RIMS Kôkyûroku Bessatsu **B6**
(2008), 129–137

Hölder continuity of harmonic functions associated to pseudo differential operators with negative definite symbol

By

Walter Hoh *

§ 1. Introduction

It is well-known that the generator of a Markov process in $\mathbb{R}^d$ admits a representation as a pseudo differential operator $-p(x, D)$ where

$$p(x,D)\varphi(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} p(x,\xi)\hat{\varphi}(\xi)\, đ\xi, \qquad \varphi \in C_0^\infty(\mathbb{R}^d) \tag{1.1}$$

and the symbol

$$p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$$

has the property that for each $x \in \mathbb{R}^d$

$$\xi \mapsto p(x,\xi)$$

is a continuous, negative definite function. This is equivalent to say that for every $x \in \mathbb{R}^d$ the symbol has the Lévy-Khinchin representation

$$p(x,\xi) = q(x,\xi) + ib(x)\cdot\xi + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{iy\cdot\xi} + i\frac{y\cdot\xi}{1+|y|^2}\right)\mu(x,dy), \tag{1.2}$$

where $q(x,\xi)$ is a nonnegative quadratic form w.r.t. ξ, $b(x)$ is a vector in $\mathbb{R}^d$ and $\mu(x,dy)$ is a kernel of Lévy measures having the standard property that

$$\int_{\mathbb{R}^d\setminus\{0\}} \frac{|y|^2}{1+|y|^2}\mu(x,dy) < \infty.$$

See [3] for further results on negative definite functions. We refer to [7] for a comprehending presentation on pseudo differential operators generating Markov processes.

2000 Mathematics Subject Classification(s):

*Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany

Besides the representation (1.1) the generator has an equivalent representation as a so-called Lévy-type operator

$$-p(x,D)\varphi(x) = L\varphi(x) + \int_{\mathbb{R}^d\setminus\{0\}} \left(\varphi(x+y) - \varphi(x) - \frac{y\cdot\nabla\varphi(x)}{1+|y|^2}\right)\mu(x,dy),$$

where L is a second order differential operator with coefficients given by $q(x,\xi)$ and $b(x)$ in (1.2), whereas the nonlocal integro-differential part is defined in terms of the Lévy kernel $\mu(x,dy)$ and governs the jumps of an associated process. In this article we shall focus on this nonlocal part only and assume the local part L to be absent.

The question of Hölder-continuity and Harnack inequality of harmonic functions for Lévy-type operators was first considered by Bass, Levin [2] who studied the case of Lévy-kernels which are comparable to the jump kernel of an α-stable process. As a simple consequence they in addition were able to prove Hölder continuity for solutions of the equation

$$\lambda f - Af = h$$

(A being the generator) given by the probabilistic resolvent

$$f(x) = E^x\left[\int_0^\infty e^{-\lambda t}h(X_t)\,dt\right].$$

Regularity results of this type are of interest in particular when dealing with the question of well-posedness of the corresponding martingale problem.

In [10] Song and Vondraček generalized the results of [2] by precisely specifying the assumptions that are need to make the argument of [2] work. Thereby they are able to treat some cases of Lévy processes with more general jumping mechanism but the α-stable case. Moreover, in [1] Bass and Kassmann considered the case of α-stable processes of variable order.

Whereas [2] and [10] are based on the Lévy-type representation of the generator, the purpose of this article is to emphasis the representation (1.1) as a pseudo differential operator. While in [10] conditions on the Lévy-kernel are required, we shall try to formulate our assumptions in terms of the symbol $p(x,\xi)$ itself. See also [9] where the representation (1.1) is used to derive in particular Harnack inequalities. As an outcome this approach will allow to prove Hölder-continuity in cases not covered by the results in [10], see section 3 for examples.

§ 2. Hölder-continuity

Throughout this paper we make the following assumptions: We consider a real-valued symbol

$$p : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$$

such that for fixed $x \in \mathbb{R}^d$ the mapping $\xi \mapsto p(x,\xi)$ is a continuous negative definite function. We assume that the corresponding pseudo differntial operator is of purely nonlocal type, i.e. the symbol has a Lévy-Khinchin representation

$$p(x,\xi) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(y \cdot \xi))\, \mu(x,dy) \tag{2.1}$$

with a Lévy-kernel $\mu(x,dy)$. Let

$$\psi : \mathbb{R}^d \mapsto \mathbb{R}$$

be a fixed continuous negative definite function. This function serves as a reference function for the symbol $p(x,\xi)$, i.e. we assume that there are constants $c, \tilde{c} > 0$ such that

$$c\psi(\xi) \leq p(x,\xi) \leq \tilde{c}\psi(\xi). \tag{2.2}$$

This means that the Lévy-process with characteristic exponent $\psi(\xi)$ should be regarded as a reference process for the process generated by $-p(x,D)$ in the same way as Brownian motion is the reference process for a general diffusion process.

Furthermore, we assume that there exists a conservative strong Markov process $((X_t)_{t\geq 0}, (P^x)_{x\in\mathbb{R}^d})$ associated to $-p(x,D)$ in the sense of the martingale problem: $(X_t)_{t\geq 0}$ is a càdlàg-process such that

$$P^x(X_0 = x) = 1$$

and for all $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\varphi(X_t) - \int_0^t (-p(x,D)\varphi)(X_s)\, ds, \qquad t \geq 0,$$

is a martingale under P^x w.r.t. the canonical filtration of $(X_t)_{t\geq 0}$. See also [4], [5], [6] for various results on the martingale problem for pseudo differential operators with negative definite symbol. Recall that by right-continuity of the paths it is allowed to enlarge the filtration to the right-continuous filtration.

A bounded measurable function h on $\mathbb{R}^d$ is called harmonic in a domain $D \subset \mathbb{R}^d$ w.r.t. $((X_t)_{t\geq 0}, (P^x)_{x\in\mathbb{R}^d})$ if for all open bounded sets $B \subset \overline{B} \subset D$

$$h(x) = E^x\left[h(X_{\tau_B}) \cdot 1_{\tau_B < \infty}\right] \qquad \forall x \in D$$

where τ_B is the first exit time from B. Moreover we consider for $\lambda > 0$ and f bounded measurable on $\mathbb{R}^d$ the associated probabilistic resolvent operator

$$R_\lambda f(x) = E^x\left[\int_0^\infty e^{-\lambda t} f(X_t)\, dt\right].$$

The following result is taken from [10]. In [10] a number of assumptions are made for the process which also guarantee a Harnack inequality for harmonic functions. Since we are only interested in Hölder-continuity not all of the assumptions are needed. More precisely the proofs of Theorems 4.9 and 4.11 in [10] yield (note that the argument is actually taken from [2]):

Theorem 2.1. *Assume that the following conditions hold:*

(i) There is a constant $c_1 \geq 0$ and $\alpha > 0$ such that for $x \in \mathbb{R}^d$ and $0 < 2r < s < 1$

$$\textit{(C1)} \qquad P^x\left(X_{\tau_{B(x,r)}} \notin B(x,s)\right) \leq c_1 \frac{r^\alpha}{s^\alpha}.$$

(ii) There is a constant $c_2 > 0$ such that for $0 < r < 1$ and $A \subset B(x,r)$ we have

$$\textit{(C2)} \qquad P^x\left(T_A < \tau_{B(x,3r)}\right) \geq c_2 \frac{|A|}{|B(x,r)|}$$

where T_A denotes the first entrance time into A and $|A|$, $|B(x,r)|$ are the Lebesgue measures of A and $B(x,r)$ respectively.

Then the following holds true:
If h is harmonic in a ball $B(x_0,r)$, $r < 1$, then h is Hölder-continuous in $B(x_0, r/2)$.
If f is bounded and measurable on $\mathbb{R}^d$, then $R_\lambda f$ is locally Hölder-continuous.

Our goal is to find conditions on the symbol that imply (C1) and (C2). We start with condition (C1). The following object turns out to be an important notion. Recall that ψ is the fixed reference function for the symbol. Let

$$A_\psi(r) := \sup_{|\eta| \leq \frac{1}{r}} \psi(\eta), \qquad r > 0.$$

For $r \to 0$ the function $A_\psi(r)$ measures the maximal growth of ψ at infinity.

Proposition 2.1. *Under the condition (2.2) we have*

$$P^x\left(X_{\tau_{B(x,r)}} \notin B(x,s)\right) \leq C \cdot \frac{A_\psi(s)}{A_\psi(\kappa r)}, \qquad 0 < 2r < s < 1,$$

where κ is an intrinsic constant ($\kappa = \frac{1}{2\pi}$ will do).

Proof. For simplicity we assume $x = 0$ and let $\tau = \tau_{B(0,r)}$. Define a cut-off function $\varphi_{r,s} \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \varphi_{r,s} \leq 1$ and

$$\varphi_{r,s}(x) = \begin{cases} 1 & |x| \leq r + \frac{s}{2} \\ 0 & |x| \geq s. \end{cases}$$

Then for all $y \in B(0,r)$ we have $1-\varphi_{r,s}(y+z)=0$ for $|z|\le \frac{s}{2}$. and hence we find a constant C independent of r and s such that

$$\sup_{y\in B(0,r)}(1-\varphi_{r,s}(y+z))\le C\left[1-\exp\left(-\left|\frac{z}{s}\right|^2\right)\right],\qquad z\in\mathbb{R}^d. \tag{2.3}$$

Since $(X_t)_{t\ge 0}$ solves the martingale problem

$$M_t := \varphi_{r,s}(X_t)-1-\int_0^t(-p(x,D)\varphi_{r,s})(X_u)\,du,\qquad t\ge 0,$$

is a martingale under P^0 with expectation $E^0[M_t]=0$. Thus we find by optional sampling, the Lévy-type representation, (2.3), (2.1) and the upper estimate in (2.2)

$$\begin{aligned}
&P^0(X_\tau\notin B(0,s))\le E^0(1-\varphi_{r,s}(X_\tau))=E^0\left[\int_0^\tau p(x,D)\varphi_{r,s}(X_u)\,du\right]\\
&\le E^0[\tau]\cdot\sup_{y\in B(0,r)}p(x,D)\varphi_{r,s}(y)\\
&=E^0[\tau]\cdot\sup_{y\in B(0,r)}\int_{\mathbb{R}^d\setminus\{0\}}(1-\varphi_{r,s}(y+z))\mu(y,dz)\\
&\le C\cdot E^0[\tau]\cdot\sup_{y\in B(0,r)}\int_{\mathbb{R}^d\setminus\{0\}}\left[1-\exp(-\left|\frac{z}{s}\right|^2)\right]\mu(y,dz)\\
&=C\cdot E^0[\tau]\cdot\sup_{y\in B(0,r)}\int_{\mathbb{R}^d\setminus\{0\}}\int_{\mathbb{R}^d}(1-\cos(z\cdot\xi))\cdot\left[1-\exp(-\left|\frac{\cdot}{s}\right|^2)\right]^\wedge(\xi)\,đ\xi\,\mu(y,dz)\\
&=C\cdot E^0[\tau]\cdot\sup_{y\in B(0,r)}\int_{\mathbb{R}^d}p(y,\xi)\cdot\left[\exp(-\left|\frac{\cdot}{s}\right|^2)\right]^\wedge(\xi)\,đ\xi\\
&\le C'\cdot E^0[\tau]\cdot\int_{\mathbb{R}^d}\psi(\xi)\cdot s^d\exp(-|s\xi|^2)\,đ\xi\\
&\le C'\cdot E^0[\tau]\cdot\int_{\mathbb{R}^d}\psi(\frac{\xi}{s})\cdot\exp(-|\xi|^2)\,đ\xi.
\end{aligned}$$

Next note that the square root of a negative definite function is subadditive which implies, see [3], Cor. 7.16 and its proof or [6], Theo. 2.7,

$$\psi(\frac{\xi}{s})\le 2\sup_{|\eta|\le\frac{1}{s}}\psi(\eta)\cdot(1+|\xi|^2)=2\cdot A_\psi(s)\cdot(1+|\xi|^2)$$

which gives

$$P^0(X_\tau\notin B(0,s))\le c\cdot E^0[\tau]\cdot A_\psi(s)$$

On the other hand it is known from [8], Theo. 4.7, that

$$E^0[\tau]\le\frac{C}{\inf_{|y|\le r}\sup_{|\xi|\le\frac{1}{\kappa r}}p(y,\xi)}\le C\cdot\frac{1}{A_\psi(\kappa r)},$$

where we used the lower bound in (2.2). Combining the last two estimates gives the result. □

From Proposition 2.1 it is obvious that the following is an appropriate assumption: There is $\alpha > 0$ such that

$$\textbf{(A1)} \qquad A_\psi(r) \sim r^{-\alpha} \qquad \text{for small } r.$$

Corollary 2.1. *Condition (A1) implies condition (C1).*

Remark: Condition (A1) means that for large values of R

$$\sup_{B(0,R)} \psi \sim R^\alpha.$$

Therefore it is sufficient that $\psi(\xi)$ possesses a maximal growth behaviour like $|\xi|^\alpha$ in a single direction, whereas in other space directions different behaviour not necessarily of power type is admissible. In this way many more general jumping mechanisms apart from the α-stable case are covered by condition (A1).

Let us now turn to condition (C2). Since here the Lebesgue measure of A is involved, possible singularities of the jump measure $\mu(x, dy)$ cause difficulties. Thus we now will assume $\mu(x, dy)$ to be absolutely continuous w.r.t Lebesgue measure and a lower bound on the density. Therefore let now

$$\mu(x, dy) = n(x, y) \cdot dy$$

and define

$$N(r) := \inf_{x \in \mathbb{R}^d, |y| \le r} n(x, y). \tag{2.4}$$

$N(r)$ is again related to A_ψ:

Lemma 2.1.

$$r^d \cdot N(4r) \le c \cdot A_\psi(r).$$

Proof. We have

$$\begin{aligned}
&r^d \cdot N(4r) = c \int_r^{4r} N(4r)\, dy \le c \int_r^{4r} n(x, y)\, dy \\
&\le c \int_{\mathbb{R}^d \setminus \{0\}} \left[1 - \exp(-\left|\frac{y}{r}\right|^2)\right] n(x, y)\, dy \\
&= c \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d} (1 - \cos(y \cdot \xi)) \cdot \left[1 - \exp(-\left|\frac{\cdot}{r}\right|^2)\right]\hat{}(\xi)\, d\xi\, \mu(x, dy) \\
&= c \int_{\mathbb{R}^d} p(x, \xi) \left[1 - \exp(-\left|\frac{\cdot}{r}\right|^2)\right]\hat{}(\xi)\, d\xi \le \int_{\mathbb{R}^d} \psi(\xi) \left[1 - \exp(-\left|\frac{\cdot}{r}\right|^2)\right]\hat{}(\xi)\, d\xi \\
&\le c \cdot A_\psi(r)
\end{aligned}$$

by the same argument as in the proof of Prop. 2.1. □

We next apply an argument taken fron [10].

Proposition 2.2. *There is a constant $c > 0$ such that for all $r \in (0;1)$, $x \in \mathbb{R}^d$ and $A \subset B(x,r)$ we have*

$$P^x\left(T_A < \tau_{B(x,3r)}\right) \geq c|A| \cdot \frac{N(4r)}{A_\psi(r)}.$$

Proof. We simply have to repeat the argument in the proof of [10], Lemma 3.4 step by step, but replace the function $\Phi(r)$ in there by $A_\psi(r)$. Then the estimate in the beginning of the proof follows from Lemma 2.1 and furthermore the estimate taken from [10], Lemma 3.1 has to be replaced by

$$P^x(\tau_{B(x,r)} \leq t) \leq c \cdot A_\psi(t) \cdot t$$

which is again a consequence of [8], Lemma 4.1, and (2.2). □

For the validity of condition (C2) it is therefore reasonable to assume for some $c > 0$:

$$\textbf{(A2)} \qquad \frac{N(4r)}{A_\psi(r)} \geq \frac{c}{r^d}$$

and we immediately obtain

Proposition 2.3. *The condition (A2) implies (C2).*

The final results therefore reads as follows:

Theorem 2.2. *Let the Lévy-kernel $\mu(x,dy)$ have a density w.r.t. Lebesgue measure and $N(r)$ is given by (2.4). Assume that the conditions (A1) and (A2) are satisfied. Then:*
If h is harmonic in a ball $B(x_0,r)$, $r < 1$, then h is Hölder-continuous in $B(x_0,r/2)$.
If f is bounded and measurable on $\mathbb{R}^d$, then $R_\lambda f$ is locally Hölder-continuous.

§ 3. Example

As a starting point we can take symbols that do not depend on x, so they are just continuous negative definite functions $\psi(\xi)$. In this case examples satisfying (A1) and (A2) can be found in [10], Examples 3.6, where the Lévy case is considered. Among others these examples include the α-stable case as well as fractional powers of the relativistic Hamiltonian.

Now assume we are given continuous negative definite functions $\psi_1, \dots, \psi_k$ all satisfying conditions (A1) and (A2) and consider the variable coefficient case, i.e a symbol

$$p(x,\xi) = \sum_{j=1}^{k} b_j(x) \cdot \psi_j(\xi),$$

where $b_j : \mathbb{R}^d \to \mathbb{R}^+$ are coefficient function which are bounded from below and above by positive constants. Then clearly

$$\psi(\xi) := \psi_1(\xi) + \ldots + \psi_k(\xi)$$

is a suitable reference function for $p(x,\xi)$ such that (2.2) holds. We want to verify that also conditions (A1) and (A2) remain valid:

Condition (A1) is almost immediate, since the growth of ψ at infinity is given by the maximal growth of the functions ψ_j.

Concerning (A2) observe that the Lévy kernel corresponding to $p(x,\xi)$ is the sum of the Lévy-kernels corresponding to $b_j(x) \cdot \psi_j(\xi)$ and therefore, since each ψ_j satisfies (A2),

$$\begin{aligned} N(4r) &\geq \frac{c}{r^d} \max_j A_{\psi_j}(r) \geq \frac{1}{k} \frac{c}{r^d} \sum_{j=1}^{k} A_{\psi_j}(r) \\ &\geq \frac{c}{k} \frac{1}{r^d} A_\psi(r), \end{aligned}$$

where the last inequality follows from the subadditivity of the sup in the definition of A_ψ. Thus also (A2) holds true.

References

[1] R.F. Bass, M. Kassmann, Harnack inequalities for non-local operators of variable order. Trans.Amer.Math.Soc. 357, 837–850, 2004

[2] R.F. Bass, D.A. Levin, Harnack inequalties for jump processes. Potential Analysis 17, 375–388, 2002.

[3] C.Berg, G. Forst, Potential theory on locally compact Abelian groups. Ergebnisse der Mathematik und ihrer Grenzgebiete, II. Serie, Bd. 87, Springer Verlag (1975)

[4] W. Hoh, The martingale problem for a class of pseudo differential operators. Math.Ann. 300 (1994) 121–147.

[5] W. Hoh, Pseudo differential operators with negative definite symbols and the martingale problem. Stoch. and Stoch.Rep. 55 (1995) 225–252.

[6] W. Hoh, Pseudo differential operators generating Markov processes. Habilitationsschrift, Bielefeld 1998.

[7] N. Jacob Pseudo-Differential Operators and Markov Processes. Vol. 1–3, Imperial College Press, London (2001) – (2005)

[8] R.L. Schilling, Growth and Hölder conditions for sample paths of Feller processes. Probab. Theory Relat. Fields 112, 565-611 (1998)

[9] R.L. Schilling, T. Uemura, On the Feller property of Dirichlet forms generated by pseudo differential operators. To appear in Tohoku Math. J.

[10] R. Song, Z. Vondraček. Harnack inequality for some classes of Markov processes. Math.Z. 246, 177–202 (2004).