Dilation-stable-like Processes on Fractals

By

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Abstract

In this note, we report some concerns on Markov processes on fractals which allow different stability indices in different “directions”. We report the simplest case, the processes on product fractals with independent components. The main tools are multivariate subordinations and time-changes. This is a preliminary report of an on-going project.

§ 1. Dilation-stable processes

At 1993 annual probability meeting at Keio University, H. Kunita gave a lecture on stable Markov processes on manifolds (in Japanese); the views revealed in the lecture are

1. stability index could/should be different in different “direction”.
2. “independent components” assumption could/should be too strong.

He described the process to be stable w.r.t. dilations \( \{ \gamma_t \}_{t>0} \) which is a semigroup of transformations characterized by, mainly, the on-diagonal entries \( t^{1/\alpha_1}, \ldots, t^{1/\alpha_n} \).

In case the state space is \( \mathbb{R}^n \), this is a class of operator-stable Lévy processes; we refer to Sato(1999) for an excellent book on the topic. In a much earlier paper, Pruitt-Taylor(1969) concerned with Lévy processes in \( \mathbb{R}^n \) with independent components and each component process is with different stability index, with the viewpoint from the collision of stable processes and the behavior of Blumenthal-Getoor indices. There have been studies on dimension formulae of such dilation-stable Lévy processes (with or without components independence): for image points we cite Hendrick(1973), Lin(1995), Meerschaert-Xiao(2005), and for multiple points we cite Hendricks(1974), Shieh(1998). See Xiao(2004) for an intensive survey on these fractal properties.

The constructions of dilation-stable Lévy processes can be proceeded by using Lévy-Khintchine formula of \( X(1) \) in polar coordinate, based on \( \gamma_t \). Here we mention another

2000 Mathematics Subject Classification(s): 60G17, 60G18, 60J75.
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view which was described in Shieh(2000, §4). Let $B^n = (B_1, \cdots, B_n)$ be Brownian motion in $\mathbb{R}^n$, and let $\xi_j$ be (strictly) stable subordinator of index $\beta_j$. Assume $\xi_j, 1 \leq j \leq n$ are independent on $B$, then consider the subordination $Y(t) = (B_1(\xi_1(t)), \cdots, B_n(\xi_n(t))$.

It has been exploited in full strength in BardorffNielsen-Peterson-Sato(2001), who called this to be a multivariate subordination. We note that $Y$ is of independent components if and only if the $n$-variate process $\xi = (\xi_j)_{j=1}^n$ itself is of independent components.

\section{The object of the work}

We would like to see how the views of Kunita could be carried out for Markov processes on fractals, especially after significant works have been done for a stable-like process on a d-set. Such a process is characterized by a unique stability index; see Kumagai (2002, 2004). However, one immediate problem may be: what should be the “direction” of a process on a fractal? for example, for Sierpinski Gasket(SG) in $\mathbb{R}^2$ the “natural direction” is not $\mathbb{R}^2$-direction. Moreover, the construction of Brownian motion on Sierpinski Carpet(SC), see for example Bass(1997), shows it is not of independent components, though the fractal and the process look like to follow $\mathbb{R}^2$-direction.

In this note we proceed the simplest model, independent product; yet it still may have some interesting turn-outs.

\section{Diffusions on product fractals}

At least there are two papers mentioning Markov processes on product fractals. However the resulting diffusions still have a certain incrementally isotropic property.

In the seminal paper by Barlow(1998), he mentioned(p27 and p45) the following, let $(F_j, \rho_j, \mu_j), j = 1, 2,$ be two fractional metric spaces(FMSs) and $X^j$ be a fractional diffusion $FD(d_f(j), d_w(j))$ on $F_j$. Consider the product FMS $F = F_1 \times F_2$. When $d_w(1) = d_w(2)$, the product process $X = (X^1, X^2)$ is a $FD(d_f, d_w)$ on $F$, with $d_f = d_f(1) + d_f(2), d_w = d_w(1) = d_w(2)$. He also remarked that, the product process is not a fractional diffusion( in his definition of FDs) if $d_w(1) \neq d_w(2)$.

Very recently Strichartz(2005) studies analysis on product fractals and mentions(p574) that “In fact, the full strength of our theory only applies to products with identical factors, and the scaling factors must be homogeneous throughout the fractal.”

\section{Multivariate subordinators}

Following BardorffNielsen-Peterson-Sato(2001), we say a Lévy process $\xi = (\xi_1, \cdots, \xi_n)$ in $[0, \infty)^n$ a n-variate subordinator when each component $\xi_j$ is a (uni-variate) subordinator, i.e. a Lévy process with increasing paths.
When each $\xi_j$ is (strictly) $\beta_j$-stable, $0 < \beta_j < 1$, and $\xi$ is stable w.r.t. the dilations $\delta_t = (t^{1/\beta_1}, \cdots, t^{1/\beta_n})$, then we may call $\xi$ a *dilation-stable subordinator* with index $(\beta_1, \cdots, \beta_n)$. The characteristic function of $\xi(1)$ is given by

$$\exp\left[\int_0^\infty \int_{\mathbb{S}_+} (\exp(i(z, \delta_r x)) - 1) \frac{\lambda(dx)dr}{r^2}\right],$$

where $\lambda$ is a finite Borel measure on $\mathbb{S}_+$, the unit spherical surface in $\mathbb{R}^n$ with non-negative coordinates. Note that $\xi$ is of independent components if and only if $\lambda$ concentrates on the coordinate axes.

We note that the above $\xi(t)$ has a continuous density function $\eta_t(u)$, $u \in \mathbb{R}_+^n$, which has a certain scaling property inherited from $\delta_t$. When $\xi$ has independent components then $\eta_t(u) = \prod \eta_{j,t}(u_j)$, where $\eta_{j,t}$ denotes the density of $\xi_j(t)$.

§ 5. *Multivariate subordination based on 3,4*

Suppose that we are given a product diffusion $X = (X^1, X^2)$ on $F = F_1 \times F_2$ as in §3, with $F_j$ a $d_f(j)$-set in $\mathbb{R}^{n_j}$, and a 2-variate dilation-stable subordinator $\xi = (\xi_1, \xi_2)$ as in §4. Assume that $X, \xi$ are independent, then we have a subordinated process

$$Y(t) = (X_1(\xi_1(t)), X_2(\xi_2(t))).$$

Note that $X_j(\xi_j(\cdot))$ is a stable-like process on $F_j$ with stability index $\alpha_j = \beta_j d_w$, where $d_w$ is the common walk dimension of $X_j$.

Let $p_t^j(x_j, y_j)$ be the heat kernel of $X^j$, then the heat kernel of $Y$ is, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$q_t(x, y) = \int \int_{\mathbb{R}_+^2} \prod_{j=1}^2 p_{u_j}^j(x_j, y_j) \eta_t(u_1, u_2) du_1 du_2.$$

When $\xi_1, \xi_2$ are independent, then $Y$ is of independent components, and

$$q_t(x, y) = q_t^1(x_1, y_1) q_t^2(x_2, y_2),$$

$q_t^j$ is the heat kernel of $X^j(\xi_j)$; see Kumagai (2004, p227). Then, a straightforward calculation gives that $q_t(x, y)$ has the following estimate:

$$q_t(x, y) \approx t^{-\sum_{j=1}^2 \frac{d_f(j)}{\alpha_j} } \wedge \frac{t}{\prod_{j=1,2} |x_j - y_j|^{d_f(j) + \alpha_j}},$$

We simply note that $q_t(x, y)$ is *not* of the form $q_t(|x - y|)$. 


§ 6. Hausdorff dimension of \(Y[0, 1]\)

Henceforth the notation \(\dim\) means Hausdorff dimension w.r.t. the Euclidean metric. Firstly we recall that an \(\alpha\)-stable-like process on a \(d\)-set \(F \subset \mathbb{R}^n, 0 < d < n, 0 < \alpha < 2, n \geq 2\), is a Markov process on \(F\) which is determined by the following Dirichlet form:

\[
\mathcal{E}^\alpha(u, v) = \int_F \int_F \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} \mu(dx) \mu(dy),
\]

where \(\mu\) is the defining \(d\)-measure on \(F\).

**Theorem 6.1.** (Chen-Kumagai 2003). For an \(\alpha\)-stable-like process \(X\) on a \(d_f\)-set \(F \subset \mathbb{R}^n, 0 < d_f < n, 0 < \alpha < 2, n \geq 2\), \(\dim X[0, 1] = \min\{d_f, \alpha\}\).

The theorem may say to deviate from the classical \(\alpha\)-stable Lévy process in the Euclidean \(\mathbb{R}^n\), where \(\dim X[0, 1]\) is always to be \(\alpha < n\).

We may have, for the subordinated \(Y(t) = (X_1(\xi_1(t)), X_2(\xi_2(t)), \ldots)\), where \(X_i\) is a fractional diffusion on a \(d_f(j)\)-set \(F_j \subset \mathbb{R}^{n_j}\), and \((\xi_1, \xi_2)\) is a 2-variate dilation-stable subordinator with index \((\beta_1, \beta_2)\). We expect that the following dimension formula could be proved:

**Theorem 6.2.** (proposed). Let \(\beta_2 < \beta_1\), so that \(\alpha_2 = \beta_2 d_w < \alpha_1 = \beta_1 d_w\).

\[
\dim Y[0, 1] = \alpha_1, \quad if \quad \alpha_1 \leq d_f(1),
\]

\[
\dim Y[0, 1] = d_f(1) + \frac{\alpha_2}{\alpha_1}(\alpha_1 - d_f(1)), \quad if \quad \alpha_1 > d_f(1).
\]

Theorem 6.2 includes the Lévy process in \(\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}\) as that appeared in Pruitt-Taylor(1969), in which the second formula appears only for \(n_1 = 1\). Theorem 6.2 applies well to dilation-stable-like processes on SG×SG and on SC×SC, regarding as subsets of \(\mathbb{R}^2 \otimes \mathbb{R}^2\). We also remark that the present formula illustrates well how the second stability index \(\alpha_2\) gets involved in the dimension formula.

Barlow(1998, Lemma 3.4(c)) tells that for his fractional diffusions it is necessary that \(d_f(j) \geq 1\), thus the term

\[
\frac{\alpha_2}{\alpha_1}(\alpha_1 - d_f(1)) < 1 \leq d_f(2).
\]

Thus, when we consider the the 3-variate subordination of the triple product of Barlow’s fractional diffusions, in addition to the double product, the third stability index seems should not be involved in the dimension formula, and this is perhaps a “good interpretation” for what happens for Pruitt-Taylor processes. However, if we consider “Brownian motions” on Cantor-like sets, see §7 below, it is possible to obtain similar detailed heat kernel estimate (Barlow’s fractional diffusions exclude such processes on
disconnected sets), and then the third stability index then could be involved. I am indebted to a question by Kumagai at the Symposium, which leads to the above thinking.

The proof of Theorem 6.2 could be proceeded as a blending of those techniques in Chen-Kumagai and Pruitt-Taylor.

§7. Restriction of BM to product fractals

Let $B^n(t)$ be the Brownian motion in $\mathbb{R}^n$, and let $\mathbb{R}^n = \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$. Let $F_j \subset \mathbb{R}^{n_j}$ be $d_j$-set, $0 < n_j - d_j < 2$. We may restrict $B^n$ to $F_1 \times F_2$ by the time-change w.r.t. product positive continuous additive functional (PCAF) $(A^1, A^2)$ in a marginal way as follows. We break $B^n$ into $(B^{n_1}, B^{n_2})$ and proceed the restriction of $B^{n_j}$ to $F_j$ by time-change w.r.t. the associated $A^j$. We remark that the PCAF $A^j$ is a natural version of the local time of $B^{n_j}$ on $F_j$; see Kumagai(2002) and Hanson-Zähle(2006). The resulting processes is of independent components, and the heat kernel estimate is also obtained by direct applications of Chen-Kumagai(2003) and Kumagai(2002, Proposition 3.1). As it is mentioned in Kumagai(2002) and Hanson-Zähle(2006), we may use an isotropic $\alpha$-stable motion in $\mathbb{R}^n$ instead of $B^n$; however the resulting process is not of independent components. The above construction can be proceeded for the triple product, and we then have the following concern.

We consider processes on a Cantor “dust” $C \subset \mathbb{R}^3 = \mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R}$. Let $C := C_1 \times C_2 \times C_3$, and each $C_j$ is a Cantor subset of $\mathbb{R}$ with Hausdorff dimension $s_j$, $0 < s_j < 1$ so that $C$ is a $s$-set in $\mathbb{R}^3$ with $s = s_1 + s_2 + s_3$. We may have two processes on $C$. One $Y_1$ is constructed by restricting $B^3$ to $C$ as that is done in Kumagai(2002, §2.3), w.r.t. a single PCAF, whenever $s > 1$. Another $Y_2$ is constructed as above, by time-change w.r.t. to the triple product PCAF $(A^1, A^2, A^3)$. Are these two processes $Y_1, Y_2$ different in view of the dimension formulae? We may expect from Theorem 6.1 and Kumagai(2002, Proposition 3.1) that $\dim Y_1[0, 1] = s - 1$. On the other hand in view of the proposed Theorem 6.2 we may conjecture that $\dim Y_2[0, 1] = s$. Thus these two $Y_1, Y_2$ should be different, even when $s_1 = s_2 = s_3$. However both the above formulae need to justify, since the full kernel estimate is not easy to estimate (we may only establish, for example, a certain Nash inequality).

§8. Product Dirichlet forms

The Dirichlet form of the product fractal is proposed by Strichartz (2005) as, for $u, v \in L^2(F_1 \times F_2, \mu_1 \times \mu_2),

\mathcal{E}(u, v) = \int_{F_2} \mathcal{E}_1(u(\cdot, x_2), v(\cdot, x_2))d\mu_2(dx_2) + \int_{F_1} \mathcal{E}_2(u(x_1, \cdot), v(x_1, \cdot))d\mu_1(x_1),

\mathcal{E}_1(u, v), \mathcal{E}_2(u, v) \in L^2(F_1 \times F_2, \mu_1 \times \mu_2).$
where $\mathcal{E}_j$ means the Dirichlet form on $F_j$.

In view of this and Kumagai (2002, Proposition 3.1), the subordinated process $Y$ on $F = F_1 \times F_2$ in §5, in case it is of independent components, is comparable to the following Dirichlet form: for $x = (x_1, x_2), y = (y_1, y_2), \mu = \mu_1 \times \mu_2$,

$$
\mathcal{E}^{\alpha_1, \alpha_2}(u, v) = \int_F \int_F \frac{(u(x) - u(y))(v(x) - v(y))}{\prod_{j=1,2} |x_j - y_j|^{d_f(j) + \alpha_j}} \mu(dx) \mu(dy).
$$

§9. Possible perspective

The following discussion with Kumagai shows that we may still have far distance to what we want to follow the viewpoints of Hunita, mentioned in §1, for processes on fractals. In Hambly-Kumagai (2004), they construct a type of diffusion on SG such that the walk dimensions along the diagonal and along the horizontal are different. We subordinate such a diffusion by a single subordinator, and ask, say, the Hausdorff dimension of the resulting stable-like processes. With respect to the resistance metric, the dimension formula should be the same form as Theorem 6.1. However, usually we prefer, at least in view of dimension, the Euclidean metric, and in Hambly-Kumagai construction the resistance and the Euclidean metrics are not comparable explicitly.

Acknowledgement. I thank Takashi Kumagai to invite me to attend both Conference of Mathematics on Fractals 2006 and German-Japan Symposium of Stochastic Analysis and Applications.

References