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Summability of formal solutions to partial differential equations

To Professor T. Aoki on the occasion of his 60th birthday

By

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Abstract

We give a survey of results on convergence and summability of formal power series solutions to the initial value problem for non-Kowalevskian partial differential equations. A special attention is paid on heat type equations.

§ 1. Introduction

One of the main problems arising in the analytic theory of partial differential equations is a characterization of data given on a manifold S for which a solution of a boundary value problem is an analytic function in a variable normal to S. In general, one can easily construct a formal power series solution in the normal to S variable, and by the Cauchy-Kowalevski theorem it is convergent if S is not the characteristic variety of the equation. In other cases formal solutions need not to be convergent. At this point there arise natural questions:

- under which conditions on the data the formal solution is convergent?
- what is the meaning of a divergent formal solution?
- is it an asymptotic expansion of an actual solution?

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- can and how the actual solution be constructed from the formal one?

In the case of ordinary differential equations answers to those questions were given in 80-ties and 90-ties of the XX century by the (multi-)summability theory. On the other hand in the case of partial differential equations the study of those problems started at the end of the XX century and, besides linear equations with constant coefficients in two variables, practically there are no general results.

In the paper we shall give a survey of solutions to those problems for some classes of partial differential equations. Namely under the assumption that $S \subset \{t = 0\}$, we shall consider equations of the form

$$P(t, z, \partial_t, \nabla_z, \nabla_z^2, \ldots)\, u(t, z) = f(t, z),$$

where P is a differential operator (not necessary linear) with holomorphic coefficients. We shall assume that the order of P with respect to z is at least 2.

§ 2. One dimensional case.

The starting point in the study of summability of formal solutions to PDE's is the paper by Lutz, Miyake and Schäfke [LMS-99]. They studied the initial value problem to the one dimensional heat equation

$$\begin{cases} \partial_t u - \partial_z^2 u = 0, \\ u_{|t=0} \qquad = u_0 \in \mathcal{A}(B), \end{cases} \tag{1}$$

where B is a ball in $\mathbb{R}$. Its formal power series solution $\widehat{u}$ is given by

$$\widehat{u}(t, z) = \sum_{j=0}^{\infty} \frac{\partial^{2j} u_0(z)}{j!} t^j. \tag{2}$$

In general the series $\widehat{u}$ is divergent, but Gevrey of order $k = 1$, i.e.,

$$\Big|\frac{\partial^{2j} u_0(z)}{j!}\Big| \leq C^{j+1} (j!)^k \quad \text{with} \quad k = 1,$$

locally uniformly in B. The problem of a characterization of the initial data u_0 ensuring convergence of the formal solution (2) was already solved in 1875 by Kowalevskaya ([Kow]). She proved that the solution $\widehat{u}$ is convergent if and only if the initial data u_0 can be analytically extended to an entire function of exponential order 2.

To state the main result of [LMS-99] recall

Definition 2.1. Let $d \in \mathbb{R} \bmod 2\pi$ be a direction in $\mathbb{C}$, U an open subset of $\mathbb{C}^n$ and $\varphi_j \in \mathcal{O}(U)$ for $j \in \mathbb{N}_0$. A formal power series

$$\widehat{\varphi}(t, z) = \sum_{j=0}^{\infty} \frac{\varphi_j(z)}{j!} t^j$$

is said to be *k-summable* (*Borel summable* if $k = 1$) with respect to t in the direction d if its *k-Borel transform* defined on $B_\epsilon \times U$ with some $\epsilon > 0$ by

$$\big(\widehat{\mathcal{B}}_k\widehat{\varphi}\big)(s,z) = \sum_{j=0}^{\infty} \frac{\varphi_j(z)}{j! \cdot \Gamma(1+j/k)} s^j$$

extends holomorphically to a domain $\big(B_\epsilon \cup S(d,\epsilon)\big) \times U$ and the extension satisfies for any $U_1 \Subset U$ and $0 < \epsilon_1 < \epsilon$,

$$\sup_{z\in U_1} |\big(\widehat{\mathcal{B}}_k\widehat{\varphi}\big)(s,z)| \leq Ae^{B|s|^k} \quad \text{for} \quad s \in S(d,\epsilon_1)$$

with some $A, B < \infty$. Here B_ϵ is a ball in $\mathbb{C}$ of radius ϵ centered at the origin and $S(d,\epsilon)$ is a sector $\{z \in \mathbb{C} : |\arg z - d| < \epsilon\}$. If so, then the function

$$\varphi^\theta(t,z) = \frac{1}{t^k}\int_0^{\infty(\theta)} \widehat{\mathcal{B}}_k\widehat{\varphi}(s,z)e^{-(s/t)^k} d(s^k), \quad |\theta - d| < \epsilon,$$

is called the *k-Borel sum* of $\widehat{\varphi}$.

Theorem 2.2. ([LMS-99]). *Let u_0 be a function real analytic in a ball B in $\mathbb{R}$ centered at the origin. The formal power series solution* (2) *of* (1) *is Borel summable in a direction d locally uniformly in B iff u_0 extends analytically to a function holomorphic on a domain*

$$D(d,\varepsilon) \supset S(d/2,\varepsilon) \cup S(d/2+\pi,\varepsilon)$$

with some $\varepsilon > 0$ which has in $D(d,\varepsilon)$ at most exponential growth of order at most 2 *locally uniformly in B.*

The above result was extended to the case of multisummable solutions of (1) by Balser [B-99]; to formal power series satisfying certain differential recursion formulas by Balser and Miyake [BMi-99]; to the equation $\partial_t^p u = \partial_z^q u$, $p < q$, by Miyake [Mi-99] and by Ichinobe [I-01], who also gave explicite integral representations of the Borel sums of solutions in terms of the Barnes hypergeometric series ${}_qF_{p-1}$.

General linear partial differential equations with constant coefficients in one space variable

$$\partial_t^m p(\partial_z)\, u - \sum_{i=1}^{m} \partial_t^{m-i} p_i(\partial_z)\, u = 0,$$

where p and p_i are polynomials, were investigated by Balser. In [B-02] he studied the case when the Newton polygon of the equation has only one slope and proved k-summability of a (unique) normalized solution. While in [B-04] he proved multisummability of normalized solutions to equations with Newton polygon having several slopes. The results were further extended in [B-05] to solutions of some integral-differential equations in two variables. Another proof of Balser's results in a more general framework of fractional equations was given by Michalik [M-10].

In [I-03] Ichinobe studied the following problem

$$\begin{cases} \partial_t^{p\nu} u &= \sum_{j=1}^{\nu} a_j \partial_t^{p(\nu-j)} \partial_z^{jq} u, \quad q > p \geq 1, \\ \partial_t^k u_{|t=0} &= 0 \quad \text{for} \quad k = 0, \dots, p\nu - 2, \\ \partial_t^{p\nu-1} u_{|t=0} &= u_0 \in \mathcal{A}(B). \end{cases}$$

He proved that its formal power series solution $\widehat{u}$ is $p/(q-p)$-summable in a direction d (also in d' with $d' = d \bmod (2\pi/p)$) iff u_0 extends holomorphically to a domain D containing union of some sectors in $\mathbb{C}$ and has in D at most exponential growth of order at most $q/(p-q)$ locally uniformly in B. He also gave an explicite integral representation of the Borel sum of $\widehat{u}$ in terms of the Meijer function $G_{p,q}^{m,n}$. Ichinobe also studied the Cauchy problem to the equation

$$\partial_t u = P(t, \partial_z) u \quad \text{where} \quad P(t, \partial_z) = \sum_{i,\alpha} a_{i\alpha} t^i \partial_z^\alpha.$$

Assuming that the Newton polygon of P has only one slope he proved that the formal solution is k-summable if the initial data are holomorphic in a sum of sectors with suitable exponential growth.

From the above papers it follows that formal solutions of non-Kowalevskian PDEs are summable only if the initial data satisfy quite restrictive conditions.

§ 3. Multidimensional case

The study of the multidimensional equations was initiated by Ōuchi [O-02]. He studied the summability of formal solutions to linear PDEs of the form

$$(3) \qquad \begin{cases} \partial_t^m u \quad + \sum_{(j,\alpha)\in\Lambda} a_{j,\alpha}(t) \partial_t^j \partial_z^\alpha u = f(t,z), \\ \partial_t^i u_{|t=0} = \varphi_i \quad \text{for} \quad i = 0, \dots, m-1. \end{cases}$$

If $\operatorname{ord}_t a_{j,\alpha} \geq \max(0, j - m + 1)$ for $(j,\alpha) \in \Lambda$, then the problem has a unique formal solution, which is convergent if $j + |\alpha| \leq m$ for all $(j,\alpha) \in \Lambda$. Ōuchi defined the Newton polygon $N(E)$ and proved that if $(j + |\alpha|, \operatorname{ord}_t a_{j,\alpha} - j) \in \operatorname{int} N(E)$ for $(j,\alpha) \in \Lambda$ with $\alpha \neq 0$ (this condition guaranties that the equation (3) can be treated as a perturbation of an ordinary differential equation), then the formal solution of (3) is multisummable in a suitable multidirection; the levels of summability are the slopes of $N(E)$.

In [Y-12] Yamazawa studied the equation

$$\partial_t u = \partial_z^2 u + t(t\partial_t)^3 u.$$

He proved that if initial data is an entire function of exponential order 2, then the solution is Borel summable in directions $d \notin \{0, \pi\}$. Later he showed that the same conclusion holds for functions of finite exponential order. Motivated by this and similar examples Tahara posed the following problem.

Assuming that initial data and f are entire functions of exponential order γ determine minimal γ guaranteeing summability of a formal solution to (3).

To solve this problem he and Yamazawa introduced in [TY-13]: t-Newton polygon $N_t(E)$, the set of admissible exponents $\mathcal{C}$ and the set of singular directions $\mathcal{Z}$. They proved that under some conditions if initial data and f are entire functions of exponential order $\gamma \in \mathcal{C}$, then the formal solution of (3) is $(k_p, \dots, k_1)$-multisummable in any direction $d \notin \mathcal{Z}$ where k_i are the slopes of $N_t(E)$. This result is in accordance with previous ones.

§ 4. Multidimensional heat equation

In the case of the multidimensional heat equation

$$\begin{cases} \partial_t u - \Delta_z u = 0, \\ u_{|t=0} \qquad = u_0 \in \mathcal{A}(\Omega), \quad \Omega \subset \mathbb{R}^n, \end{cases} \tag{4}$$

where Δ is the n-dimensional Laplace operator, conditions for k-summability of formal solutions were obtained by Balser and Malek [BM-04]. However the conditions were stated in terms of some auxiliary function expressed in terms of the formal solution itself and not directly in terms of the initial data.

Using the modified Borel transformation, which transforms the heat equation into the wave equation, Michalik obtained in [M-06] conditions for Borel summability in terms of the initial data, only. He proved that the (unique) formal power series solution of (4) is Borel summable in a direction d iff the auxiliary function

$$\Phi_n(z, \tau) = \begin{cases} \int_{\partial B(1)} u_0(z + \tau y)\, dS(y) & \text{if } n \text{ is odd}, \\ \int_{B(1)} \frac{u_0(z+\tau y)}{\sqrt{1-|y|^2}}\, dy & \text{if } n \text{ is even} \end{cases}$$

is holomorphic at the origin in z variable and can be analytically continued with respect to τ in sectors in directions $d/2$ and $\pi + d/2$, and this continuation is of exponential order at most 2.

§ 5. Mean values

Studying the paper [M-06] we have arrived at the idea that conditions for convergence and Borel summability of solutions to the heat equation can be expressed in terms of integral means of the initial data over balls or spheres.

§ 5.1. Spherical and solid means.

Let Ω be a domain in $\mathbb{R}^n$ and $x \in \Omega$. For $0 < R < \operatorname{dist}(x, \partial\Omega)$ define solid and spherical means of a continuous function $u \in C^0(\Omega)$ by

$$M(u;\, x, R) = \frac{1}{\sigma(n)R^n} \int_{B(x,R)} u(y)\, dy,$$
$$N(u;\, x, R) = \frac{1}{n\sigma(n)R^{n-1}} \int_{S(x,R)} u(y)\, dS(y),$$

where $\sigma(n)$ is the volume of the unit ball in $\mathbb{R}^n$. The relations between $M(u;\, x, R)$ and $N(u;\, x, R)$ are given by

Lemma 5.1. ([Ł-11, Lemma 1]). *Let $u \in C^0(\Omega)$. Then for any $x \in \Omega$ and $0 < R < \operatorname{dist}(x, \partial\Omega)$,*

$$\Big(\frac{R}{n}\frac{\partial}{\partial R} + 1\Big) M(u;\, x, R) = N(u;\, x, R).$$

If $u \in C^2(\Omega)$, then

$$\frac{n}{R}\frac{\partial}{\partial R} N(u;\, x, R) = M(\Delta u;\, x, R).$$

§ 5.2. Characterization of real analyticity

It appears that real analyticity of a function can be characterized in terms of its integral means by the so-called Pizzetti series.

Theorem 5.2. (Mean-value property, [Ł-12, Theorem 3.1]). *Let $u \in \mathcal{A}(\Omega)$, $x \in \Omega$. Then $M(u;\, x, R)$ and $N(u;\, x, R)$ are analytic functions at the origin and for small R,*

$$M(u;\, x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \big(\frac{n}{2} + 1\big)_k k!} R^{2k}, \tag{5}$$

$$N(u;\, x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \big(\frac{n}{2}\big)_k k!} R^{2k}. \tag{6}$$

Here $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhamer symbol.

Proof. If $x = 0$ the proof of (5) is done by expending u into Taylor series

$$u(y) = \sum_{\ell \in \mathbb{N}_0^n} \frac{1}{\ell_1! \cdots \ell_n!} \frac{\partial^{|\ell|}}{\partial y^\ell} u(0) y^\ell,$$

and then computing the integral of $y^\ell = y_1^{\ell_1} \cdots y_n^{\ell_n}$ over $B(R)$. Next, applying Lemma 5.1 we get (6). □

Theorem 5.3. (Converse to the mean value property, [Ł-12, Theorem 3.2]). *Let $\rho \in C^0(\Omega, \mathbb{R}^+)$ and $u \in C^\infty(\Omega)$. If*

$$\widetilde{N}(x,R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}\right)_k k!} R^{2k}$$

$$\Big(\textit{respectively,} \qquad \widetilde{M}(x,R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}+1\right)_k k!} R^{2k} \quad \Big)$$

is convergent locally uniformly in $\{(x,R) : x \in \Omega, |R| < \rho(x)\}$, then $u \in \mathcal{A}(\Omega)$ and $N(u;\, x,R) = \widetilde{N}(x,R)$ (respectively, $M(u;\, x,R) = \widetilde{M}(x,R)$) for $x \in \Omega$, $R < \min\big(\rho(x), \operatorname{dist}(x, \partial\Omega)\big)$.

Proof. We first derive that for any compact set $K \Subset \Omega$ one can find $C < \infty$ such that for $k \in \mathbb{N}_0$,

$$\sup_{x \in K} |\Delta^k u(x)| \leq C^{2k+1}(2k)!.$$

But by [ACL, Theorem 2.2 in Chapter II] this inequality implies that $u \in \mathcal{A}(\Omega)$. Finally, by Theorem 5.2 we get $\widetilde{N}(x,R) = N(u;\, x,R)$ and $M(u;\, x,R) = \widetilde{M}(x,R)$ □

§ 5.3. Functions of Laplacian growth.

In order to control the growth of iterated Laplacians of smooth functions Aronszajn, Creese and Lipkin introduced the notion of the Laplacian growth.

Definition 5.4. ([ACL]). Let $\varrho > 0$ and $\tau \geq 0$. A function u smooth on $\Omega \subset \mathbb{R}^n$ is of *Laplacian growth* (ϱ, τ) if for every $K \Subset \Omega$ and $\varepsilon > 0$ one can find $C = C(K, \varepsilon) < \infty$ such that for $k \in \mathbb{N}_0$,

$$\sup_{x \in K} |\Delta^k u(x)| \leq C(\tau + \varepsilon)^{2k}(2k)!^{1-1/\varrho}. \tag{7}$$

Recall also the definition of functions of exponential growth.

Definition 5.5. ([Boas]). Let $\varrho > 0$ and $\tau \geq 0$. An entire function F is said to be of *exponential growth* (ϱ, τ) if for every $\varepsilon > 0$ one can find C_ε such that for any $R < \infty$

$$\sup_{|z| \leq R} |F(z)| \leq C_\varepsilon \exp\{(\tau + \varepsilon) R^\varrho\}.$$

The exponential growth of an entire function can be also expressed in terms of estimations of its Taylor coefficients.

It appears that a function u of Laplacian growth (ϱ, τ) on Ω is in fact real-analytic on Ω (see [ACL, Theorem 2.2 in Chapter II]). So the spherical and solid means $N(u;\, x,R)$

and $M(u;\, x, R)$ are expressed by the Pizzetti series valid for $x \in \Omega$ and R small enough. However due to estimation (7) both functions $N(u;\, x, R)$ and $M(u;\, x, R)$ can be extended to entire functions of exponential growth.

Theorem 5.6. ([Ł-12, Theorem 4.5]). *Let $u \in \mathcal{A}(\Omega)$, $\varrho > 0$ and $\tau \geq 0$. If u is of Laplacian growth (ϱ, τ), then $N(u;\, x, R)$ and $M(u;\, x, R)$ extend holomorphically to entire functions of exponential growth $(\varrho, \tau^{\varrho}/\varrho)$ locally uniformly in Ω.*

Theorem 5.7. ([Ł-12, Theorem 4.6]). *Let $u \in \mathcal{A}(\Omega)$. If $M(u;\, x, R)$ defined for $x \in \Omega$ and $0 \leq R < \operatorname{dist}(x, \partial\Omega)$ extends as a function of R to an entire function $\widetilde{M}(u;\, x, z)$ of exponential growth (ϱ, τ) locally uniformly in Ω, then u is of Laplacian growth $\big(\varrho, (\varrho\tau)^{1/\varrho}\big)$. Analogous result holds for $N(u;\, x, R)$.*

§ 5.4. Application to the heat equation

Using Theorems 5.6 and 5.7 with $\rho = 2$ we get

Theorem 5.8. ([Ł-12, Theorem 5.1]). *Let $0 < T \leq \infty$, $u_0 \in \mathcal{A}(\Omega)$. The formal power series solution*

$$\widehat{u}(t, z) = \sum_{j=0}^{\infty} \frac{\Delta^j u_0(z)}{j!} t^j \tag{8}$$

of the n-dimensional heat equation (4) is convergent for $|t| < T$ locally uniformly in Ω iff $N(u_0;\, z, R)$ and/or $M(u_0;\, z, R)$ extend to an entire function of exponential growth $(2, 1/(4T))$ locally uniformly in Ω.

Proof. Assume that $\widehat{u}(t, z)$ is convergent for $|t| < T$ locally uniformly in Ω. Then for any compact set $K \Subset \Omega$ and $\varepsilon > 0$ there exist $C = C(K, \varepsilon) < \infty$ such that for all $k \in \mathbb{N}_0$,

$$\begin{aligned} \sup_{z \in K} |\Delta^k u_0(z)| &\leq C\Big(\frac{1}{T} + \varepsilon\Big)^k \cdot k! \\ &\leq C_\varepsilon\Big((2T)^{-1/2} + \varepsilon\Big)^{2k} \cdot (2k)!^{1/2}. \end{aligned}$$

Hence u_0 is of Laplacian growth $(2, 1/\sqrt{2T})$ and by Theorem 5.6, $N(u_0;\, z, R)$ and $M(u_0;\, z, R)$ extend to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in Ω.

On the other hand assume that $N(u_0;\, z, R)$ or $M(u_0;\, z, R)$ extends to an entire function of exponential growth $(2, 1/(4T))$ locally uniformly in Ω. Then by Theorem

5.7, u_0 is of Laplacian growth $(2, 1/\sqrt{2T})$ locally uniformly in Ω. Hence for $|t| < T$ and small $\varepsilon > 0$,

$$\sup_{z\in K}\sum_{k=0}^{\infty}\frac{|\Delta^k u_0(z)|}{k!}|t|^k \le C_\varepsilon \sum_{k=0}^{\infty}\frac{\big(1/\sqrt{2T}+\varepsilon\big)^{2k}\cdot(2k!)^{1/2}|t|^k}{k!},$$

which can be estimated by

$$C_\varepsilon \sum_{k=0}^{\infty}\Big[\Big(\frac{1}{T}+\varepsilon\Big)|t|\Big]^k < \infty.$$

So $\widehat{u}(t,z)$ is convergent for $|t| < T$ locally uniformly in Ω. □

Using the above ideas and results from his previous paper [M-06] S. Michalik obtained a characterization of Borel summable solutions of the heat equation (4).

Theorem 5.9. ([M-12, Theorem 4.1]). *Let $d \in \mathbb{R}$, $U \subset \mathbb{C}^n$ and let $\widehat{u}$ be the formal power series solution* (8) *of the heat equation* (4) *with $u_0 \in \mathcal{O}(U)$. Then the following conditions are equivalent:*

1. *$\widehat{u}$ is* 1*-summable in the direction d;*
2. *The solid mean $M(u_0;\, z, R)$ extends to $U \times \big(S(d/2,\epsilon)\cup S(d/2+\pi,\epsilon)\big)$ with $0<\epsilon$ and for any $U_1 \Subset U$, $0 < \epsilon_1 < \epsilon$ and $R \in S(d/2,\epsilon_1)\cup S(d/2+\pi,\epsilon_1)$,*

   $\sup_{z\in U_1}|M(u_0;\, z, R)| \le Ae^{B|R|^2}$;
3. *The spherical mean $N(u_0;\, z, R)$ satisfies the same condition as in 2.*

Furthermore, if the above conditions hold, then the 1*-sum of $\widehat{u}$ is given by*

$$u^d(t,z) = \frac{1}{(4\pi t)^{n/2}}\int_{(e^{id/2}\mathbb{R})^n}\exp\Big\{\frac{-e^{id}|x|^2}{4t}\Big\}u_0(x+z)\,dx$$

provided that the integral is well defined.

§ 5.5. A perturbed heat equation

Set $\Delta^{a,b} = \Delta + \langle a, \nabla\rangle + b$ for $a \in \mathbb{R}^n$, $b \in \mathbb{R}$. Then the unique formal power series solution $\widehat{w}(t,z)$ to the initial value problem

$$\begin{cases} \partial_t w - \Delta^{a,b} w = 0, \\ w_{|t=0} \qquad\; = w_0 \in \mathcal{A}(\Omega), \end{cases} \tag{9}$$

is given by

$$\widehat{w}(t,z) = \sum_{k=0}^{\infty}\frac{(\Delta^{a,b})^k w_0(z)}{k!}t^k. \tag{10}$$

On the other hand $w(t,z)$ satisfies (9) iff $u(t,z) = \exp\{\frac{1}{2}\langle a,z\rangle - ct\}\, w(t,z)$ with $c = \frac{1}{4}a^2 - \frac{1}{2}\sum_{i=1}^n a_i + b$ is a solution of the heat equation (4).

Set

$$M^a(w_0;\, z,R) = \int_{B(z,R)} w_0(\xi)\exp\{\tfrac{1}{2}\langle a,\xi\rangle\}\, d\xi,$$
$$N^a(w_0;\, z,R) = \int_{S(z,R)} w_0(\xi)\exp\{\tfrac{1}{2}\langle a,\xi\rangle\}\, dS(\xi).$$

Since the multiplication by an exponential function has no influence on convergence and divergence properties of its Taylor series by Theorems 5.6 and 5.7 we get

Corollary 5.10. *Let $0 < T \le \infty$. The formal power series solution (10) of the initial value problem (9) is convergent for $|t| < T$ locally uniformly in Ω iff $M^a(w_0;\, z,R)$ and/or $N^a(w_0;\, z,R)$ as functions of R extend holomorphically to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in Ω.*

§ 6. Heat equation with variable coefficients

The general one dimensional heat equation $\partial_t u - a(z)\partial_z^2 u = \widehat{q}(t,z)$ with a variable coefficient $a(z)$ and inhomogeneity $\widehat{q}(t,z)$ was studied by Balser and Loday-Richaud [BLR-09]. In fact they stated results for the equivalent equation

$$\big(1 - a(z)\partial_t^{-1}\partial_z^2\big)\,\widehat{u}(t,z) = \widehat{f}(t,z)$$

with $a(z) \in \mathcal{O}(D_\rho)$, $\widehat{f}(t,z) \in \mathcal{O}(D_\rho)[[t]]$. Then its formal solution $\widehat{u}(t,z) = \sum_{j=0}^\infty \frac{u_j(z)}{j!}t^j$ satisfies the recurrence equations

$$u_j(z) = f_j(z) + a(z)u''_{j-1}(z), \quad j \in \mathbb{N}.$$

Balser and Loday-Richaud proved that the map $\widehat{u} \mapsto \big(1 - a(z)\partial_t^{-1}\partial_z^2\big)\,\widehat{u}$ gives a linear isomorphism of $\mathcal{O}(D_\rho)[[t]]$ onto itself. Furthermore, assuming that $a(0) \ne 0$ or $a'(0) \ne 0$, the formal solution $\widehat{u}$ is 1-summable in a direction d iff $\widehat{f}$ and the first two terms $\widehat{u}_{*,0}$ and $\widehat{u}_{*,1}$ of $\widehat{u}(t,z) = \sum_{n=0}^\infty \widehat{u}_{*,n}(t)\frac{z^n}{n!}$ are 1-summable in d. In special cases $a(z) = a$ and $a(z) = bz$ the conditions on 1-summability of $\widehat{u}$ are given in terms of the initial data, only.

Costin, Park and Takei in [CPT-12] studied Borel summability of the IVP

$$\begin{cases} \partial_t u &= a(z)\partial_z^2 u, \\ u(0,z) &= \frac{1}{1+z^2}\,, \end{cases}$$

where $a(z)$ is a quartic polynomial with 4 distinct roots. Setting $y(z) = \int_{z_0}^z a(\xi)^{-1/2} d\xi$ and $z = \varphi(y)$, the inverse to $y(z)$, they obtained the integral equation for the function $g(y,s)$ (related to the Borel transform of $\widehat{u}$),

$$g(y,s)=\frac{g_0(y+s)+g_0(y-s)}{2}+\frac{1}{2}\int_0^s\int_{y-(s-\tilde{s})}^{y+(s-\tilde{s})}\eta(\tilde{y})g(\tilde{y},\tilde{s})\,d\tilde{y}d\tilde{s},$$

where

$$g_0(y)=u_0\big(\varphi(y)\big)a\big(\varphi(y)\big)^{-1/4},\quad \eta(y)=\frac{a''(\varphi(y))}{4}-\frac{3(a'(\varphi(y)))^2}{16a(\varphi(y))}.$$

Since $\varphi(y)$ can be expressed in terms of the Weierstrass elliptic function $\mathcal{P}(y;g_2,g_3)$ with parameters g_2 and g_3 related to coefficients of $a(z)$, they were able to describe singularities of $g(y,s)$ and prove its Laplace transformability. They also treated the case $a(z)=z$ and obtained the detailed resurgence structure of the singular manifold (which even in this simple case is quite intricate), see [CPT-12, Theorem 10].

§ 7. Heat type equations on manifolds

Let $\mathcal{M}$ be a real analytic manifold of dimension n and $X_1,\dots,X_d$ real analytic linearly independent vector fields on $\mathcal{M}$. Define a Laplace type operator on $\mathcal{M}$ by $\widetilde{\Delta}=X_1^2+\cdots+X_n^2$ and consider the initial value problem

$$\begin{cases}\partial_t v-\widetilde{\Delta}v=0,\\ v_{|t=0}\quad\ =v_0,\qquad v_0\in\mathcal{A}(\mathcal{M}).\end{cases}\tag{11}$$

The formal power series solution of (11) is given by

$$\widehat{v}(t,y)=\sum_{k=0}^{\infty}\frac{\widetilde{\Delta}^k v_0(y)}{k!}t^k.\tag{12}$$

It is well known that if vector fields X_i commute,

$$[X_i,X_j]=0\quad\text{for}\quad i,j=1,\dots,n,\tag{13}$$

then for a fixed $\mathring{y}\in\mathcal{M}$ one can find a real analytic diffeomorphism $\Phi:\mathbb{R}^n\supset\Omega\xrightarrow{onto}V\subset\mathcal{M}$ such that $\mathring{y}\in V=\Phi(\Omega)$ and $\Phi_*^{-1}(X_i)=\frac{\partial}{\partial z_i}$ for $i=1,\dots,n$. Set $B_\Phi(y,R)=\Phi\big(B(z,R)\big)$, $S_\Phi(y,R)=\Phi\big(S(z,R)\big)$ with $z=\Phi^{-1}(y)$, $0<R<\operatorname{dist}(z,\partial\Omega)$. Let μ_Φ (respectively, dS_Φ) be a measure on V (respectively, on S_Φ) defined by $\mu_\Phi(A)=\int_{\Phi^{-1}(A)}d\xi$ for a Borel measurable set $A\subset V$ (respectively, $dS_\Phi(A)=\int_{\Phi^{-1}(A)}dS(\xi)$ for a Borel measurable set $A\subset S_\Phi$).

Theorem 7.1. ([Ł-14, Theorem 2]). *Let $0 < T \leq \infty$. The formal power series solution* (12) *of the heat type equation* (11) *is convergent for $|t| < T$ locally uniformly in V if and only if the solid integral mean*

$$M_\Phi(v_0;\, y, R) = \frac{1}{\mu_\Phi\big(B_\Phi(y,R)\big)} \int_{B_\Phi(y,R)} v_0(\eta)\, d\mu_\Phi(\eta)$$

and/or the spherical integral mean

$$N_\Phi(v_0;\, y, R) = \frac{1}{dS_\Phi\big(S_\Phi(y,R)\big)} \int_{S_\Phi(y,R)} v_0(\eta) dS_\Phi(\eta)$$

extends to an entire function of exponential growth $(2, 1/(4T))$ *locally uniformly in V.*

Proof. Assume that the formal power series solution (12) of (11) is convergent for $|t| < T$ locally uniformly in V. Denote its sum as $v(t, y)$ and set $u(t, z) = v\big(t, \Phi(z)\big)$, $u_0(z) = v_0\big(\Phi(z)\big)$ for $|t| < T$, $z \in \Omega$. Then u satisfies the heat equation (4) and is given by (8) with the series convergent for $|t| < T$ locally uniformly in Ω. Hence by Theorem 5.8, $M(u_0;\, z, R)$ extends to entire functions of exponential growth $\big(2, 1/(4T)\big)$ locally uniformly in Ω. But for $z \in \Omega$ and $y = \Phi(z)$ we have

$$\sigma(n)R^n = \int_{\Phi^{-1}(B_\Phi(y,R))} d\xi = \mu_\Phi\big(B_\Phi(y,R)\big),$$

$$\int_{B(z,R)} u_0(\xi)\, d\xi = \int_{\Phi^{-1}(B_\Phi(y,R))} v_0\big(\Phi(\xi)\big)\, d\xi = \int_{B_\Phi(y,R)} v_0(\eta)\, d\mu_\Phi(\eta)$$

and

$$n\sigma(n)R^{n-1} = \int_{\Phi^{-1}(S_\Phi(y,R))} dS(\xi) = dS_\Phi\big(S_\Phi(y,R)\big),$$

$$\int_{S(x,R)} u_0(\xi) dS(\xi) = \int_{\Phi^{-1}(S_\Phi(y,R))} v_0\big(\Phi(\xi)\big) dS(\xi) = \int_{S_\Phi(y,R)} v_0(\eta) dS_\Phi(\eta).$$

So

$$M(u_0;\, z, R) = M_\Phi(v_0;\, y, R) \quad \text{ and } \quad N(u_0, x; R) = N_\Phi(v_0; y, R).$$

Hence $M_\Phi(v_0;\, y, R)$ and $N_\Phi(v_0; y, R)$ as functions of R extend to entire functions of exponential growth $\big(2, 1/(4T)\big)$ locally uniformly in V.
The proof of the converse statement is done in the same way with *and* replaced by *or.* □

Remark 1. An analogue of Theorem 7.1 holds for solutions of heat type equations with $\widetilde{\Delta}$ perturbed by $\sum_{i=1}^n a_i X_i + b$. In that case the measure μ_Φ should be replaced by

$$\mu^a_\Phi(A) = \int_{\Phi^{-1}(A)} \exp\{\tfrac{1}{2}\langle a, \xi\rangle\}\, d\xi, \qquad A \subset V.$$

Repeating the proof of Theorem 7.1, but now using Theorem 5.9 in place of Theorem 5.8, we get

Theorem 7.2. ([Ł-14, Theorem 4]). *Let $\mathcal{M}$ be a real analytic manifold, $v_0 \in \mathcal{A}(\mathcal{M})$ and $X_1, \dots, X_n$ real analytic linearly independent commuting vector fields on $\mathcal{M}$. Fix $\mathring{y} \in \mathcal{M}$ and let Φ, Ω, V, B_Φ, μ_Φ and dS_Φ be as in Theorem 7.1. Set $u_0 = v_0 \circ \Phi$ and assume that u_0 and Φ extend to a complex neighborhood $U \subset \mathbb{C}^n$ of Ω. Then v_0 extends to the neighborhood $\Phi(U)$ of V in the complexification of $\mathcal{M}$. Let $d \in \mathbb{R}$ and let $\widehat{v}$ be the formal solution* (12) *of the heat type equation* (11)*. Then the following conditions are equivalent:*

1. *$\widehat{v}$ is Borel summable in d locally uniformly in $\Phi(U)$;*
2. *$M_\Phi(v_0; z, R)$ extends to $\Phi(U) \times \big(D_\epsilon \cup S(d/2, \epsilon) \cup S(d/2+\pi, \epsilon)\big)$ with $0 < \epsilon$ and for any $U_1 \Subset U$, $0 < \epsilon_1 < \epsilon$ and $R \in S(d/2, \epsilon_1) \cup S(d/2+\pi, \epsilon_1)$,*

   $\sup_{z \in \Phi(U_1)} |M_\Phi(v_0; z, R)| \leq Ae^{B|R|^2}$*;*
3. *The function $N_\Phi(v_0; z, R)$ satisfies the same condition as in 2..*

§ 7.1. Remarks and open problems

1. The results on convergence and Borel summability are local. It would be interesting to obtain global analogues. In case of the one dimensional heat equation on S^1 the solution to the problem can be expressed in terms of estimations of Fourier coefficients of the initial data. The general case seems to be open.
2. It would be also interesting to obtain conditions for convergence and Borel summability of formal solutions to (11) in cases when vector fields X_i do not commute and/or are not linearly independent. Of special interest here are the cases when $\widetilde{\Delta}$ is the Grushin operator $\partial_x^2 + x^2\partial_y^2$ or the Laplace operator on the Heisenberg group.

§ 8. Nonlinear equations

Until now there are only few papers devoted to the study of summability of formal solutions to nonlinear partial differential equations. Ōuchi in [O-06] considered a class of singular partial differential equations with polynomial nonlinearity which can be considered as a perturbation (in some sense) of ODEs. He proved that under some technical conditions the null formal power series solutions are multisummable.

Costin and Tanveer in [CT-07] considered the Cauchy problem for a system of quasilinear PDEs

$$\mathbf{u}_t + \mathcal{P}(\partial_z^j)\mathbf{u} + g\big(t, z, \{\partial_z^j \mathbf{u}\}_{|j| \leq n}\big) = 0, \quad \mathbf{u}(0, z) = \mathbf{u_I}(z), \tag{14}$$

with $\mathbf{u}$ in $\mathbb{C}^r$ for small t and large $|z|$ in a poly-sector S in $\mathbb{C}^r$. Assuming that the principal part of $\mathcal{P}$ satisfies a cone condition and that the nonlinearity g and the initial data $\mathbf{u_I}$ are analytic and satisfy some decay conditions on S, they proved existence and uniqueness of solutions to (14). Under further regularity conditions on g and $\mathbf{u_I}$ ensuring the existence of formal power series solutions for large $z \in S$ they showed that formal series solutions are Borel summable to actual solutions, see [CT-07, Theorem 2].

In a subsequent paper [CT-09] Costin and Tanveer studied the initial value problem to the 3-dimensional Navier-Stokes system

$$\begin{cases} \partial_t u - \Delta u = -\mathcal{P}(u \cdot \nabla u) + f(x), \\ u(0,x) \qquad = u_0(x), \quad x \in \mathbb{R}^3, \end{cases}$$

where $\mathcal{P}$ is the Hodge-projection operator to the space of divergence free vector fields. Assuming that the initial data u_0 and the force f are divergence free and analytic they proved [CT-09, Theorem 1.2] that the solution u is Borel summable in $1/t$, i.e., there exists $U(p,x)$, analytic in p in a neighborhood of $\mathbb{R}_+$ and exponentially bounded as $p \to \infty$, and analytic in x for $|\operatorname{Im} x_i| < \varepsilon$, $i = 1,2,3$, so that

$$u(t,x) = u_0(x) + \int_0^\infty U(p,x) e^{-p/t}\, dp$$

for $t > 0$ small enough.

8.1. Burgers equation

In [Ł-09] we considered the IVP for the Burgers equation

$$\begin{cases} \partial_t u - \partial_z^2 u = \partial_z\big(u^2\big), \\ u_{|t=0} \qquad = u_0. \end{cases} \tag{15}$$

The formal power series solution is given by

$$\widehat{u}(t,z) = \sum_{k=0}^{\infty} \frac{u_k(z)}{k!} t^k,$$

where

$$u_{k+1} = \partial^2 u_k + v_k \quad \text{with} \quad v_k = \sum_{\substack{\kappa \in \mathbb{N}_0^2, \\ \kappa_1 + \kappa_2 = k}} \frac{k!}{\kappa_1! \kappa_2!} \partial\big(u_{\kappa_1} u_{\kappa_2}\big), \quad k \in \mathbb{N}_0.$$

Applying the Cole-Hopf transformation

$$u(t,z) \mapsto v(t,z) = \exp\Big\{ \int^z u(t,y) dy \Big\}$$

which transforms (15) into the heat equation and its inverse $v(t,z) \mapsto u(t,z) = \big(\ln v(t,z)\big)'_z$ we proved

Theorem 8.1. ([Ł-09, Theorem 1]). *Let B be a ball at $\{0\}$ and let $u_0 \in \mathcal{A}(B)$. If the formal power series solution of the Burgers equation* (15) *is convergent locally uniformly in B, then u_0 extends to a meromorphic function on $\mathbb{C}$ of the form*

$$u_0(z) = 2az + b + \sum_{n=1}^{\infty} \Big(\frac{1}{z - z_n} + \frac{1}{z_n} + \frac{z}{z_n^2} \Big), \tag{16}$$

where $a, b \in \mathbb{C}$ and $\{z_n\}_{n\in\mathbb{N}}$ is a sequence of $z_n \in \mathbb{C}^* \cup \{\infty\}$ with nondecreasing modulus such that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{2+\varepsilon}} < \infty \quad \textit{for} \quad \textit{any} \quad \varepsilon > 0. \tag{17}$$

Conversely, if u_0 extends to a meromorphic function of the form (16) *and* (17) *holds, then the formal solution of* (15) *is convergent in a neighborhood of $\{0\}$.*

Theorem 8.2. ([Ł-09, Theorem 2]). *Let $u_0 \in \mathcal{A}(B)$ and $d \in \mathbb{R}$. If the formal power series solution of* (15) *is Borel summable in the direction d locally uniformly in a neighborhood of $\{0\}$, then u_0 extends analytically to a function meromorphic on a domain*

$$D(d, \varepsilon) \supset S(d/2, \varepsilon) \cup S(d/2 + \pi, \varepsilon)$$

with some $\varepsilon > 0$ which has in $D(d, \varepsilon)$ at most simple poles with residua in $\mathbb{N}$. Conversely, if u_0 extends to a meromorphic function on $D(d, \varepsilon)$ of the form

$$u_0(z) = \sum_{n=1}^{\infty} \Big(\frac{1}{z - z_n} + \frac{1}{z_n} + \frac{z}{z_n^2} \Big) + v(z),$$

where $0 \neq z_n \in D(d, \varepsilon)$ satisfy (17)*, v is holomorphic on $D(d, \varepsilon)$ and $|v(z)| \le a|z| + b$, then the formal power series solution of* (15) *is Borel summable in the direction d locally uniformly in a neighborhood of the origin.*

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