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# A short note on Lyons-Zheng decomposition in the non-sectorial case

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**Abstract**

Let $E$ be a Hausdorff space such that its Borel $\sigma$-algebra is generated by the set of all continuous functions on $E$. Given a conservative generalized Dirichlet form $\mathcal{E}$ with associated diffusion $(X_t, P_x)$, reference measure $m$, and domain $\mathcal{F}$, on $L^2(E;m)$. Suppose that the co-form with domain $\hat{\mathcal{F}}$ is also conservative and associated to a diffusion. Then

$$\widetilde{u}(X_t) - \widetilde{u}(X_t) = \frac{1}{2}M_t^{[u]} - \frac{1}{2}\left\{\hat{M}_T^{[u]}(r_T) - \hat{M}_{T-t}^{[u]}(r_T)\right\} + \frac{1}{2}\left\{N_t^{[u]} - \hat{N}_t^{[u]}\right\}; \quad 0 \le t \le T, \quad P_m\text{-}a.e. \tag{1}$$

Here $r_T$ is the time reversal operator, $M_t^{[u]}$ (resp. $\hat{M}_t^{[u]}$) is the MAF of finite energy, and $N_t^{[u]}$ (resp. $\hat{N}_t^{[u]}$) is the CAF of zero energy appearing in the Fukushima decomposition corresponding to the generalized Dirichlet form (resp. co-form) and $u$ is in some extended range $\mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$. In the symmetric case of course $M^{[u]} = \hat{M}^{[u]}$, and $N^{[u]} = \hat{N}^{[u]}$, and furthermore $\mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$ contains the domain of the form, so that we obtain the forward and backward martingale decomposition of T.J. Lyons and W. Zheng (cf. [3]). In the non-symmetric, but still sectorial case, the structure of decomposition (1) was first pointed out by M. Takeda through a typical example (cf. [9, Theorem 6.3.]).

## § 1. Notice

The following is an extension to a result in [9]. Decomposition (1) is obtained in [9, Theorem 6.3.] for a special, but typical example of finite-dimensional non-symmetric sectorial process, and we will make use of the main ideas presented there. However, since

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we are in the non-sectorial case we are confronted with structural problems. Especially, the range of functions $\widetilde{u}$ for which (1) is valid, isn't clear at all in the non-sectorial case. More precisely, in the sectorial case the domain of the form and the co-form coincides, so that the intersection is again the whole domain, but in the non-sectorial case we do not even know in general whether the intersection is non-empty. In this short note we will provide a suitable range $\mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$ of functions for the validity of decomposition (1) (see section 4). For notations and introduced objects see [10], [11], [12].

## § 2. Framework

Let $E$ be a Hausdorff space such that its Borel $\sigma$-algebra $\mathcal{B}(E)$ is generated by the set of all continuous functions on $E$. We assume that we are given a conservative diffusion $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_z)_{z\in E})$ which is associated to a generalized Dirichlet form $\mathcal{E}$ on $L^2(E; m)$. Then $\mathcal{E}$ is quasi-regular by [7, IV.Theorem 3.1]. The class of generalized Dirichlet forms (see [7]) is quite large and contains as special subclasses symmetric, sectorial, and time dependent Dirichlet forms (see e.g. [2], [4], [1], [5], [6]). Note that in contrast to the classical theory it isn't known whether regularity or quasi-regularity alone implies the existence of an associated process. An additional structural assumption on $\mathcal{F}$ is made in [7, IV.2,D3] in order to construct explicitly an associated process. Since we do not make use of this technical assumption and since it may be subject to some further progress, we instead prefer to assume merely the existence of $\mathbb{M}$. Since our $\mathbb{M}$ is a conservative diffusion we may assume that the path space $\Omega$ is given as the space of all contionuous paths $C([0,\infty) \to E)$, so that $X_t(w) = w(t)$.
Next we suppose that the co-form $\hat{\mathcal{E}}$ is also associated with a conservative diffusion $\hat{\mathbb{M}} = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\hat{P}_z)_{z\in E})$, and is hence also quasi-regular.
We will use the same notations as in [10], [11], [12]. In particular, notations with a superposed hat, such as e.g. $\hat{E}_m, \hat{P}_m, \hat{P}_x, \hat{G}_\alpha$, correspond to the co-process, or equivalently to the co-form. Since the capacities correponding to the form and co-form are the same we need not to precise whether quasi-continuous (q.c.), quasi-everywhere (q.e.), etc., is meant w.r.t. to $\mathcal{E}$ or $\hat{\mathcal{E}}$. In particular $\widetilde{u}$ (if it exists) always denotes a q.c. $m$-version of a given function $u : E \to \mathbb{R}$. By [10, Theorem 4.5(i)] we know that for $u \in \mathcal{F}$

$$A_t^{[u]} := \widetilde{u}(X_t) - \widetilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad P_x\text{-}a.s.\ q.e.\ x, \tag{2}$$

where $M_t^{[u]}$ is a martingale additive functional (of $\mathbb{M}$) of finite energy, and $N_t^{[u]}$ is a continuous additive functional (of $\mathbb{M}$) of zero energy. Here the energy of an additive functional $A_t$ (of $\mathbb{M}$) is defined as

$$e(A) = \frac{1}{2} \lim_{\alpha\to\infty} \alpha^2 E_m \left[ \int_0^\infty e^{-\alpha\, t} A_t^2 dt \right]$$

whenever the limit exists. We write $\overline{e}(A)$ if the lim in the above expression is replaced by $\overline{\lim}$. Since $\mathcal{E}$ satisfies the same assumptions than the co-form $\hat{\mathcal{E}}$ we also have a decomposition relative to the co-process and $u \in \hat{\mathcal{F}}$

$$\widetilde{u}(X_t) - \widetilde{u}(X_0) = \hat{M}_t^{[u]} + \hat{N}_t^{[u]}, \quad \hat{P}_x\text{-}a.s.\ q.e.\ x. \tag{3}$$

The main problem will be to specify a suitable range $\mathcal{F}^{ext}$ (resp. $\hat{\mathcal{F}}^{ext}$) of quasi-continuous functions $\widetilde{u}$ in $L^2(E;m)$ for which (2) (resp. for (3)) holds. Suitable means that the class of functions $\mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$ for which (1) will then hold covers all previous cases and is large enough for new applications.

We continue to proceed as in [9]. The corresponding result to the next lemma is only stated in [9]. We therefore present a proof. The proof doesn't depend on the sector condition, only on the conservativity of both forms. So we do not present anything new. Finally, before stating the next lemma, let us just define the time reversal operator:

We fix an arbitrary $T > 0$ and let $\Omega_T$ be the space of all continuous functions from $[0, T]$ to $E$. On $\Omega_T$ we consider the $\sigma$-algebra $\mathcal{F}_t^0 = \sigma\{X_s, 0 \le s \le t\}, 0 \le t \le T$. By natural restriction from $\Omega$ to $\Omega_T$ we may regard any probability measure $P_\mu$, $\mu \in \mathcal{P}(E)$, on $(\Omega, \mathcal{F}_\infty^0)$ as a probability measure on $(\Omega_T, \mathcal{F}_T^0)$. Let us denote by $\mathcal{F}_T^\mu$ (resp. $\mathcal{F}_t^\mu, 0 \le t \le T$) the completion of $\mathcal{F}_T^0$ (resp. completion of $\mathcal{F}_t^0$ in $\mathcal{F}_T^\mu$) w.r.t. $P_\mu$ and let $\mathcal{F}_t = \cap_{\mu \in \mathcal{P}(E)} \mathcal{F}_t^\mu$. (2) resp. (3) remain valid on $(\Omega_T, (\mathcal{F}_t)_{0 \le t \le T}, (X_t)_{0 \le t \le T}, P_x)$ resp. $(\Omega_T, (\mathcal{F}_t)_{0 \le t \le T}, (X_t)_{0 \le t \le T}, \hat{P}_x)$. The time reversal operator $r_T$ on $\Omega_T$ is now defined by

$$r_T(\omega)(t) = \omega(T - t); \quad 0 \le t \le T.$$

**Lemma 2.1.** *(i) If $0 < t_1 < ... < t_{n-1} < t_n$, and $f_0, ..., f_n \in \mathcal{B}(E)^+$, then*

$$E_m[f_0(X_0) \cdot ... \cdot f_n(X_{t_n})] = \hat{E}_m[f_n(X_0) f_{n-1}(X_{t_n - t_{n-1}}) \cdot ... \cdot f_1(X_{t_n - t_1}) f_0(X_{t_n})].$$

*(ii) For any $\mathcal{F}_T$-measurable set $A$ on $\Omega_T$ we have*

$$P_m[r_T \in A] = \hat{P}_m[A].$$

*In other words, the time reversed process and the co-process are identical in law.*

*Proof.* (i) (cf. also [2, Lemma 4.1.2.]) Since by conservativity of both forms $m = P_m \circ X_t^{-1} = \hat{P}_m \circ X_t^{-1}$ for any $t$, the statement is clear for $n = 0$. Suppose the statement is true for given $n$. Then, using the simple Markov property we obtain

$$\begin{aligned} &E_m[f_0(X_0) \cdot ... \cdot f_{n+1}(X_{t_{n+1}})] \\ &= \hat{E}_m[f_0(X_0) \cdot ... \cdot f_{n-1}(X_{t_{n-1}})(f_n \cdot p_{t_{n+1} - t_n} f_{n+1})(X_{t_n})]. \end{aligned}$$

By assumption the r.h.s. equals

$$\hat{E}_m[(f_n \cdot p_{t_{n+1} - t_n} f_{n+1})(X_0) f_{n-1}(X_{t_n - t_{n-1}}) \cdot ... \cdot f_1(X_{t_n - t_1}) f_0(X_n)].$$

Using normality and duality, and then again the simple Markov property but for the co-process, the last is equal to

$$\int_E f_{n+1}(x)\hat{p}_{t_{n+1}-t_n}\hat{E}_\cdot[(f_n(X_0)f_{n-1}(X_{t_n-t_{n-1}})\cdot\ldots\cdot f_1(X_{t_n-t_1})f_0(X_n)](x)\,m(dx)$$
$$= \hat{E}_m[f_{n+1}(X_0)f_n(X_{t_{n+1}-t_n})\cdot\ldots\cdot f_1(X_{t_{n+1}-t_1})f_0(X_{t_{n+1}})].$$

(ii) By (i), (ii) follows similarly to [2, Lemma 5.7.1.]. □

## § 3. Extension

In this section we will present a practicable domain for the validity of (1). Again we emphasize that this is in principle our sole really new contribution.

We start this section with a general remark about the energy of the continuous additive functional $A^{[u]}$ where $u$ is in $L^2(E;m)$ and admits a q.c. $m$-version $\widetilde{u}$. Let $(G_\alpha)_{\alpha>0}$ be the $L^2(E;m)$-resolvent associated to $\mathcal{E}$, and $(\hat{G}_\alpha)_{\alpha>0}$ the $L^2(E;m)$-resolvent associated to $\hat{\mathcal{E}}$. If $(\cdot,\cdot)$ denotes the inner product in $L^2(E;m)$ then (due to conservativity)

$$e(A^{[u]}) = \lim_{\alpha\to\infty} \alpha(u-\alpha G_\alpha u, u) = \lim_{\alpha\to\infty} \alpha(u-\alpha\hat{G}_\alpha u, u) = \hat{e}(A^{[u]}),$$

whenever one of the limits exists. The same equalities hold true if we replace lim by $\overline{\lim}$, and accordingly $e$ by $\overline{e}$, and $\hat{e}$ by $\overline{\hat{e}}$ in the above equation. Hence, we could restrict our attention to $e$, and $\overline{e}$, when looking at the energy of $A^{[u]}$. In particular, if $u$ admits decomposition (2) and decomposition (3), then

$$e(M^{[u]}) = e(A^{[u]}) = \overline{e}(A^{[u]}) = \overline{\hat{e}}(A^{[u]}) = \hat{e}(A^{[u]}) = \hat{e}(\hat{M}^{[u]}).$$

Now, let us briefly recall a procedure how to check whether (3) can be extended to a given $u \in L^2(E;m)$ with q.c. version $\widetilde{u}$. Let $u_n \in \hat{\mathcal{F}}$, $n\in\mathbb{N}$. Let $(S_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{n\to\infty} S_n = 0$. Suppose that there exists for each $\mu \in S_{00}$, and $\overline{T} > 0$, a constant $C^{\overline{T},\mu}$, such that

**(A)** $\hat{P}_\mu(\sup_{0\le t\le\overline{T}} |\widetilde{u}(X_t) - \widetilde{u}_n(X_t)| > \varepsilon) \le \frac{C^{\overline{T},\mu}}{\varepsilon} S_n$ for any $\varepsilon > 0$,

**(B)** $\overline{\hat{e}}(A^{[u_n-u]}) \longrightarrow 0$ as $n\to\infty$.

Then the decomposition (3) extends to $A^{[u]}$ by Theorem 4.5(ii) in [11], and remains of course valid if the time is restricted to $[0,T]$. Condition (A) ensures that some subsequence $A_t^{[u_{n_k}]}$ converges (in the sense of additive functionals) locally uniformly in $t$ against $A_t^{[u]}$. Condition (B) ensures that $\hat{M}_t^{[u_{n_k}]}$ converges locally uniformly against $\hat{M}_t^{[u]}$. The same is then also true for $\hat{N}_t^{[u]}$. Furthermore, condition (B) ensures that $\hat{N}_t^{[u]}$ is of zero energy.

We denote by $\hat{\mathcal{F}}^{ext}$ the linear space of all $u$ which can be obtained by the given procedure. By Theorem 4.5(i) in [11] $\hat{\mathcal{F}} \subset \hat{\mathcal{F}}^{ext}$. Accordingly, let $\mathcal{F}^{ext} \supset \mathcal{F}$ denote the linear space obtained w.r.t. the corresponding co-notions.
The basic idea for the proof of the following lemma can also be found in [9].

**Lemma 3.1.** *Let $u \in \hat{\mathcal{F}}^{ext}$. Then*

$$\hat{N}_t^{[u]}(r_T) = \hat{N}_T^{[u]} - \hat{N}_{T-t}^{[u]}; \quad 0 \le t \le T, \quad P_m\text{-a.e.}$$

*Proof.* Let first $u \in \hat{\mathcal{F}}$, and $u_n = \hat{G}_1 g_n$, $g_n \in L^2(E;m)$, such that $\lim_{n\to\infty} u_n = u$ in $\hat{\mathcal{F}}$. Then

$$\hat{P}_x[\hat{\Gamma}_T] = 1 \quad for\ q.e.\ x,$$

where

$$\hat{\Gamma}_T = \{\omega \in \Omega | \hat{N}_t^{[u_{n_k}]}(\omega)\ converges\ uniformly\ in\ t\ to\ \hat{N}_t^{[u]}(\omega)\ on\ [0,T]\}.$$

Since

$$\begin{aligned}\hat{N}_t^{[u_{n_k}]}(r_T) &= \int_0^t (\hat{G}_1 g_{n_k} - g_{n_k})(X_{T-s})ds \\ &= \int_{T-t}^T (\hat{G}_1 g_{n_k} - g_{n_k})(X_s)ds = \hat{N}_T^{[u_{n_k}]} - \hat{N}_{T-t}^{[u_{n_k}]}\end{aligned}$$

the set $\hat{\Gamma}_T$ is $r_T$-invariant, i.e. $\{r_T \in \hat{\Gamma}_T\} = \hat{\Gamma}_T$, and so is $\hat{\Gamma}_T^c = \Omega \setminus \hat{\Gamma}_T$. It follows

$$P_m[\hat{\Gamma}_T^c] = \hat{P}_m[r_T \in \hat{\Gamma}_T^c] = \hat{P}_m[\hat{\Gamma}_T^c] = 0,$$

and the assertion follows for $u \in \hat{\mathcal{F}}$. If $u \in \hat{\mathcal{F}}^{ext}$, we can find $u_{n_k} \in \hat{\mathcal{F}}$ such that $\hat{\Gamma}_T$ is again $r_T$-invariant, and such that again $P_m[\hat{\Gamma}_T^c] = 0$. Therefore the assertion follows. $\square$

We are ready to state our main result (cf. [9, Theorem 6.3]).

**Theorem 3.1.** *Let $u \in \mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$. Then*

$$\begin{aligned}\widetilde{u}(X_t) - \widetilde{u}(X_0) = &\frac{1}{2}M_t^{[u]} - \frac{1}{2}\left\{\hat{M}_T^{[u]}(r_T) - \hat{M}_{T-t}^{[u]}(r_T)\right\} \\ &+\frac{1}{2}\left\{N_t^{[u]} - \hat{N}_t^{[u]}\right\}; \quad 0 \le t \le T, \quad P_m\text{-a.e.}\end{aligned}$$

*Proof.* Applying Lemma 3.1 with $t$ replaced by $T-t$, resp. $t$ replaced by $T$, we obtain $\hat{N}_{T-t}^{[u]}(r_T) = \hat{N}_T^{[u]} - \hat{N}_t^{[u]}$, resp. $\hat{N}_T^{[u]}(r_T) = \hat{N}_T^{[u]} - \hat{N}_0^{[u]} = \hat{N}_T^{[u]}$; $0 \le t \le T$, $P_m$-a.e.

Therefore

$$
\begin{aligned}
&-\frac{1}{2}\left\{\hat{M}^{[u]}_{T}(r_T)-\hat{M}^{[u]}_{T-t}(r_T)\right\}\\
&=-\frac{1}{2}\left\{A^{[u]}_{T}(r_T)-A^{[u]}_{T-t}(r_T)-(\hat{N}^{[u]}_{T}(r_T)-\hat{N}^{[u]}_{T-t}(r_T))\right\}\\
&=\frac{1}{2}\left\{A^{[u]}_{t}+\hat{N}^{[u]}_{t}\right\};\quad 0\le t\le T,\ P_m\text{-}a.e.
\end{aligned}
$$

Applying additionally (2) for $u \in \mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$, the result now follows immediately. □

## § 4. Examples

In this section we present typical classes of examples for the decomposition of Theorem 3.1. We always, if not explicitely mentioned, assume that the given generalized Dirichlet form satisfies the assumtions needed for the theorem (see summary), and illustrate how large at least $\mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$ is.

### (i) Sectorial forms

(a) Dirichlet forms:

If $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(E;m)$, then $\mathcal{F} = \hat{\mathcal{F}}$, and therefore $\mathcal{F} \subset \mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$. If $\mathcal{E}$ is symmetric, i.e. $\mathcal{E} = \hat{\mathcal{E}}$, then $M^{[u]} = \hat{M}^{[u]}$, and $N^{[u]} = \hat{N}^{[u]}$, so that the decomposition of Theorem 3.1 coincides with the one presented in [3].

### (ii) Non-sectorial forms

(a) Time-dependent Dirichlet forms:

Let $E$ be a locally compact separable metric space and $m$ a positive Radon measure on $E$ with full support. Consider a time dependent Dirichlet form with domain $\mathcal{W}(= \mathcal{F} = \hat{\mathcal{F}}$ in our notation) in the sense of [6] on $L^2(E;m)$ (N.B.: in contrary to our notation in [6] the domain of the coercive part of a time dependent Dirichlet form is denoted by $\mathcal{F}$). By [6, Theorem 7.2.] or (2) we have $\mathcal{W} \subset \mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$.

(b) Divergence free vector fields with boundary conditions in finite dimensions:

Consider $\mathbb{R}^d$ equipped with the usual Euclidean norm $|\cdot| = \langle \cdot, \cdot \rangle^{1/2}$. Let $G \subset \mathbb{R}^d$ be a bounded Lipschitz domain with euclidean closure $\overline{G}$. Let $\rho \in H^{1,1}(G, dx)$, $\rho > 0$ $dx$-a.e, $m := \rho dx$. Let $A = (a_{ij})_{1 \le i,j \le d}$ be measurable, symmetric, and uniformly globally strictly elliptic on $G$. Then

$$\mathcal{E}^r(u,v) := \frac{1}{2}\int_G \langle A\nabla u, \nabla v\rangle\, dm\,; \quad u,v \in C^\infty(\overline{G}),$$

is closable in $L^2(G,m)$ (see [12, Lemma 1.1.]). Since $m(\overline{G}\setminus G) = 0$ we may regard $\mathcal{E}^r$ on $L^2(\overline{G},m)$ by obvious identifications. The closure $(\mathcal{E}^r, D(\mathcal{E}^r))$ is regular on $\overline{G}$. In general it is not regular on $G$. We denote by $(L^r, D(L^r))$ the generator associated to $(\mathcal{E}^r, D(\mathcal{E}^r))$ and consider a measurable vector field $B : G \longrightarrow \mathbb{R}^d$, which is $m$-square integrable on $G$, i.e. $\int_G |B|^2\, dm < \infty$, and such that

$$\int_G \langle B, \nabla u\rangle\, dm = 0 \text{ for all } u \in C^\infty(\overline{G}).$$

From [12, Proposition 1.4.] we know that the operator $L^r u + \langle B, \nabla u\rangle$, $u \in D(L^r)_b$, is dissipative, hence in particular closable in $L^1(G,m)$. Moreover, the closure $(\overline{L}, D(\overline{L}))$ generates a sub-Markovian $C_0$-semigroup of contractions $(\overline{T}_t)_{t\geq 0}$. The part $(L, D(L))$ of $(\overline{L}, D(\overline{L}))$ on $L^2(G,m)$, i.e. $D(L) = \{u \in D(\overline{L}) \cap L^2(G,m) | \overline{L}u \in L^2(G,m)\}$, and $Lu := \overline{L}u$, $u \in D(L)$, is by [7, Examples 4.9.(ii)] associated to a generalized Dirichlet form. The adjoint operator $(\hat{L}, D(\hat{L}))$ in $L^2(G,m)$ is associated to the co-form, so that $\mathcal{F} = D(L)$, $\hat{\mathcal{F}} = D(\hat{L})$. Both forms are associated to conservative diffusions (see [12]). In particular by [12, Theorem 4.1.] we have $D(\mathcal{E}^r)_b \subset \mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$.

(c) Divergence free vector fields in infinite dimensions:

Let $E$ be a separable real Banach space and $(H, \langle\cdot,\cdot\rangle_H)$ a separable real Hilbert space such that $H \subset E$ densely and continuously. Identifying $H$ with its topological dual $H'$ we obtain that $E' \subset H \subset E$ densely and continuously. Define the linear space of finitely based smooth functions on $E$ by

$$\mathcal{F}C_b^\infty := \{f(l_1,...,l_m) \mid m \geq 1, f \in C_b^\infty(\mathbb{R}^m), l_1,...,l_m \in E'\}.$$

Here $C_b^\infty(\mathbb{R}^m)$ denotes the set of all infinitely differentiable (real-valued) functions on $\mathbb{R}^m$ with all partial derivatives bounded. For $u \in \mathcal{F}C_b^\infty$, $k \in E$ let

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds}u(z+sk)\mid_{s=0},\ z \in E,$$

be the Gâteaux derivative of $u$ in direction $k$. Since $k \mapsto \frac{\partial u}{\partial k}(z)$ is continuous on $H$ one can define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), k\rangle_H = \frac{\partial u}{\partial k}(z).$$

Let $m$ be a finite positive measure on $(E, \mathcal{B}(E))$ with full support. An element $k$ in $E$ is called *well-m-admissible* if there exist $\beta_k^m \in L^2(E;m)$ such that for all $u$, $v$ in $\mathcal{F}C_b^\infty$

$$\int \frac{\partial u}{\partial k}\, dm = -\int u\beta_k^m\, dm.$$

Let us assume:

($\star$) There exists a dense linear subspace $K$ of $E'$ consisting of well-$m$-admissible elements.

It is well known that under the assumption ($\star$)

$$\mathcal{E}^0(u,v) := \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_H \, dm \quad u,v \in \mathcal{F}\mathcal{C}_b^\infty$$

is closable on $L^2(E;m)$ and that the closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric quasi-regular Dirichlet form. Let $(L^0, D(L^0))$ be the associated generator. Let $\overline{\beta} : E \to E$ be measurable, $\overline{\beta}(E) \subset H$, and $\|\overline{\beta}\|_H \in L^2(E;m))$ such that

$$\int \langle \overline{\beta}, \nabla u\rangle_H \, dm = 0 \ \text{ for all } u \in \mathcal{F}\mathcal{C}_b^\infty.$$

It follows from [8, Proposition 4.1.] that

$$Lu := L^0 u + \langle \overline{\beta}, \nabla u\rangle_H \ , \quad u \in D(L^0)_b.$$

is closable on $L^1(E;\mu)$ and that the closure $(\overline{L}, D(\overline{L}))$ generates a Markovian $C_0$-semigroup of contractions. Similarly to (ii)(b) the part $(L, D(L))$ of $(\overline{L}, D(\overline{L}))$ on $L^2(E;m)$ is associated to a generalized Dirichlet form, and the adjoint $(\hat{L}, D(\hat{L}))$ of $(L, D(L))$ in $L^2(E;m)$ is associated to the co-form, so that $\mathcal{F} = D(L)$, $\hat{\mathcal{F}} = D(\hat{L})$. Both forms are associated to conservative diffusions (see [10]). In particular exactly as in the case of (ii)(b) one can show that $D(\mathcal{E}^0)_b \subset \mathcal{F}^{ext} \cap \hat{\mathcal{F}}^{ext}$.

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