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On the fourth order PI equation and coalescing phenomena of nonlinear turning points

Dedicated to Professor Takashi AOKI for his sixtieth birthday

By

Yoshitsugu Takei*

Abstract

In this report we present a conjecture for the fourth order PI equation with a large parameter to show its importance in the exact WKB analysis. The conjecture is related to coalescing phenomena of turning points and can be regarded as a nonlinear analogue of Hirose's result ([9]) for the Pearcey system. We also discuss some relations between the conjecture and Dubrovin's result ([7]) for the KdV equation.

§ 1. Introduction

The purpose of this report is to show the importance of the fourth order PI equation

$$(P_{\mathrm{I}})_2 \qquad \eta^{-4}\frac{d^4u}{dt^4} - 10\eta^{-2}\left[2u\frac{d^2u}{dt^2} + \left(\frac{du}{dt}\right)^2\right] + 40u^3 + 8cu - 8t = 0,$$

where η (> 0) denotes a large parameter and $c \in \mathbb{C}$ is a constant, in the exact WKB analysis by presenting a conjecture for $(P_{\mathrm{I}})_2$.

Recently Hirose ([8, 9]) studies the exact WKB analysis for a completely integrable system of linear differential equations with two independent variables and proves that the Pearcey system, that is, the most degenerate hypergeometric system with two variables, gives the normal form at a critical point where two turning points coalesce. Our conjecture can be regarded as a nonlinear analogue of Hirose's result: As is well-known (e.g., [18, 19]), higher order Painlevé equations can be extended to completely integrable systems of nonlinear differential equations. For example, $(P_{\mathrm{I}})_2$ is extended to the

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*RIMS, Kyoto University, Kyoto 606-8502, Japan. e-mail: takei@kurims.kyoto-u.ac.jp

most degenerate Garnier system with two variables. Then the conjecture claims that in the study of completely integrable systems of nonlinear differential equations (with two variables) the fourth order PI equation $(P_{\mathrm{I}})_2$ gives the normal form at a critical point where two nonlinear turning points coalesce. To show the validity of the conjecture we discuss the transformation between the fourth order PII equation and $(P_{\mathrm{I}})_2$ (to be more precise, between the completely integrable systems of nonlinear equations corresponding to them) in this report.

On the other hand, Dubrovin ([7]) shows that the fourth order PI equation is a universal model at a point of gradient catastrophe for a solution of the KdV equation. In fact, Dubrovin's result is one of the motivations of our study for the fourth order PI equation. At the end of this report we also discuss some possible relations between our conjecture and Dubrovin's result.

The paper is organized as follows: In Section 2 we briefly review the exact WKB analysis and then recall Hirose's result for the Pearcey system in Section 3. The main claim (conjecture) will be given in Section 4. Section 5 is devoted to the discussion of the transformation between the fourth order PII equation and the fourth order PI equation. Finally in Section 6 we discuss relations between our conjecture and Dubrovin's result.

§ 2. Brief review of the exact WKB analysis

In this section we briefly review the exact WKB analysis for ordinary differential equations.

First, let us consider linear ordinary differential equations with a large parameter $\eta > 0$:

$$\left(\left(\eta^{-1}\frac{d}{dx}\right)^{m} + a_1(x)\left(\eta^{-1}\frac{d}{dx}\right)^{m-1} + \cdots + a_m(x)\right)\psi = 0. \tag{2.1}$$

Equation (2.1) has a formal solution $\psi_j(x,\eta)$ $(j = 1,\ldots,m)$, called a WKB solution, of the form

$$\psi_j(x,\eta) = \exp\left(\eta\int^x \zeta_j(x)\,dx\right)\sum_{n=0}^{\infty}\eta^{-(n+1/2)}\psi_{j,n}(x), \tag{2.2}$$

where $\zeta_j(x)$ is a root of the characteristic equation of (2.1):

$$\zeta^m + a_1(x)\zeta^{m-1} + \cdots + a_m(x) = 0. \tag{2.3}$$

In the exact WKB analysis we use the Borel resummation technique to endow a WKB solution with analytic meaning. Several important properties of the Borel sum of a WKB solution are described in terms of turning points and Stokes curves, which are defined as follows:

Definition 2.1. (i) A zero of the discriminant of (2.3) is called a turning point of (2.1). In particular, a simple zero of the discriminant is called a simple turning point. When two characteristic roots $\zeta_j(x)$ and $\zeta_{j'}(x)$ merge at a turning point $x = a$, we say that $x = a$ is of type (j, j').
(ii) Let $x = a$ be a turning point of type (j, j'). Then a curve defined by

$$\mathrm{Im} \int_a^x (\zeta_j(x) - \zeta_{j'}(x))\, dx = 0 \tag{2.4}$$

is called a Stokes curve of type (j, j').

Note that there exist $m(m-1)/2$ types of turning points and Stokes curves for an m-th order equation. Thus it is not necessary to specify their types for second order equations, while it is essentially important to specify their types for higher order equations.

As is discussed in [22], [6], [15], etc., in the case of second order equations a WKB solution is Borel summable except on Stokes curves and Stokes phenomena for WKB solutions occur only on Stokes curves. Furthermore, the structure of Stokes phenomena for WKB solutions on a Stokes curve emanating from a simple turning point is well understood by using a WKB theoretic transformation to the Airy equation near a simple turning point. See [20] for the Borel summability of WKB solutions and see [2], [15, Chapter 2] and [11] for WKB theoretic transformations to the Airy equation near a simple turning point.

On the other hand, in the case of higher order equations, we encounter a problem of "new Stokes curves" and "virtual turning points".

Example 2.2 (BNR equation; cf. [5]).

$$\left(\left(\eta^{-1}\frac{d}{dx}\right)^3 + 3\eta^{-1}\frac{d}{dx} + 2ix\right)\psi = 0. \tag{2.5}$$

Equation (2.5) has turning points at $x = \pm 1$. After numbering the characteristic roots suitably, we find that $x = -1$ is of type $(1, 2)$ and $x = 1$ is of type $(2, 3)$. As the types of these two turning points differ, Stokes curves emanating from them cross. Berk et al ([5]) pointed out that Stokes phenomena for WKB solutions of (2.5) occur not only on Stokes curves emanating from the turning points $x = \pm 1$ but also on a new Stokes curve (of type $(1, 3)$) passing through the crossing points of Stokes curves. (To be more precise, Stokes phenomena occur only on a solid portion, not on a broken portion, of a new Stokes curve.) See Figure 1. Later Aoki et al ([3]) introduced the notion of virtual turning points through microlocal study of the Borel tranform of (2.5) and interpreted this new Stokes curve as a Stokes curve emanating from a virtual turning point at $x = 0$.

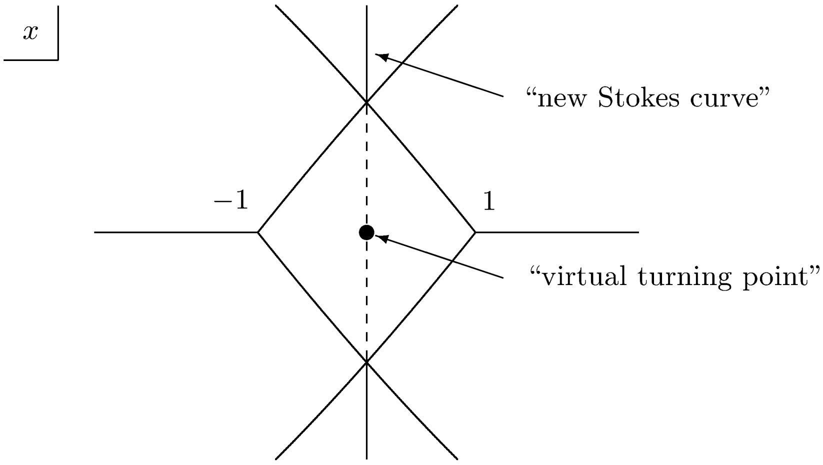

Figure 1. Stokes curves and a new Stokes curve of the BNR equation (2.5)

Although it is a troublesome problem to determine the complete structure of new Stokes curves and virtual turning points for a generic higher order equation, we have a concrete procedure, which is practically satisfactory, to determine it. For details see [10] and references cited there.

Next, let us consider nonlinear ordinary differential equations of Painlevé type, that is, Painlevé equations with a large parameter $\eta > 0$:

$$(P_J) \qquad \frac{d^2\lambda}{dt^2} = \eta^2 F_J(\lambda, t) + G_J\left(\lambda, \frac{d\lambda}{dt}, t\right) \quad (J = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI})$$

(where $F_J(\lambda, t)$ and $G_J(\lambda, \mu, t)$ are some rational functions; for example, $F_{\mathrm{I}} = 6\lambda^2 + t$, $F_{\mathrm{II}} = 2\lambda^3 + t\lambda + c$, $G_{\mathrm{I}} = G_{\mathrm{II}} = 0$, etc) and their higher order analogues $(P_J)_m$ (i.e., $2m$-th order J-th Painlevé equations). In the case of nonlinear equations such as (P_J) there exists a formal power series solution, called a 0-parameter solution, of the form

$$\lambda^{(0)}(t, \eta) = \lambda_0^{(0)}(t) + \eta^{-1}\lambda_1^{(0)}(t) + \eta^{-2}\lambda_2^{(0)}(t) + \cdots . \tag{2.6}$$

There also exists a formal solution, called an instanton-type solution, with sufficiently many free parameters for (P_J) and $(P_J)_m$. Using 0-parameter solutions and instanton-type solutions instead of WKB solutions, we can develop the exact WKB theory for nonlinear equations of Painlevé type. In particular, turning points and Stokes curves for $(P_J)_m$ are defined as follows:

Definition 2.3. Let $(\Delta P_J)_m$ be the Fréchet derivative (or the linearized equation) of $(P_J)_m$ at a 0-parameter solution. Then a turning point and a Stokes curve of $(P_J)_m$ are, by definition, a turning point and a Stokes curve of $(\Delta P_J)_m$, respectively.

Although the (generalized) Borel summability of 0-parameter solutions and instanton-type solutions are not in general verified yet, the framework of the exact WKB theory for (P_J) and $(P_J)_m$ has been constructed in a parallel way to the exact WKB analysis for linear equations (2.1). See, e.g., [13], [4], [14], [21], [12], [16] and [17]. For example, as a nonlinear analogue of the transformation to the Airy equation near a simple turning point, a formal WKB theoretic transformation to the second order PI equation (P_{I}) near a nonlinear simple turning point is constructed in [13], [14], [16] and [17]. The appearance of a new Stokes curve for higher order Painlevé equations is also confirmed in [12].

§ 3. Hirose's result for the Pearcey system

Recently in [8] and [9], following the pioneering work of Aoki ([1]), Hirose discusses the exact WKB analysis of a 3×3 completely integrable system of linear differential equations with two independent variables

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x)\Psi, \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x)\Psi, \end{cases} \qquad (P(x),\, Q(x) \,:\, 3\times 3 \text{ matrices}). \tag{3.1}$$

In particular, he studies the Pearcey system, i.e., the most degenerate hypergeometric system with two variables:

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x)\Psi, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -x_1/4 & -x_2/2 & 0 \end{pmatrix}, \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x)\Psi, \quad Q = P^2 + \dfrac{x_2}{3} - \dfrac{\eta^{-1}}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \end{cases} \tag{3.2}$$

which is equivalent to

$$\begin{cases} \left(\eta^{-3}\dfrac{\partial^3}{\partial x_1^3} + \dfrac{x_2}{2}\eta^{-1}\dfrac{\partial}{\partial x_1} + \dfrac{x_1}{4}\right)\psi = 0, \\ \left(\eta^{-1}\dfrac{\partial}{\partial x_2} - \eta^{-2}\dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0. \end{cases} \tag{3.3}$$

Note that the restriction of the Pearcey system (3.2) or (3.3) to a complex line $x_2 =$ Const. is equivalent to the BNR equation (2.5).

A completely integrable system (3.1) has a WKB solution of the form

$$\Psi_j(x,\eta) = \exp\left(\eta\int^x \omega_j\right)\sum_{n=0}^{\infty}\eta^{-(n+1/2)}\Psi_{j,n}(x) \tag{3.4}$$

with ω_j being a closed 1-form defined by

$$\omega_j = \zeta_{1,j}(x)dx_1 + \zeta_{2,j}(x)dx_2, \tag{3.5}$$

where $\zeta_{1,j}(x)$ and $\zeta_{2,j}(x)$ are eigenvalues of $P(x)$ and $Q(x)$, respectively. (Note that, thanks to the complete integrability of (3.1), $P(x)$ and $Q(x)$ commute and hence they are simultaneously diagonalizable.) In the case of a completely integrable system (3.1) turning points and Stokes surfaces are defined as follows:

Definition 3.1. (i) A turning point of (3.1) is a common zero of the discriminant of the characteristic equation of $P(x)$ and that of $Q(x)$, that is, a point where two different 1-forms $\omega_j = \zeta_{1,j}(x)dx_1 + \zeta_{2,j}(x)dx_2$ and $\omega_{j'} = \zeta_{1,j'}(x)dx_1 + \zeta_{2,j'}(x)dx_2$ merge. When ω_j and $\omega_{j'}$ merge at a turning point $x = a$, we say that $x = a$ is of type (j, j').
(ii) Let $x = a$ be a turning point of type (j, j'). Then a surface defined by

$$\mathrm{Im} \int_a^x (\omega_j - \omega_{j'}) = 0 \tag{3.6}$$

is called a Stokes surface of type (j, j').

For example, the set of turning points of the Pearcey system (3.2) is given by

$$27x_1^2 + 8x_2^3 = 0. \tag{3.7}$$

This set of turning points of (3.2) has a cuspidal singularity at the origin and two turning points with different types coalesce there (cf. Figure 2).

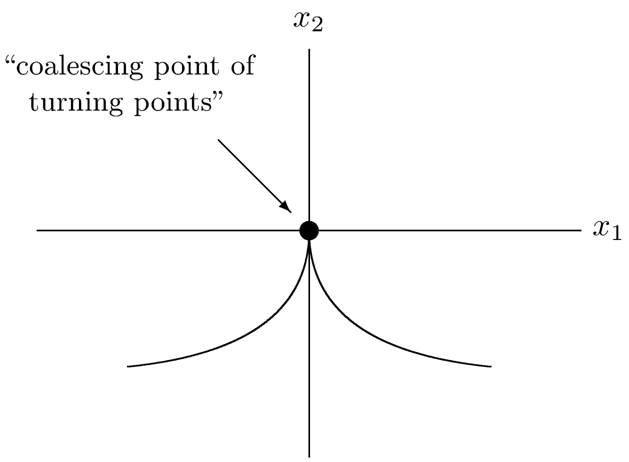

Figure 2. The set of turning points of the Pearcey system (3.2)

As is discussed in [8], a coalescing point of turning points plays an important role in the exact WKB analysis of completely integrable systems and Hirose proves the following intriguing result in [9].

Theorem 3.2 (Hirose [9]). *The Pearcey system* (3.2) *gives the normal form at a coalescing point of turning points for a completely integrable system* (3.1).

To be more specific, let us assume that coalescence of turning points occurs at $\tilde{x} = (0, 0)$ for a completely integrable system

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial \tilde{x}_1}\tilde{\Psi} = \tilde{P}(\tilde{x})\tilde{\Psi}, \\ \eta^{-1}\dfrac{\partial}{\partial \tilde{x}_2}\tilde{\Psi} = \tilde{Q}(\tilde{x})\tilde{\Psi}. \end{cases} \tag{3.8}$$

Then under some genericity condition we can find a coordinate transform

$$x(\tilde{x}) = (x_1(\tilde{x}_1, \tilde{x}_2), x_2(\tilde{x}_1, \tilde{x}_2)) \tag{3.9}$$

and a formal gauge transform of unknown functions

$$T(\tilde{x}, \eta) = \sum_{n=0}^{\infty} \eta^{-n} T_n(\tilde{x}) \qquad (T_n(\tilde{x}) : 3 \times 3 \text{ matrices}) \tag{3.10}$$

near $\tilde{x} = (0, 0)$ so that the following relation holds:

$$\tilde{\Psi}(\tilde{x}, \eta) = T(\tilde{x}, \eta)\Psi(x(\tilde{x}), \eta), \tag{3.11}$$

where $\Psi(x, \eta)$ and $\tilde{\Psi}(\tilde{x}, \eta)$ are unknown functions of the Pearcey system (3.2) and a completely integrable system (3.8), respectively.

§ 4. Main claim (conjecture)

Now a natural question we want to ask in this report is

Question. *What is the nonlinear analogue of Hirose's result ?*

Our answer to this question is the following

Claim (Conjecture) 1. *The fourth order PI equation* $(P_{\mathrm{I}})_2$ *gives the normal form at a coalescing point of nonlinear turning points for a higher order Painlevé equation.*

In what follows we explain some fundamental facts about the fourth order PI equation to state the above claim in a more specific manner.

It is shown in [19] that the fourth order PI equation $(P_{\mathrm{I}})_2$ can be obtained by restricting the most degenerate Garnier system $G(9/2; 2)$ of two variables

$$\eta^{-1}\frac{\partial \lambda_j}{\partial t_k} = \frac{\partial H_k}{\partial \mu_j}, \quad \eta^{-1}\frac{\partial \mu_j}{\partial t_k} = -\frac{\partial H_k}{\partial \lambda_j} \qquad (j, k = 1, 2) \tag{4.1}$$

with the Hamiltonian

$$H_1 = \frac{\mu_1^2 - \mu_2^2}{\lambda_1 - \lambda_2} - \frac{(\lambda_1^5 - \lambda_2^5) + t_2(\lambda_1^3 - \lambda_2^3) + t_1(\lambda_1^2 - \lambda_2^2)}{\lambda_1 - \lambda_2}, \tag{4.2}$$

$$\begin{aligned} H_2 = \frac{\lambda_1\mu_2^2 - \lambda_2\mu_1^2}{\lambda_1 - \lambda_2} - \eta^{-1}\frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2} \\ + \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}\left\{(\lambda_1^4 - \lambda_2^4) + t_2(\lambda_1^2 - \lambda_2^2) + t_1(\lambda_1 - \lambda_2)\right\} \end{aligned} \tag{4.3}$$

onto the complex line $\{t_2 = c\}$. As a matter of fact, introducing the symmetric variables (u_1, u_2, v_1, v_2) of $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ by

$$\begin{aligned} &u_1 = \lambda_1 + \lambda_2, \quad u_2 = -\lambda_1\lambda_2, \\ &v_1 = \frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2}, \quad v_2 = \frac{\lambda_1\mu_2 - \lambda_2\mu_1}{\lambda_1 - \lambda_2}, \end{aligned} \tag{4.4}$$

we find that (4.1) is expressed as

$$\begin{cases} \eta^{-1}\dfrac{\partial u_1}{\partial t_1} = 2v_1, \\ \eta^{-1}\dfrac{\partial u_2}{\partial t_1} = 2v_2, \\ \eta^{-1}\dfrac{\partial v_1}{\partial t_1} = 3u_1^2 + 2u_2 + t_2, \\ \eta^{-1}\dfrac{\partial v_2}{\partial t_1} = u_1^3 + 4u_1u_2 - v_1^2 + t_2u_1 + t_1, \end{cases} \tag{4.5}$$

$$\begin{cases} \eta^{-1}\dfrac{\partial u_1}{\partial t_2} = \dfrac{2}{3}v_2, \\ \eta^{-1}\dfrac{\partial u_2}{\partial t_2} = \dfrac{2}{3}(v_1u_2 - u_1v_2) - \dfrac{1}{3}\eta^{-1}, \\ \eta^{-1}\dfrac{\partial v_1}{\partial t_2} = \dfrac{1}{3}(u_1^3 + 4u_1u_2 - v_1^2 + t_2u_1 + t_1), \\ \eta^{-1}\dfrac{\partial v_2}{\partial t_2} = -\dfrac{1}{3}(u_1^4 + u_2u_1^2 - 2u_2^2 - u_1v_1^2 + t_2(u_1^2 - u_2) + t_1u_1), \end{cases} \tag{4.6}$$

and that the restriction of (4.1) onto $\{t_2 = c\}$ exactly coincides with the fourth order PI equation $(P_{\mathrm{I}})_2$ for $u = u_1$. Note that the first Hamiltonian H_1 of (4.1) gives the Hamiltonian structure for $(P_{\mathrm{I}})_2$.

For the Garnier system (4.1), or its symmetric form (4.5) and (4.6), there exists a 0-parameter solution

$$\begin{cases} u_j^{(0)} = u_j^{(0)}(t_1, t_2) = u_{j,0}^{(0)}(t_1, t_2) + \eta^{-1}u_{j,1}^{(0)}(t_1, t_2) + \cdots \\ v_j^{(0)} = v_j^{(0)}(t_1, t_2) = v_{j,0}^{(0)}(t_1, t_2) + \eta^{-1}v_{j,1}^{(0)}(t_1, t_2) + \cdots \end{cases} \quad (j = 1, 2), \tag{4.7}$$

where its top order term $(u_{1,0}^{(0)}, u_{2,0}^{(0)}, v_{1,0}^{(0)}, v_{2,0}^{(0)})$, abbreviated to $(\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2)$ in what follows, satisfies a system of algebraic equations

$$5\hat{u}_1^3 + t_2\hat{u}_1 - t_1 = 0, \quad 3\hat{u}_1^2 + 2\hat{u}_2 + t_2 = 0, \quad \hat{v}_1 = \hat{v}_2 = 0 \tag{4.8}$$

and its higher order terms $(u_{1,l}^{(0)}, u_{2,l}^{(0)}, v_{1,l}^{(0)}, v_{2,l}^{(0)})$ $(l \geq 1)$ are recursively determined. In particular, its first component $u_1^{(0)}$ gives a 0-parameter solution of the fourth order PI equation $(P_{\mathrm{I}})_2$.

For a system of nonlinear differential equations such as the Garnier system (4.1), combining Definitions 2.3 and 3.1, we can define a turning point and a Stokes surface in terms of this 0-parameter solution. That is, a turning point and a Stokes surface of (4.1) are, by definition, a turning point and a Stokes surface of the Fréchet derivative of (4.1) at a 0-parameter solution. Note that the Fréchet derivative is a system of linear differential equations and consequently its turning point and Stokes surface can be defined by applying Definition 3.1. Let $(\Delta P_{\mathrm{I}})_2$ denote the Fréchet derivative of (4.1) at the above 0-parameter solution. (It may be an abuse of notation but, we think, there will be no fear of confusions caused by this.) Then by straightforward computations we see that the characteristic equation of $(\Delta P_{\mathrm{I}})_2$ in the t_1 direction is

$$\nu_1^4 - 20\hat{u}_1\nu_1^2 + 16\left(6\hat{u}_1^2 - \hat{u}_2\right) = 0. \tag{4.9}$$

Hence zeros of the discriminant of (4.9) are given by

$$\begin{cases} 6\hat{u}_1^2 - \hat{u}_2 = 0 & (\text{“1st kind”}), \\ (10\hat{u}_1)^2 - 16\left(6\hat{u}_1^2 - \hat{u}_2\right) = 4\left(\hat{u}_1^2 + 4\hat{u}_2\right) = 0 & (\text{“2nd kind”}). \end{cases} \tag{4.10}$$

(When we consider the Stokes geometry of (4.1), zeros of $6\hat{u}_1^2 - \hat{u}_2$ and those of $\hat{u}_1^2 + 4\hat{u}_2$ have different characters. Thus, to distinguish them, we call a zero of the former equation and the corresponding turning point of (4.1) “1st kind” and a zero of the latter equation and the corresponding turning point of (4.1) “2nd kind”. See [12, Section 2] for details.) Similarly the characteristic equation of $(\Delta P_{\mathrm{I}})_2$ in the t_2 direction is

$$\nu_2^4 - \frac{4}{9}\hat{u}_1\left(2\hat{u}_1^2 + 3\hat{u}_2\right)\nu_2^2 + \frac{16}{81}\hat{u}_2^2\left(6\hat{u}_1^2 - \hat{u}_2\right) = 0 \tag{4.11}$$

and zeros of its discriminant are given by

$$\begin{cases} \hat{u}_2^2\left(6\hat{u}_1^2 - \hat{u}_2\right) = 0 & (\text{“1st kind”}), \\ \dfrac{4}{81}\left(2\hat{u}_1^2 - \hat{u}_2\right)^2\left(\hat{u}_1^2 + 4\hat{u}_2\right) = 0 & (\text{“2nd kind”}). \end{cases} \tag{4.12}$$

Hence the set of turning points of the most degenerate Garnier system (4.1) is explicitly described by

$$\begin{aligned} &6\hat{u}_1^2 - \hat{u}_2 = 0, \quad \text{i.e.,} \quad 135t_1^2 + 4t_2^3 = 0 \qquad (\text{“1st kind”}), \\ &\hat{u}_1^2 + 4\hat{u}_2 = 0, \quad \text{i.e.,} \quad 5t_1^2 + 2t_2^3 = 0 \qquad (\text{“2nd kind”}). \end{aligned} \tag{4.13}$$

Note that, since the turning points of the fourth order PI equation $(P_{\rm I})_2$ are defined by the characteristic equation of $(\Delta P_{\rm I})_2$ in the t_1 direction, they exactly coincide with those of the most degenerate Garnier system (4.1). Thus we now observe that the set of turning points of (4.1) or $(P_{\rm I})_2$ has a cuspidal singularity at the origin and a coalescing phenomenon of turning points occurs there. More precisely, two turning points (with different types) of the 1st kind and those of the 2nd kind simultaneously coalesce at the origin in this case (cf. Figure 3).

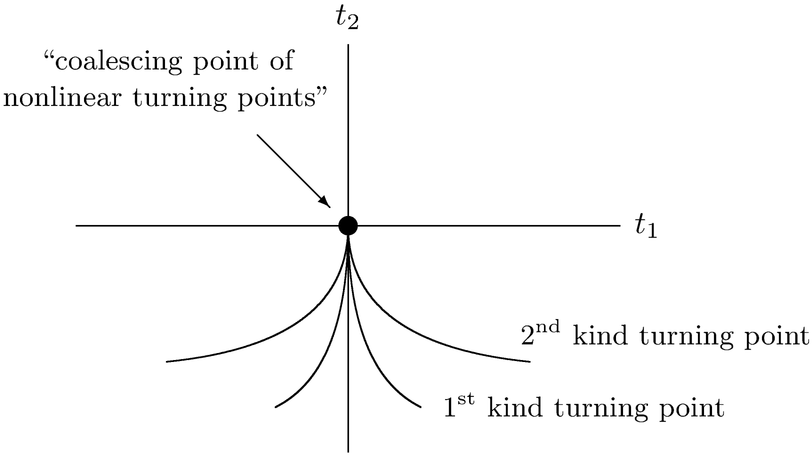

Figure 3. The set of turning points of the fourth order PI equation $(P_{\rm I})_2$

In a manner similar to the case of $(P_{\rm I})_2$ a higher order Painlevé equation can be in general extended to a completely integrable system of nonlinear differential equations (cf. [18, 19]) and we conjecture that the most degenerate Garnier system (4.1) corresponding to the fourth order PI equation $(P_{\rm I})_2$ gives the normal form at a coalescing point of nonlinear turning points in such a class of systems of nonlinear equations; this is a more precise statement of our conjecture.

§ 5. Toward the proof of the main claim

In this section, to show the validity of our conjecture, we discuss the transformation between a degenerate Garnier system with two variables corresponding to the fourth order PII equation and the most degenerate Garnier system (4.1).

Let us consider the following degenerate Garnier system with two variables:

$$\eta^{-1}\frac{\partial\tilde{\lambda}_j}{\partial\tilde{t}_k}=\frac{\partial\tilde{H}_k}{\partial\tilde{\mu}_j},\quad \eta^{-1}\frac{\partial\tilde{\mu}_j}{\partial\tilde{t}_k}=-\frac{\partial\tilde{H}_k}{\partial\tilde{\lambda}_j}\qquad (j,k=1,2) \tag{5.1}$$

with the Hamiltonian

$$\tilde{H}_1 = \frac{1}{2}\frac{\tilde{\mu}_1^2 - \tilde{\mu}_2^2}{\tilde{\lambda}_1 - \tilde{\lambda}_2} - \frac{\tilde{\lambda}_1^3\tilde{\mu}_1 - \tilde{\lambda}_2^3\tilde{\mu}_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2} - \tilde{t}_2\frac{\tilde{\lambda}_1\tilde{\mu}_1 - \tilde{\lambda}_2\tilde{\mu}_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2} - \frac{\tilde{t}_1}{2}\frac{\tilde{\mu}_1 - \tilde{\mu}_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2} - \alpha(\tilde{\lambda}_1 + \tilde{\lambda}_2) + \frac{1}{2}\tilde{t}_1\tilde{t}_2, \tag{5.2}$$

$$\tilde{H}_2 = \frac{1}{2}\frac{\tilde{\lambda}_1\tilde{\mu}_2^2 - \tilde{\lambda}_2\tilde{\mu}_1^2}{\tilde{\lambda}_1 - \tilde{\lambda}_2} + \tilde{\lambda}_1\tilde{\lambda}_2\frac{\tilde{\lambda}_1^2\tilde{\mu}_1 - \tilde{\lambda}_2^2\tilde{\mu}_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2} + \tilde{t}_2\tilde{\lambda}_1\tilde{\lambda}_2\frac{\tilde{\mu}_1 - \tilde{\mu}_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2} - \frac{\tilde{t}_1}{2}\frac{\tilde{\lambda}_1\tilde{\mu}_2 - \tilde{\lambda}_2\tilde{\mu}_1}{\tilde{\lambda}_1 - \tilde{\lambda}_2} - \frac{\eta^{-1}}{2}\frac{\tilde{\mu}_1 - \tilde{\mu}_2}{\tilde{\lambda}_1 - \tilde{\lambda}_2} - \alpha\tilde{\lambda}_1\tilde{\lambda}_2 + \frac{1}{8}\tilde{t}_1^2 + \frac{1}{2}\tilde{t}_2, \tag{5.3}$$

where $\alpha \in \mathbb{C}$ is a complex constant. The restriction of (5.1) onto the complex line $\{t_2 = c\}$ becomes the fourth order PII equation $(P_{\mathrm{II}})_2$.

In what follows we assume $\alpha \neq 0$. We then find that coalescence of nonlinear turning points for (5.1) occurs at four points determined by

$$9\tilde{t}_2^2 + 10\alpha = 0, \quad 135\tilde{t}_1^2 + 512\tilde{t}_2^3 = 0. \tag{5.4}$$

In this situation our conjecture can be stated in a specific manner as follows:

Claim (Conjecture) 2. *Let $(\tilde{t}_1^\dagger, \tilde{t}_2^\dagger)$ be a point satisfying* (5.4)*, that is, a coalescing point of nonlinear turning points for* (5.1)*. Then near $(\tilde{t}_1^\dagger, \tilde{t}_2^\dagger)$ there exist a formal coordinate transformation $(t_1, t_2) = (t_1(\tilde{t}_1, \tilde{t}_2, \eta), t_2(\tilde{t}_1, \tilde{t}_2, \eta))$ of the form*

$$t_1(\tilde{t}_1, \tilde{t}_2, \eta) = \sum_{n=0}^{\infty} \eta^{-n} t_{1,n}(\tilde{t}_1, \tilde{t}_2), \tag{5.5}$$

$$t_2(\tilde{t}_1, \tilde{t}_2, \eta) = \sum_{n=0}^{\infty} \eta^{-n} t_{2,n}(\tilde{t}_1, \tilde{t}_2), \tag{5.6}$$

and a formal transformation of unknown functions

$$x(\tilde{x}, \tilde{t}_1, \tilde{t}_2, \eta) = \sum_{n=0}^{\infty} \eta^{-n} x_n(\tilde{x}, \tilde{t}_1, \tilde{t}_2), \tag{5.7}$$

such that the following relations hold:

$$\lambda_j^{(0)}(t_1(\tilde{t}_1, \tilde{t}_2, \eta), t_2(\tilde{t}_1, \tilde{t}_2, \eta), \eta) = x(\tilde{\lambda}_j^{(0)}(\tilde{t}_1, \tilde{t}_2, \eta), \tilde{t}_1, \tilde{t}_2, \eta) \qquad (j = 1, 2), \tag{5.8}$$

where $\lambda_j^{(0)}(t_1, t_2, \eta)$ and $\tilde{\lambda}_j^{(0)}(\tilde{t}_1, \tilde{t}_2, \eta)$ denote (λ*-component of*) *the* 0*-parameter solution* (*i.e., the formal power series solution*) *of* (4.1) *and* (5.1)*, respectively.*

In a sense our conjecture can be thought of a generalization of a transformation to the second order PI equation (P_{I}) near a nonlinear simple turning point developed in [13], [14], [16] and [17], and our strategy for the proof of the conjecture follows the strategy employed in these previous works. That is, to construct the transformations (5.5)–(5.7), we make full use of the isomonodromic deformation theory for (4.1) and (5.1) (or $(P_{\mathrm{I}})_2$ and $(P_{\mathrm{II}})_2$). Otherwise stated, we use the Lax pairs of linear differential equations associated to (4.1) and (5.1).

For example, let us first consider the Lax pair for (4.1), which is given as follows:

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x}\Psi = A\Psi, \\ \eta^{-1}\dfrac{\partial}{\partial t_1}\Psi = B_1\Psi, \\ \eta^{-1}\dfrac{\partial}{\partial t_2}\Psi = B_2\Psi. \end{cases} \tag{5.9}$$

Here $A = A(x, t_1, t_2, \lambda, \mu, \eta)$ and $B_k = B_k(x, t_1, t_2, \lambda, \mu, \eta)$ $(k = 1, 2)$ are 2×2 matrices given by

(5.10)

$$A = \begin{pmatrix} v_1x + v_2 & 2(x^2 - u_1x - u_2) \\ \dfrac{1}{2}\left[x^3 + u_1x^2 + (t_2 + u_2 + u_1^2)x + (t_2u_1 + 2u_1u_2 + u_1^3 + t_1 - v_1^2)\right] & -(v_1x + v_2) \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 2 \\ \dfrac{1}{2}(x + 2u_1) & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \dfrac{v_1}{3} & \dfrac{2}{3}(x - u_1) \\ \dfrac{1}{6}\left[x^2 + u_1x + (t_2 + 2u_2 + u_1^2)\right] & -\dfrac{v_1}{3} \end{pmatrix},$$

where (u_1, u_2, v_1, v_2) are the symmetric variables of $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ introduced by (4.4). Note that the compatibility condition of (5.9) is equivalent to (4.1).

We now substitute a 0-parameter solution (4.7) of (4.5) and (4.6) into the coefficients A and B_k of the Lax pair (5.9). We then find the following

Proposition 5.1 ([12, Proposition 2.1.1]). *The first equation $\eta^{-1}\dfrac{\partial}{\partial x}\Psi = A\Psi$ of the Lax pair* (5.9) *has two double turning points at $x = \hat{\lambda}_j$ $(j = 1, 2)$ and one simple turning point at $x = -2(\hat{\lambda}_1 + \hat{\lambda}_2)$ (denoted by $\hat{a}$ in what follows), where $\hat{\lambda}_j$ denotes the top order term of a 0-parameter solution of* (4.1) *corresponding to* (4.7)*.*

Furthermore, as is also verified in [12], the Stokes geometry, i.e., turning points and Stokes surfaces, of the nonlinear system (4.1) has a close relationship with that of its Lax pair (5.9) to the following effect:

Proposition 5.2 ([12, Propositions 2.1.4 and 2.1.5]). (i) *At a turning point of the 1st kind* (*resp., the 2nd kind*) *of* (4.1) *two double turning points* (*resp., one double turning point and one simple turning point*) *of the first equation of* (5.9) *merge.*
(ii) *If* $t = (t_1, t_2)$ *lies on a Stokes surface of* (4.1) *emanating from a turning point of the 1st kind* (*resp., the 2nd kind*), *two double turning points* (*resp., one double turning point and one simple turning point*) *of the first equation of* (5.9) *are connected by its Stokes curve.*

Note that a key relation for the proof of Proposition 5.2 is the following integral relation.

$$\frac{1}{2}\int_{(0,0)}^{(t_1,t_2)} \left((\nu_{1,j}^{+} - \nu_{1,j}^{-})\, dt_1 + (\nu_{2,j}^{+} - \nu_{2,j}^{-})\, dt_2 \right) = \int_{\hat{a}}^{\hat{\lambda}_j} (\alpha^{+} - \alpha^{-})\, dx \quad (j = 1, 2), \tag{5.11}$$

where $\alpha^{\pm}$ denote the eigenvalues of the top order term of A and $\nu_{1,j}^{\pm}$ (resp., $\nu_{2,j}^{\pm}$) denote the characteristic roots of the Fréchet derivative of (4.5) (resp., (4.6)) at a 0-parameter solution (4.7). As a consequence of Proposition 5.2, at a coalescing point of nonlinear turning points of (4.1) or $(P_{\mathrm{I}})_2$, we can also observe the following

Proposition 5.3. *At the coalescing point* $(t_1, t_2) = (0, 0)$ *of nonlinear turning points of* $(P_{\mathrm{I}})_2$, *the three turning points* $x = \hat{\lambda}_1$, $x = \hat{\lambda}_2$ *and* $x = \hat{a}$ *of the first equation of* (5.9) *merge to one point.*

One important point is that these propositions also hold for the Garnier system (5.1) corresponding to $(P_{\mathrm{II}})_2$ (cf. [12]). Having this resemblance between the Stokes geometry of the Lax pair for $(P_{\mathrm{I}})_2$ and that for $(P_{\mathrm{II}})_2$ in mind, we now construct a formal coordinate transformation

$$\begin{cases} x = x(\tilde{x}, \tilde{t}_1, \tilde{t}_2, \eta) = \displaystyle\sum_{n=0}^{\infty} \eta^{-n} x_n(\tilde{x}, \tilde{t}_1, \tilde{t}_2), \\ t_1 = t_1(\tilde{t}_1, \tilde{t}_2, \eta) = \displaystyle\sum_{n=0}^{\infty} \eta^{-n} t_{1,n}(\tilde{t}_1, \tilde{t}_2), \\ t_2 = t_2(\tilde{t}_1, \tilde{t}_2, \eta) = \displaystyle\sum_{n=0}^{\infty} \eta^{-n} t_{2,n}(\tilde{t}_1, \tilde{t}_2) \end{cases} \tag{5.12}$$

that transforms the Lax pair associated to $(P_{\mathrm{II}})_2$ to (5.9), the Lax pair associated to $(P_{\mathrm{I}})_2$, in an open set $\tilde{\Omega}$ containing three turning points $\widehat{\tilde{\lambda}}_1$, $\widehat{\tilde{\lambda}}_2$ and $\widehat{\tilde{a}}$ of (the first equation of) the Lax pair associated to $(P_{\mathrm{II}})_2$ (cf. Figure 4). Then, in view of the relationship between the Stokes geometry of the Garnier system (4.1) or $(P_{\mathrm{I}})_2$ (as well as (5.1) or $(P_{\mathrm{II}})_2$) and that of its underlying Lax pair ensured by Proposition 5.2, we can expect that the transformation (5.12) thus constructed gives a transformation from $(P_{\mathrm{II}})_2$ to $(P_{\mathrm{I}})_2$ required in our claim (conjecture) 2.

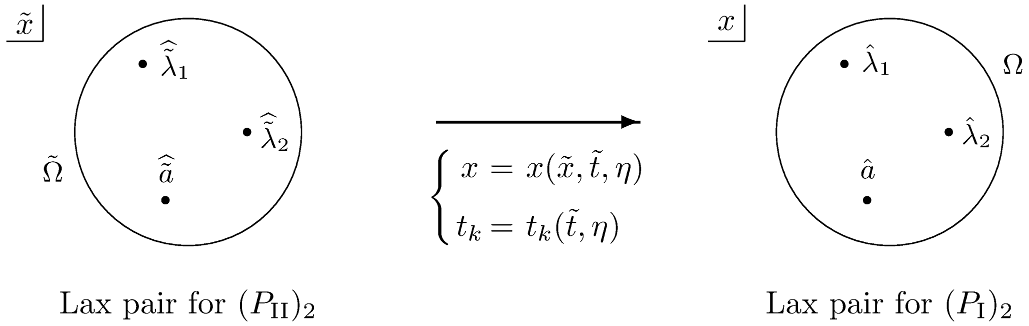

Figure 4. Formal transformation between two Lax pairs

This is our strategy to prove the conjecture. We hope we can provide a complete proof of the conjecture based on this strategy in our forthcoming paper.

§ 6. Discussion — relation with Dubrovin's result for the KdV equation

In this final section we discuss some relation between our conjecture and Dubrovin's result for the KdV equation.

In [7] Dubrovin showed the following intriguing result: Let

$$u(t, \epsilon) = u_0(t) + \epsilon^2 u_2(t) + \cdots \tag{6.1}$$

be a perturbative solution of the KdV equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\epsilon^2 \frac{\partial^3 u}{\partial x^3} = 0, \tag{6.2}$$

where ϵ (> 0) is a small parameter and α is a constant. The top order term $u_0(t)$ satisfies a first order nonlinear wave equation

$$\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} = 0. \tag{6.3}$$

It is well known that a shock wave occurs with a solution of (6.3) since the propagation speed of a solution u_0 of (6.3) depends on the modulus $|u_0|$ of u_0. Let (t_0, x_0) be a point where a shock occurs with a solution u_0 of (6.3). (Dubrovin called such a point "a point of gradient catastrophe"; cf. Figure 5.) Then at a point of gradient catastrophe Dubrovin showed the following

Theorem 6.1 (Dubrovin [7]). *Under some genericity condition the behavior of the perturbative solution* (6.1) *of the KdV equation near a point of gradient catastrophe is described by a* (*special*) *solution of* $(P_{\mathrm{I}})_2$.

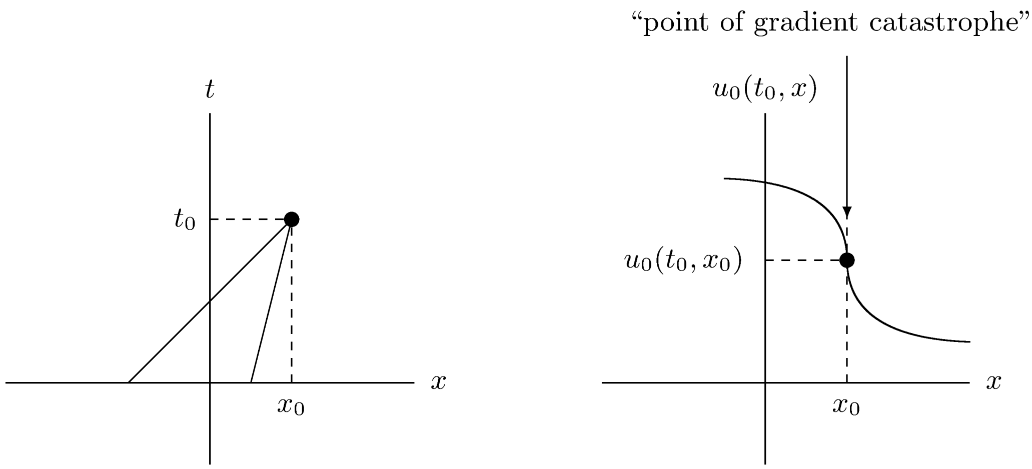

Figure 5. Wave propagation for (6.3) and appearance of gradient catastrophe

Note that the independent variables x and t of the KdV equation correspond to the independent variable t and the parameter c of $(P_{\mathrm{I}})_2$ (or the independent variables t_1 and t_2 of the most degenerate Garnier system (4.1)), respectively.

Remark. It is also shown in [7] that the above result holds universally for any Hamiltonian perturbations of the equation

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0. \tag{6.4}$$

In view of Theorem 6.1, a natural question arises: *Why does* $(P_{\mathrm{I}})_2$ *appear in the description of the behavior of solutions of the KdV equation near a point of gradient catastrophe ?* This question is one of the motivations of our study on the fourth order PI equation. Our tentative answer to this question is that some coalescing phenomenon of nonlinear turning points may be occurring at a point of gradient catastrophe of the KdV equation and consequently Dubrovin's result can be deduced from our conjecture. However, the definition of a turning point is not known yet for the KdV equation and at the present stage this is still just a guess. We hope this guess will be appropriately formulated and rigorously verified in some future.

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