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Semi-formal solution and monodromy of some confluent hypergeometric equations

Dedicated to Professor Takashi AOKI for his sixtieth birthday

By

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Abstract

This paper studies the monodromy of some class of confluent hypergeometric equations. By using the convergent semi-formal solution introduced in [1] we will show the concrete formula of the monodromy for some class of confluent hypergeometric equations.

§ 1. Introduction

In this note we will study the monodromy of some class of confluent hypergeometric equations which can be written in a Hamiltonian system. In [1] it was shown that monodromy in the class of formal power series can be calculated if one uses semi-formal solutions in expressing monodromy. We will use the convergent semi-formal solutions defined by first integrals of the Hamiltonian system which are identical to the ones given in [1]. More precisely, a convergent semi-formal solution is defined in terms of sufficiently many functionally independent first integrals. There are similarities between our idea and the so-called KAM theory. The definition of the monodromy via first integrals enables us to calculate the monodromy in an elementary way. We will give examples for which one can calculate the monodromy concretely. We hope that our method may be extended to more general class of equations in a future paper.

This paper is organized as follows. In section 2 we study the convergent semi-formal solutions. In section 3 we introduce a class of confluent hypergeometric system written

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in a Hamiltonian form. In section 4 we construct functionally independent first integrals and calculate the monodromy for a certain example.

§ 2. Semi-formal solution via first integrals

Let $n \geq 2$ and $\sigma \geq 1$ be integers. Consider the Hamiltonian system

$$z^{2\sigma}\frac{dq}{dz} = \nabla_p \mathcal{H}(z,q,p), \quad z^{2\sigma}\frac{dp}{dz} = -\nabla_q \mathcal{H}(z,q,p), \tag{2.1}$$

where $q = (q_2, \ldots, q_n)$, $p = (p_2, \ldots, p_n)$, and where $\mathcal{H}(z,q,p)$ is analytic in $z \in \mathbb{C}$ in some neighborhood of the origin and entire in $(q,p) \in \mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$. We note that, by taking $q_1 = z$ as a new unknown function (2.1) is written in an equivalent form for the Hamiltonian function H, $H(q_1, q, p_1, p) := p_1 q_1^{2\sigma} + \mathcal{H}(q_1, q, p)$

$$\begin{aligned} \dot{q_1} = H_{p_1} = q_1^{2\sigma}, \quad \dot{p_1} = -H_{q_1} = -2\sigma p_1 q_1^{2\sigma-1} - \partial_{q_1}\mathcal{H}(q_1,q,p), \\ \dot{q} = \nabla_p H = \nabla_p \mathcal{H}(q_1,q,p), \quad \dot{p} = -\nabla_q H = -\nabla_q \mathcal{H}(q_1,q,p). \end{aligned} \tag{2.2}$$

The solution of (2.1) is given in terms of that of (2.2) by taking $q_1 = z$ as an independent variable.

Semi-formal solution. We define the semi-formal solution of (2.1) following [1]. Let $\mathcal{O}(\tilde{S}_0)$ be the set of holomorphic functions on $\tilde{S}_0$, where $\tilde{S}_0$ is the universal covering space of the punctured disk of radius r, $S_0 = \{|z| < r\} \setminus 0$ for some $r > 0$. The $(2n-2)$-vector $\check{x}(z,c)$ of formal power series of c

$$\check{x}(z,c) := \sum_{|\nu|\geq 0} \check{x}_\nu(z)c^\nu = \check{x}_0(z) + X(z)c + \sum_{|\nu|\geq 2} \check{x}_\nu(z)c^\nu \tag{2.3}$$

is said to be a *semi-formal solution* of (2.1) if $\check{x}_\nu \in (\mathcal{O}(\tilde{S}_0))^{2n-2}$ and $(q(z,c), p(z,c)) := \check{x}(z,c)$ is the formal power series solution of (2.1). As for the properties of the semi-formal series (2.3) we refer to [1]. Here $X(z)$ is a $(2n-2)$ square matrix with component belonging to $\mathcal{O}(\tilde{S}_0)$. If $X(z)$ is invertible, then we say that $(q(z,c), p(z,c))$ is a *complete semi-formal solution*. We say that a semi-formal solution is a convergent semi-formal solution (at the origin) if the following condition holds. For every compact set K in $\tilde{S}_0$ there exists a neighborhood U such that the formal series converges for $q_1 \in K$ and $c \in U$. The semi-formal solution at the general point $z_0 \in \mathbb{C}$ is defined similarly.

Monodromy function. We consider (2.1). Let z_0 be any point in $\mathbb{C}$ and let q and p be semi-formal solutions of (2.1) around z_0. We define the monodromy function $v(c)$ around z_0 by

$$(q,p)((z-z_0)e^{2\pi i} + z_0, v(c)) = (q,p)(z,c), \tag{2.4}$$

where $v(c) = (v_j(c))$. The existence of $v(c)$ is proved in [1]. If we denote the linear part of $v(c)$ by $M^{-1}c$, then by considering the linear part of the monodromy relation we have $X((z-z_0)e^{2\pi i}+z_0) = X(z)M$. Hence M is the so-called monodromy factor.

In the following we will show that the convergent semi-formal solutions of (2.1) can be obtained by solving certain system of nonlinear equations given by first integrals. We consider (2.2). Given functionally independent first integrals $H(q_1, q, p_1, p)$ and $\psi_j \equiv \psi_j(q_1, q, p)$ $(j = 1, 2, \ldots, 2n-2)$ of (2.2), where the functional independentness means that there exists a neighborhood V of the origin of (q, p, p_1) such that the matrix

$$(2.5) \qquad {}^t\left(\nabla_{q,p,p_1} H, \nabla_{q,p,p_1} \psi_j\right)_{j\downarrow 1,2,\ldots,2n-2}$$

has full rank $2n-1$ on $(q_1, p_1, q, p) \in \tilde{S}_0 \times V$. We assume that every coefficient of ψ_j expanded in the power series of q, p is holomorphic with respect to q_1 on $\tilde{S}_0$.

Let the point $(q_{1,0}, p_{1,0}, q_0, p_0)$ and the values $c_{j,0}$ $(j = 1, 2, \ldots, 2n-2)$ satisfy that

$$(2.6) \qquad H(q_{1,0}, p_{1,0}, q_0, p_0) = 0, \ \psi_j(q_{1,0}, q_0, p_0) = c_{j,0}, \ (j = 1, 2, \ldots, 2n-2).$$

For $c_j = \tilde{c}_j + c_{j,0}$, $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_{2n-2}) \in \mathbb{C}^{2n-2}$ we consider the system of equations of p_1, q and p

$$(2.7) \qquad H(q_1, p_1, q, p) = 0, \ \psi_j(q_1, q, p) = c_j, \ (j = 1, 2, \ldots, 2n-2).$$

If (2.7) has a solution, then we denote it by $q \equiv q(q_1, c)$, $p \equiv p(q_1, c)$, $p_1 \equiv p_1(q_1, c)$. We see that q, p and p_1 are holomorphic functions of q_1 in $\tilde{S}_0$ and $\tilde{c}$ in some neighborhood of the origin if we assume (2.5). We have

Theorem 2.1. *Suppose that $H(q_1, q, p_1, p)$ and $\psi_j \equiv \psi_j(q_1, q, p)$ $(j = 1, 2, \ldots, 2n-2)$ be functionally independent. Assume (2.6). Then the solution of (2.7) gives the convergent complete semi-formal solution $(q(z, c), p(z, c))$ $(q_1 = z)$ of (2.1) provided q or p is not a constant function.*

Proof. Define $\tilde{q} = (q_1, q)$, $\tilde{p} = (p_1, p)$ and write $G^{(j)} := \psi_j$. For the sake of simplicity we write q and p instead of $\tilde{q}$ and $\tilde{p}$, respectively. By assumption and the implicit function theorem q, p and p_1 are convergent semi-formal series in some neighborhood of $c = c^0$. In order to show that they are the solution of (2.2) we differentiate (2.7) with respect to the time variable. Then we have

$$(2.8) \qquad \dot{q}H_q + \dot{p}H_p = 0, \quad \dot{q}G_q^{(j)} + \dot{p}G_p^{(j)} = 0, \quad (j = 1, 2, \ldots, 2n-2),$$

where $\dot{q} = dq/dt$ and so on. Because $G^{(j)}$ is the first integral it follows that

$$(2.9) \qquad H_p G_q^{(j)} - H_q G_p^{(j)} = 0, \quad (j = 1, 2, \ldots, 2n-2).$$

By assumption on the functional independentness we see, from (2.8) and (2.9), that the vectors $(\dot{q}, \dot{p})$ and $(H_p, -H_q)$ are contained in some two dimensional plane Π. Note that these vectors are orthogonal to $(H_q, H_p) \neq 0$. If $(H_q, H_p) \in \Pi$, then there exists $c(t)$ such that $\dot{q} = c(t)H_p$, $\dot{p} = -c(t)H_q$. In order to show that the assertion holds in case $(H_q, H_p) \notin \Pi$, we note that the orthogonal projection of (H_q, H_p) to Π, $(\tilde{H}_q, \tilde{H}_p)$ does not vanish by the assumption on (2.5). By (2.8) we have that $\dot{q}\tilde{H}_q + \dot{p}\tilde{H}_p = 0$. On the other hand, by (2.9) $(H_p, -H_q)$ is orthogonal to $(H_q, H_p) - (\tilde{H}_q, \tilde{H}_p)$. Since $(H_p, -H_q)$ is orthogonal to (H_q, H_p), it follows that $H_p\tilde{H}_q - H_q\tilde{H}_p = 0$. Hence we have the same assertion.

If $|\dot{q}|^2 + |\dot{p}|^2 \neq 0$, then $c(t)$ does not vanish. Because $H_p, H_q \in \mathcal{O}(\tilde{S}_0)$ do not vanish similutaneously, we see that $c(t) \in \mathcal{O}(\tilde{S}_0)$. If we introduce s by $\dot{s} = c(t)$, then

$$dq/ds = H_p, \quad dp/ds = -H_q. \tag{2.10}$$

We will remove the assumption $|\dot{q}|^2 + |\dot{p}|^2 \neq 0$. If q and p are not a constant function, then either q or p does not vanish except for a discrete set because they are analytic functions. Hence, by continuity we see that q and p satisfy (2.10). We note that the invertibility of X in (2.3) is verifed because (2.5) has a full rank. If we come back to the original notation, then by definition and the relation between (2.1) and (2.2) $(q(z, c), p(z, c))$ $(z = q_1)$ is the convergent semi-formal solution of (2.1). □

§ 3. Confluent hypergeometric equation

We consider a class of hypergeometric system introduced by Okubo (cf. [2])

$$(z - C)\frac{dv}{dz} = Av, \tag{3.1}$$

where C is a diagonal matrix and A is a constant matrix. The system has only regular singular points on $\mathbb{C} \cup \{\infty\}$. Set $v = {}^t(q, p) \in \mathbb{C}^n$ and assume that C and A are block diagonal matrices

$$C = \mathrm{diag}(\Lambda_1, \Lambda_1), \; A = \mathrm{diag}(A_1, -{}^tA_1) \tag{3.2}$$

where Λ_1 and A_1 are $n - 1$ square diagonal and constant matrices, respectively such that

$$(z - \Lambda_1)A_1 = A_1(z - \Lambda_1), \quad \forall z \in \mathbb{C}. \tag{3.3}$$

Define

$$H := \langle (z - \Lambda_1)^{-1}p, A_1q\rangle. \tag{3.4}$$

Then one can write (3.1) in the Hamiltonian form

$$\frac{dq}{dz} = H_p(z,q,p), \quad \frac{dp}{dz} = -H_q(z,q,p). \tag{3.5}$$

We will introduce the irregular singularity by the confluence of singularities. Let λ_j $(j = 2,\dots,n)$ be the diagonal elements of Λ_1. We assume $\lambda_j \neq 0$ for all j. Take nonempty sets J and J' such that $J \cup J' = \{2,3,\dots,n\}$ and $\lambda_i \neq \lambda_j$ for every $i \in J$ and $j \in J'$. Without loss of generality one may assume $J = \{2,3,\dots,n_0\}$ for some $n_0 \geq 2$. We merge all regular singular points $z = \lambda_j$ $(j \in J')$ to the infinity. First, by setting $z = 1/\zeta$ in (3.5) we have

$$-\zeta^2\frac{dq}{d\zeta} = (\frac{1}{\zeta} - \Lambda_1)^{-1}A_1 q, \quad -\zeta^2\frac{dp}{d\zeta} = -{}^tA_1(\frac{1}{\zeta} - \Lambda_1)^{-1}p. \tag{3.6}$$

Subsitute $\zeta = \varepsilon^{-1}\eta$ in (3.6). Replace λ_ν with $\varepsilon\lambda_\nu$ if $\nu \in J$ and multiply the μ-th row of A_1 with ε^{-1} if $\mu \in J'$. Then we let $\varepsilon \to 0$. Define the diagonal matrix $\mathfrak{A}$ by $\mathfrak{A} := \mathrm{diag}\ (\mathfrak{A}_1,\dots,\mathfrak{A}_n)$ where $\mathfrak{A}_\nu$ is given by $-\lambda_\nu^{-1}$ if $\nu \in J'$ and $(\eta^{-1} - \lambda_\mu)^{-1}$ if $\mu \in J$, respectively. Then we obtain

$$-\eta^2\frac{dq}{d\eta} = \mathfrak{A}A_1 q, \quad -\eta^2\frac{dp}{d\eta} = -{}^tA_1\mathfrak{A}p. \tag{3.7}$$

We will write (3.7) in a Hamiltonian form. Set $\eta = q_1$, and define H by

$$H(q_1,p_1,q,p) := p_1 q_1^2 - \langle \mathfrak{A}(q_1)A_1 q, p\rangle. \tag{3.8}$$

One can easily see that $\dot{q} = \eta^2\frac{dq}{d\eta}$ and $\dot{p} = \eta^2\frac{dp}{d\eta}$. Because $-\mathfrak{A}A_1 q = H_p$ and $-{}^tA_1\mathfrak{A}p = H_q$, one easily sees that (3.7) is equivalent to the Hamiltonian system with the Hamiltonian function (3.8).

If λ_j's are mutually different, then it follows from (3.3) that A_1 is a diagonal matrix. Denote the diagonal entries of A_1 by τ_j. Then we have

$$H(q_1,p_1,q,p) = p_1 q_1^2 + \sum_{j=2}^{n}\frac{\tau_j}{\lambda_j}q_j p_j + \sum_{j\in J}\frac{\tau_j}{\lambda_j^2}\frac{q_j p_j}{q_1 - \lambda_j^{-1}}. \tag{3.9}$$

§ 4. Calculation of monodromy

In this section we will calculate the monodromy for the Hamiltonian (3.9) via first integrals. We assume that λ_j's are mutually different. First, we construct first integrals of the Hamiltonian vector field

$$\begin{aligned}\chi_H := q_1^2\frac{\partial}{\partial q_1} - 2q_1p_1\frac{\partial}{\partial p_1} - \sum_{j\in J}\frac{\tau_j}{\lambda_j^2}\frac{q_j p_j}{(q_1 - \lambda_j^{-1})^2}\frac{\partial}{\partial p_1} \\ + \sum_{j=2}^{n}\frac{\tau_j}{\lambda_j}\left(q_j\frac{\partial}{\partial q_j} - p_j\frac{\partial}{\partial p_j}\right) + \sum_{j\in J}\frac{\tau_j}{\lambda_j^2}\frac{1}{q_1 - \lambda_j^{-1}}\left(q_j\frac{\partial}{\partial q_j} - p_j\frac{\partial}{\partial p_j}\right).\end{aligned} \tag{4.1}$$

For $k = 2, \dots, n$ we will construct the first integrals in the form $q_k w_k(q_1)$. We see that w_k satisfies

$$\left\{ \begin{aligned} &\left(q_1^2 \frac{\partial}{\partial q_1} + \frac{\tau_k}{\lambda_k} + \frac{\tau_k}{\lambda_k^2} \frac{1}{q_1 - \lambda_k^{-1}} \right) w_k = 0 \quad \text{if } k \in J \\ &\left(q_1^2 \frac{\partial}{\partial q_1} + \frac{\tau_k}{\lambda_k} \right) w_k = 0 \quad \text{if } k \notin J. \end{aligned} \right. \tag{4.2}$$

Hence we have

$$w_k(q_1) = \begin{cases} \left(\dfrac{q_1}{q_1 - \lambda_k^{-1}} \right)^{\tau_k} & \text{if } k \in J \\ \exp\left(\dfrac{\tau_k}{\lambda_k q_1} \right) & \text{if } k \notin J. \end{cases} \tag{4.3}$$

Next we consider the first integrals $w := p_k u_k(q_1)$. By (4.1) the equation $\chi_H w = 0$ can be written in the form

$$\left(q_1^2 \frac{d}{dq_1} - \frac{\tau_k}{\lambda_k} - \sum_{j \in J} \frac{\tau_j}{\lambda_j^2} \frac{\delta_{k,j}}{q_1 - \lambda_j^{-1}} \right) u_k(q_1) = 0, \tag{4.4}$$

where $\delta_{k,j}$ is the Kronecker's delta, namely $\delta_{k,j} = 1$ if $k = j$ and $=0$ if otherwise. By solving the equation we have $u_k(q_1) = \left(\frac{q_1}{q_1 - \lambda_k^{-1}} \right)^{-\tau_k}$ if $k \in J$, and $= \exp\left(-\frac{\tau_k}{\lambda_k q_1} \right)$ if $k \notin J$. Hence we have

$$u_k(q_1) = w_k(q_1)^{-1}, \quad k = 2, \dots, n. \tag{4.5}$$

By (4.3) and (4.5) we have the first integrals ψ_j $(j = 1, 2, \dots, 2n-2)$

$$\psi_j = \begin{cases} q_{j+1} w_{j+1}(q_1) & (j = 1, 2, \dots, n-1) \\ p_{j-n+2} w_{j-n+2}(q_1)^{-1} & (j = n, n+1, \dots, 2n-2). \end{cases} \tag{4.6}$$

Summing up the above we have

Theorem 4.1. *Assume $\lambda_j \neq 0$ for all j and that λ_j's are mutually different. Then the Hamitonian vector field (4.1) has $2n-1$ functionally independent first integrals H and ψ_j's $(j = 1, 2, \dots, 2n-2)$ given by (4.6).*

We will determine monodromy using first integral. We take the convergent non constant semi-formal solution $q(q_1, c)$, $p(q_1, c)$ and $p_1(q_1, c)$ defined by (2.7). The monodromy function $v(c)$ around z_0 is defined by (2.4). In view of the argument in section 2, we will study the monodromy around the origin $z_0 = 0$ or around $z_0 = \lambda_k^{-1}$ for some $k \in J$. Note that λ_k^{-1} is a regular singular point of the our equation which remains unchanged under the confluence procedure.

First we consider the case $z_0 = 0$. In order to determine the monodromy function $v(c)$, we first note $H(q_1 e^{2\pi i}, p_1, q, p) = H(q_1, p_1, q, p)$. On the other hand, for $1 \le j \le n-1$ we have

$$\begin{aligned}\psi_j(q_1 e^{2\pi i}, q, p) &= q_{j+1} w_{j+1}(q_1 e^{2\pi i}) = \\ &= \begin{cases} e^{2\pi i \tau_{j+1}} q_{j+1} w_{j+1}(q_1) = c_j e^{2\pi i \tau_{j+1}} & \text{if } j+1 \in J \\ q_{j+1} w_{j+1}(q_1) = c_j & \text{if } j+1 \notin J. \end{cases}\end{aligned} \tag{4.7}$$

If $n \le j \le 2n-2$, then we have

$$\begin{aligned}&\psi_j(q_1 e^{2\pi i}, q, p) = q_{j-n+2} w_{j-n+2}(q_1 e^{2\pi i})^{-1} = \\ &= \begin{cases} e^{-2\pi i \tau_{j-n+2}} p_{j-n+2} w_{j-n+2}(q_1)^{-1} = c_j e^{-2\pi i \tau_{j-n+2}} & \text{if } j-n+2 \in J \\ p_{j-n+2} w_{j-n+2}(q_1)^{-1} = c_j & \text{if } j-n+2 \notin J. \end{cases}\end{aligned} \tag{4.8}$$

We define $v(c) = (v_j(c))_j$ by

$$v_j(c) = \begin{cases} c_j e^{2\pi i \tau_{j+1}} & \text{if } 1 \le j \le n-1,\ j+1 \in J \\ c_j & \text{if } 1 \le j \le n-1,\ j+1 \notin J \\ c_j e^{-2\pi i \tau_{j-n+2}} & \text{if } n \le j \le 2n-2,\ j-n+2 \in J \\ c_j & \text{if } n \le j \le 2n-2,\ j-n+2 \notin J. \end{cases} \tag{4.9}$$

Similarly we define $\tilde{v}(c) = (\tilde{v}_j(c))_j$ by the right-hand side of (4.9) with τ_{j+1} and τ_{j-n+2} in (4.9) replaced by $-\tau_{j+1}\delta_{k,j+1}$ and $-\tau_{j-n+2}\delta_{k,j-n+2}$, respectively. Here $\delta_{k,j+1}$ and $\delta_{k,j-n+2}$ are Kronecker's delta.

Let q and p satisfy (2.7) with ψ_j's given by (4.6). Then we easily see that

$$H(q_1 e^{2\pi i}, p_1, q, p) = 0, \quad \psi_j(q_1 e^{2\pi i}, q, p) = v_j(c), \quad 1 \le j \le 2n-2. \tag{4.10}$$

By the uniqueness of semi-formal solution we obtain $q(q_1 e^{2\pi i}, v(c)) = q(q_1, c)$ and $p(q_1 e^{2\pi i}, v(c)) = p(q_1, c)$. This implies that $v(c)$ is the monodromy function as desired. In the case of other regular singular points we may argue in the same way as in the case of the origin. Thus we have proved

Theorem 4.2. *Assume $\lambda_j \ne 0$ for all j and that λ_j's are mutually different. Then the monodromy functions $v(c)$ around the origin and λ_k^{-1} $(k \in J)$ corresponding to the semi-formal solution of (2.1) defined by (2.7) are given by (4.9) and $\tilde{v}(c)$, respectively.*

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