}^{-1}(P)\right)$ with respect to the Lie-Poisson bracket $\{\cdot,\cdot\}$ defined through $\{F,G\}(P) = P \cdot (\nabla F(P) \times \nabla G(P))$, $P \in \mathbb{R}^3$, where F, G are smooth functions on $\mathbb{R}^3$ and $\nabla F(P)$ denotes the derivative of F at P.

An important feature of Euler equation (5.1) is its first integral $L(P) = \dfrac{1}{2}P \cdot P$. Thus, the system (5.1) can be restricted to the level surfaces of L: $L(P) = \ell(\text{constant})$, which is a two-dimensional sphere. The Lie-Poisson bracket $\{\cdot,\cdot\}$ naturally induces the symplectic form

$$\omega = \frac{\mathsf{d}p_1 \wedge \mathsf{d}p_2}{3p_3} = \frac{\mathsf{d}p_2 \wedge \mathsf{d}p_3}{3p_1} = \frac{\mathsf{d}p_3 \wedge \mathsf{d}p_1}{3p_2}$$

on $L = \ell$. The restricted system on the level surface $L = \ell$ is, in fact, a Hamiltonian system for the Hamiltonian H restricted to $L = \ell$ with respect to the symplectic form ω. Therefore, we have a Hamiltonian system of one degree of freedom, for which we can apply the results in Section 3.

The phase portrait of the restricted system on $L = \ell$ is well known and can be found e.g. in [15]. Assume that $I_1 < I_2 < I_3$. Then, there are six equilibria on the level sphere $L = \ell$, which are the intersection of $L = \ell$ and the three principal axes. Four of these equilibria on the p_1- and p_3-axes are elliptic, while the other two on the p_2-axis are hyperbolic.

Now, we calculate the derivative of the inverse for Birkhoff normal form around each equilibrium. We first consider that around the elliptic equilibrium $(p_1, p_2, p_3) = (\sqrt{2\ell}, 0, 0)$ on the p_1-axis, where (p_2, p_3) can be regarded as the local coordinates of $L = \ell$. We take the one-form

$$\eta_s = (1-s)\frac{\mathsf{d}p_2}{3\left(\frac{1}{I_3} - \frac{1}{I_1}\right)p_3 p_1} + s\frac{\mathsf{d}p_3}{3\left(\frac{1}{I_1} - \frac{1}{I_2}\right)p_1 p_2},$$

where s is an arbitrary parameter. One can easily verify that $\eta_s \wedge \mathsf{d}H = \omega$.

We denote the inverse function of Birkhoff normal form $H - \dfrac{\ell}{I_1} = \mathcal{H}_1$ around the elliptic equilibrium $(\sqrt{2\ell}, 0, 0)$ by Φ_1. From Theorem 3.1, we have the following expression of its derivative.

Theorem 5.1. *The derivative of the inverse Birkhoff normal forms around the elliptic equilibria* $(p_1, p_2, p_3) = (\pm\sqrt{2\ell}, 0, 0)$ *on the* p_1*-axis can be given by*

$$\Phi_1'(h) = -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{(d-c)(a-b)}}\mathcal{K}\left(\frac{(d-a)(b-c)}{(d-c)(b-a)}\right), \tag{5.2}$$

where $a = \dfrac{1}{I_1}$, $b = \dfrac{1}{I_2}$, $c = \dfrac{1}{I_2}$, $d = \dfrac{h}{\ell}$ *and where* $\mathcal{K}(\lambda) = \displaystyle\int_0^1 \frac{\mathsf{d}x}{\sqrt{(1-x^2)(1-\lambda x^2)}}$ *is the complete elliptic integral of the first kind.*

For the proof of this theorem, see [10]. We denote the right hand side of (5.2) by $S(a,b,c,d)$. Note that the two elliptic equilibria $(p_1,p_2,p_3) = (\pm\sqrt{2\ell},0,0)$ on the p_1-axis have the same Birkhoff normal form.

Similarly, we have the expression of the derivative of the inverse Φ_3 for Birkhoff normal form $H - \dfrac{\ell}{I_3} = \mathcal{H}_3$ around the elliptic equilibria $(p_1,p_2,p_3) = (0,0,\pm\sqrt{2\ell})$ on the p_3-axis.

Theorem 5.2. *The derivative of the inverse Birkhoff normal forms around the elliptic equilibria $(p_1,p_2,p_3) = (0,0,\pm\sqrt{2\ell})$ on the p_3-axis can be given by*

$$\Phi_3'(h) = -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{(d-a)(c-b)}}\mathcal{K}\left(\frac{(d-c)(b-a)}{(d-a)(b-c)}\right) = S(c,b,a,d),$$

where $a = \dfrac{1}{I_1}$, $b = \dfrac{1}{I_2}$, $c = \dfrac{1}{I_2}$, $d = \dfrac{h}{\ell}$ and $\mathcal{K}(\lambda)$ is the complete elliptic integral of the first kind.

We consider the derivatives of the inverse for Birkhoff normal forms around the hyperbolic equilibria $(p_1,p_2,p_3) = (0,\sqrt{2\ell},0)$ on the p_2-axis, where (p_1,p_3) serves as the local coordinate systems of $L = \ell$. We take the one-form

$$\eta_s' = (1-s)\frac{\mathsf{d}p_3}{3\left(\frac{1}{I_1}-\frac{1}{I_2}\right)p_1p_2} + s\frac{\mathsf{d}p_1}{3\left(\frac{1}{I_2}-\frac{1}{I_3}\right)p_2p_3},$$

with an arbitrary parameter s as before, and the closed arc

$$\gamma : p_3 = \sqrt{2\ell\frac{\frac{1}{I_2}-\frac{h}{\ell}}{\frac{1}{I_2}-\frac{1}{I_3}}}\cos\theta, p_1 = \sqrt{2\ell\frac{\frac{1}{I_2}-\frac{h}{\ell}}{\frac{1}{I_2}-\frac{1}{I_1}}}\sin\theta, \quad \theta\in[0,2\pi],$$

around the equilibrium $(p_1,p_2,p_3) = (0,\sqrt{2\ell},0)$. Note that the arc γ is a cycle in the complexified integral curve $H = h$, where h is the energy level, in the complexified phase space $L = \ell$. From Theorem 3.2, we can calculate the derivative of the inverse Φ_2 for Birkhoff normal form $H - \dfrac{\ell}{I_2} = \mathcal{H}_2$ around the equilibrium $(p_1,p_2,p_3) = (0,\sqrt{2\ell},0)$ as $\Phi_2'(h) = \dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_\gamma \eta_s'$. As a result, we have the following expression.

Theorem 5.3. *The derivative of the inverse of Birkhoff normal forms around the hyperbolic equilibria $(p_1,p_2,p_3) = (0,\pm\sqrt{2\ell},0)$ on the p_2-axis can be given by*

$$\Phi_2'(h) = \frac{\sqrt{-1}}{3\pi}\sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{(d-c)(b-a)}}\mathcal{K}\left(\frac{(d-b)(a-c)}{(d-c)(a-b)}\right) = -\sqrt{-1}S(b,a,c,d),$$

where $a = \dfrac{1}{I_1}$, $b = \dfrac{1}{I_2}$, $c = \dfrac{1}{I_2}$, $d = \dfrac{h}{\ell}$ and $\mathcal{K}(\lambda)$ is the complete elliptic integral of the first kind.

Note that the two hyperbolic equilibria on the p_2-axis have the same Birkhoff normal form.

Remark 5.1. The above computation of the derivative for the inverse Birkhoff normal forms in Theorems 5.1, 5.2, 5.3 agrees with the results in [9].

Thus, we have the concrete expression of the derivatives for the inverse Birkhoff normal forms around each pair of the six equilibria on the three principal axes in terms of complete elliptic integral of the first kind. Since the phase space $\mathbb{R}^3$ with the affine coordinates (p_1, p_2, p_3) can be naturally complexified to $\mathbb{C}^3$ with (p_1, p_2, p_3) being regarded as the complex affine coordinates, and since the restricted phase space $L = \ell$ as well as its symplectic form ω are naturally complexified, all the systems are now considered to be complex analytic Hamiltonian systems. Then, the derivatives of the inverse for Birkhoff normal forms can be seen as complex analytic functions in the energy level h around any point of the open set $h \neq \frac{\ell}{I_1}, \frac{\ell}{I_2}, \frac{\ell}{I_3}$ in $\mathbb{C}$.

On the basis of the previous results in Theorems 5.1, 5.2, 5.3, we consider the analytic continuation of the derivatives of the inverse Birkhoff normal forms around the three pairs of equilibria on the principal axes with respect to the energy level h. As in the case of the simple pendulum dynamics, we use the relation between the complete elliptic integral $\mathcal{K}(\lambda)$ of the first kind and the special Gauß hypergeometric differential equation (4.2).

By means of the connection formula of the equation (4.2) [5, 7.405-7.406, pp.167-169], we can show the following formula of the derivatives $\Phi_1'(h), \Phi_2'(h), \Phi_3'(h)$ for the inverse Birkhoff normal forms around the equilibria.

Proposition 5.4. *The analytic continuation of the function*

$$S(a, b, c, d) = -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{(d-c)(a-b)}}\mathcal{K}\left(\frac{(d-a)(b-c)}{(d-c)(b-a)}\right)$$

satisfies the formula

$$S(a, b, c, d) + S(b, a, c, d) = S(c, b, a, d),$$

which means

$$\Phi_1'(h) + \sqrt{-1}\Phi_2'(h) = \Phi_3'(h).$$

As in the case of the simple pendulum dynamics, the derivatives $\Phi_1'(h)$, $\Phi_2'(h)$, $\Phi_3'(h)$ for the inverse Birkhoff normal forms can be analytically extended with respect to the energy level h on the open set $h \neq \frac{\ell}{I_1}, \frac{\ell}{I_2}, \frac{\ell}{I_3}$ in $\mathbb{C}$. One can observe that the analytic continuation of these functions has the monodromy, which can be calculated

as follows:
The closed real arc which counterclockwise encloses the point $h = \dfrac{\ell}{I_1}$ corresponds to the monodromy

$$S_1 \mapsto S_1, \quad S_2 \mapsto 2S_1 + S_2, \quad S_3 \mapsto 2S_1 + S_3,$$

where $S_1 = \Phi_1'(h) = S(a,b,c,d)$, $S_2 = \sqrt{-1}\Phi_2'(h) = S(b,a,c,d)$, and $S_3 = \Phi_3'(h) = S(c,b,a,d)$. The closed real contour which counterclockwise goes around the point $h = \dfrac{\ell}{I_2}$ gives rise to the monodromy

$$S_1 \mapsto S_1 - 2S_2, \quad S_2 \mapsto S_2, \quad S_3 \mapsto S_3 - 2S_2.$$

The closed real arc which encloses the point $h = \dfrac{\ell}{I_3}$ corresponds to the monodromy

$$S_1 \mapsto S_1 - 2S_3, \quad S_2 \mapsto S_2 + 2S_3, \quad S_3 \mapsto S_3.$$

Thus, we have observed the monodromy as the global behavior of the derivatives for the inverse Birkhoff normal forms around the equilibria of the free rigid body dynamics. The detailed proofs of the above results can be found in [10].

As to the free rigid body dynamics, one can consider some elliptic fibration which naturally arises from the dynamics. In fact, the integral curve of Euler equation for the free rigid body dynamics is given as the intersection of the two quadric level surfaces of the first integrals:

$$\begin{cases} H = \dfrac{1}{2}\left(\dfrac{p_1^2}{I_1} + \dfrac{p_2^2}{I_2} + \dfrac{p_3^2}{I_3}\right) = h, \\ L = \dfrac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) = \ell, \end{cases}$$

where (p_1, p_2, p_3) is the angular momentum of the rigid body. This quadrics intersection can be complexified and projectified by the projective space curve

$$\begin{cases} ax^2 + by^2 + cz^2 + dw^2 = 0, \\ x^2 + y^2 + z^2 + w^2 = 0, \end{cases} \tag{5.3}$$

where $(x : y : z : w)$ are the homogeneous coordinates of the projective space $P_3(\mathbb{C})$ which are related to the original affine coordinates of the angular momentum as $p_1 = \sqrt{-2\ell}\dfrac{x}{w}$, $p_2 = \sqrt{-2\ell}\dfrac{y}{w}$, $p_3 = \sqrt{-2\ell}\dfrac{z}{w}$. And $a = \dfrac{1}{I_1}$, $b = \dfrac{1}{I_2}$, $c = \dfrac{1}{I_2}$, $d = \dfrac{h}{\ell}$, as before. It is known that the curve (5.3) is a smooth elliptic curve, if the parameters a, b, c, d are distinct. See [16] or [1] for the proof of this fact. Furthermore, the same equation (5.3) defines a smooth four-dimensional variety F in the product of two projective spaces $P_3(\mathbb{C}) \times P_3(\mathbb{C})$ with the coordinates $((x : y : z : w), (a : b : c : d))$. The restriction of the projection $P_3(\mathbb{C}) \times P_3(\mathbb{C}) \ni ((x : y : z : w), (a : b : c : d)) \mapsto (a : b : c : d) \in P_3(\mathbb{C})$ onto

F gives rise to an elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$, whose fibres can be seen as the compactified and complexified integral curves of the dynamics.

Although the four-dimensional variety F is smooth and rational, the elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$ is not trivial at all. It does not admit either holomorphic or meromorphic section and it is further not flat, i.e. it has a two-dimensional fibre. See [16] for these basic facts on the elliptic fibration π_F. Here, we mention the classification of the singular fibres of the fibration π_F.

- If only two of the parameters a, b, c, d are equal, the fibre consists of two smooth rational curves intersecting at two points. This is a singular fibre of type I_2 in Kodaira's notation [14, 2].
- If two of a, b, c, d are equal and the other two are also equal without further coincidence, the fibre consists of four smooth rational curves intersecting cyclically. This is a singular fibre of type I_4 in Kodaira's notation.
- If three of a, b, c, d are equal without further coincidence, the fibre is a smooth rational curve, as a point set, but with multiplicity two. This singular fibre is not in the list of singular fibres of elliptic surfaces by Kodaira.
- If $a = b = c = d$, the fibre is a space quadric surface $x^2 + y^2 + z^2 + w^2 = 0$.

The regular locus of the fibration π_F is the open set $R := P_3(\mathbb{C}) \setminus \{a = b, a = c, a = d, b = c, b = d, c = d\}$.

The fundamental group $\pi_1(R, *)$ of the regular locus for the fibration π_F is calculated in [10] by means of the arguments on the fundamental groups of the complements of hyperplane arrangements (cf. [17]). We denote the generators h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34} of $\pi_1(R, *)$ which are respectively represented by real closed arcs enclosing the irreducible components $a = b$, $a = c$, $a = d$, $b = c$, $b = d$, $c = d$ of the singular locus.

Theorem 5.5. *The fundamental group $\pi_1(R, *)$ of the regular locus for the fibration π_F is generated by h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34}, with the relations*

$$\begin{aligned}
h_{12}h_{23}h_{13} &= h_{23}h_{13}h_{12} = h_{13}h_{12}h_{23},\\
h_{23}h_{34}h_{24} &= h_{34}h_{24}h_{23} = h_{24}h_{23}h_{34},\\
h_{12}h_{24}h_{14} &= h_{24}h_{14}h_{12} = h_{14}h_{12}h_{24},\\
h_{34}h_{14}h_{13} &= h_{14}h_{13}h_{34} = h_{13}h_{34}h_{14},\\
h_{12}h_{34} &= h_{34}h_{12},\\
h_{13}h_{23}^{-1}h_{24}h_{23} &= h_{23}^{-1}h_{24}h_{23}h_{13},\\
h_{23}h_{14} &= h_{14}h_{23},\\
h_{13}h_{12}h_{23}h_{34}h_{24}h_{14} &= 1.
\end{aligned}$$

It is shown in [10] that the derivatives S_3, S_1 of the inverse Birkhoff normal forms around the equilibria on the p_3- and p_1-axes form a basis of the first cohomology group of the regular fibre of the fibration π_F. From this, we see that the monodromy of the inverse Birkhoff normal forms for the free rigid body dynamics corresponds exactly to the monodromy of the elliptic fibration π_F. In [10], the following theorem is obtained.

Theorem 5.6. *The basis of the first cohomology group for the regular fibre of the fibration $\pi_F : F \to P_3(\mathbb{C})$ is given by the analytic continuations of S_3 and of S_1, which are proportional to the derivative of the inverse Birkhoff normal forms around the p_3- and p_1-axes, respectively. The monodromy of the fibration π_F with respect to S_3 and S_1 is given by the correspondence of the generators h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34} of the fundamental group $\pi_1(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D))$ to the matrices in $SL(2, \mathbb{Z})$ as follows:*

$$h_{14}, h_{23} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, h_{13}, h_{24} \mapsto \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, h_{12}, h_{34} \mapsto \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

The monodromy of the elliptic fibration π_F described here is essentially the same as the monodromy which we have above calculated for the analytic extension of the derivatives for the inverse Birkhoff normal forms. The detail of these geometric aspects of the elliptic fibration π_F which naturally arises from the free rigid body dynamics, including the calculation of the fundamental group of the regular locus of π_F, are described in [10] from the viewpoint of the close relation to the analytic extension of the derivatives of the inverse Birkhoff normal forms.

§ 6. Concluding remarks

Based on the expression of the derivatives of the inverse functions for Birkhoff normal forms around the elliptic and the hyperbolic equilibria, their analytic extension has been considered for two basic examples in analytical mechanics, the simple pendulum and the free rigid body dynamics. In both cases, the derivatives of the inverse Birkhoff normal forms are expressed in terms of the complete elliptic integral of the first kind, and by means of its relation with the special Gauß hypergeometric differential equation, the monodromy has been calculated explicitly. This result indicates that the problem on the global behavior of Birkhoff normal forms which has been posed in Section 1 has been answered with the nontrivial monodromy of the analytic extension for the derivative of their inverse functions. In the case of the free rigid body dynamics, the monodromy of the analytic extension for the derivatives of the inverse Birkhoff normal forms is closely related with the monodromy of the elliptic fibration which naturally arises from the dynamics.

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