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Dilation-stable-like Processes on Fractals

By

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Abstract

In this note, we report some concerns on Markov processes on fractals which allow different stability indices in different "directions". We report the simplest case, the processes on product fractals with independent components. The main tools are multivariate subordinations and time-changes. This is a preliminary report of an on-going project.

§ 1. Dilation-stable processes

At 1993 annual probability meeting at Keio University, H. Kunita gave a lecture on stable Markov processes on manifolds(in Japanese); the views revealed in the lecture are

1. stability index could/should be different in different "direction".
2. "independent components" assumption could/should be too strong.

He described the process to be *stable w.r.t. dilations* $\{\gamma_t\}_{t>0}$ which is a semigroup of transformations characterized by, mainly, the on-diagonal entries $t^{1/\alpha_1}, \cdots, t^{1/\alpha_n}$.

In case the state space is $\mathbb{R}^n$, this is a class of operator-stable Lévy processes; we refer to Sato(1999) for an excellent book on the topic. In a much earlier paper, Pruitt-Taylor(1969) concerned with Lévy processes in $\mathbb{R}^n$ with independent components and each component process is with different stability index, with the viewpoint from the collision of stable processes and the behavior of Blumenthal-Getoor indices. There have been studies on dimension formulae of such *dilation-stable* Lévy processes(with or without components independence): for image points we cite Hendrick(1973), Lin(1995), Meerschaert-Xiao(2005), and for multiple points we cite Hendricks(1974), Shieh(1998). See Xiao(2004) for an intensive survey on these fractal properties.

The constructions of dilation-stable Lévy processes can be proceeded by using Lévy-Khintchine formula of $X(1)$ in polar coordinate, based on γ_t. Here we mention another

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view which was described in Shieh(2000, §4). Let $B^n = (B_1, \cdots, B_n)$ be Brownian motion in $\mathbb{R}^n$, and let ξ_j be (strictly) stable subordinator of index β_j. Assume $\xi_j, 1 \le j \le n$ are independent on B, then consider the subordination $Y(t) = (B_1(\xi_1(t)), \cdots, B_n(\xi_n(t))$. It has been exploited in full strength in BardorffNielsen-Peterson-Sato(2001), who called this to be a *multivariate subordination.* We note that Y is of independent components if and only if the n-variate process $\xi = (\xi_j)_{j=1}^n$ itself is of independent components.

§ 2. The object of the work

We would like to see how the views of Kunita could be carried out for Markov processes on fractals, especially after significant works have been done for a *stable-like process* on a *d-set.* Such a process is characterized by a unique stability index; see Kumagai (2002, 2004). However, one immediate problem may be: what should be the "direction" of a process on a fractal? for example, for Sierpinski Gasket(SG) in $\mathbb{R}^2$ the "natural direction" is *not* $\mathbb{R}^2$-direction. Moreover, the construction of Brownian motion on Sierpinski Carpet(SC), see for example Bass(1997), shows it is *not* of independent components, though the fractal and the process look like to follow $\mathbb{R}^2$-direction.

In this note we proceed the simplest model, independent product; yet it still may have some interesting turn-outs.

§ 3. Diffusions on product fractals

At least there are two papers mentioning Markov processes on product fractals. However the resulting diffusions still have a certain incrementally isotropic property.

In the seminal paper by Barlow(1998), he mentioned(p27 and p45) the following, let $(F_j, \rho_j, \mu_j), j = 1, 2$, be two fractional metric spaces(FMSs) and X^j be a fractional diffusion $FD(d_f(j), d_w(j))$ on F_j. Consider the product FMS $F = F_1 \times F_2$. When $d_w(1) = d_w(2)$, the product process $X = (X^1, X^2)$ is a $FD(d_f, d_w)$ on F, with $d_f = d_f(1) + d_f(2), d_w = d_w(1) = d_w(2)$. He also remarked that, the product process is not a fractional diffusion(in his definition of FDs) if $d_w(1) \ne d_w(2)$.

Very recently Strichartz(2005) studies analysis on product fractals and mentions(p574) that "In fact, the full strength of our theory only applies to products with identical factors, and the scaling factors must be homogeneous throughout the fractal."

§ 4. Multivariate subordinators

Following BardorffNielsen-Peterson-Sato(2001), we say a Lévy process $\xi = (\xi_1, \cdots, \xi_n)$ in $[0, \infty)^n$ a *n-variate subordinator* when each component ξ_j is a (uni-variate) subordinator, i.e. a Lévy process with increasing paths.

When each ξ_j is (strictly) β_j-stable, $0 < \beta_j < 1$, and ξ is stable w.r.t. the dilations $\delta_t = (t^{1/\beta_1}, \cdots, t^{1/\beta_n})$, then we may call ξ a *dilation-stable subordinator* with index $(\beta_1, \cdots, \beta_n)$. The characteristic function of $\xi(1)$ is given by

$$exp\Big[\int_0^\infty \int_{S_+} \big(exp(i(z, \delta_r x)) - 1\big) \frac{\lambda(dx)dr}{r^2}\Big],$$

where λ is a finite Borel measure on S_+, the unit spherical surface in $\mathbb{R}^n$ with non–negative coordinates. Note that ξ is of independent components if and only if λ concentrates on the coordinate axes.

We note that the above $\xi(t)$ has a continuous density function $\eta_t(u), u \in \mathbb{R}^n_+$, which has a certain scaling property inherited from δ_t. When ξ has independent components then $\eta_t(u) = \prod \eta_{j,t}(u_j)$, where $\eta_{j,t}$ denotes the density of $\xi_j(t)$.

§ 5. Multivariate subordination based on 3,4

Suppose that we are given a product diffusion $X = (X^1, X^2)$ on $F = F_1 \times F_2$ as in **§3**, with F_j a $d_f(j)$-set in $\mathbb{R}^{n_j}$, and a 2-variate dilation-stable subordinater $\xi = (\xi_1, \xi_2)$ as in **§4**. Assume that X, ξ are independent, then we have a subordinated process

$$Y(t) = (X_1(\xi_1(t)), X_2(\xi_2(t)).$$

Note that $X_j(\xi_j(\cdot))$ is a stable-like process on F_j with stability index $\alpha_j = \beta_j d_w$, where d_w is the common walk dimension of X_j.

Let $p^j_t(x_j, y_j)$ be the heat kernel of X^j, then the heat kernel of Y is, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$q_t(x, y) = \int\int_{R^2_+} \prod_{j=1}^{2} p^j_{u_j}(x_j, y_j)\eta_t(u_1, u_2)du_1du_2.$$

When ξ_1, ξ_2 are independent, then Y is of independent components, and

$$q_t(x, y) = q^1_t(x_1, y_1)q^2_t(x_2, y_2),$$

q^j_t is the heat kernel of $X^j(\xi_j)$; see Kumagai (2004, p227). Then, a a straightforward calculation gives that $q_t(x, y)$ has the following estimate:

$$q_t(x, y) \approx t^{-\sum_{j=1}^{2} \frac{d_f(j)}{\alpha_j}} \wedge \frac{t}{\prod_{j=1,2} |x_j - y_j|^{d_f(j)+\alpha_j}},$$

We simply note that $q_t(x, y)$ is *not* of the form $q_t(|x - y|)$.

§ 6. Hausdorff dimension of $Y[0,1]$

Henceforth the notation *dim* means Hausdorff dimension w.r.t. the Euclidean metric. Firstly we recall that an α-stable-like process on a d-set $F \subset \mathbb{R}^n, 0 < d < n, 0 < \alpha < 2, n \geq 2$, is a Markov process on F which is determined by the the following Dirichlet form:

$$\mathcal{E}^\alpha(u,v) = \int_F \int_F \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}} \mu(dx)\mu(dy),$$

where μ is the defining d-measure on F.

Theorem 6.1. *(Chen-Kumagai 2003). For an α-stable-like process X on a d_f-set $F \subset \mathbb{R}^n, 0 < d_f < n, 0 < \alpha < 2, n \geq 2$, $\dim X[0,1] = \min\{d_f, \alpha\}$.*

The theorem may say to deviate from the classical α-stable Lévy process in the Euclidean $\mathbb{R}^n$, where $\dim X[0,1]$ is always to be $\alpha < n$.

We may have, for the subordinated $Y(t) = (X_1(\xi_1(t)), X_2(\xi_2(t)))$, where X_i is a fractional diffusion on a $d_f(j)$-set $F_j \subset \mathbb{R}^{n_j}$, and (ξ_1, ξ_2) is a 2-variate dilation-stable subordinator with index (β_1, β_2). We expect that the following dimension formula could be proved:

Theorem 6.2. *(proposed). Let $\beta_2 < \beta_1$, so that $\alpha_2 = \beta_2 d_w < \alpha_1 = \beta_1 d_w$.*

$$\dim Y[0,1] = \alpha_1, \quad if \quad \alpha_1 \leq d_f(1),$$
$$\dim Y[0,1] = d_f(1) + \frac{\alpha_2}{\alpha_1}(\alpha_1 - d_f(1)), \quad if \quad \alpha_1 > d_f(1).$$

Theorem 6.2 includes the Lévy process in $\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$ as that appeared in Pruitt-Taylor(1969), in which the second formula appears only for $n_1 = 1$. Theorem 6.2 applies well to dilation-stable-like processes on SG×SG and on SC×SC, regarding as subsets of $\mathbb{R}^2 \otimes \mathbb{R}^2$. We also remark that the present formula illustrates well how the second stability index α_2 gets involved in the dimension formula.

Barlow(1998, Lemma 3.4(c)) tells that for his fractional diffusions it is necessary that $d_f(j) \geq 1$, thus the term

$$\frac{\alpha_2}{\alpha_1}(\alpha_1 - d_f(1)) < 1 \leq d_f(2).$$

Thus, when we consider the the 3-variate subordination of the triple product of Barlow's fractional diffusions, in addition to the double product, the third stability index seems *should not be* involved in the dimension formula, and this is perhaps a "good interpretation" for what happens for Pruitt-Taylor processes. However, if we consider "Brownian motions" on Cantor-like sets, see §7 below, it is possible to obtain similar detailed heat kernel estimate(Barlow's fractional diffusions exclude such processes on disconnected sets), and then the third stability index then *could be* involved. I am indebted to a question by Kumagai at the Symposium, which leads to the above thinking.

The proof of Theorem 6.2 could be proceeded as a blending of those techniques in Chen-Kumagai and Pruitt-Taylor.

§ 7. Restriction of BM to product fractals

Let $B^n(t)$ be the Brownian motion in $\mathbb{R}^n$, and let $\mathbb{R}^n = \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$. Let $F_j \subset \mathbb{R}^{n_j}$ be d_j-set, $0 < n_j - d_j < 2$. We may restrict B^n to $F_1 \times F_2$ by the time-change w.r.t. product positive continuous additive functional (PCAF) (A^1, A^2) in a marginal way as follows. We break B^n into (B^{n_1}, B^{n_2}) and proceed the restriction of B^{n_j} to F_j by time-change w.r.t. the associated A^j. We remark that the PCAF A^j is a natural version of the local time of B^{n_j} on F_j ; see Kumagai(2002) and Hanson-Zähle(2006). The resulting processes is of independent components, and the heat kernel estimate is also obtained by direct applications of Chen-Kumagai(2003) and Kumagai(2002, Proposition 3.1). As it is mentioned in Kumagai(2002) and Hanson-Zähle(2006), we may use an isotropic α-stable motion in $\mathbb{R}^n$ instead of B^n; however the resulting process is *not* of independent components. The above construction can be proceeded for the triple product, and we then have the following concern.

We consider processes on a Cantor "dust" $C \subset \mathbb{R}^3 = \mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R}$. Let $C := C_1 \times C_2 \times C_3$, and each C_j is a Cantor subset of $\mathbb{R}$ with Hausdorff dimension s_j, $0 < s_j < 1$ so that C is a s-set in $\mathbb{R}^3$ with $s = s_1 + s_2 + s_3$. We may have *two* processes on C. One Y_1 is constructed by restricting B^3 to C as that is done in Kumagai(2002, §2.3), w.r.t. a single PCAF, whenever $s > 1$. Another Y_2 is constructed as above, by time-change w.r.t. to the triple product PCAF (A^1, A^2, A^3). Are these two processes Y_1, Y_2 different in view of the dimension formulae? We may expect from Theorem 6.1 and Kumagai(2002, Proposition 3.1) that $\dim Y_1[0,1] = s - 1$. On the other hand in view of the proposed Theorem 6.2 we may conjecture that $\dim Y_2[0,1] = s$. Thus these two Y_1, Y_2 should be different, even when $s_1 = s_2 = s_3$. However both the above formulae need to justify, since the full kernel estimate is *not* easy to estimate(we may only establish, for example, a certain Nash inequality).

§ 8. Product Dirichlet forms

The Dirichlet form of the product fractal is proposed by Strichartz (2005) as, for $u, v \in L^2(F_1 \times F_2, \mu_1 \times \mu_2)$,

$$\mathcal{E}(u,v) =$$
$$\int_{F_2} \mathcal{E}_1(u(\cdot, x_2), v(\cdot, x_2)) d\mu_2(dx_2) + \int_{F_1} \mathcal{E}_2(u(x_1, \cdot), v(x_1, \cdot)) d\mu_1(x_1),$$

where $\mathcal{E}_j$ means the Dirichlet form on F_j.

In view of this and Kumagai(2002, Proposition 3.1), the subordinated process Y on $F = F_1 \times F_2$ in **§5**, in case it is of independent components, is comparable to the following Dirichlet form: for $x = (x_1, x_2), y = (y_1, y_2), \mu = \mu_1 \times \mu_2$,

$$\mathcal{E}^{\alpha_1,\alpha_2}(u,v) = \int_F \int_F \frac{(u(x)-u(y))(v(x)-v(y))}{\prod_{j=1,2} |x_j - y_j|^{d_f(j)+\alpha_j}} \mu(dx)\mu(dy).$$

§ 9. Possible perspective

The following discussion with Kumagai shows that we may still have far distance to what we want to follow the viewpoints of Hunita, mentioned in §1, for processes on fractals. In Hambly-Kumagai(2004), they construct a type of diffusion on SG such that the walk dimensions along the diagonal and along the horizontal are *different*. We subordinate such a diffusion by a single subordinator, and ask, say, the Hausdorff dimension of the resulting stable-like processes. With respect to the resistance metric, the dimension formula should be the same form as Theorem 6.1. However, usually we prefer, at least in view of dimension, the Euclidean metric, and in Hambly-Kumagai construction the resistance and the Euclidean metrics are not comparable explicitly.

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