RIMS Kôkyûroku Bessatsu **B6**
(2008), 221–236

Some Topics connected with Gaugeability for Feynman-Kac Functionals

By

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Abstract

For symmetric Markov processes, an analytic criterion for the integrability of Feynman-Kac functionals was obtained in Z.-Q.Chen [6], M. Takeda [37] and M. Takeda and T. Uemura [46]. We explain how it is used in the study of the Brownian motions on Riemannian manifolds and the symmetric stable processes. We discuss the stability of heat kernels, the ultracontractivity of Feynman-Kac semigroups, the expectations of the number of branches hitting closed sets in branching Brownian motions, the differentiability of spectral functions, and the L^p-independence of spectral bounds for Schrödinger type operators.

§ 1. Introduction

Let $\mathbf{M} = (\mathbb{P}_x, X_t)$ be an m-symmetric Markov process on a locally compact separable metric space X. Let μ be a smooth measure and A_t^μ the positive continuous additive functional (PCAF) in the Revuz correspondence to μ. Then the measure μ is said to be *gaugeable* on an open set $D \subset X$ if

$$\sup_{x\in D} \mathbb{E}_x\left(\exp(A^\mu_{\tau_D})\right) < \infty, \tag{1.1}$$

where τ_D is the first exit time from D. The objective of this paper is to survey the topics connected with the gaugeability.

We treat two classes of positive Radon measures, $\mathcal{K}_\infty$ and $\mathcal{S}_\infty$; a measure in $\mathcal{K}_\infty$ (resp. in $\mathcal{S}_\infty$) is said to be *Green-tight* (resp. *conditional Green-tight*) (see Definition 2.1 below). For a Brownian motion, Zhao [50] introduced a class of Green-tight measures and Chen [6] generalized the notions of Green-tightness and conditional Green-tightness for more general transient Markov processes. Let $\mathbf{M}^D$ be the absorbing process killed upon leaving $D \subset X$ and assume that $\mathbf{M}^D$ is transient. We denote by $\mathcal{S}^D_\infty$ the class

2000 Mathematics Subject Classification(s): 60J45, 60J40, 35J10
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of conditional Green-tight measures associated with $\mathbf{M}^D$. Chen [6] and Takeda [37] established an analytic condition for $\mu \in \mathcal{S}_\infty^D$ being gaugeable on D; define

$$\lambda(\mu; D) = \inf\left\{\mathcal{E}^D(u,u) : \ u \in \mathcal{D}(\mathcal{E}^D), \ \int_D u^2(x)\mu(dx) = 1\right\}$$

where $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$ is the Dirichlet space generated by the symmetric Markov process $\mathbf{M}^D$ (e.g. [13, Theorem 4.4.2]). Then the gaugeability of μ on D is equivalent to that of $\lambda(\mu; D) > 1$, which is also equivalent to the subcriticality of Schrödinger operators (Theorem 2.2). Employing this fact, we showed the stability of heat kernel estimates of Schrödinger operators ([39], [40]), the L^p-independence of spectral bounds of Schrödinger semigroups ([36], [43]), and the differentiability of spectral functions with applications to large deviations for additive functionals ([38], [42], [44], [45], [47]). Furthermore, it was applied to branching Markov processes ([8], [27], [28], [39], [41]). In this paper, we explain these results by using a Brownian motions on a Riemaniann manifold or a symmetric α-stable process as a typical example of diffusion processes or of pure jump Markov processes respectively.

§ 2. Notations and some facts

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $\mathbf{M} = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, \mathbb{P}_x, X_t, \zeta)$ be an m-symmetric Markov process on X. Here $\{\mathcal{M}_t\}_{t\geq 0}$ is the minimal (augmented) admissible filtration, θ_t, $t \geq 0$, are the shift operators satisfying $X_s(\theta_t) = X_{s+t}$ identically for $s, t \geq 0$. Let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of M, and ζ is the lifetime of $\mathbf{M}$, $\zeta = \inf\{t \geq 0 : X_t = \infty\}$. Throughout this paper, we assume that the Markov process $\mathbf{M}$ is transient. We denote by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ the Dirichlet form generated by $\mathbf{M}$.

We treat mainly the Brownian motion on a Riemaniann manifold M, that is, the diffusion process generated by half the Laplace-Beltrami operator, $(1/2)\Delta$, and the symmetric α-stable process on $\mathbb{R}^d$, that is, the pure jump process generated by $-(1/2)(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$. The symmetrizing measure m is the Riemannian volume on M for the Brownian motion and the Lebesgue measure for the symmetric stable process. The Dirichlet form generated by the Brownian motion is written as

$$\mathcal{E}(u,v) = (-\frac{1}{2}\Delta u, v)_m = \frac{1}{2}\int_M (\nabla u, \nabla v)\ dm, \ u, \ v \in C_0^\infty(M),$$

where $(\cdot,\cdot)_m$ is the inner product on $L^2(M; m)$ and $C_0^\infty(M)$ denotes the set of all smooth functions with compact support. The domain $\mathcal{D}(\mathcal{E})$ is the closure of $C_0^\infty(M)$ with respect to the norm $\sqrt{\mathcal{E}(\ ,\ ) + (\ ,\ )_m}$. The Dirichlet space $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$ generated by the symmetric α-stable process is given by

$$\mathcal{E}^{(\alpha)}(u,v) = K \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}dxdy,$$

$$\mathcal{D}(\mathcal{E}^{(\alpha)}) = \left\{u \in L^2(\mathbb{R}^d) : \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}dxdy < \infty\right\}$$

($K = \alpha 2^{\alpha-3}\pi^{-\frac{d+2}{2}}\sin(\frac{\alpha\pi}{2})\Gamma(\frac{d+\alpha}{2})\Gamma(\frac{\alpha}{2})$). Every function u in $\mathcal{D}(\mathcal{E})$ admits a quasi-continuous version $\tilde{u}$ (see [13, Theorem 2.1.3]). In the sequel we always assume that every function $u \in \mathcal{D}(\mathcal{E})$ is represented by its quasi-continuous version.

Let $D \subset X$ be an open set and $\mathbf{M}^D = (P_x, X_t^D)$ be the absorbing process:

$$X_t^D = \begin{cases} X_t & t < \tau_D \\ \infty & t \geq \tau_D \end{cases}$$

($\tau_D = \inf\{t > 0 : X_t \notin D\}$). Let $G_\beta^D(x,y)$ ($\beta \geq 0$) be the β-Green function of $\mathbf{M}^D$. We simply write $G^D(x,y)$ for $G_0^D(x,y)$. Following [6], we make

Definition 2.1.

(1) (i) *A positive Radon measure μ on D is said to be in the Kato class* (*in notation, $\mu \in \mathcal{K}^D$*)*, if*

$$\lim_{\beta\to\infty}\sup_{x\in D}\int_D G_\beta^D(x,y)d\mu(y) = 0.$$

(ii) *A measure $\mu \in \mathcal{K}^D$ is said to be Green-tight* (*in notation, $\mu \in \mathcal{K}_\infty^D$*)*, if for any $\epsilon > 0$, there exists a compact set $K \subset D$ such that*

$$\sup_{x\in D}\int_{K^c} G^D(x,y)d\mu(y) \leq \epsilon.$$

(2) *A measure μ on D is said to be conditionally Green-tight* (*in notation, $\mu \in \mathcal{S}_\infty^D$*)*, if for any $\epsilon > 0$, there exist a compact set $K \subset D$ and $\delta > 0$ such that*

$$\sup_{(x,z)\in D\times D\setminus\triangle}\int_{K^c}\frac{G^D(x,y)G^D(y,z)}{G^D(x,z)}\mu(dy) \leq \epsilon$$

and for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{(x,z)\in D\times D\setminus\triangle}\int_B\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}\mu(dy) \leq \epsilon$$

When $D = X$, we remove D in the notations; we simply denote $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{S}_\infty$ for $\mathcal{K}^X$, $\mathcal{K}_\infty^X$ and $\mathcal{S}_\infty^X$. It is known in the remark after [6, Definition 3.1] that

$$\mathcal{S}_\infty^D \subset \mathcal{K}_\infty^D \subset \mathcal{K}^D. \tag{2.1}$$

For $\mu \in \mathcal{K}$, let A_t^μ be the positive continuous additive functional of $\mathbf{M}$ in the Revuz correspondence to the measure μ: for any non-negative Borel function $f \in \mathcal{B}_+(X)$ and γ-excessive function h,

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}_{hm}\left(\int_0^t f(X_s)dA_s^\mu\right) = \int_X f(x)h(x)d\mu(x) \tag{2.2}$$

(see [13, p.188]). Then it is shown in [6] that for $\mu \in \mathcal{K}_\infty^D$

$$\sup_{x\in D}\mathbb{E}_x(A_{\tau_D}^\mu) = \sup_{x\in D}\int_D G_D(x,y)\mu(dy) < \infty. \tag{2.3}$$

For a measure μ in $\mathcal{K}^D$, define

$$\lambda(\mu; D) = \inf\left\{\mathcal{E}(u,u) :\ u \in C_0^\infty(D),\ \int_D u^2(x)\mu(dx) = 1\right\}, \tag{2.4}$$

where $C_0^\infty(D)$ is the set of smooth functions with compact support in D. On account of Lemma 3.1 in [35], we see that $\lambda(\mu; D)$ is the principal eigenvalue of the time changed process of $\mathbf{M}^D$ by $A_{\tau_D\wedge t}^\mu$. We abbreviate $\lambda(\mu; X)$ as $\lambda(\mu)$.

Let $p_t^{\mu,D}(x,y)$ be the integral kernel of the Feynman-Kac semigroup,

$$\mathbb{E}_x(\exp(A_t^\mu)f(X_t); t < \tau_D) = \int_D p_t^{\mu,D}(x,y)f(y)dy,$$

and $G^{\mu,D}(x,y)$ its Green function, $G^{\mu,D}(x,y) = \int_0^\infty p_t^{\mu,D}(x,y)dt$. We then have

Theorem 2.2. ([6], [37]) *Let $\mu \in \mathcal{S}_\infty^D$. Then the following conditions are equivalent*:

(i) (gaugeability) $\sup_{x\in D}\mathbb{E}_x(e^{A_{\tau_D}^\mu}) < \infty$;

(ii) (subcriticality) $G^{\mu,D}(x,y) < \infty$ *for* $x, y \in D$, $x \neq y$;

(iii) $\lambda(\mu; D) > 1$.

Theorem 2.2 suggests us that $\lambda(\mu; D)$ accurately measures the size of the pair (μ, D). This is a key idea of the proofs of the theorems in this paper.

§ 3. Stability of heat kernels

Let M be a complete, non-compact, Riemannian manifold with dimension d. Let $d(x,y)$ be the geodesic distance. We suppose that the heat kernel $p_t(x,y)$ associated with $(1/2)\Delta$ satisfies global Gaussian lower and upper bounds: for every $x, y \in M$ and $t > 0$,

$$\frac{C_1 \exp\left(-c_1 \frac{d^2(x,y)}{t}\right)}{m(B(x,\sqrt{t}))} \le p_t(x,y) \le \frac{C_2 \exp\left(-c_2 \frac{d^2(x,y)}{t}\right)}{m(B(x,\sqrt{t}))}, \tag{3.1}$$

where C_1, c_1, C_2, and c_2 are positive constants and $B(x,r)$ is the geodesic ball of radius r centered at $x \in M$. Following [16], we say that the heat kernel $p_t(x,y)$ satisfies the *Li-Yau estimate*, if it has the estimate (3.1). For a measure μ in $\mathcal{S}_\infty$, let $p_t^\mu(x,y)$ be the heat kernel associated with the Schrödinger operator, $(1/2)\Delta + \mu$. We establish a necessary and sufficient condition on the potential μ for the heat kernel $p_t^\mu(x,y)$ also to satisfy the Li-Yau estimate.

Theorem 3.1. ([40]) *Suppose that $\mu \in \mathcal{S}_\infty$. Then $p_t^\mu(x,y)$ satisfies the Li-Yau estimate if and only if $\lambda(\mu) > 1$.*

Theorem 3.1 is an extension of [49, Theorem C] and [16, Theorem 10.5], where they considered the case that the potential μ is absolutely continuous with respect to the Riemannian volume. We explain how to prove the "if" part; let $h(x) = \mathbb{E}_x(e^{A_\infty^\mu})$. If $\lambda(\mu) > 1$, then

$$1 \le h(x) \le \sup_{x \in M} \mathbb{E}_x(e^{A_\infty^\mu}) < \infty$$

by Theorem 2.2, and h satisfies

$$\frac{1}{2}\Delta h + \mu h = 0.$$

Define a multiplicative functional L_t^h by

$$L_t^h = \frac{h(X_t)}{h(X_0)} \exp(A_t^\mu),$$

and denote by $\mathbf{M}^h$ the transformed process by L_t^h. Then $\mathbf{M}^h$ is an h^2m-symmetric Markov process and its transition density $p_t^h(x,y)$ with respect to h^2m is given by

$$p_t^h(x,y) = \frac{p_t^\mu(x,y)}{h(x)h(y)}.$$

Hence we see that the Li-Yau estimate of $p_t^\mu(x,y)$ is equivalent to that of $p_t^h(x,y)$. For the proof of the Li-Yau estimate of $p_t^h(x,y)$, it is enough to show that the Dirichlet form of $\mathbf{M}^h$ is equivalent to the one of the Brownian motion. Indeed, the Li-Yau estimate of $p_t^h(x,y)$ follows by employing a celebrated theorem proved independently by Grigor'yan [15] and Saloff-Coste [25]; the heat kernel satisfies the Li-Yau estimate if and only if its Dirichlet form satisfies the Poincaré inequality and its symmetrizing measure satisfies the volume doubling condition.

This approach for the proof of Theorem 3.1 is exactly the same as that in [16]; however the identification of the transformed Dirichlet form becomes difficult because of the singularity of the potential μ. We overcome this by applying a theorem in [20] on the uniqueness of Silverstein extensions and Theorem 2.8 in [7] on the identification of Girsanov transformed Dirichlet forms.

Example 3.2. *Let M be a spherically symmetric Riemannian manifold with a pole o. Let $B_r = \{x \in M : d(o,x) < r\}$ and ∂B_r its boundary. Let σ_r be the surface measure of ∂B_r and $S(r)$ the area of ∂B_r, $S(r) = \sigma_r(\partial B_r)$. The measure σ_r belongs to $\mathcal{S}_\infty$. It is known in* [14] *that M is non-parabolic if and only if*

$$\int_1^\infty \frac{dr}{S(r)} < \infty.$$

and in [37] *that*

$$\lambda(\sigma_r) > 1 \Longleftrightarrow S(r)\int_r^\infty \frac{1}{S(u)}du < \frac{1}{2}. \tag{3.2}$$

In particular, for $M = \mathbb{R}^d$ $(d \geq 3)$

$$\lambda(\sigma_r) > 1 \Longleftrightarrow \frac{r}{d-2} < \frac{1}{2}, \tag{3.3}$$

and thus $p_t^{\sigma_r}(x,y)$ satisfies the Li-Yau estimate, if and only if $r < (d-2)/2$.

Suppose that $d = 3$ and $\lambda(\mu) = 1$. Then $(1/2)\Delta + \mu$ is critical, that is, $(1/2)\Delta + \mu$ does not admit the minimal positive Green function but admits a positive continuous $(1/2)\Delta + \mu$-harmonic function. This harmonic function is called a ground state and is uniquely determined up to constant multiplication. We proved in [45] *that the ground state h satisfies*

$$\frac{c}{|x|} \leq h(x) \leq \frac{C}{|x|}, \quad |x| \geq 1$$

Hence we see from Example 10.15 *in* [16] *that*

$$p_t^\mu(x,y) \asymp \frac{1}{t^{3/2}}\left(1 + \frac{\sqrt{t}}{\langle x\rangle}\right)\left(1 + \frac{\sqrt{t}}{\langle y\rangle}\right)\exp\left(-c\frac{|x-y|^2}{t}\right)$$

$(\langle x\rangle := 1 + |x|)$.

Theorem 3.1 can be extended to signed measures. Let $\mu = \mu^+ - \mu^-$ be a signed measure such that $\mu^+ \in \mathcal{S}_\infty$ and μ^- is Green-bounded,

$$\sup_{x \in M} \int_M G(x,y) d\mu^-(y) < \infty.$$

Then the Li-Yau estimate of the heat kernel $p_t^\mu(x,y)$ is equivalent to

$$\inf \left\{ \mathcal{E}(u,u) + \int_M u^2 d\mu^- : u \in \mathcal{D}(\mathcal{E}), \int_M u^2 d\mu^+ = 1 \right\} > 1.$$

For a signed measure μ we also denote by $\lambda(\mu)$ the left hand side above.

For the symmetric α-stable process, we have

Theorem 3.3. ([39]) *Suppose that $d > \alpha$. Let $\mu \in \mathcal{S}_\infty$ with finite energy integral, that is, $\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} |x-y|^{\alpha-d} d\mu(x) d\mu(y) < \infty$. Then*

$$\lambda(\mu) > 1 \Longleftrightarrow \|p_t^\mu\|_{1,\infty} \leq \frac{c}{t^{d/\alpha}}, \quad t > 0. \tag{3.4}$$

Using the argument in [29, Theorem B.1.1], we see that if 2μ is gaugeable on $\mathbb{R}^d$, the ultracontractivity in the right hand side of (3.4) holds. In the proof in [29], the Schwarz inequality in the Feynman-Kac formula and the duality argument were used, which is the reason why the gaugeability of 2μ is required.

Bass and Levin [3] proved that if the Lévy measure of a symmetric pure jump process is equivalent to the one of the symmetric α-stable process, then the transition probability density is also equivalent to the one of the symmetric α-stable process. Employing this theorem in [3] instead of the theorem due to Grigor'yan and Saloff-Coste, we can show a stronger result than Theorem 3.3 in the same way as that in Theorem 3.1:

Theorem 3.4. *Suppose that $\mu \in \mathcal{S}_\infty$ is of finite energy integral. Then the heat kernel of $-1/2(-\Delta)^{\alpha/2} + \mu$ satisfies*

$$C_1 \left(\frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x-y|} \right) \leq p^\mu(t,x,y) \leq C_2 \left(\frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x-y|} \right), \tag{3.5}$$

if and only if $\lambda(\mu) > 1$.

§ 4. Branching symmetric Markov processes

In [21], R. Z. Khas'minskii gave a sufficient condition for the integrability of Feynman-Kac functionals, so called, Khas'minskii lemma. He applied it to branching diffusion processes and obtained a sufficient condition that the expectation of the number of branches hitting the outside of a bounded domain is finite. We consider in [41] branching Brownian motions on Riemannian manifolds and give a necessary and sufficient condition for the expectation of the number of branches hitting a closed set being finite.

Let M be a complete, non-compact Riemannian manifold and $\bar{\mathbf{M}} = (\bar{B}_t, \bar{\mathbb{P}}_x)$ be a *branching Brownian motion* on M with *branching rate* k and *branching mechanism* $\{p_n(x)\}_{n\geq 0, n\neq 1}$. The branching rate k is a positive measure on M and the branching mechanism satisfies $\sum_{n=0}^{\infty} p_n(x) = 1$. We define the *intensity of population growth* by $\mu(dx) = (Q(x)-1)k(dx)$, $Q(x) = \sum_{n=2}^{\infty} np_n(x)$. Assume that $\sup_{x\in M} Q(x) < \infty$. For a closed subset K of positive capacity, we denote by N_K the number of branches of $\bar{B}_t$ ever hitting K. We then have

Theorem 4.1. ([41]) *Assume that $\mu \in \mathcal{S}_\infty^D$ $(D = M \setminus K)$. Then it holds that*

$$\sup_{x\in D} \bar{\mathbb{E}}_x(N_K) < \infty \Longleftrightarrow \lambda(\mu; D) > 1.$$

Lalley and Sellke [22], A. Grigor'yan and M. Kelbert [17] considered branching Brownian motions on Riemannian manifolds. In particular, in [17], they used *gauge functions* to study the recurrence and transience of branching Brownian motions. Theorem 4.1 is motivated by [17]. A new point is that we treat branching Brownian motions with singular branching rate and establish a necessary and sufficient condition for the finiteness of expectations, while we restrict the branching rate to the class $\mathcal{S}_\infty$. For the proof of Theorem 4.1, we show that for $\mu \in \mathcal{S}_\infty^D$ with $\mathrm{Cap}(M \setminus D) > 0$

$$\sup_{x\in D} \mathbb{E}_x(\exp(A^\mu_{\tau_D})) < \infty \Longleftrightarrow \sup_{x\in D} \mathbb{E}_x(\exp(A^\mu_{\tau_D}); \tau_D < \infty) < \infty. \tag{4.1}$$

In [9], the gaugeability of μ on D is defined by the right hand side in (4.1). The equation (4.1) tells us that for a measure in $\mathcal{S}_\infty^D$, two definitions of the gaugeability are equivalent.

We discussed in [39] the same topic for branching symmetric stable processes.

Example 4.2. ([26]) *Suppose that $d = 1$ and $1 < \alpha < 2$. Let $k = \delta_a$, $a \neq 0$, the Dirac measure at a and $p_2(x) = 1$. Then*

$$\lambda(\delta_a; \mathbb{R} \setminus \{0\}) = -\frac{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}{2|a|^{\alpha-1}}.$$

and thus

$$\sup_{x\in\mathbb{R}\setminus\{0\}} \bar{\mathbb{E}}(N_{\{0\}}) < \infty \Longleftrightarrow 0 < |a| < \left(-\frac{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}{2}\right)^{1/(\alpha-1)}.$$

§ 5. L^p-independence of spectral bounds

Let $\mathcal{L}$ be an m-symmetric Markov generator on X and μ a certain signed Kato measure. We study the Schrödinger type operator $\mathcal{H}^\mu = \mathcal{L} + \mu$ on $L^p(X;m)$. In particular, we prove that the growth of the operator norm of its semigroup $p_t^\mu := \exp(t\mathcal{H}^\mu)$ is independent of p. More precisely, we define the spectral bound of p_t^μ by

$$\alpha_p(\mu) = -\lim_{t\to\infty} \frac{1}{t} \log \|p_t^\mu\|_{p,p}, \quad 1 \leq p \leq \infty,$$

where $\|p_t^\mu\|_{p,p}$ is the operator norm of p_t^μ from $L^p(X;m)$ to $L^p(X;m)$. Then our aim is to show that $\alpha_p(\mu)$ is independent of p. Needless to say, it is impossible to show the independence for all symmetric Markov processes and associated Kato measures. In fact, let us consider the Brownian motion on a hyperbolic space $\mathbf{H}^d$ and the zero measure as μ. Then, $\alpha_\infty(\mu)$ equals zero because of the conservativeness of the Brownian motion, while $\alpha_2(\mu)$ equals $(d-1)^2/8$ (§5.7 in [10]). Hence we suppose that the Markov semigroup $p_t := \exp(t\mathcal{L})$ satisfies the four conditions:

(I) (Irreducibility) If a Borel set A is $\{p_t\}$-invariant, i.e., $p_t(I_A f)(x) = I_A(x) p_t f(x)$ m-a.e. for any $f \in L^2(X;m) \cap \mathcal{B}_b(X)$ and $t > 0$, then A satisfies either $m(A) = 0$ or $m(X \setminus A) = 0$. Here $\mathcal{B}_b(X)$ is the space of bounded Borel functions on X.

(II) (Conservativeness) $P_x(\zeta = \infty) = 1$ for all $x \in X$.

(III) (Feller Property) $p_t(C_\infty(X)) \subset C_\infty(X)$ for each $t > 0$ and $\lim_{t\to 0} p_t f(x) = f(x)$, $x \in X$, for $f \in C_\infty(X)$, where $C_\infty(X)$ is the space of continuous functions on X vanishing at infinity.

(IV) (Regularity of Transition Density) There exists a continuous transition density $p_t(x,y) \in C((0,\infty) \times X \times X)$ such that

$$p_t f(x) = \int_X p_t(x,y) f(y) dm(y), \qquad f \in \mathcal{B}_b(X).$$

For example, the semigroup of the Brownian motion on the hyperbolic space satisfies the four conditions. We also assume that the potential μ is in $\mathcal{K}_\infty$.

For a classical Schrödinger operator $(1/2)\Delta + V$, B. Simon [30] proved the p-independence, and K.-Th. Sturm [32],[33] extended it to Schrödinger operators on Riemannian manifolds with non-negative Ricci curvature. For the proof of the p-independence, they used heat kernel estimates. Our approach is completely different; we use the arguments in Donsker-Varadhan's large deviation theory. Let $\mathbf{M} = (\mathbb{P}_x, X_t)$ be the m-symmetric Markov process generated by $\mathcal{L}$ and assume that it satisfies the four conditions (I)$\sim$(IV). We extend $\mathbf{M}$ to the Markov process $\widetilde{\mathbf{M}}$ on the one-point compactification X_∞ by making the adjoined point ∞ a trap. Then $\widetilde{\mathbf{M}}$ has the Feller property, $\widetilde{p}_t(C(X_\infty)) \subset C(X_\infty)$, while it is no longer strong Feller. Here $\widetilde{p}_t$ is the transition semigroup of $\widetilde{\mathbf{M}}$. In the proof of the large deviation upper bound for Markov processes with compact state space, we need only the Feller property. Thus we have the following upper bound; let $\mathcal{P}(X_\infty)$ be the set of probability measures on X_∞ and define a function I_μ on $\mathcal{P}(X_\infty)$ by

$$I_\mu(\nu) = - \inf_{\phi \in \mathcal{D}_{++}(\mathcal{H}^\mu)} \int_X \frac{\mathcal{H}^\mu \phi}{\phi}(x) d\nu(x),$$

where $\mathcal{D}_{++}(\mathcal{H}^\mu)$ is a suitable domain of the operator $\mathcal{H}^\mu$ (See [43, Section 3]). Then

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \mathbb{E}_x\left(\exp(A_t^\mu)\right) \leq - \inf_{\nu \in \mathcal{P}(X_\infty)} I_\mu(\nu). \tag{5.1}$$

We would like to make two remarks on the equation (5.1). First, since A_t^μ is not generally regarded as a function of the empirical measure,

$$L_t(A) = \frac{1}{t}\int_0^t I_A(X_s) ds, \quad A \in \mathcal{B}(X),$$

we can not directly use the Donsker-Varadhan large deviation theory ([12]) for the proof of the equation (5.1); however in [34] we extended it to Markov processes with Feynman-Kac functionals. We here apply the upper bound established in [34]. Second, the function I_μ is defined on the space of probability measures on X_∞ not on X. Hence, it happens that $\nu(\{\infty\}) > 0$ and, in this sense, the point ∞ makes a contribution to the function I_μ. We learn this idea from [4] and [19]; accounting the contribution to I_μ-function from ∞, A. Budhiraja and P. Dupuis proved large deviation principles of empirical measures for Markov processes without stability property and H. Kaise and S. J. Sheu studied the asymptotic of Feynman-Kac functionals.

We prove in [43] that if $\alpha_2(\mu) \leq 0$, then

$$\inf_{\nu \in \mathcal{P}(X_\infty)} I_\mu(\nu) = \alpha_2(\mu), \tag{5.2}$$

which implies that $\alpha_\infty(\mu) \geq \alpha_2(\mu)$ because the left hand side of (5.1) is equal to $-\alpha_\infty(\mu)$. The inequality, $\alpha_\infty(\mu) \leq \alpha_2(\mu)$, holds generally by the symmetry and the positivity of p_t^μ. Hence we see that if $\alpha_2(\mu) \leq 0$, then $\alpha_p(\mu) = \alpha_2(\mu)$, $\ 1 \leq p \leq \infty$. On the other hand, if $\alpha_2(\mu) > 0$, then $\alpha_\infty(\mu) = 0$. Indeed, it follows from the same argument as that in the paragraph under [35, Corollary 4.1] that if $\alpha_2(\mu) > 0$, then $\lambda(\mu) > 1$. Combining Theorem 2.2 with [6, Corollary 2.9], we have

$$\sup_{x\in X} \mathbb{E}_x\left(\exp(A_t^\mu)\right) \leq \sup_{x\in X} \mathbb{E}_x\left(\sup_{0\leq t<\infty} \exp(A_t^\mu)\right) < \infty.$$

Moreover, since the measure μ^- in $\mathcal{K}_\infty$ is Green-bounded, $\sup_{x\in X}\mathbb{E}_x(A_\infty^{\mu^-}) = \sup_{x\in X} G\mu^-(x) < \infty$,

$$\sup_{x\in X}\mathbb{E}_x\left(\exp(A_t^{\mu})\right) \geq \exp\left(-\sup_{x\in X}\mathbb{E}_x(A_\infty^{\mu^-})\right) > 0,$$

Hence if $\alpha_2(\mu) > 0$, then $\alpha_\infty(\mu) = 0$.

Theorem 5.1. ([43]) *Assume* (I)$\sim$(IV). *Let $\mu \in \mathcal{K}_\infty - \mathcal{K}_\infty$.*

(i) *If $\alpha_2(\mu) \leq 0$, then $\alpha_p(\mu) = \alpha_2(\mu)$, $\quad 1 \leq p \leq \infty$;*

(ii) *If $\alpha_2(\mu) > 0$, then $\alpha_\infty(\mu) = 0$.*

Example 5.2. *We use the notations in Example 3.2. Let $M = \mathbb{H}^d$, d-dimensional hyperbolic space. We then see that*

$$\alpha_2(\theta\sigma_r) \leq 0 \Longleftrightarrow \lambda(\theta\sigma_r) \leq 1 \left(\Longleftrightarrow \theta \geq \frac{1}{2S(r)\int_r^\infty \frac{dr}{S(r)}}\right).$$

Put $G(r) = 2S(r)\displaystyle\int_r^\infty \frac{1}{S(r)}dr$. Then

$$G(r) = \begin{cases} (e^r - e^{-r})\log\left(\dfrac{e^r+1}{e^r-1}\right) & d = 2 \\ \dfrac{e^{2r}-1}{e^{2r}} & d = 3 \\ 2(e^r - e^{-r})^{d-1}\displaystyle\int_r^\infty \frac{1}{(e^r - e^{-r})^{d-1}}dr & d \geq 4. \end{cases}$$

For $d = 2$, $G(r)$ is strictly increasing, $\lim_{r\to 0} G(r) = 0$, and $\lim_{r\to\infty} G(r) = 2$. For $d \geq 3$, $G(r) < 1$. Theorem 5.1 tells us that if $\theta \geq 1/G(r)$, then $\alpha_p(\theta\sigma_r) = \alpha_2(\theta\sigma_r)$ for $1 \leq p \leq \infty$, and if $\theta \leq 1/G(r)$, then $\alpha_\infty(\theta\sigma_r) = 0$ and $\alpha_2(\theta\sigma_r) > 0$. This says that the p-independence of $\alpha_p(\mu)$ is recovered by adding a negative Green-tight potential μ such that $\alpha_2(\mu) \leq 0$.

Consider a spatially homogeneous symmetric Lévy process with Lévy measure J. The Lévy measure J is said to be *exponentially localized* if there exists a positive constant δ such that

$$\int_{|x|>1} e^{\delta|x|}J(dx) < \infty. \tag{5.3}$$

For example, the Lévy measure of the *relativistic Schrödinger process*, the symmetric Lévy process generated by $\sqrt{-\Delta + m^2} - m$, $m > 0$, satisfies (5.3) ([5]). We proved in

[34] that if J is exponentially localized, then $\alpha_p(\mu)$ is independent of p for $\mu \in \mathcal{K} - \mathcal{K}$. On the other hand, the symmetric α-stable process does not satisfies (5.3). However, Theorem 5.1 implies that for $\mu \in \mathcal{K}_\infty - \mathcal{K}_\infty$, $\lambda_p(\mu)$ is independent of p. Indeed, if $\mu \in \mathcal{K}_\infty$, then the embedding of $\mathcal{D}(\mathcal{E}^{(\alpha)})$ into $L^2(\mu)$ is compact ([34]), which implies that $\lambda_2(\mu) \leq 0$ for any $\mu \in \mathcal{K}_\infty - \mathcal{K}_\infty$. Therefore we see the existence of the moment generating function of A_t^μ, $\mu \in \mathcal{K}_\infty - \mathcal{K}_\infty$;

Corollary 5.3. *For any* $\mu \in \mathcal{K}_\infty - \mathcal{K}_\infty$

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left(\exp(\theta A_t^\mu)\right) = -\alpha_2(\theta\mu), \quad \theta \in \mathbb{R}^1. \tag{5.4}$$

§ 6. Differentiability of spectral functions

In this section we denote by $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$, $0 < \alpha \leq 2$, the Dirichlet form generated by a symmetric α-stable process and μ a positive Radon measure in the Kato class. Denote by $\mathcal{H}^{\theta\mu}$ the Schrödinger type operator $-(1/2)(-\Delta)^{\alpha/2} + \theta\mu$, $\theta \in \mathbb{R}^1$ and define its *spectral function* $C(\theta)$ by

$$\begin{aligned} C(\theta) &= -\inf\{\lambda : \lambda \in \sigma(\mathcal{H}^{\theta\mu})\} \\ &= -\inf\left\{\mathcal{E}^{(\alpha)}(u,u) - \theta \int_{\mathbb{R}^d} u^2 d\mu : u \in \mathcal{D}(\mathcal{E}^{(\alpha)}),\ \int_{\mathbb{R}^d} u^2 dx = 1\right\}, \end{aligned}$$

where $\sigma(\mathcal{H}^{\theta\mu})$ is the spectrum of $\mathcal{H}^{\theta\mu}$. By the spectral theorem, $C(\theta)$ is identical to $-\alpha_2(\theta\mu)$. We consider the differentiability of the function $C(\theta)$. When $\alpha = 2$ and the potential μ is a function in a certain Kato class, Arendt and Batty [2] proved that the spectral function is differentiable at $\theta = 0$ and its derivative equals zero ([2, Corollary 2.10]). Using a large deviation for additive functionals of the Brownian motion, Wu [48] obtained a necessary and sufficient condition for the spectral function being differentiable at $\theta = 0$. In [38] and [44] we extended Wu's result to measures in the Kato class. Furthermore, we showed that if $d \leq 4$, then the spectral function is differentiable on $\mathbb{R}^1$ for $\mu \in \mathcal{K}_\infty$.

Theorem 6.1. ([45]) *If* $d \leq 2\alpha$ *and* $\mu \in \mathcal{K}_\infty$*, then the spectral function* $C(\theta)$ *is differentiable for all* $\theta \in \mathbb{R}^1$.

To prove the differentiability of the spectral function at $\theta = 0$, one of authors used in [38] a well-known property of the Brownian motion; if $d \leq 2$, the Brownian motion is a Harris recurrent process with infinite invariant measure, the Lebesgue measure. However, since the symmetric α-stable process is transient for $\alpha < d$, the arguments in

[38] can not be used immediately for the proof of Theorem 6.1. To overcome this, we prepared criticality theory for the Schrödinger type operator $\mathcal{H}^{\theta\mu}$. More precisely, let

$$\theta^+ = \inf\{\theta > 0 : C(\theta) > 0\}. \tag{6.1}$$

We proved that if $\alpha < d$, then the operator $\mathcal{H}^{\theta^+\mu}$ is *critical*, that is, $\mathcal{H}^{\theta^+\mu}$ does not admit the minimal positive Green function (i.e. non-subcriticality) but admits a positive continuous $\mathcal{H}^{\theta^+\mu}$-harmonic function (this function is called a *ground state* and uniquely determined up to constant multiplication.). Moreover, we proved that $\mathcal{H}^{\theta^+\mu}$ is *null critical*, that is, the ground state does not belong to L^2 if and only if $d \le 2\alpha$. In fact, denoting by h the ground state, we showed in [45] that there exist positive constants c, C such that

$$\frac{c}{|x|^{d-\alpha}} \le h(x) \le \frac{C}{|x|^{d-\alpha}}, \quad |x| > 1. \tag{6.2}$$

When $\mathcal{H}^{\theta^+\mu}$ is null critical, the arguments in [38] still work for $\alpha < d \le 2\alpha$ through h-transform. This is a key idea of the proof of Theorem 6.1.

The equation (6.2) was shown by Murata [23] for Schrödinger operators on $\mathbb{R}^d$ and extended by Pinchover [24] to second order elliptic operators in a domain of $\mathbb{R}^d$. If $\mu = 0$, the criticality and the null criticality are equivalent to the recurrence and the null recurrence respectively. The equation (6.2) says that if $d > 2\alpha$, $\mathcal{H}^{\theta^+\mu}$ is *positive critical*, that is, the ground state belongs to L^2. Hence the transformed process has a finite invariant measure h^2dx and the argument in [38] does not work. In fact, using the argument in [31], we can show that $C(\theta)$ is not differentiable at $\theta = \theta^+$.

Our motivation of Theorem 6.1 lies in the proof of a large deviation principle for the continuous additive functional A_t^μ. The function $C(\theta)$ is regarded as a *logarithmic moment generating function* of the additive functional A_t^μ by Corollary 5.3 and its differentiability follows from Theorem 6.1. The Gärtner-Ellis Theorem (see [11]) yields the large deviation principle for additive functional A_t^μ; let $I(\lambda)$ be the Legendre transform of $C(\theta)$,

$$I(\lambda) = \sup_{\theta\in\mathbb{R}^1} \{\lambda\theta - C(\theta)\}, \quad \lambda \in \mathbb{R}^1.$$

We then have:

Theorem 6.2. ([42]) *Assume that $d \le 2\alpha$. Then for $\mu \in \mathcal{K}_\infty$, A_t^μ/t obeys the large deviation principle with rate function $I(\lambda)$.*

(i) *For any closed set $K \in \mathbb{R}^1$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x \left(\frac{A_t^\mu}{t} \in K\right) \le - \inf_{\lambda\in K} I(\lambda).$$

(ii) *For any open set $G \subset \mathbb{R}^1$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x \left(\frac{A_t^\mu}{t} \in G \right) \geq - \inf_{\lambda \in G} I(\lambda).$$

Example 6.3. *Let $d = 1$ and $\alpha > 1$. When $\mu = \delta_0$, the Dirac measure at the origin, the corresponding additive functional is identical to the local time at the origin. For $\theta > 0$, the principal eigenvalue of $-\frac{1}{2}(-\Delta)^{\alpha/2} - \theta\delta_0$ is calculated in* [27]*:*

$$C(\theta) = \begin{cases} \left(\dfrac{2^{1/\alpha}}{\alpha \sin\left(\frac{\pi}{\alpha}\right)} \right)^{\frac{\alpha}{\alpha-1}} \theta^{\frac{\alpha}{\alpha-1}} & \theta > 0 \\ 0 & \theta \leq 0. \end{cases}$$

As a result, we have

$$I(x) = \begin{cases} \dfrac{(\alpha-1)^{(\alpha-1)}}{2} \left(\sin \dfrac{\pi}{\alpha} \right)^\alpha x^\alpha & x > 0 \\ 0 & x \leq 0. \end{cases}$$

The fact in this example is due to J. Hawkes [18].

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