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Multiple-scale analysis for some class of systems of non-linear differential equations

Dedicated to Professor Takashi Aoki on his sixtieth birthday

By

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Abstract

We consider a construction of instanton-type solutions for some class of systems of non-linear differential equations by multiple-scale analysis. We also investigate some problems associated with the construction of instanton-type solutions of $(P_{\mathrm{I}})_m$.

§ 1. Introduction

T. Kawai and Y. Takei ([8], [9]) established structure theorem for instanton-type solutions of Painlevé hierarchies $(P_{\mathrm{J}})_m$ (J$=$ I, 34, II-2 or IV) with a large parameter η. They explained the Stokes phenomenon for instanton-type solutions of $(P_{\mathrm{J}})_m$ by the changes of parameters (See [11] for more details). Instanton-type solutions are formal solutions with sufficiently many free parameters. For example, the instanton-type solution $(u, v) = (u_1, \ldots, u_m, v_1, \ldots, v_m)$ of $(P_{\mathrm{I}})_m$ has the following form (See [12] and [3]).

$$u_j = u_{j,0}(t) + \sum_{|k|\geq 1} \eta^{k\alpha} \left(\sum_{p\in\mathbb{Z}^m,\, |p|\in\{k,\, k-2,\, k-4,\, \ldots\}} u_{j,k,p}(t) e^{p\cdot\tau} \right),$$
$$v_j = v_{j,0}(t) + \sum_{|k|\geq 1} \eta^{k\alpha} \left(\sum_{p\in\mathbb{Z}^m,\, |p|\in\{k,\, k-2,\, k-4,\, \ldots\}} v_{j,k,p}(t) e^{p\cdot\tau} \right),$$

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where $u_{j,0}(t)$, $v_{j,0}(t)$ denote the leading term of a 0-parameter solution of $(P_{\rm I})_m$ and $\alpha = -1/2$ and $\tau = (\tau_1, \ldots, \tau_m)$ (We refer the reader to §2 for the details on τ_j's), and $u_{j,k,p}(t)$, $v_{j,k,p}(t)$ are multi-valued holomorphic functions with a finite number of branching points and poles. When $\alpha = -1/2$, the solution (u, v) contains $2m$ free parameters $(\beta_1^+, \ldots, \beta_m^+, \beta_1^-, \ldots, \beta_m^-)$ of the form

$$\beta_j^+ = \eta^\alpha \sum_{k=0}^{\infty} \beta_{j,k}^+ \eta^{2k\alpha}, \quad \beta_j^- = \eta^\alpha \sum_{k=0}^{\infty} \beta_{j,k}^- \eta^{2k\alpha}.$$

Here $\beta_{j,k}^\pm$ are free complex constants.

We have two methods for the construction of instanton-type solutions. Y. Takei ([10], [12]) established an effective method for a system of non-linear ordinary differential equations which can be written in the form of a Hamiltonian system. The other method is based on multiple-scale analysis. Multiple-scale analysis is also an effective method in obtaining the concrete forms of instanton-type solutions. We refer the reader to [1], [2], [4], [5] and [7]. The latest results about the construction of instanton-type solutions of $(P_{\rm J})_m$ (J = I,II,IV,34) by multiple-scale analysis are given in [3] and [13]. As a next problem, we want to analyze locations of singularities of coefficients of instanton-type solutions constructed in [3] and [13]. The following conjecture is given in [3].

Conjecture: The singularities of the coefficients of instanton-type solutions of $(P_{\rm I})_m$ constructed by multiple-scale analysis are located only in the set of turning points. By computing some terms of instanton-type solutions for $(P_{\rm I})_2$ in the case of $\alpha = -1/2$, the conjecture is given. The author wants to confirm whether the conjecture is expected to be valid as we change the value of α. Further we have another question: What kind of classes of differential equations is multiple-scale analysis effective for? Specifically, in the procedure of the construction of instanton-type solutions by multiple-scale analysis, we need to see the solvability of non-secularity conditions and the coefficients of instanton-type solutions are determined by the non-secularity conditions. We want to specify classes of differential equations with solvable non-secularity conditions.

Motivated by these problems, in this paper we investigate the following. When we change the value of α, what kind of influence do we have in the construction of solutions by multiple-scale analysis? The content of this paper is as follows. In §2, we generally explain multiple-scale analysis for some class of systems of non-linear differential equations. By Lemma 2.3 in §2, we see that the value of α is specified by the form $\alpha = -\frac{1}{\ell}$ $(2 \leq \ell \in \mathbb{N})$. In §3, following the method given in §2, we consider our problems in the case of $(P_{\rm I})_m$. When $\ell = 2$, [3] proved that a solvable system of non-linear differential equations with $2m$ unknown functions appears as the first member of the non-secularity conditions associated with $(P_{\rm I})_m$ and instanton-type solutions with $2m$ free parameters are constructed. Here we particularly consider the following questions:

(i) In response to the change of ℓ, how do the non-secularity conditions change?
(ii) If the value of ℓ is changed, is the construction of instanton-type solutions with $2m$ free parameters possible?
At the end of §3, we report some interesting results and a certain conjecture concerned on (i) and (ii).

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§ 2. Instanton-type solutions and multiple-scale analysis

We first give the definition of an instanton-type solution for some class of systems of non-linear equations, and then, we give an outline of multiple-scale analysis by which we construct a formal solution of instanton type with sufficiently many free parameters.

Let us consider the system of non-linear differential equations with a large parameter η for unknown functions $u(t) := (u_1(t), u_2(t), \ldots, u_{2m}(t))$ of the form

$$\eta^{-1}\frac{du}{dt} = F(u, t), \tag{2.1}$$

where $F(u, t)$ is a vector valued function $(F_1(u, t), \ldots, F_{2m}(u, t))$ and each $F_i(u, t)$ is a polynomial of $u_1, \ldots, u_{2m}$ with coefficients in holomorphic functions of t.

We assume the existence of a solution $u_0(t)$ of the equation $F(u_0, t) = 0$ and in what follows we use the solution $u_0(t)$. Let $\Lambda(\lambda, t)$ denote the characteristic polynomial of λ for the Fréchet derivative of (2.1) at $(u_0(t), t)$, i.e.,

$$\Lambda(\lambda, t) = \det\left(\lambda E_{2m} - \partial_u F(u_0(t), t)\right). \tag{2.2}$$

Here E_{2m} is the identity matrix of size $2m$, and $\partial_u F$ is the Jacobian matrix of $F(u, t)$ with respect to the variables $u_1, \ldots, u_{2m}$.

We only consider the system whose $\Lambda(\lambda, t)$ is an even polynomial of λ with coefficients in functions of t. Then the equation $\Lambda(\lambda, t) = 0$ has m-pairs $(\nu_i^+(t), \nu_i^-(t))$ of roots with $\nu_i^+(t) = -\nu_i^-(t)$ $(i = 1, \ldots, m)$. For convenience, we set $\nu_i := \nu_i^+$ and $\nu_{-i} := \nu_i^-$ $(i = 1, \ldots, m)$.

Let Ω be an open subset in $\mathbb{C}_t$. In what follows, the following two conditions are always assumed.

(A1) The roots $\nu_i(t)$'s $(1 \le |i| \le m)$ are mutually distinct for each $t \in \Omega$.

(A2) The function $p_1\nu_1(t) + \cdots + p_m\nu_m(t)$ does not vanish identically on Ω for any $(p_1, \ldots, p_m) \in \mathbb{Z}^m \setminus \{0\}$.

Recall that $t_0 \in \mathbb{C}_t$ is said to be a turning point if the discriminant of the characteristic polynomial $\Lambda(\lambda, t)$ vanishes at t_0. As $\Lambda(\lambda, t)$ is an even polynomial of λ, we have two kinds of turning points (cf. [12]).

Definition 2.1.

(i) A point t_0 is said to be a turning point of the first kind if two roots ν_i and ν_{-i} merge at t_0 for an index i.

(ii) If there exist mutually distinct indices i and j for which $\nu_i = \nu_j$ or $\nu_i = \nu_{-j}$ holds at t_0, then t_0 is said to be a turning point of the second kind.

By the definition, the assumption (A1) implies that each point in Ω is neither a turning point of the first kind nor a turning point of the second kind. Note that, as t_0 is a turning point of the first kind if and only if $\det \partial_u F(u_0(t), t) = 0$ at $t = t_0$, it follows from (A1) that $\det \partial_u F(u_0(t), t) \neq 0$ holds at any point t in Ω.

Let α be a negative real number and let $\tau := (\tau_1, \ldots, \tau_m)$ be m-independent variables. Then we define the rings

$$\begin{aligned} \mathcal{A}_\alpha(\Omega) &:= \mathcal{M}(\Omega)\left[\left[\eta^\alpha e^{\tau_1}, \ldots, \eta^\alpha e^{\tau_m}, \eta^\alpha e^{-\tau_1}, \ldots, \eta^\alpha e^{-\tau_m}\right]\right], \\ \mathcal{A}^{\mathcal{O}}_\alpha(\Omega) &:= \mathcal{O}(\Omega)\left[\left[\eta^\alpha e^{\tau_1}, \ldots, \eta^\alpha e^{\tau_m}, \eta^\alpha e^{-\tau_1}, \ldots, \eta^\alpha e^{-\tau_m}\right]\right], \end{aligned} \tag{2.3}$$

where $\mathcal{M}(\Omega)$ (resp. $\mathcal{O}(\Omega)$) denotes the set of multi-valued holomorphic functions with a finite number of branching points and poles (resp. holomorphic functions) on Ω. An element in $\mathcal{A}_\alpha(\Omega)$ can be written in the form

$$\sum_{p,\, k} f_{p,k}(t)\eta^{(|p|+2k)\alpha} e^{p\cdot\tau}, \tag{2.4}$$

where $(p, k) = (p_1, \ldots, p_m, k)$ runs through $\mathbb{Z}^m \times \mathbb{Z}_{\geq 0}$, the $f_{p,k}(t)$ belongs to $\mathcal{M}(\Omega)$, and $|p| := |p_1| + \cdots + |p_m|$. Note that, as $\eta^\alpha e^{\tau_i} \times \eta^\alpha e^{-\tau_i} = \eta^{2\alpha}$ and α is strictly negative, the multiplication of formal power series in $\mathcal{A}_\alpha(\Omega)$ or $\mathcal{A}^{\mathcal{O}}_\alpha(\Omega)$ is well-defined.

Let φ be a formal Puiseux series of η in the form

$$\varphi = \varphi_{\beta_0}(\tau, t)\eta^{\beta_0} + \varphi_{\beta_1}(\tau, t)\eta^{\beta_1} + \varphi_{\beta_2}(\tau, t)\eta^{\beta_2} + \cdots .$$

Here $\varphi_{\beta_0} \neq 0$, $0 \geq \beta_0 > \beta_1 > \beta_2 > \ldots$ and each φ_{β_i} does not contain a large parameter η. We say that the order of φ with respect to η is β_0, and denote it by $\mathrm{ord}(\varphi) = \beta_0$.

Note that we set $\operatorname{ord}(0) := -\infty$ as usual. We also denote by $\sigma_\beta(\varphi)$ the coefficient of η^β in φ, for example, $\sigma_{\beta_0}(\varphi) = \varphi_{\beta_0}(\tau, t)$. When $\psi = \sigma_\beta(\psi)\eta^\beta$ holds for some β, we say that ψ is a homogeneous element of order β with respect to η.

For $\beta \le 0$, we define the following subset in $\mathcal{A}_\alpha(\Omega)$:

$$\mathcal{A}_\alpha(\Omega)(\beta) := \{\psi \in \mathcal{A}_\alpha(\Omega);\ \operatorname{ord}(\psi) \le \beta\}. \tag{2.5}$$

In a similar manner, we define $\mathcal{A}^{\mathcal{O}}_\alpha(\Omega)(\beta)$. For simplicity, we set $\hat{\mathcal{A}}_\alpha(\Omega) := \mathcal{A}_\alpha(\Omega)(\alpha)$ (resp. $\hat{\mathcal{A}}^{\mathcal{O}}_\alpha(\Omega) := \mathcal{A}^{\mathcal{O}}_\alpha(\Omega)(\alpha)$), i.e., the subset of formal power series of η^α in $\mathcal{A}_\alpha(\Omega)$ (resp. $\mathcal{A}^{\mathcal{O}}_\alpha(\Omega)$) containing no constant terms.

Recall that $u_0(t)$ is a solution of the equation $F(u_0, t) = 0$. We take the following change of vectors of unknown functions u and $U = (U_1, \ldots, U_{2m})$ in (2.1):

$$u = u_0 + U. \tag{2.6}$$

Then we obtain the system of non-linear differential equations for U of the form

$$\Big(\hat{D}_t - \partial_u F(u_0, t)\Big)U - \Big(F(u_0 + U, t) - \partial_u F(u_0, t)U\Big) = -\hat{D}_t u_0 \tag{2.7}$$

with $\hat{D}_t := \eta^{-1}\dfrac{d}{dt}$. Let $\varphi(\tau, t)$ be an element in $\hat{\mathcal{A}}^{2m}_\alpha(\Omega) := (\hat{\mathcal{A}}_\alpha(\Omega))^{2m}$. We define the system of partial differential equations associated with (2.7) by

$$\Big(\chi_\tau - \partial_u F(u_0, t)\Big)\varphi - \Big(F(u_0 + \varphi, t) - \partial_u F(u_0, t)\varphi\Big) + \eta^{-1}\frac{\partial}{\partial t}\varphi = -\eta^{-1}\frac{\partial u_0}{\partial t}, \tag{2.8}$$

where χ_τ is the first-order differential operator with respect to the variables τ given by

$$\nu_1(t)\frac{\partial}{\partial \tau_1} + \nu_2(t)\frac{\partial}{\partial \tau_2} + \cdots + \nu_m(t)\frac{\partial}{\partial \tau_m}. \tag{2.9}$$

For $\psi(\tau_1, \ldots, \tau_m, t) \in \hat{\mathcal{A}}^{2m}_\alpha(\Omega)$, we define the morphism ι by

$$\iota(\psi)(t) = \psi\left(\eta\int^t \nu_1(s)ds,\ \eta\int^t \nu_2(s)ds,\ \ldots,\ \eta\int^t \nu_m(s)ds,\ t\right). \tag{2.10}$$

Then, clearly, we have

$$\hat{D}_t\iota(\psi) = \iota\left(\chi_\tau\psi + \eta^{-1}\frac{\partial}{\partial t}\psi\right).$$

Hence, for a solution $\varphi(\tau, t) \in \hat{\mathcal{A}}^{2m}_\alpha(\Omega)$ of the system (2.8), the $U := \iota(\varphi)(t)$ becomes a formal solution of the system (2.7).

Definition 2.2. We say that a formal solution u on Ω of the system (2.1) is of instanton type if u has the form $u_0(t) + \iota(\varphi)(t)$ for which $u_0(t)$ is a solution of $F(u_0, t) = 0$ and $\varphi(\tau, t) \in \hat{\mathcal{A}}^{2m}_\alpha(\Omega)$ is a solution of the system (2.8).

For existence of a solution of instanton type, the possible values of α are specified by the following lemma.

Lemma 2.3. *Suppose that the $u_0(t)$ is not a constant function and that the system* (2.8) *has a solution $\varphi \in \hat{\mathcal{A}}^{2m}_\alpha(\Omega)$. Then there exists an integer $k \geq 2$ with $\alpha = -\dfrac{1}{k}$.*

Proof. The first term in the left-hand side of (2.8) is an element in $\hat{\mathcal{A}}^{2m}_\alpha(\Omega)$. The second term in the left-hand side of (2.8) also belongs to $\mathcal{A}^{2m}_\alpha(\Omega)(2\alpha)$. Hence the term $\eta^{-1}\dfrac{\partial u_0}{\partial t}$ in the right-hand side of (2.8) is in $\hat{\mathcal{A}}^{2m}_\alpha(\Omega)$, from which we have $k\alpha = -1$ for some $k \in \mathbb{N}$. Now assume $\alpha = -1$. Then the second and third terms in the left-hand side of (2.8) are of order less than -1, and it follows from (2.4) that a coefficient of η^{-1} in an element of $\mathcal{A}_{-1}(\Omega)$ is a linear combination of e^{τ_i}'s over $\mathcal{M}(\Omega)$. This contradicts the fact that the right-hand side of (2.8) is non-zero and independent of the variables τ. Hence we have $\alpha \neq -1$. $\square$

By taking the lemma into account, we assume $\alpha = -\dfrac{1}{2}$ from now on. We set $\mathcal{A}(\Omega) := \mathcal{A}_\alpha(\Omega)$ and $\mathcal{A}^{\mathcal{O}}(\Omega) := \mathcal{A}^{\mathcal{O}}_\alpha(\Omega)$ for simplicity. Note that $\mathcal{A}(\Omega)$ (resp. $\mathcal{A}^{\mathcal{O}}(\Omega)$) contains the ring $\mathcal{M}(\Omega)[[\eta^{-1}]]$ (resp. $\mathcal{O}(\Omega)[[\eta^{-1}]]$), and that an element in $\mathcal{A}(\Omega)$ can be written uniquely in the form

$$\sum_{p\in\mathbb{Z}^m} f_p(t;\,\eta)\eta^{|p|\alpha}e^{p\cdot\tau} \tag{2.11}$$

with $f_p(t;\,\eta) \in \mathcal{M}(\Omega)[[\eta^{-1}]]$.

Let $A(\nu_i) \in \mathcal{O}^{2m}(\Omega)$ $(1 \leq |i| \leq m)$ be an eigenvector of the matrix $\partial_u F(u_0(t),\,t)$ corresponding to the eigenvalue $\nu_i(t)$. Let $\mathcal{H}(\Omega)$ be the subspace in $\mathcal{A}^{2m}(\Omega)$ generated by the vectors $\eta^\alpha e^{\tau_i}A(\nu_i)$ $(1 \leq |i| \leq m)$ over $\mathcal{M}(\Omega)[[\eta^{-1}]]$, i.e.,

$$\mathcal{H}(\Omega) = \bigoplus_{1\leq|i|\leq m} \mathcal{M}(\Omega)[[\eta^{-1}]]\Big(\eta^\alpha e^{\tau_i}A(\nu_i)\Big) \subset \mathcal{A}^{2m}(\Omega). \tag{2.12}$$

Here we set $\tau_{-i} = -\tau_i$ $(i = 1, 2, \ldots, m)$ for convenience. As every element in $\mathcal{A}^{2m}(\Omega)$ is uniquely expressed by

$$\psi = \sum_{1\leq|i|\leq m,\, p\in\mathbb{Z}^m} f_{i,p}(t;\,\eta)\eta^{|p|\alpha}e^{p\cdot\tau}A(\nu_i) \tag{2.13}$$

with $f_{i,p} \in \mathcal{M}(\Omega)[[\eta^{-1}]]$, we can define the projection $\pi_{\mathcal{H}} : \mathcal{A}^{2m}(\Omega) \to \mathcal{H}(\Omega)$ by

$$\pi_{\mathcal{H}}(\psi) = \sum_{1\leq|i|\leq m} f_{i,e_i}(t;\,\eta)\eta^\alpha e^{\tau_i}A(\nu_i), \tag{2.14}$$

where $e_i \in \mathbb{Z}^m$ is the vector with $|e_i| = 1$ and its $|i|$-th component being $\frac{i}{|i|}$.

Lemma 2.4. *Let $T : \mathcal{A}^{2m}(\Omega) \longrightarrow \mathcal{A}^{2m}(\Omega)$ denote the linear operator $\chi_\tau - \partial_u F(u_0(t), t)$. Then we have*

1. $\operatorname{Ker} T = \mathcal{H}(\Omega)$.
2. *T is bijective from $\pi_{\mathcal{H}}^{-1}(0)$ onto itself. In particular, we have* $\operatorname{Im} T = \pi_{\mathcal{H}}^{-1}(0)$.

Proof. Let ψ be an element in $\mathcal{A}^{2m}(\Omega)$ given by (2.13). Then, as $\partial_u F(u_0(t), t)A(\nu_i) = \nu_i A(\nu_i)$ holds, we have

$$T(\psi) = \sum_{1\le|i|\le m,\, p\in\mathbb{Z}^m} (p_1\nu_1 + \cdots + p_m\nu_m - \nu_i) f_{i,p}(t;\, \eta)\eta^{|p|\alpha} e^{p\cdot\tau} A(\nu_i).$$

The claims of the lemma easily follow from this. $\square$

Remark. If ψ is a homogeneous element of order β with respect to η in $\pi_{\mathcal{H}}^{-1}(0)$, then we can find a homogeneous element $\tilde{\psi}$ of order β in $\pi_{\mathcal{H}}^{-1}(0)$ with $T\tilde{\psi} = \psi$.

Now we describe a recipe to obtain a solution $\varphi \in \hat{\mathcal{A}}^{2m}(\Omega) := \mathcal{A}^{2m}_\alpha(\Omega)(\alpha)$ to (2.8) which has sufficiently many free parameters. Set

$$\varphi = \sum_{k\ge1} \varphi_k(\tau,\, t;\, \eta) \in \hat{\mathcal{A}}^{2m}(\Omega), \tag{2.15}$$

where each term $\varphi_k(\tau,\, t;\, \eta)$ is a homogeneous element of order $k\alpha$ in $\hat{\mathcal{A}}^{2m}(\Omega)$, that is, φ_k has the form

$$\eta^{k\alpha}\left(\sum_{p\in\mathbb{Z}^m,\, |p|\in\{k,\, k-2,\, k-4,\, \dots\}} \varphi_{k,p}(t) e^{p\cdot\tau}\right) \tag{2.16}$$

with $\varphi_{k,p} \in \mathcal{M}^{2m}(\Omega)$. Note that, if k is even, we have $\varphi_k \in \pi_{\mathcal{H}}^{-1}(0)$ as terms containing e^{τ_i}'s $(1 \le |i| \le m)$ never appear in φ_k. Generally, by the same reasoning as above, a homogeneous element in $\hat{\mathcal{A}}^{2m}(\Omega)$ of order $k\alpha$ for an even k belongs to $\pi_{\mathcal{H}}^{-1}(0)$.

We put (2.15) into the system (2.8). Then both sides of (2.8) belong to $\hat{\mathcal{A}}^{2m}(\Omega)$ because $\mathcal{A}(\Omega)$ is a ring and an $\mathcal{M}(\Omega)[[\eta^{-1}]]$ module.

By looking at homogeneous terms of order α in both sides of (2.8), as the second and third terms of the left-hand side of (2.8) are in $\mathcal{A}^{2m}(\Omega)(2\alpha)$, we have $T(\varphi_1) = 0$. Hence, by Lemma 2.4, we obtain

$$\varphi_1 = \eta^\alpha \sum_{1\le|i|\le m} \omega_i^{(1)}(t) e^{\tau_i} A(\nu_i) \tag{2.17}$$

with $\omega^{(1)} = (\omega_{-m}^{(1)}(t),\, \dots,\, \omega_m^{(1)}(t))$ being arbitrary functions in $\mathcal{M}^{2m}(\Omega)$. Then by comparing homogeneous terms of order $2\alpha\,(= -1)$ in both sides of (2.8), we have

$$T(\varphi_2) = F^{(2)}(\tau,\, t,\, \omega^{(1)};\, \eta) \tag{2.18}$$

for some vector function $F^{(2)}$ which is a polynomial of $\omega^{(1)}$ with coefficients in homogeneous elements of order 2α in $\mathcal{A}^{2m}(\Omega)$. Since $F^{(2)}(\tau, t, \omega^{(1)}; \eta)$ belongs to $\pi_{\mathcal{H}}^{-1}(0)$ as noted above, it follows from Lemma 2.4 that we have the unique homogeneous element φ_2 of order 2α in $\mathcal{A}^{2m}(\Omega)$.

Now, by comparing homogeneous terms of order $3\alpha\,(= -\frac{3}{2})$ in both sides of (2.8), we get

$$T\,(\varphi_3) = F^{(3)}(\tau, t, \omega^{(1)}; \eta) \tag{2.19}$$

for some vector function $F^{(3)}$ which is a polynomial of $\omega^{(1)}$ with coefficients in homogeneous elements of order 3α in $\mathcal{A}^{2m}(\Omega)$. It follows from Lemma 2.4 that (2.19) has a solution if and only if the right-hand side of (2.19) satisfies the condition $\pi_{\mathcal{H}}(F^{(3)}(\tau, t, \omega^{(1)}; \eta)) = 0$. And this condition is reduced to a system of non-linear differential equations for $\omega^{(1)}$. As a matter of fact, by taking the term $\eta^{-1}\frac{\partial}{\partial t}\varphi$ in (2.8) into account, we have the system

$$\frac{d\omega^{(1)}}{dt} = H^{(1)}(t, \omega^{(1)}), \tag{$\mathcal{E}_1$}$$

where $H^{(1)}$ is a polynomial of $\omega^{(1)}$ with coefficients in $\mathcal{M}^{2m}(\Omega)$. The system $(\mathcal{E}_1)$ has a solution defined locally with $2m$ free parameters $(a_{-m}, \ldots, a_m) \in \mathbb{C}^{2m}$. However, as it is a non-linear system, existence of a solution on the whole Ω is uncertain. Further its solution may have movable singularities depending on the $2m$ free parameters like the non-linear equation $\dfrac{df}{dt} + f^2 = 0$ with $f(0) = a$, whose solution is give by $f(t) = \dfrac{1}{t + a^{-1}}$. In [3], we showed the fact that the system $(\mathcal{E}_1)$ associated with $(P_I)_m$ $(m = 1, 2, \ldots)$ has a solution on the whole Ω without movable singularities.

Now we assume that the system $(\mathcal{E}_1)$ has a solution on Ω without movable singularities. Then φ_3 is given by

$$\tilde{\varphi}_3(\tau, t, \omega^{(1)}; \eta) + \eta^{3\alpha} \sum_{1 \le |i| \le m} \omega_i^{(3)}(t) e^{\tau_i} A(\nu_i)$$

where $\tilde{\varphi}_3$ is a homogeneous solution of order 3α in $\pi_{\mathcal{H}}^{-1}(0)$ to (2.19) and $\omega^{(3)} = (\omega_{-m}^{(3)}(t), \ldots, \omega_m^{(3)}(t))$ are arbitrary functions in $\mathcal{M}^{2m}(\Omega)$. Then we repeat the same arguments as above, and we obtain the system $(\mathcal{E}_3)$ of differential equations for $\omega^{(3)}$ by comparing homogeneous terms of order 5α in (2.8).

$$\frac{d\omega^{(3)}}{dt} = H^{(3)}(t, \omega^{(1)}, \omega^{(3)}). \tag{$\mathcal{E}_3$}$$

Here $H^{(3)}$ is a polynomial of $\omega^{(1)}$ and $\omega^{(3)}$ with coefficients in $\mathcal{M}^{2m}(\Omega)$. However, on the contrary to $H^{(1)}$ in $(\mathcal{E}_1)$, the $H^{(3)}$ is a first-order polynomial with respect to $\omega^{(3)}$

because a higher-order monomial of $\omega^{(3)}$ appears in a term of order less than or equal to $2 \times 3\alpha = 6\alpha\,(= -3)$. Therefore $(\mathcal{E}_3)$ is a system of linear differential equations for $\omega^{(3)}$, that always has a (possibly multi-valued) solution on Ω with $2m$ free parameters in $\mathbb{C}^{2m}$.

For an odd k greater than 3, comparing terms of order $k\alpha$ in (2.8) and using the same argument as that for $(\mathcal{E}_3)$, we successively obtain the system $(\mathcal{E}_k)$ of linear differential equations for $\omega^{(k)}$.

$$\frac{d\omega^{(k)}}{dt} = H^{(k)}(t, \omega^{(1)}, \omega^{(3)}, \ldots, \omega^{(k)}), \tag{$\mathcal{E}_k$}$$

where $H^{(k)}$ is a polynomial of $\omega^{(1)}, \ldots, \omega^{(k)}$ with coefficients in $\mathcal{M}^{2m}(\Omega)$ and, in particular, a first-order polynomial with respect to $\omega^{(k)}$.

Definition 2.5. A family $\{(\mathcal{E}_k)\}_{k=1,3,\ldots}$ is called the non-secularity condition for the system (2.1).

Summing up, if the first member $(\mathcal{E}_1)$ of the non-secularity condition has a solution with $2m$ free parameters in $\mathbb{C}^{2m}$ on the whole Ω without movable singularities, then we obtain a solution $\varphi \in \hat{\mathcal{A}}^{2m}(\Omega)$ for (2.8) with $2m$ free parameters in $\mathbb{C}^{2m}[[\eta^{-1}]]$.

§ 3. On the construction of instanton-type solutions for $(P_{\mathrm{I}})_m$ in case of $\alpha = -\frac{1}{\ell}$ $(\ell \geq 3)$

In case of $\alpha = -\frac{1}{2}$, the paper [3] showed that the first member $(\mathcal{E}_1)$ of non-secularity conditions associated with $(P_{\mathrm{I}})_m$ is a system of non-linear differential equations with $2m$ unknown functions $(\omega_{-m}, \ldots, \omega_m)$ (see Theorem 4.9 in [3]):

$$\frac{d\omega_k}{dt} = \left(\frac{1}{\nu_k}\sum_{j=1}^{m} \varphi(k, j)\omega_j\omega_{-j} - h_k\right)\omega_k \qquad (1 \leq k \leq m). \tag{3.1}$$

$$\frac{d\omega_{-k}}{dt} = \left(-\frac{1}{\nu_k}\sum_{j=1}^{m} \varphi(-k, j)\omega_j\omega_{-j} - h_{-k}\right)\omega_{-k} \quad (1 \leq k \leq m). \tag{3.2}$$

Here $\frac{1}{\nu_k}\varphi(k, j)$ and h_k will be given by (3.18) and (3.28) later. By solving the system $(\mathcal{E}_1)$ globally, we proved the existence of instanton-type solutions with $2m$ free parameters. From now on, in case of $\alpha = -\frac{1}{\ell}$ $(\ell \geq 3)$, we study the existence of instanton-type solutions for $(P_{\mathrm{I}})_m$.

§ 3.1. Preparations

Let us first recall results in [3] which are needed in subsequent discussions. Throughout the paper, θ denotes an independent variable and the notation $A \equiv B$ means that $A - B$ is zero modulo θ^{m+2}. For any formal power series x of θ, we define $\sigma_i^\theta(x)$ by the coefficient of θ^i in x. According to [3], we can represent $(P_{\mathrm{I}})_m$ (discussed in [8]) in terms of generating functions:

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} 2V\theta \\ -(1+2u_1\theta)(1-U) + \dfrac{1+2C-\theta V^2}{1-U} \end{pmatrix}. \tag{3.3}$$

Here U, V and C are generating functions of unknown functions u_k, v_k and constants c_k as follows.

$$U(\theta) := \sum_{k=1}^{\infty} u_k\theta^k, \;\; V(\theta) := \sum_{k=1}^{\infty} v_k\theta^k, \;\; C(\theta) := \sum_{k=1}^{\infty}(c_k + \delta_{km}t)\theta^{k+1} \tag{3.4}$$

with the conditions $\sigma_{m+1}^\theta(U) = \sigma_{m+1}^\theta(V) = 0$ and $c_{m+1} = 0$. Note that the solution space for (3.3) is defined in the same way as that of $\mathcal{A}_\alpha(\Omega)$ where $\mathcal{M}(\Omega)$ is replaced by $\mathcal{M}(\Omega)[[\theta]]$ (Here $\mathcal{A}_\alpha(\Omega)$ was defined by (2.3)).

To obtain the equation corresponding to (2.8) in §2, we prepare several notations. Let Θ denote the set of formal power series of θ without constant terms and let $Q : (\Theta\theta)^2 \longrightarrow \Theta^2$ be the map defined by

$$Q\begin{pmatrix} x\,\theta \\ y\,\theta \end{pmatrix} := 2\begin{pmatrix} y\theta \\ (1+2\hat{u}_{1,0}\theta)\,x - \sigma_1^\theta(x)\theta \end{pmatrix} \tag{3.5}$$

for any $x = \sum_{i=1}^{\infty} x_i\theta^i$ and $y = \sum_{i=1}^{\infty} y_i\theta^i$ in Θ. We define ν_k and $A(\nu_k)$ by the eigenvalue and the corresponding eigenvector of Q in the sense of $Q(A(\nu_k)\theta) = \nu_k A(\nu_k)\theta$. Let $\hat{u}_0$ and $\hat{v}_0$ denote the generating functions of the leading term $\hat{u}_{i,\,0}, \hat{v}_{i,\,0}$ of a 0-parameter solution to $(P_{\mathrm{I}})_m$ in the form (see (11), (12) in [3] for more explicit forms of $\hat{u}_0$ and $\hat{v}_0$)

$$\hat{u}_0(\theta) := \sum_{i=1}^{\infty} \hat{u}_{i,\,0}\theta^i, \quad \hat{v}_0(\theta) := \sum_{i=1}^{\infty} \hat{v}_{i,\,0}\theta^i. \tag{3.6}$$

By taking the change of unknown functions

$$U = \hat{u}_0 + (1-\hat{u}_0)u, \qquad V = \hat{v}_0 + (1-\hat{u}_0)v \qquad (u,\, v) \in \hat{\mathcal{A}}_\alpha^2(\Omega),$$

we have the partial differential equations associated with (3.3) of the form

(3.7)

$$P\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} \equiv \left(\begin{pmatrix} \eta^{-1}\rho\theta \\ S(u,\, v) \end{pmatrix} + u\ P\begin{pmatrix} u\theta \\ v\theta \end{pmatrix}\right) - \left(u\begin{pmatrix} \eta^{-1}\rho \\ 2\sigma_1^\theta(u)u \end{pmatrix} + \eta^{-1}\left(\rho + \frac{\partial}{\partial t}\right)\begin{pmatrix} u \\ v \end{pmatrix}\right)\theta$$
$$+\,\eta^{-1}u\left(\rho + \frac{\partial}{\partial t}\right)\begin{pmatrix} u \\ v \end{pmatrix}\theta.$$

Here the operator P is given by $P := \chi_\tau - Q$ and $S(u,\, v)$ and ρ are defined by

$$S(u,\, v) := \frac{1}{2}(-v,\, u)Q\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} + 3\sigma_1^\theta(u)u\theta \ \text{ and } \ \rho := \frac{d}{dt}(\log(1-\hat{u}_0)). \tag{3.8}$$

Recall that the solution $(u,\, v)$ to (3.7) takes a form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{j=1}^{\infty}\left(\sum_{1\le |k|\le m} f_{k,\,j\alpha}(\tau,\, t)A(\nu_k)\right)\eta^{j\alpha}. \tag{3.9}$$

Here $f_{k,\,j\alpha}$'s are independent of θ. As is shown in Lemma 2.3, α must be $\alpha = -\frac{1}{\ell}$ $(\ell \ge 2)$ so that we have a solution $(u,\, v) \in \hat{\mathcal{A}}^2_\alpha(\Omega)$ of (3.9) for (3.7).

In the next subsection, when $\alpha = -\frac{1}{\ell}$ $(\ell \ge 3)$, we give the explicit forms of $f_{k,\,j\alpha}$ $(j = 1,\, 2,\, 3)$ by the method described in §2.

§ 3.2. The case of $\alpha = -\frac{1}{\ell}$ $(\ell \ge 3)$

We define $\sigma^\eta_{j\alpha}(u)$ (resp. $\sigma^\eta_{j\alpha}(v)$) by the coefficient of $\eta^{j\alpha}$ in u (resp. v) and we set $u_{j\alpha} := \sigma^\eta_{j\alpha}(u)$, $v_{j\alpha} := \sigma^\eta_{j\alpha}(v)$ $(j \ge 1)$. In what follows, we use the Kronecker's delta $\delta_{3\alpha,\,-1}$. Putting (3.9) into (3.7), we have

$$P\begin{pmatrix} u_\alpha\,\theta \\ v_\alpha\,\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.10}$$

$$P\begin{pmatrix} u_{2\alpha}\,\theta \\ v_{2\alpha}\,\theta \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \frac{1}{2}\,(-v_\alpha,\ u_\alpha)\,Q\begin{pmatrix} u_\alpha\,\theta \\ v_\alpha\,\theta \end{pmatrix} + 3\sigma_1^\theta(u_\alpha)u_\alpha\theta \end{pmatrix}, \tag{3.11}$$

(3.12)

$$P\begin{pmatrix} u_{3\alpha}\theta \\ v_{3\alpha}\theta \end{pmatrix} \equiv \begin{pmatrix} \delta_{3\alpha,\,-1}\times\rho\theta \\ (-v_\alpha,\ u_\alpha)\,Q\begin{pmatrix} u_{2\alpha}\,\theta \\ v_{2\alpha}\,\theta \end{pmatrix} + \frac{u_\alpha}{2}\,(-v_\alpha,\ u_\alpha)\,Q\begin{pmatrix} u_\alpha\,\theta \\ v_\alpha\,\theta \end{pmatrix} + (u_\alpha)^2\,\sigma_1^\theta(u_\alpha)\theta \end{pmatrix}$$
$$+\begin{pmatrix} 0 \\ 2\sigma_1^\theta(u_\alpha)u_{2\alpha} + 4\sigma_1^\theta(u_{2\alpha})u_\alpha \end{pmatrix}\theta.$$

By (3.10) and Lemma 2.4, we obtain the lemma below.

Lemma 3.1. *For any k $(1 \le |k| \le m)$, we have*

$$f_{k,\,\alpha} = \omega_k^{(1)} e^{\tau_k}. \tag{3.13}$$

Here $\omega_k^{(1)}$'s are arbitrary functions of t.

From now on, we abbreviate $\omega_k^{(1)}$ to ω_k. An easy computation shows:

Lemma 3.2. *For any k $(1 \le |k| \le m)$, the $f_{k,\,2\alpha}$ is given by*

$$\begin{aligned} f_{k,\,2\alpha}(t,\,\tau) = & \sum_{\substack{1\le|j|\le m,\\ j\ne -k}} \frac{2}{(\nu_k+\nu_j)\nu_k\nu_j}\left((2\nu_k+\nu_j)\omega_k\omega_j e^{\tau_k+\tau_j} - \nu_j\omega_{-k}\omega_{-j}e^{-\tau_k-\tau_j}\right) \\ & - \frac{1}{\nu_k}\left(\sum_{j=1}^{m}\frac{\nu_j^2}{\nu_k}h_{j,\,k}\omega_j\omega_{-j} + \frac{6}{\nu_k}\omega_k\omega_{-k}\right). \end{aligned} \tag{3.14}$$

Here $h_{j,\,|k|}$ are defined by

$$h_{j,\,|k|} := \frac{4\displaystyle\prod_{\substack{1\le l\le m,\\ l\ne j,\,|k|}}\left(\nu_j^2-\nu_l^2\right)}{\displaystyle\prod_{\substack{1\le l\le m,\\ l\ne |k|}}\left(\nu_k^2-\nu_l^2\right)} \quad (j\ne|k|), \qquad h_{j,\,j} := \sum_{\substack{l=1,\\ l\ne j}}^{m}\frac{4}{\nu_j^2-\nu_l^2} \tag{3.15}$$

and $h_{j,\,k} := h_{|j|,\,|k|}$.

It follows from Lemmas 3.1 and 3.2 that we have the following.

Lemma 3.3. *The equation* (3.12) *is written in the form*

$$P\begin{pmatrix} u_{3\alpha}\theta \\ v_{3\alpha}\theta\end{pmatrix} = \delta_{3\alpha,\,-1}\times\frac{1}{2}\sum_{1\le|k|\le m}\gamma_k A(\nu_k)\theta + \sum_{1\le|k|\le m}\frac{1}{\nu_k}\varphi_k A(\nu_k)\theta \tag{3.16}$$

with

$$\begin{aligned} \varphi_k := & \sum_{\substack{1\le|i|\le m,\\ i\ne -k}}\ \sum_{\substack{1\le|j|\le m,\\ j\ne -k,\\ j\ne -i}} \frac{4(2\nu_k+\nu_i+\nu_j)(\nu_k+\nu_i+\nu_j)}{\nu_i\nu_j(\nu_k+\nu_i)(\nu_k+\nu_j)} \\ & \qquad \times\left(\omega_k\omega_j\omega_i e^{\tau_k+\tau_j+\tau_i} + \omega_{-k}\omega_{-j}\omega_{-i}e^{-\tau_k-\tau_j-\tau_i}\right) \\ & + \sum_{j=1}^{m}\varphi(k,\,j)\omega_j\omega_{-j}(\omega_k e^{\tau_k}+\omega_{-k}e^{-\tau_k}) + \sum_{\substack{1\le|j|\le m,\\ j\ne\pm k}}\tilde{\varphi}(k,\,j)\omega_j e^{\tau_j}, \end{aligned} \tag{3.17}$$

where $\varphi(k,\,j)$ and $\tilde{\varphi}(k,\,j)$ are given by

$$
\begin{aligned}
\varphi(k,\,j) &:= -\Bigg(\frac{16}{\nu_k^2-\nu_j^2}+\frac{48}{\nu_j^2}+\frac{12\nu_j^2}{\nu_k^2}h_{j,\,k}+\sum_{\substack{1\le r\le m,\\ r\neq |k|}}\frac{8\nu_j^2}{\nu_r^2}h_{j,\,r}\Bigg) \quad (j\neq |k|),\\
\varphi(k,\,k) &:= -\Bigg(\frac{60}{\nu_k^2}+12h_{k,\,k}+\sum_{\substack{1\le r\le m,\\ r\neq |k|}}\frac{8\nu_k^2}{\nu_r^2}h_{k,\,r}\Bigg),\quad \varphi(-k,\,k):=\varphi(k,\,k),\\
\tilde{\varphi}(k,\,j) &:= -\left(\frac{16}{\nu_k^2-\nu_j^2}+\frac{24}{\nu_k^2}+4h_{k,\,k}\right)\omega_k\omega_{-k}-4\sum_{\substack{i=1,\\ i\neq |k|}}^{m}\frac{\nu_i^2}{\nu_k^2}h_{i,\,k}\omega_i\omega_{-i},
\end{aligned}
\tag{3.18}
$$

and γ_k's are functions of t which are determined by

$$
\rho \equiv \sum_{k=1}^{m}\gamma_k(t)\frac{\theta}{1-\theta g(\nu_k)} \quad \textit{with} \quad g(\nu_k):=\frac{\nu_k^2-8\hat{u}_{1,\,0}}{4}. \tag{3.19}
$$

See Eq.(45) *in* [*3*] *for the complete forms of γ_k's.*

Proof. By using (3.13) and (3.14), we have

$$
(-v_\alpha,\ u_\alpha)\,Q\begin{pmatrix} u_{2\alpha}\,\theta\\ v_{2\alpha}\,\theta\end{pmatrix}+\frac{u_\alpha}{2}\,(-v_\alpha,\ u_\alpha)\,Q\begin{pmatrix} u_\alpha\,\theta\\ v_\alpha\,\theta\end{pmatrix}+(u_\alpha)^2\,\sigma_1^\theta(u_\alpha)\theta\equiv 0. \tag{3.20}
$$

Hence (3.12) is equivalent to

$$
P\begin{pmatrix} u_{3\alpha}\theta\\ v_{3\alpha}\theta\end{pmatrix}\equiv\begin{pmatrix}\delta_{3\alpha,\,-1}\times\rho\\ 2\sigma_1^\theta(u_\alpha)u_{2\alpha}+4\sigma_1^\theta(u_{2\alpha})u_\alpha\end{pmatrix}\theta. \tag{3.21}
$$

Noticing that $\sigma_\alpha^\eta(u)$ (resp. $\sigma_{2\alpha}^\eta(u)$) ($\alpha=-\frac{1}{\ell}$, $\ell\ge 3$) is the same as $\sigma_\alpha^\eta(u)$ (resp. $\sigma_{2\alpha}^\eta(u)$) in the case of $\alpha=-1/2$, we have the assertion of Lemma 3.3 by (D2) in Appendix D [3]. □

In order to solve (3.16), by Lemma 2.4 we need the non-secularity condition below.

$$
\frac{1}{\nu_k}\sum_{j=1}^{m}\varphi(k,\,j)\omega_j\omega_{-j}\omega_k=0 \qquad (1\le |k|\le m). \tag{3.22}
$$

As we consider our problem outside turning points of the first kind, we have

Lemma 3.4. *Under the non-secularity condition*

$$
\sum_{j=1}^{m}\varphi(k,\,j)\omega_j\omega_{-j}\omega_k=0 \quad (1\le |k|\le m), \tag{3.23}
$$

we have

(3.24)
$$f_{k,\,3\alpha}(t,\,\tau)=\sum_{\substack{1\le|i|\le m,\\ i+k\neq 0}}\ \sum_{\substack{1\le|j|\le m,\\ j+k\neq 0,\\ j+i\neq 0}}\frac{4(\nu_k+\nu_i+\nu_j)}{\nu_k\nu_i\nu_j(\nu_k+\nu_i)(\nu_k+\nu_j)(\nu_i+\nu_j)}$$
$$\times\,((2\nu_k+\nu_i+\nu_j)\omega_k\omega_i\omega_j e^{\tau_k+\tau_i+\tau_j}-(\nu_i+\nu_j)\omega_{-k}\omega_{-i}\omega_{-j}e^{-\tau_k-\tau_i-\tau_j})$$
$$-\frac{1}{2\nu_k^2}\sum_{j=1}^{m}\varphi(k,\,j)\omega_j\omega_{-j}\omega_{-k}e^{-\tau_k}+\sum_{\substack{1\le|j|\le m,\\ j\neq\pm k}}\frac{1}{(\nu_j-\nu_k)\nu_k}\tilde{\varphi}(k,\,j)\omega_j e^{\tau_j}$$
$$-\,\delta_{3\alpha,\,-1}\times\frac{1}{2\nu_k}\gamma_k+\omega_k^{(3)}(t)e^{\tau_k}$$

for any k $(1\le|k|\le m)$. Here $\omega_k^{(3)}(t)$ is defined by the subsequent members of the non-secularity conditions and γ_k's are defined in Lemma 3.3.

By looking at terms of $\eta^{\alpha-1}$ in the right-hand side of (3.7), the first member $(\mathcal{E}_1)$ of the non-secularity conditions is determined. The difference between cases of $\alpha=-\frac{1}{2}$ and $\alpha=-\frac{1}{\ell}$ $(\ell\ge 3)$ is that ω_k's must satisfy not only $(\mathcal{E}_1)$ determined by the terms of $\eta^{\alpha-1}$ but also (3.23) when $\ell\ge 3$. Furthermore, the form of $(\mathcal{E}_1)$ differs according to the parity of ℓ. In fact, when $\ell=2$ and $\ell=4$, $(\mathcal{E}_1)$ is a system of non-linear differential equations. On the other hand, when $\ell=3$, $(\mathcal{E}_1)$ is a system of linear differential equations (see §3.3).

§ 3.3. A concrete calculation in case of $\alpha=-\frac{1}{3}$

In case of $\alpha=-\frac{1}{3}$, let us write down $(\mathcal{E}_1)$ obtained by looking at terms of $\eta^{\alpha-1}$ in the right-hand side of (3.7). By the straightforward computations, we have

$$P\begin{pmatrix}u_{4\alpha}\theta\\ v_{4\alpha}\theta\end{pmatrix}\equiv\begin{pmatrix}0\\ H(u,\,v)\end{pmatrix}-\left(\rho+\frac{\partial}{\partial t}\right)\begin{pmatrix}u_\alpha\\ v_\alpha\end{pmatrix}\theta, \tag{3.25}$$

where $H(u,\,v)$ is defined by

$$\begin{aligned}H(u,\,v):=&\,(-v_\alpha,\,u_\alpha)Q\begin{pmatrix}u_{3\alpha}\theta\\ v_{3\alpha}\theta\end{pmatrix}+\frac{1}{2}(-v_{2\alpha},\,u_{2\alpha})Q\begin{pmatrix}u_{2\alpha}\theta\\ v_{2\alpha}\theta\end{pmatrix}\\ &+\frac{u_{2\alpha}}{2}(-v_\alpha,\,u_\alpha)Q\begin{pmatrix}u_\alpha\theta\\ v_\alpha\theta\end{pmatrix}+(\sigma_1^\theta(u_\alpha)u_\alpha u_{2\alpha}+2\sigma_1^\theta(u_{2\alpha})u_\alpha^2)\theta\\ &+(4\sigma_1^\theta(u_{3\alpha})u_\alpha+3\sigma_1^\theta(u_{2\alpha})u_{2\alpha}+2\sigma_1^\theta(u_\alpha)u_{3\alpha})\theta.\end{aligned} \tag{3.26}$$

By existence of the terms containing $e^{\tau_k}A(\nu_k)$ $(1\le|k|\le m)$ in the right-hand side of (3.25), we see that $(\mathcal{E}_1)$ is expressed as

$$-h_k\omega_k-\frac{d\omega_k}{dt}=0\quad(1\le|k|\le m) \tag{3.27}$$

with

$$h_k := \frac{\nu_k'}{2\nu_k} + g(\nu_k)' h_{k,\,k} + \sum_{\substack{1\le r\le m,\\ r\neq |k|}} \frac{2(\gamma_r+\gamma_k)}{\nu_k^2-\nu_r^2}. \tag{3.28}$$

Here h_k is the same as the one defined by Eq.(77) in [3]. The following proposition follows from (3.23) and (3.27).

Proposition 3.5. *When $\alpha = -\frac{1}{3}$, we have the explicit forms of ω_k's in* (3.13):

$$\omega_k = \beta_k \exp\left(-\int h_k dt\right), \quad \omega_{-k} = \beta_{-k} \exp\left(-\int h_k dt\right) \quad (1 \le k \le m) \tag{3.29}$$

with free parameters $(\beta_{-m},\,\ldots,\,\beta_m) \in E$ and E is defined by

$$E := \left\{ (\beta_{-m},\,\ldots,\,\beta_{-1},\,\beta_1,\,\ldots,\,\beta_m) \in \mathbb{C}^{2m} \,\middle|\, \sum_{j=1}^{m} \varphi(k,\,j)\beta_j\beta_{-j}\beta_k = 0 \ (1 \le |k| \le m) \right\}.$$

Remark that there exist indices k and j for which $\varphi(k,\,j) \neq 0$ holds, and hence we have $\dim E \le 2m-1$. Therefore we have the following.

Theorem 3.6. *When $\alpha = -\frac{1}{3}$, there is no instanton-type solution with $2m$ free parameters in $\mathcal{A}^2_{-\frac{1}{3}}(\Omega)$ for* (3.3).

Remark. Let M be the $m \times m$ matrix defined by $M := (\varphi(k,\,j))_{1\le k,\,j\le m}$. Then there is a possibility that an arbitrary minor determinant of M does not vanish and $\det M \neq 0$. Hence we might not be able to add parameters more than $m+1$.

We give some comments on instanton-type solutions with m free parameters. Taking parameters which satisfy $(*)$ below, we see that the leading term of (3.9) (with respect to η) contains m free parameters.

$$(*) \quad \beta_j\beta_{-j} = 0 \text{ for any } 1 \le j \le m.$$

Next, let us consider the second member $(\mathcal{E}_3)$ of the non-secularity conditions which determines $\omega_k^{(3)}$'s in (3.24). By the right-hand side of the equation for $(u_{6\alpha},\,v_{6\alpha})$, we confirm that $(\mathcal{E}_3)$ is a system of first-order linear inhomogeneous differential equations for $\omega_k^{(3)}$. Since there exist the terms containing $\omega_j^{(3)}$ and $A(\nu_k)e^{\tau_k}$ simultaneously (for example, $\omega_k^{(3)}\omega_j\omega_{-j}A(\nu_k)e^{\tau_k}$, $\omega_j^{(3)}\omega_k\omega_{-j}A(\nu_k)e^{\tau_k}$) in the right-hand side of the equation for $(u_{5\alpha},\,v_{5\alpha})$, the $\omega_k^{(3)}$'s must satisfy similar non-secularity conditions as those in (3.23). A similar argument holds for higher-order terms. Summing up, taking parameters suitably, we expect that there exists an instanton-type solution with m free parameters in $\mathcal{A}^2_{-\frac{1}{3}}(\Omega)$ for (3.3).

§ 3.4. A certain conjecture in case of $\alpha = -\frac{1}{\ell}$ $(\ell \geq 4)$

In the case of $\alpha = -\frac{1}{4}$, we note that $(\mathcal{E}_1)$ is a system of fifth-order non-linear differential equations for ω_k's and ω_k's must satisfy (3.23). By the same reasoning as $\alpha = -\frac{1}{3}$, we can't expect the existence of instanton-type solutions with $2m$ free parameters in $\mathcal{A}^2_{-\frac{1}{4}}(\Omega)$. Hence instanton-type solutions in $\mathcal{A}^2_\alpha(\Omega)$ for (3.3) seem to have $2m$ free parameters only when $\alpha = -\frac{1}{2}$ and, by (3.23), the following conjecture is expected in general cases:

Conjecture: When $\alpha = -\frac{1}{\ell}$ $(\ell \geq 3)$, there is no instanton-type solutions with $2m$ free parameters in $\mathcal{A}^2_\alpha(\Omega)$ for (3.3).

Finally, we remark that the conjecture given in page 523, [3] is also valid for the second and the third terms (except for $\omega_k^{(3)}$'s) of (u, v) in the case of $\alpha = -\frac{1}{\ell}$ $(\ell \geq 3)$.

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