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On the Laplace-type asymptotics and the stochastic Taylor expansion for Itô functionals of Brownian rough paths

By

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Abstract

In this article we will give a survey on the Laplace-type asymptotics for the laws of solutions of formal Stratonovich-type stochastic differential equations in Banach spaces. The rigorous meaning of the solutions is given by the rough path theory. The key of the proof is the (stochastic) Taylor expansion, which is, in fact, deterministic in this context. The main example we have in mind is the Brownian motion over loop groups.

§ 1. Introduction

Let (X, H, μ) be an abstract Wiener space, i.e., X is a real separable Banach space, H is the Cameron-Martin space and μ is the Wiener measure on X. Let Y be another real separable Banach space and $w := (w_t)_{0\le t\le 1}$ be the X-valued Brownian motion on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ associated with μ. We denote by $L(X, Y)$ the space of bounded linear operators from X to Y. In this paper, we consider a class of Y-valued Wiener functionals $X^\varepsilon := (X_t^\varepsilon)_{0\le t\le 1}$ defined through the following formal Stratonovich type stochastic differential equation (SDE) on Y:

$$dX_t^\varepsilon = \sigma(X_t^\varepsilon) \circ \varepsilon dw_t + b(\varepsilon, X_t^\varepsilon)dt, \qquad X_0^\varepsilon = 0, \tag{1.1}$$

where the coefficients σ and b take values in $L(X, Y)$ and Y, respectively, with a suitable regularity condition. Here, we note that the equation (1.1) cannot be interpreted through the usual theory of SDEs when X and Y are infinite dimensional Banach

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spaces, because the diffusion coefficient σ takes values in $L(X, Y)$. The main results in this paper are taken from Inahama and Kawabi [11] (also from [9, 10, 12]). See Section 2 for the precise formulation of our Wiener functionals X^ε. The main objective of this paper is to discuss the asymptotic behavior of the Laplace type functional integral $\mathbb{E}\big[G(X^\varepsilon)\exp(-F(X^\varepsilon)/\varepsilon^2)\big]$ as $\varepsilon \searrow 0$, which is called Laplace's method. In this paper, we interpret our Wiener functionals X^ε as Itô functionals of Brownian rough paths, and show that these asymptotics hold for wider classes of (infinite dimensional) Banach space-valued Wiener functionals by using the fact that the rough path theory of T. Lyons works on any Banach space.

The main example we have in mind is the loop group-valued Brownian motion, which was introduced by Malliavin [18]. Although this process is rather simple, it is not easy to prove the Laplace type asymptotics. The main reason is, perhaps, lack of nice SDE theory on continuous loop spaces with the sup-norm. However, the rough path theory provides a nice "SDE" on loop spaces and we can treat this process as a Itô functional of Brownian rough paths. See Section 4 below.

To establish the large deviation principle for X^ε, due to the lack of the continuity of the Itô map $w \mapsto X^\varepsilon$, Schilder's theorem and the contraction principle may not be used directly. To overcome this difficulty, Freidlin and Wentzell developed refined techniques involving the exponential continuity. On the other hand, recently, Ledoux, Qian, and Zhang [16] gave a new proof for the large deviation principle by using the rough path theory. The basic idea in [16] is summarized as follows: First, they showed that the laws of Brownian rough paths satisfy the large deviation principle. Next, they used the contraction principle since the Itô map is continuous in the framework of the rough path theory. Hence their approach seems straightforward and much simpler than conventional proofs. In [9], it is shown that their approach is also applicable to stochastic processes on infinite dimensional spaces (including X^ε above).

As an application of the large deviation principle, Laplace's method is investigated in many research fields of probability theory and mathematical physics. In finite dimensional settings, Schilder [21] initiated the study in the case of $X^\varepsilon = \varepsilon w$. The problem of [21] is easier because each term of the expansion is continuous, which comes from the fact that X^ε is nothing but the scaled Brownian motion. Azencott [3] and Ben Arous [4] continued this study for the solution of (1.1). (For results concerning with more general Wiener functionals, see Kusuoka and Stroock [13, 14] and Takanobu and Watanabe [22].) In these papers, the stochastic Taylor expansion for X^ε plays an important role. However, things are not very simple since each term of this expansion is not a continuous Wiener functional in the sense of conventional stochastic analysis.

On the other hand, Aida [1, 2] proposed a new approach with the rough path theory for this problem recently and recovered the results in [3, 4]. In [2], he obtained the stochastic Taylor expansion with respect to the topology of the space of geometric rough paths for finite dimensional cases. Since the Itô map is continuous in the rough path sense, each term of the expansion is also continuous. Hence, his method is quite straightforward. The authors extended this method to the infinite dimensional setting in [12, 11, 10], which is the main content of this paper.

The organization of this paper is as follows: In Section 2, we give the precise formulation of our problem and give the main result of the paper. In Section 3, we give a brief explanation of the (stochastic) Taylor expansion, which is the key to prove the main result. In Section 4, we give an example and explain the advantage of the rough path theory in analysis of this example.

§ 2. Setting and the Main Results

In this section, we set notations, introduce our Wiener functionals through the Itô map in the rough path sense and state our results.

Let (X, H, μ) be an abstract Wiener space and Y be a real Banach space. The Brownian motion on X is denoted by $(w_t)_{t\geq 0}$. We equip the tensor products of Banach spaces with the projective norm, which is the strongest among natural tensor norms.

Let $p \in (2, 3)$ be the roughness and let $G\Omega_p(X)$ be the space of geometric rough path over X. In this paper, the time interval is always $[0, 1]$. Let $P(X)$ be the Banach space of continuous path in X which starts at $0 \in X$. Let $\mathrm{BV}(X)$ be the Banach space of all continuous and bounded variational paths in X, which start at $0 \in X$. For $\gamma \in \mathrm{BV}(X)$, an geometric rough path $\overline{\gamma} = (1, \overline{\gamma}_1, \overline{\gamma}_2) \in G\Omega_p(X)$ is naturally defined by

$$\overline{\gamma}_1(s,t) = \gamma_t - \gamma_s, \qquad \overline{\gamma}_2(s,t) = \int_s^t (\gamma_u - \gamma_s) \otimes d\gamma_u, \qquad (0 \leq s \leq t \leq 1).$$

$\overline{\gamma}$ is called a rough path lying above γ. In this way, $\mathrm{BV}(X)$ is continuously and densely imbedded in $G\Omega_p(X)$. Note that the Cameron-Martin space $\mathcal{H}$ of the X-valued Brownian motion w is continuously imbedded in $\mathrm{BV}(X)$. For details of the rough path theory, see Lyons and Qian [17]. (Notations in this paper are mainly taken from [9].)

Under the *exactness condition* **(EX):** below, the Brownian rough paths $\overline{w}$ exist as the almost sure limit in $G\Omega_p(X)$ of the rough paths lying above the dyadic piecewise linear approximations of w. (See Ledoux, Lyons, and Qian [15].) The law of $\overline{w}$ is a probability measure on $G\Omega_p(X)$. Note that **(EX)** holds with $\alpha = 1/2$ if $\dim(X) < \infty$.

(EX): We say that the Gaussian measure μ and the projective norm on $X \otimes X$ satisfies the *exactness condition* if there exist $C > 0$ and $1/2 \leq \alpha < 1$ such that, for all $n = 1, 2, \ldots,$

$$\mathbb{E}\Big[\Big|\sum_{i=1}^{n} \eta_{2i-1} \otimes \eta_{2i}\Big|\Big] \leq C n^{\alpha}.$$

Here, $\{\eta_i\}_{i=1}^{\infty}$ are an i.i.d. on X such that the law of η_i is μ.

We set notations for coefficients. Let $\sigma \in C_b^{\infty}(Y, L(X,Y))$ and $b_1, \ldots, b_N \in C_b^{\infty}(Y,Y)$, $N \in \mathbb{N}$. Here, $L(X,Y)$ denotes the Banach space of all bounded linear maps from X to Y. For $k \in \mathbb{N}$, $\nabla^k \sigma$ and $\nabla^k b$ are maps from Y to $L^k(Y, \ldots, Y; L(X,Y))$ and $L^k(Y, \ldots, Y; Y)$, respectively. Here, ∇ denotes the Fréchet derivative on Y. We set $\tilde{X} := X \oplus \mathbb{R}^N$ and define $\tilde{\sigma} \in C_b^{\infty}(Y, L(\tilde{X}, Y))$ by

$$\tilde{\sigma}(y)\big[(x,u)\big]_{\tilde{X}} := \sigma(y)x + \sum_{i=1}^{N} b_i(y)u_i, \quad y \in Y, x \in X, u = (u_1, \ldots, u_N) \in \mathbb{R}^N.$$

Next, we consider the following differential equation in the rough path sense:

$$dy_t = \tilde{\sigma}(y_t)d\tilde{x}_t \quad \text{with } y_0 = 0. \tag{2.1}$$

Then for any $\overline{\tilde{x}} \in G\Omega_p(\tilde{X})$, there exists a unique solution $\overline{z} \in G\Omega_p(\tilde{X} \oplus Y)$ in the rough path sense. Note that the natural projection of $\overline{z}$ onto the first component is $\overline{\tilde{x}}$. Projection of $\overline{z}$ onto the second component is denoted by $\overline{y} \in G\Omega_p(Y)$ and we write $\overline{y} = \Phi(\overline{\tilde{x}})$ and call it a solution of (2.1). The map $\Phi : G\Omega_p(\tilde{X}) \to G\Omega_p(Y)$ is called the Itô map and is locally Lipschitz continuous in the sense of Lyons and Qian. See Theorem 6.2.2 in [17] for details. If $\tilde{x}_t = (\gamma_t, \lambda_t^{(1)}, \ldots, \lambda_t^{(N)})$ is a $\tilde{X}$-valued continuous path of finite variation, the map $t \mapsto \Phi(\overline{\tilde{x}})_1(0,t)$ is the solution of

$$dy_t = \sigma(y_t)d\gamma_t + \sum_{i=1}^{N} b_i(y_t)d\lambda_t^{(i)} \quad \text{with } y_0 = 0$$

in the usual sense and $\overline{z}$ is the smooth rough path lying above $(\tilde{x}, \Phi(\overline{\tilde{x}})_1(0,\,\cdot\,))$.

For $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)}) \in \mathrm{BV}(\mathbb{R}^N)$ and $\overline{x} \in G\Omega_p(X)$, we set $\iota(\overline{x}, \lambda) \in G\Omega_p(\tilde{X})$ by $\iota(\overline{x}, \lambda)_1(s,t) = (\overline{x}_1(s,t), \lambda_t - \lambda_s)$ and

$$\begin{aligned} &\iota(\overline{x}, \lambda)_2(s,t) \\ &= \Big(\overline{x}_2(s,t), \int_s^t \overline{x}_1(s,u) \otimes d\lambda_u, \int_s^t (\lambda_u - \lambda_s) \otimes \overline{x}_1(s,du), \int_s^t (\lambda_u - \lambda_s) \otimes d\lambda_u\Big). \end{aligned}$$

Here the second and the third component are Young integrals. If $\overline{h}$ is a smooth rough path lying above $h \in \mathrm{BV}(X)$, then $\iota(\overline{h}, \lambda)$ is a smooth rough path lying above $(h, \lambda) \in \mathrm{BV}(\tilde{X})$. Note that the map $\iota : G\Omega_p(X) \times \mathrm{BV}(\mathbb{R}^N) \to G\Omega_p(\tilde{X})$ is continuous.

For $\varepsilon \in [0,1]$, we define $\lambda^{\varepsilon} \in \mathrm{BV}(\mathbb{R}^N)$ by $\lambda^{\varepsilon}(t) := (a_1(\varepsilon)t, \ldots, a_N(\varepsilon)t)$, where $a = (a_1, \ldots, a_N) : [0,1] \to \mathbb{R}^N$ is a $\mathbb{R}^N$-valued smooth curve. In what follows, we usually use the notation

$$a^j(\varepsilon) \cdot \nabla^k b(y) := \sum_{i=1}^{N} \frac{d^j}{d\varepsilon^j} a_i(\varepsilon) \nabla^k b_i(y), \qquad j,k \in \mathbb{N} \cup \{0\}.$$

Next, we regard the Itô map defined above as a map from $\mathrm{BV}(X)$ to $\mathrm{BV}(Y)$. We define $\Psi_\varepsilon : \mathrm{BV}(X) \to \mathrm{BV}(Y)$ by $\Psi_\varepsilon(h)_t := \Phi\big(\iota(\overline{h}, \lambda^\varepsilon)\big)_1(0,t)$ for $0 \le t \le 1$. That is, $y := \Psi_\varepsilon(h)$ is the unique solution of the ordinary differential equation

$$dy_t = \sigma(y_t)dh_t + a(\varepsilon) \cdot b(y_t)dt \quad \text{with } y_0 = 0. \tag{2.2}$$

For the X-valued Brownian motion w, let $\overline{w}$ be the Brownian rough path over X. For $\varepsilon \in [0,1]$, we define a Wiener functional $X^\varepsilon \in P(Y)$ by

$$X^\varepsilon_t := \Phi\big(\iota(\varepsilon\overline{w}, \lambda^\varepsilon)\big)_1(0,t), \qquad 0 \le t \le 1.$$

Note that, in the finite dimensional case, $(X^\varepsilon_t)_{t\ge 0}$ is the solution of Stratonovich-type stochastic differential equation. We investigate the asymptotic behavior of the law of X^ε as $\varepsilon \searrow 0$.

In the finite dimensional case, the large deviation principle for X^ε is called Freidlin-Wentzell type and has been studied for a long time. Recently, Ledoux, Qian, and Zhang [16] gave a new proof by using the rough path theory. Their method is as follows: First they showed the large deviation principle for Brownian rough paths. Then, since the Itô map is continuous is the rough path theory, they can use the contraction principle to obtain the large deviation principle of Freidlin-Wentzell type.

In [9] the authors extended Ledoux, Qian, and Zhang's method to the infinite dimensional case under the exactness condition to give a new proof for Fang and Zhang's large deviation for Brownian motion on loop groups (See Fang and Zhang [8] and Section 4 below). The following is essentially Theorem 4.9 of Inahama and Kawabi [9].

Theorem 2.1. *For $\varepsilon > 0$, we denote by $\mathcal{V}_\varepsilon$ the law of the process X^ε. Then, $\{\mathcal{V}_\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle as $\varepsilon \searrow 0$ with the good rate function Λ, where*

$$\Lambda(\phi) = \begin{cases} \frac{1}{2}\inf\big\{\|\gamma\|^2_{\mathcal{H}} \mid \phi = \Psi_0(\gamma)\big\}, & \text{if } \phi = \Psi_0(\gamma) \text{ for some } \gamma \in \mathcal{H}, \\ \infty, & \text{otherwise.} \end{cases}$$

More precisely, for any measurable set $K \subset P(Y)$, it holds that

$$-\inf_{\phi\in K^\circ} \Lambda(\phi) \le \liminf_{\varepsilon\searrow 0} \varepsilon^2 \log \mathcal{V}_\varepsilon(K) \le \limsup_{\varepsilon\searrow 0} \varepsilon^2 \log \mathcal{V}_\varepsilon(K) \le -\inf_{\phi\in\overline{K}} \Lambda(\phi).$$

As a consequence of Theorem 2.1, we have the following asymptotics for every bounded continuous function F on $P(Y)$:

$$\lim_{\varepsilon\searrow 0} \varepsilon^2 \log \mathbb{E}\big[\exp\big(-F(X^\varepsilon)/\varepsilon^2\big)\big] = -\inf\big\{F(\phi) + \Lambda(\phi) \mid \phi \in P(Y)\big\}.$$

This is Varadhan's integral lemma. Our next concern is to investigate the more precise asymptotics of a generalization of the integral on the left-hand side of above equality.

That is, we aim to establish the asymptotic expansions of the integral $\mathbb{E}\big[G(X^\varepsilon)\exp\big(-F(X^\varepsilon)/\varepsilon^2\big)\big]$ as $\varepsilon \searrow 0$.

In this paper, we impose the following conditions on the functions F and G. In what follows, we especially denote by D the Fréchet derivatives on $\mathrm{BV}(X)$ and $P(Y)$.

(H1): F and G are real-valued bounded continuous functions defined on $P(Y)$.

(H2): The function $F_\Lambda := F \circ \Psi_0 + \|\cdot\|^2_{\mathcal{H}}/2$ defined on $\mathcal{H}$ attains its minimum at a unique point $\gamma \in \mathcal{H}$. For this γ, we write $\phi := \Psi_0(\gamma)$.

(H3): The functions F and G are $n+3$ and $n+1$ times Fréchet differentiable on a neighborhood $B(\phi)$ of $\phi \in P(Y)$, respectively. Moreover there exist positive constants $M_1, \dots, M_{n+3}$ such that

$$\begin{aligned} \big|D^k F(\eta)\big[y,\dots,y\big]\big| &\le M_k \|y\|^k_{P(Y)}, \quad k = 1,\dots,n+3, \\ \big|D^k G(\eta)\big[y,\dots,y\big]\big| &\le M_k \|y\|^k_{P(Y)}, \quad k = 1,\dots,n+1, \end{aligned}$$

hold for any $\eta \in B(\phi)$ and $y \in P(Y)$.

(H4): At the point $\gamma \in \mathcal{H}$, we consider the Hessian $A := D^2(F\circ\Psi_0)(\gamma)|_{\mathcal{H}\times\mathcal{H}}$. As a bounded self-adjoint operator on $\mathcal{H}$, the operator A is strictly larger than $-\mathrm{Id}_{\mathcal{H}}$ in the form sense.

Now we are in a position to state our main theorem. The explicit values of $\{\alpha_m\}_{m=0}^\infty$ are given in [11]. In the case $a'(0) = 0$, the explicit value of α_0 is written in terms of $\det_2(\mathrm{Id}_{\mathcal{H}} + A)$ as usual. (See [11] for details.) These coefficients $\{\alpha_m\}_{m=0}^\infty$ are formally in the same form as in the finite dimensional case (See Ben Arous [4]). This is not so surprising because all the formal computations are the same. The key of the proof is the (stochastic) Taylor expansion of the Itô map around the minimal point γ, which will be explained in the next section.

Theorem 2.2. *Under conditions* **(EX)**, **(H1)**, **(H2)**, **(H3)** *and* **(H4)** *we have the following asymptotic expansion:*

$$\begin{aligned} &\mathbb{E}\Big[G(X^\varepsilon)\exp\big(-F(X^\varepsilon)/\varepsilon^2\big)\Big] \\ &= \exp\big(-F_\Lambda(\gamma)/\varepsilon^2\big)\exp\big(-c(\gamma)/\varepsilon\big)\cdot\big(\alpha_0 + \alpha_1\varepsilon + \cdots + \alpha_n\varepsilon^n + O(\varepsilon^{n+1})\big), \end{aligned} \tag{2.3}$$

where the constant $c(\gamma)$ *in* (2.3) *is given by* $c(\gamma) := DF(\phi)[\Xi_1(\gamma)]$. *Here* $\Xi_j(\gamma) \in P(Y), j \in \mathbb{N}$, *is the unique solution of the differential equation*

$$d\,\Xi_t - \nabla\sigma(\phi_t)[\Xi_t, d\gamma_t] - a(0)\cdot\nabla b(\phi_t)[\Xi_t]dt = a^{(j)}(0)\cdot b(\phi_t)dt \quad \textit{with} \quad \Xi_0 = 0. \tag{2.4}$$

§ 3. Stochastic Taylor expansion

In this section, we establish the Taylor expansion for the differential equation (2.2) in the sense of rough paths. It is deterministic in the context of the rough path theory and, hence, the term "stochastic Taylor expansion" may not be appropriate anymore. Therefore, it is quite likely that this method applies to asymptotics of integrals of this kind for other probability measures, although in this article we only consider the law of the Brownian rough paths.

In this section we discuss without conditions **(EX)**, **(H1)**, **(H2)**, **(H3)** and **(H4)**. In particular, $\gamma \in \mathrm{BV}(X)$ and $\phi = \Psi_0(\gamma)$ are not the special elements as in **(H2)**. Notice also that we do not need the imbedded Hilbert space $H \subset X$, neither.

Our method of the stochastic Taylor expansion is slightly different from Aida's method in [2]. He uses the derivative equation, whose coefficient is of course of linear growth. Since it is not known whether Lyons' continuity theorem holds or not for unbounded coefficients, he extends the continuity theorem for the case of the derivative equation in [1]. On the other hand, we use the method in Azencott [3] and we only need the continuity theorem for the given equation, whose coefficients are bounded. The price we have to pay is that notations and proofs may seem slightly long. However, the strategy of this method is quite simple and straightforward.

At the beginning, we discuss in a heuristic way in order to find out what the terms in the expansion are like. Fix $\gamma \in \mathrm{BV}(X)$ and $\phi = \Psi_0(\gamma) \in \mathrm{BV}(Y)$. Suppose that we have an expansion around ϕ as

$$\Delta\phi := \Phi(\iota(\gamma + \varepsilon h, \lambda^\varepsilon))_1 - \phi \sim \varepsilon\phi^1 + \cdots + \varepsilon^n\phi^n + \cdots, \qquad \text{as } \varepsilon \searrow 0.$$

Of course, we also have

$$a(\varepsilon) \sim a(0) + \varepsilon a'(0) + \cdots + \varepsilon^n \frac{a^{(n)}(0)}{n!} + \cdots, \qquad \text{as } \varepsilon \searrow 0.$$

From the equation (2.2),

$$\begin{aligned} d(\phi + \Delta\phi) &\sim \sigma(\phi + \Delta\phi)d(\gamma + \varepsilon h) + \Big(\sum_{n=0}^{\infty} \varepsilon^n \frac{a^{(n)}(0)}{n!}\Big) \cdot b(\phi + \Delta\phi)dt, \\ &\sim \Big(\sum_{n=0}^{\infty} \frac{1}{n!}\nabla^n\sigma(\phi)\big[\Delta\phi, \dots, \Delta\phi, d(\gamma + \varepsilon h)\big]\Big) \\ &\quad + \Big(\sum_{n=0}^{\infty} \varepsilon^n \frac{a^{(n)}(0)}{n!}\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{1}{n!}\nabla^n b(\phi)\big[\Delta\phi, \dots, \Delta\phi\big]dt\Big). \end{aligned} \tag{3.1}$$

Picking up terms of order $n \in \mathbb{N}$, we see the following definition is quite natural.

Definition 3.1. For fixed $\gamma \in \mathrm{BV}(X)$, We set $\phi^0 = \phi$ by

$$d\phi_t = \sigma(\phi_t) d\gamma_t + a(0) \cdot b(\phi_t) dt \qquad \text{with } \phi_0 = 0 \tag{3.2}$$

and set ϕ^1 by

$$\begin{aligned} &d\phi^1_t - \nabla\sigma(\phi_t)[\phi^1_t, d\gamma_t] - a(0) \cdot \nabla b(\phi_t)[\phi^1_t] dt \\ &\qquad = \sigma(\phi_t) dh_t + a'(0) \cdot b(\phi_t) dt \quad \text{with } \phi^1_0 = 0. \end{aligned} \tag{3.3}$$

For $n = 2, 3, \ldots$, we set $\phi^n = \phi^n(h, \gamma)$ by

$$\begin{aligned} &d\phi^n_t - \nabla\sigma(\phi_t)[\phi^n_t, d\gamma_t] - a(0) \cdot \nabla b(\phi_t)[\phi^n_t] dt \\ &= dK^n(\phi, \phi^1, \ldots, \phi^{n-1}; h)_t + d\tilde{K}^n(\phi, \phi^1, \ldots, \phi^{n-1}; \gamma)_t \text{ with } \phi^n_0 = 0. \end{aligned} \tag{3.4}$$

Here $K^n_t = K^n(\phi, \phi^1, \ldots, \phi^{n-1}; h)_t$ and $\tilde{K}^n_t = \tilde{K}^n(\phi, \phi^1, \ldots, \phi^{n-1}; \gamma)_t$ are defined by

$$K^n_t := \int_0^t \sum_{k=1}^{n-1} \sum_{(i_1, \ldots, i_k) \in S^{n-1}_k} \frac{1}{k!} \nabla^k \sigma(\phi_s) \big[\phi^{i_1}_s, \ldots, \phi^{i_k}_s, dh_s\big], \tag{3.5}$$

$$\begin{aligned} \tilde{K}^n_t := &\int_0^t \sum_{k=2}^{n} \sum_{(i_1, \ldots, i_k) \in S^n_k} \frac{1}{k!} \nabla^k \sigma(\phi_s) \big[\phi^{i_1}_s, \ldots, \phi^{i_k}_s, d\gamma_s\big] \\ &+ \int_0^t \sum_{k=2}^{n} \sum_{(i_1, \ldots, i_k) \in S^n_k} \frac{a(0)}{k!} \cdot \nabla^k b(\phi_s) \big[\phi^{i_1}_s, \ldots, \phi^{i_k}_s\big] ds \\ &+ \int_0^t \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \sum_{(i_1, \ldots, i_k) \in S^{n-j}_k} \frac{a^{(j)}(0)}{j!k!} \cdot \nabla^k b(\phi_s) \big[\phi^{i_1}_s, \ldots, \phi^{i_k}_s\big] ds \\ &+ \int_0^t \frac{1}{n!} a^{(n)}(0) \cdot b(\phi_s) ds, \end{aligned} \tag{3.6}$$

where the sum on the right-hand side runs over

$$S^n_k := \{(i_1, \ldots, i_k) \in \mathbb{N}^k \,|\, i_j \geq 1 \text{ for all } 1 \leq j \leq k \text{ and } i_1 + \cdots + i_k = n\ \}.$$

Note that ϕ^n is given by a simple ordinary differential equation of first order. It is easy to see that the correspondence

$$(h, \gamma) \in \mathrm{BV}(X)^2 \mapsto \phi^n(h, \gamma) \in \mathrm{BV}(Y) \hookrightarrow G\Omega_p(Y), \qquad n = 1, 2, \ldots$$

is well-defined and continuous. The following proposition states that this maps extends to a (locally Lipschitz) continuous map from $G\Omega_p(X) \times \mathrm{BV}(X)$ to $G\Omega_p(Y)$. This is taken from [11]. In the following proposition, we set $\xi(\overline{h}) = \|\overline{h}_1\|_p + \|\overline{h}_2\|^{1/2}_{p/2}$. In the statement below, the choice of ω may depend on h and γ, but $\omega(0,1)$ is bounded if $\|\gamma\|_1$ stays bounded.

Proposition 3.1. *The map* $(h,\gamma)\mapsto\big(h,\phi,\phi^1,\dots,\phi^n\big)$ *extends to a continuous map from* $G\Omega_p(X)\times\mathrm{BV}(X)$ *to* $G\Omega_p(X\oplus Y^{\oplus n+1})$*. For any* $h,\gamma\in\mathrm{BV}(X)$*, there exists a control function* ω *which satisfies that* (i) *the following inequalities*

$$|\phi^j_t-\phi^j_s|\le\big(1+\xi(\overline{h})\big)^j\omega(s,t)^{1/p},$$
$$\Big|\int_s^t(\phi^j_u-\phi^j_s)\otimes dh_u\Big|\le\big(1+\xi(\overline{h})\big)^{j+1}\omega(s,t)^{2/p}$$

hold for all $0\le s<t\le1$ *and* $j=1,\dots,n$ *and* (ii) $\omega(0,1)$ *is dominated by a positive constant* $c=c(r_0)$ *which may depend on* n*, but not on* $h,\gamma\in\mathrm{BV}(X)$ *with* $\|\gamma\|_1\le r_0$*.*

Proof. We give a sketch of proof. For the detail, see [11]. Suppose that the proposition holds for $n-1$. First, let us observe that the second and the third terms on the left hand side of (3.4) does not cause a serious trouble. In the usual calculus, we can deal with those terms with Duhamel's principle. In our context we can prove the rough path version of the principle. So, we have only to consider the right hand side of (3.4).

The second term $\tilde{K}^n$ on the right hand side of (3.4) is also easy. This term is of bounded variation and it satisfies that, for some constant $c=c(r_0)$,

$$|\tilde{K}^n_t-\tilde{K}^n_s|\le c\big(1+\xi(\overline{h})\big)^n\omega(s,t),\qquad s<t.$$

Since the "addition" of an element of $G\Omega_p(Y)$ and an element of $\mathrm{BV}(Y)$ is well-defined as a continuous map, we only have to deal with K^n in (3.4).

From the explicit form in (3.5), K^n is given by an integral of $\big(h,\phi,\phi^1,\dots,\phi^{n-1}\big)$. Hence, by the integration theory for rough paths, we see that

$$(h,\gamma)\mapsto\big(h,\phi,\phi^1,\dots,\phi^{n-1}\big)\mapsto\big(h,\phi,\phi^1,\dots,\phi^{n-1},K^n\big)$$

extends to a continuous map. We can prove the necessary estimates

$$|K^n_t-K^n_s|\le\big(1+\xi(\overline{h})\big)^n\omega(s,t)^{1/p},$$
$$\Big|\int_s^t(K^n_u-K^n_s)\otimes dh_u\Big|\le\big(1+\xi(\overline{h})\big)^{n+1}\omega(s,t)^{2/p}$$

by giving estimates for the almost rough path which defines this integral.

□

Now we turn to the remainder terms. Let $\varepsilon\in(0,1]$ and define $R^{n+1}_\varepsilon=R^{n+1}_\varepsilon(h,\gamma)$ by

$$R^{n+1}_\varepsilon=\phi^{(\varepsilon)}-\phi^0-\varepsilon\phi^1-\cdots-\varepsilon^n\phi^n,\tag{3.7}$$

where $\phi^{(\varepsilon)} = \Phi(\iota(\gamma + \varepsilon h, \lambda^{\varepsilon}))_1$. When $h, \gamma \in \mathrm{BV}(X)$, this is clearly defined. The following proposition states that this extends to a continuous map from the space of geometric rough paths.

Proposition 3.2. *For any $\varepsilon \in (0,1]$, the map*

$$(h,\gamma) \mapsto \big(\varepsilon h, \phi^{(\varepsilon)}, \phi, \phi^1, \dots, \phi^{n-1}, R^n_{\varepsilon}\big)$$

extends to a continuous map from $G\Omega_p(X) \times \mathrm{BV}(X)$ to $G\Omega_p(X \oplus Y^{\oplus n+2})$. Moreover, for any $h, \gamma \in \mathrm{BV}(X)$, there exists a control function ω which satisfies that (i) *the following inequalities*

$$\begin{aligned} \big|R^n_{\varepsilon}(h)_t - R^n_{\varepsilon}(h)_s\big| &\le \big(\varepsilon + \xi(\overline{\varepsilon h})\big)^n \omega(s,t)^{1/p}, \\ \Big|\int_s^t \big(R^n_{\varepsilon}(h)_u - R^n_{\varepsilon}(h)_s\big) \otimes \varepsilon dh_u\Big| &\le \big(\varepsilon + \xi(\overline{\varepsilon h})\big)^{n+1} \omega(s,t)^{2/p} \end{aligned} \tag{3.8}$$

hold for all $0 \le s < t \le 1$ and $\varepsilon \in (0,1]$ and (ii) *for all $h \in \mathrm{BV}(X)$ with $\xi(\overline{\varepsilon h}) \le r_1$ and $\gamma \in \mathrm{BV}(X)$ with $\|\overline{\gamma}\|_1 \le r_0$, $\omega(0,1)$ is dominated by a positive constant $c = c(r_0, r_1)$ which depends only on r_0 and r_1 (not on ε).*

Proof. We give a sketch of proof. Assume that the proposition holds for n. We now prove it for $n+1$. Existence of continuous extension is almost obvious, since we can construct ϕ^n in the same way as in the previous proposition and we immediately get R^{n+1}_{ε} by (3.7).

Now we turn to the estimate of R^{n+1}_{ε}. From the differential equation for $\phi^{(\varepsilon)} = \Phi\big(\iota(\gamma + \varepsilon h, \lambda^{\varepsilon})\big)_1$ and (3.2)–(3.6), we see that R^{n+1}_{ε} satisfies the following differential equation:

$$\begin{aligned} &dR^{n+1}_{\varepsilon,t} - \nabla\sigma(\phi_t)[R^{n+1}_{\varepsilon,t}, d\gamma_t] - a(0)\cdot\nabla b(\phi_t)[R^{n+1}_{\varepsilon,t}]dt \\ &= dL^n(\phi^{(\varepsilon)}, \phi, \phi^1, \dots, \phi^{n-1}; h)_t + d\tilde{L}^n(\phi^{(\varepsilon)}, \phi, \phi^1, \dots, \phi^{n-1}; \gamma)_t \end{aligned}$$

with $R^{n+1}_{\varepsilon,0} = 0$. Here, L^n is an integral with respect to dh and $\tilde{L}^n$ is an integral with respect to $d\gamma$ or dt. Since no ϕ^n is involved in the right hand side, we can estimate R^{n+1}_{ε} in essentially the same way as in the previous proposition. □

§ 4. Example: Heat process over loop spaces

In this section, we consider a class of stochastic processes on continuous loop spaces and show that the rough path theory is applicable to them. These processes are called heat process on loop spaces and are defined by a collection of finite-dimensional SDEs.

A process of this kind was first introduced by Malliavin [18] in the case of loop groups and then was generalized by many authors.

Let $\mathcal{L}_0(\mathbb{R}^d) := \{x \in C([0,1], \mathbb{R}^d) |\ x(0) = x(1) = 0\}$ be the continuous loop group with the usual sup-norm. We set

$$H_0(\mathbb{R}^d) = \{k \in \mathcal{L}_0(\mathbb{R}^d) \mid k \text{ is absolutely continuous and } \|k\|_{H_0(\mathbb{R}^d)} < \infty \},$$

where $\|k\|^2_{H_0(\mathbb{R}^d)} := \int_0^1 |k'_t|^2 dt$. It is well-known that there exists a Gaussian measure μ^d such that the triplet $(\mathcal{L}_0(\mathbb{R}^d), H_0(\mathbb{R}^d), \mu^d)$ becomes an abstract Wiener space. Note that μ^d is the usual d-dimensional pinned Wiener measure.

For $\tau \in [0,1]$ and $j = 1, 2, \ldots, d$, we denote by δ^j_τ the element in $\mathcal{L}_0(\mathbb{R}^d)^*$ defined by $\langle \delta^j_\tau, x\rangle = x^j(\tau)$ and set $x(\tau) := (x^1(\tau), \ldots, x^d(\tau))$. Let $(w_t)_{t\geq 0}$ be a $\mathcal{L}_0(\mathbb{R}^d)$-valued Brownian motion associated with μ^d. We set $w^j_t(\tau) := \langle \delta^j_\tau, w_t\rangle$ and $w_t(\tau) := (w^1_t(\tau), \ldots, w^d_t(\tau))$.

Now we define heat processes over loop spaces. Let

$$A_j(x) = \sum_{i=1}^r a_{ij}(x)\frac{\partial}{\partial x_i}, \quad A_0(x) = \sum_{i=1}^r b_i(x)\frac{\partial}{\partial x_i}, \quad V_0(x) = \sum_{i=1}^r \beta_i(x)\frac{\partial}{\partial x_i}$$

be vector fields on $\mathbb{R}^r$, $j = 1, \ldots, d$. We assume the following regularities on the coefficients:

$$a_{ij}, b_i, \beta_i \in C_b^\infty(\mathbb{R}^r, \mathbb{R}) \qquad \text{for } 1 \leq i \leq r, 1 \leq j \leq d. \tag{4.1}$$

We write a for the $r \times d$-matrix $\{a_{ij}\}_{1\leq i\leq r, 1\leq j\leq d}$ and write b and β for the column vectors $(b_1, \ldots, b_r)^{\mathrm{T}}$ and $(\beta_1, \ldots, \beta_r)^{\mathrm{T}}$, respectively.

For each fixed space parameter $\tau \in [0,1]$ and $\varepsilon > 0$, we consider the following (finite dimensional) SDE:

$$\begin{aligned} d_t X^\varepsilon_t(\tau) &= \sum_{j=1}^r A_j(X^\varepsilon_t(\tau)) \circ \varepsilon d_t w^j_t(\tau) + A_0(X^\varepsilon_t(\tau))\varepsilon^2 dt + V_0(X^\varepsilon_t(\tau))dt \\ &= a(X^\varepsilon_t(\tau)) \circ \varepsilon d_t w_t(\tau) + b(X^\varepsilon_t(\tau))\varepsilon^2 dt + \beta(X^\varepsilon_t(\tau))dt. \end{aligned} \tag{4.2}$$

with the initial data $X^\varepsilon_0(\tau) = 0$. We will often write $X^\varepsilon(t,\tau) := X^\varepsilon_t(\tau)$. In Proposition 4.1 below, we will prove that $X^\varepsilon(t,\tau)$ has a bi-continuous modification. We call $X^\varepsilon = (X^\varepsilon(t,\cdot))_{0\leq t\leq 1}$ a heat process over $\mathcal{L}_0(\mathbb{R}^d)$. X^ε can be regarded as a random variable in $P(\mathcal{L}_0(\mathbb{R}^d))$.

Next we recall that $(\mathcal{L}_0(\mathbb{R}^d), \mu^d)$ satisfies the exactness condition for all tensor norms (including the projective norm) on $\mathcal{L}_0(\mathbb{R}^d) \otimes \mathcal{L}_0(\mathbb{R}^d)$. (See Lemma 4.1 in [9] for the proof.) Therefore the Brownian rough path $\overline{w} \in G\Omega_p(\mathcal{L}_0(\mathbb{R}^d))$ defined by $(w_t)_{t\geq 0}$ exists and we can deal with our heat process X^ε defined by (4.2) from the viewpoint of rough paths.

We define a Nemytski map $\tilde{\sigma} : \mathcal{L}_0(\mathbb{R}^r) \to L\big(\mathcal{L}_0(\mathbb{R}^d) \oplus \mathbb{R}^2, \mathcal{L}_0(\mathbb{R}^r)\big)$ by

$$\tilde{\sigma}(y)[(x,u_1,u_2)](\tau) := a\big(y(\tau)\big)x(\tau) + b\big(y(\tau)\big)u_1 + \beta\big(y(\tau)\big)u_2, \quad \tau \in [0,1]. \tag{4.3}$$

for $(x,u_1,u_2) \in \mathcal{L}_0(\mathbb{R}^d) \oplus \mathbb{R}^2$ and $y \in \mathcal{L}_0(\mathbb{R}^r)$. Note that the assumption (4.1) implies $\tilde{\sigma} \in C_b^\infty\big(\mathcal{L}_0(\mathbb{R}^r), L\big(\mathcal{L}_0(\mathbb{R}^d) \oplus \mathbb{R}^2, \mathcal{L}_0(\mathbb{R}^r)\big)\big)$. Then we can consider a random element $\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))$ in $G\Omega_p(\mathcal{L}_0(\mathbb{R}^r))$ through the differential equation in the rough path sense (2.1). The following proposition is taken from Lemma 4.8 in [9]. By this proposition, we can obtain a dynamics on $\mathcal{L}_0(\mathbb{R}^r)$. In the proof, the Wong-Zakai approximation theorem plays a crucial role.

Proposition 4.1. *For each $\varepsilon > 0$, $(t,\tau) \mapsto \Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1(0,t)(\tau)$ is a bi-continuous modification of the two-parameter process $(X^\varepsilon(t,\tau))_{0\le t\le 1, 0\le\tau\le 1}$ defined in* (4.2).

Remark 4.1. We now give several remarks on the (possible) advantage of the rough path approach for this process.

1. This kind of process was first introduced by Malliavin [18] in the loop group case as a collection of finite dimensional SDEs. (The vector fields A_j's are basis of the Lie algebra of a compact Lie group which is imbedded in $\mathbb{R}^d$ and $A_0 = V_0 = 0$). In that case the process is a diffusion process associated with one half of the Gross Laplacian. Note that there is not a very nice SDE theory on $\mathcal{L}_0(\mathbb{R}^d)$, since $\mathcal{L}_0(\mathbb{R}^d)$ is not an M-type 2 Banach space. In [7], Kolmogorov's criterion is used for the proof of the existence of continuous modifications. Hence Proposition 4.1 is regarded as a revisit via the rough path theory.

2. Even if this process is constructed as a collection of solutions of finite dimensional SDEs, many problems can be solved. For example, in the loop group case, a logarithmic Sobolev inequality was shown by Driver and Lohrenz [6]. However, for the Laplace asymptotics, an approach which is more fit for the topology of the loop space is required. The reason is as follows. To prove the logarithmic Sobolev inequality, it is enough to consider cylinder functions. Once a cylinder function is chosen, then the problem is reduced to a finite dimensional one. By contrast, the functions F and G in Theorem 2.2 are much more general. Since the rough path theory works on any Banach space, it may be a nice tool for stochastic analysis on Banach spaces.

3. In the loop group case, the large deviation principle was proved by Fang and Zhang [8]. Theorem 2.1 implies it as a special case.

4. Brzeźniak and Elworthy [5] constructed this process with SDE theory on M type-2 Banach spaces. However, $\mathcal{L}_0(\mathbb{R}^r)$ is not of M type-2. So they replace the sup-norm with a Besov-like norm and work on that Besov-like subspace. However, the

regularity of the coefficient $\tilde{\sigma}$ is lost with respect to a new norm. The authors do not know whether one can prove asymptotic theorems of this kind with this approach.

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