Let $G$ be the inductive limit group of countable direct product groups $G(j) = \prod_{k \leq j} G_k$, where $G_k$ are non-trivial type I locally compact groups.

In the previous paper [T], we proved a duality theorem for locally compact groups. That is, any locally compact group is isomorphic to the group of so-called bi-representations on its dual space which is the set of all (equivalence classes of) unitary representations of the initial group.

Obviously, our $G$ is not locally compact in general. But in this paper, we show that for the above $G$, analogous duality theorem holds too.

§ 1. Preliminary

We quote [TSH] for the definition of inductive limit group. At first we show a property of general inductive limit groups.

Lemma 1-1 Consider a set $\{K_j\}$ of countable locally compact groups $\{K_j\}$ such that
\[ \forall j, \quad K_j \subset K_{j+1} \]
as a topological subgroup.

Let $K$ be the inductive limit of $\{K_j\}$, and $C$ be any compact set in $K$.
Then there exists $n$ such that $C \subset K_n$.

Proof. Step 1. If there exists $n$ such that $K_n$ is open in $\forall K_m (m > n)$, the assertion is obvious. Therefore we can assume that for $\forall n, \exists m > n, \ K_n$ is not open in $K_m$.
Let the assertion fail, then we can take $m$ as $C \cap (K_m - K_n) \neq \phi$.

If necessary, changing the numbering of groups, we can assume $\forall n, C \cap (K_n - K_{n-1}) \neq \phi$, and take a sequence $\{g_n\}$ as $g_n \in C \cap (K_n - K_{n-1})$.

Step 2. By induction on $j$, we construct a family $\{W_j\}$, where $W_j$ is a neighborhood of $e$ in $K_j$, satisfying
\[ (1) \quad \forall k < j, \quad g_k (g_k)^{-1} \notin W_1 W_2 \cdots W_{j-1} W_j \]
\[ (2) \quad (W_j)^2 \cap K_{j-1} \subset W_{j-1} \]

Since a locally compact subgroup of a topological group is closed, $K_{j-1}$ is closed in $K_j$. 
and \( g_1(g_k)^{-1} \) does not belong to \( K_{j-1} \), we can select a neighborhood \( U \) of \( e \) in \( K_j \) as
\[
\forall \; k < j, \quad g_1(g_k)^{-1} \notin K_j \cup U \quad (\supset W_1 W_2 \cdots \cdot W_j U).
\]
Next we take neighborhood \( W_j \) of \( e \) in \( K_j \) satisfying (2) and \( (W_j)^2 \subset U \).

**Step 3.** We put \( W = \cup_{j=1}^{\infty} W_1 W_2 \cdots \cdot W_j \).

We have shown that this set gives a neighborhood of \( e \) in \( K \) named Bamboo Shoot neighborhood. [TSH, Lemma 2.2.]

Here we remark that for any \( j \), if \( m < j \), obviously. \( W_1 W_2 \cdots W_m \subset K_m \) can not contain \( g_1(g_k)^{-1} \) ( \( \forall \; k < j \) ), and when \( m \geq j \), \( g_m(g_k)^{-1} \) ( \( \forall \; k < j \) ) is not in \( W_1 W_2 \cdots \cdot W_m \) from (1).

**Step 4.** Next we consider \( W_1 W_2 \cdots \cdot W_m \cap K_j \) for the case \( j \leq m \). The condition (2) shows \( W_1 W_2 \cdots \cdot W_m \cap K_j = W_1 W_2 \cdots \cdot W_m \cap K_{j-1} \cap K_j \subset \)
\[
W_1 W_2 \cdots \cdot W_{m-2} (W_{m-1})^2 \cap K_{m-1} \cap K_j = W_1 W_2 \cdots \cdot W_{m-2} (W_{m-1})^2 \cap K_m \cap K_j \subset \]
\[
W_1 W_2 \cdots \cdot W_{m-3} (W_{m-2})^2 \cap K_{m-2} \cap K_j = W_1 W_2 \cdots \cdot W_{m-3} (W_{m-2})^2 \cap K_{m-3} \cap K_j \subset \]
\[
\cdots \cdots \subset W_1 W_2 \cdots \cdot W_{j-1} (W_j)^2 \]

The condition (1) leads us to
\[
\forall \; m, \quad \forall \; k < j, \quad g_1(g_k)^{-1} \notin W_1 W_2 \cdots \cdot W_{m-2} W_{m-1} \cap K_j
\]
i.e.
\[
g_1(g_k)^{-1} \notin W_1 W_2 \cdots \cdot W_{m-2} W_{m-1} \cap W_m.
\]

Joining the results of Step 3 and Step 4, we get \( \forall \; j, \forall \; k < j, \quad g_1(g_k)^{-1} \notin W \). But \( j \) and \( k \) are free. So \( g_1(g_k)^{-1} \notin W \) for \( \forall \; k \neq j \).

**Step 5.** After the result [TSH, Proposition 2.3], \( K \) is a topological group, so we have a symmetric open neighborhood \( V \) of \( e \) in \( K \) such that \( V^2 \subset W \). And obtain \( \forall \; k \neq j \), \( g_1(g_k)^{-1} \notin V^2 \). And from the symmetry of \( V \) this is the same as \( Vg_i \cap Vg_k = \emptyset \).

**Step 6.** Now take an open covering \( C \subset \cup g \in C \; Vg \). Since \( C \) is compact, there exists a finite sub-covering as \( C \subset \cup N^n Vg_i \). But all \( g_i \)'s are belonging to \( C \). So there exists a pair \((g_p, g_q)\) contained in the same \( Vg_i \). That is, \( g_p \in Vg_i \), \( g_q \in Vg_i \), i.e. \( g_i \in Vg_p \cap Vg_q \), so \( Vg_p \cap Vg_q \neq \emptyset \).

This contradicts the conclusion of Step 5. q.e.d.

We consider a countable family \( \{ G_k \} (k = 1,2, \ldots) \) of non-trivial locally compact groups \( G_k \). For finite number \( j \), write \( G(j) \equiv \Pi_{k \leq j} G_k \). \( G(j) \) is imbeded into \( G(j+1) \) as a subgroup \( \Pi_{k \leq j} G_k \times \{ e \} \).

By definition, the inductive limit group \( G \) of \( G(j) \)'s is equal to \( \cup j G(j) = \Pi' k G_k \) (restricted direct product) as a set. And the topology of \( G \) is given by the following.

\((*)\) A set \( E \) in \( G \) is open if and only if \( \forall \; j, \; E \cap G(j) \) is open in \( G(j) \).

As in [TSH, Proposition 2.3.], by this topology, \( G \) becomes a topological group. So we can consider unitary representations of such a group.

Apply the analogous argument to the family \( \{ G_k \} (k = j + 1, j + 2, \ldots) \) and get the
inductive limit group $G[j]^V$ in the same way. Then the following is easily shown.

**Lemma 1-2** $G = G(j) \times G[j]^V$ as a topological group.

**Proof.** Omitted.

**Definition 1-1.** (Infinite tensor product of Hilbert spaces) For a given set of Hilbert spaces $\{H(\alpha)\}$, we consider a family of vectors $\{v(\alpha) | \in H(\alpha), \|v(\alpha)\|_\alpha = 1\}$ (we call $v \equiv \otimes \alpha v(\alpha)$, the reference vector). And we define an infinite product Hilbert space $H(v) \equiv \{ \otimes \alpha H(\alpha), v \}$, which is the completion of the space of linear combinations of symbols $v \equiv \otimes \alpha u(\alpha)$ such that

$$\Sigma \alpha |\|u(\alpha)\| - 1| < \infty \quad \text{and} \quad \Sigma \alpha |<u(\alpha), v(\alpha)> - 1| < \infty,$$

with scalar product $<u, v> \equiv \Pi \alpha <u(\alpha), v(\alpha)>$.

For properties of this tensor product, we quote [G], p.148.

**Notations.** Denote by $\Omega$, the set of all unitary representations of G. The element of $\Omega$, we use the notation as $\omega \equiv \{H(\omega), Tg(\omega)\}$, where $H(\omega)$ is the representation space of $\omega$ and $Tg(\omega)$ ($g \in G$) the representation operators. For two representations $\omega_1, \omega_2, \omega_1 \sim \omega_2$ means $\omega_1$ is unitary equivalent to $\omega_2$ with the intertwining operator $A$.

A representation $\omega \equiv \{H(\omega), Tg(\omega)\}$ is called cyclic, if there exists a non-zero vector $v$ in $H(\omega)$, such that the space of linear combinations of the set $\{Tg(\omega)v; g \in G\}$ is dense in $H(\omega)$.

It is easily shown that an irreducible representation of $G$ is cyclic.

For a given cyclic representation $\omega \equiv \{H(\omega), Tg(\omega)\}$, and any non-zero vector $v$, the function $\varphi(g) \equiv <Tg(\omega)v(\omega), v(\omega)>$ is continuous and satisfies the axiom of positive definite property,

(*) For any finite pairs $\{(g_i, c_i), g_i \in G, c_i \in \mathbb{C}; j = 1, 2, 3, \ldots, n\}$,

$$\Sigma c_i \varphi(g_i^{-1}g_k) \geq 0.$$  

We call this positive definite function as associated to $\omega$.

Conversely, for any continuous positive definite function $\varphi$, we can construct a cyclic unitary representation $\omega$ which associates to $\varphi$.

If $\varphi(\omega)(=\|v\|_\varphi^2) = 1$, this positive definite function $\varphi$ is called normalized.

Of course a cyclic representation can have many positive definite functions associated to it.
Definition 1-2. (Fell-topology on the space of positive definite functions) Let \( \Omega \) be the set of all normalized continuous positive definite functions on \( G \). For any compact subset \( C \) in \( G \), we consider semi-metrics \( \mathcal{M}_C(\varphi_1, \varphi_2) \equiv \sup_{g \in C} (|\varphi_1(g) - \varphi_2(g)|) \) and topology on \( \Omega \) defined by these metrics.

Make running compact sets \( C \), we obtain the topology \( \tau \) on \( \Omega \) generated by all \( \mathcal{M}_C \).

In this paper, we call this topology on \( \Omega \) simply, as Fell-topology.

For the case where \( G \) is locally compact, this topology induces some important topology on the dual space of \( G \). In our case, given \( G \) is not locally compact in general, but we can say the following.

Lemma 1-3 For our group \( G \), \( \Omega \) is compact convex in Fell-topology.

Proof. The convexity is trivial.

For any compact subset \( C \), by Lemma 1-1 there exists an \( n \) such that \( C \subset \Omega(n) \). Now we consider the restriction \( \varphi_n \) of a given \( \varphi \in \Omega \) onto \( G(n) \), and obtain a continuous positive definite function on locally compact group \( G(n) \). We denote the space of all normalized continuous positive definite functions on \( G(n) \) by \( \Omega^* \).

General representation theory of locally compact groups taught us that \( \Omega^* \) is compact under Fell-topology. If \( n < m \), the restriction map \( \kappa_{nm}: \Omega^* \equiv \varphi_m \rightarrow \varphi_n \equiv \varphi_m |_{\varphi(n)} \in \Omega^* \) is continuous and surjective for our group.

Take the compact convex set \( \Omega^* \equiv \prod_n \Omega(n) \), then \( \Omega \) is imbedded in \( \Omega^* \) by the continuous map \( \kappa: \Omega \rightarrow (\varphi_n(\equiv \kappa_n(\varphi) \equiv (\varphi |_{\varphi(n)}) \in \Omega^* \). By the definition of topology of \( \Omega \) and \( \Omega^* \), this map must be open, that is, isomorphic.

Now we show that the image \( \kappa(\Omega) \) is closed,

For this, it is enough to see that a ultra filter \( \{ \varphi \alpha \} \) in \( \Omega \) converges to an element of \( \Omega \). On \( G(n) \), \( \forall n, \varphi \alpha \in \varphi(n) \in \Omega \) converges to some \( \varphi_\alpha \), and \( \kappa_{mn}(\varphi_\alpha) = \varphi_\alpha \).

So there exists \( \varphi \) satisfying \( \kappa_n(\varphi) = \varphi_n \) as the compact uniform limit of \( \varphi \alpha \)'s. It is easy to see that \( \varphi \) is positive definite.

We must show that \( \varphi \) is continuous. Now put

\[ E(a, b) \equiv \{ g \in G, \text{Re}(\varphi(g)) > a, \text{Im}(\varphi(g)) > b \} \quad (a, b \in \mathbb{R}) \]

Since for any \( n \), \( E(a, b) \cap G(n) = \{ g \in G(n), \text{Re}(\varphi_\alpha(g)) > a, \text{Im}(\varphi_\alpha(g)) > b \} \) is open in \( G(n) \), so \( E(a, b) \) are open for any real \( a, b \). This shows that \( \varphi \) is continuous. q.e.d.

Now we quote the following famous theorem by M.G.Krein and D.Mil'man.
**Proposition 1-1** (External Point Theorem) Non-void convex subset in a locally convex space coincides with the closed convex envelope of the set of all its terminal points.

As a result of Lemma 1-3 and Proposition 1-1, we can confirm the following.

**Proposition 1-2** (Extended I. M. Gel'fand-D. A. Raikov's Theorem) Any continuous positive definite function $\varphi$ of $G$ can be approached uniformly on any compact set by linear combinations with positive coefficients of normalized positive definite functions associated to irreducible representations.

In $\Omega$, there exist three relations, 1) unitary equivalence, 2) direct sum, 3) tensor product. Using these relations we define the following.

**Definition 1-3. (Birepresentation)** An operator field $U \equiv \{U(\omega)\}$ over $\Omega$, where $U(\omega)$ is a bounded operator in $H(\omega)$, is called a birepresentation when

1. $\forall \omega_1, \omega_2 \in \Omega$, if $\omega_1 \sim A \omega_2$ then $U(\omega_1) = A^{-1}U(\omega_2)A$,
2. $\forall \omega_1, \omega_2 \in \Omega$, $U(\omega_1 \oplus \omega_2) = U(\omega_1) \oplus U(\omega_2)$,
3. $\forall \omega_1, \omega_2 \in \Omega$, $U(\omega_1 \oplus \omega_2) = U(\omega_1) \otimes U(\omega_2)$,
4. $\forall \omega \in \Omega$, $U(\omega) \neq 0$.

In [T], to prove duality theorem for locally compact groups, in the definition of birepresentation, conditions (1)-(4) were enough, but in this paper we must add the following condition:

5. $U(\omega)$ is weak continuous (w-continuous) on $\Omega_p$ with respect to Fell-topology.

This means that if $\Omega_p \ni \varphi$ is given as $\varphi(g) \equiv \langle Tg(\omega)v(\omega), v(\omega) \rangle$, then $\forall g_0 \equiv G$, the function $\varphi \rightarrow U(\varphi)(g_0) \equiv \langle Tg_0(\omega)v(\omega), v(\omega) \rangle$ is continuous on $\Omega_p$.

For any $g \in G$, operator field $Tg \equiv \{Tg(\omega)\}$ over $\Omega$ gives a birepresentation.

**§ 2. Unitary representations**

Let $\omega_k \equiv \{H^k, T^g_k\}$ be a unitary representation of group $G_k$ for each $k$.
We consider the Hilbert space $H = \{ \oplus_k H^k, v \equiv \oplus_k v_k \}$, where $v_k \in H^k (\forall k, \|v_k\| = 1)$ and $v$ is a reference vector in $H$.

For any element $g = \{g_k\}$ in $G$, $g_k = e$ except finite $k$'s, so the operator $Tg \equiv \oplus_k T^g_k$ can be defined as a unitary operator on $H$ and $\omega \equiv \{H, Tg\}$ is an algebraic representation of $G$. It is easy to see that $G \ni g \rightarrow Tg$ is weak continuous. So $\omega$ gives
a unitary representation of $G$.

**Definition 2-1** We call the above $\omega \equiv \{H,T\}$, a direct product type representation (DPR). And denote it as $\omega (\equiv \omega (v)) = (\otimes \omega_k, v \equiv \otimes v_k)$, where $\otimes \omega_k$ means multiple of outer tensor products operation. (The notation $\otimes \omega$ shows outer tensor product.)

And we denote the set of direct sums of DPR's of $G$ by $\Omega(G)$.

**Definition 2-2** For a DPR $\omega (v) = \{\otimes \omega_k, v \equiv \otimes v_k\}$, if $\omega_k$ are the trivial representation of $G_k$ except finite $k$'s, that is, there is a finite subset $S$ in $N$ such that $\omega_k = I_k$ (the trivial representation) for $k \not\in S$, we call this direct product type representation of finite type (FT).

Especially if $S = \{k\}$ is a one point set, this $\omega (v)$ is called single type of index $k$. And we show the set of all index $k$ single type representations by $\Omega(k)$.

Easy to see that for FT-representation, every reference vector gives the same Hilbert space. Therefore hereafter, we use the notation for FT-representation without reference vector.

An index $k$ single type representation is of the following form:

$$\omega = (\otimes - 1) \otimes \omega (k) \otimes (\otimes - 1) \quad (\omega (k) \equiv \{H(k), T(k)\} \in \Omega(Gk)).$$

It is easy to see that by the correspondence $\omega (v) \rightarrow \omega (k)$, we can see the set of all index $k$ single type representations as the set of all representations of $G_k$. So we can identify $\Omega(k)$ to the weak dual $\Omega(Gk)$ of $G_k$.

Now we consider a DPR $\omega (v) = \{\otimes \omega_k, v \equiv \otimes v_k\}$, FT-representation $\omega = (\otimes - 1) \otimes \omega (k) \otimes (\otimes - 1)$ and thier inner tensor product $\omega \otimes \omega (v)$.

Take any normalized vector $u$ in $H(k)$, then we get

$$\omega \otimes \omega (v) = \{\otimes j \otimes (\omega (k) \otimes \omega_k) \otimes (\otimes - 1) \otimes \omega_j, \otimes j \otimes v \otimes (u \otimes v_k) \otimes v_j\}$$

Corresponding to arbitrarily given DPR $\omega (v) = \{\otimes \omega_k, v \equiv \otimes v_k\}$ and finite subset $S$ in $N$, we can consider a finite type DPR $\omega (v)_S \equiv \{(\otimes \omega_k (k \in S)) \otimes (\otimes I_k (k \not\in S)), (\otimes v_k (k \in S)) \otimes (\otimes I)\}$, and the representation

$$(\omega (v)_S)^{\otimes} \equiv \{(\otimes I_k (k \in S)) \otimes (\otimes \omega_k (k \not\in S)), (\otimes v_k (k \in S)) \otimes (\otimes I)\}.$$ Then $\omega (v) = \omega (v)_S \otimes (\omega (v)_S)^{\otimes}$. ( $\otimes$ shows inner tensor product.)
**Definition 2-3** The case that for any \( k, \omega_k = \mathcal{R}_k \) (the right regular representation of \( G_k \)), we call such \( \omega (v) \) the **full regular representation** of \( G \), and denote it by \( \mathcal{R} (v) \).

As well known, for a locally compact group its regular representation is unique up to unitary equivalence, but in our present case there exist many \( \mathcal{R} (v) \)'s depending on the reference vectors \( v \equiv \otimes_k v_k \), and in general they are not equivalent mutually.

**Example 2-1.** Consider the case where all \( G_k \) are compact. \( \mathcal{R}_k \) has trivial component \( I_k \) with multiplicity 1. Denote the normalized vector in the component \( I_k \) as \( 1_k \). Take another irreducible component \( \omega_k \), and a normalized vector \( v_k \) in \( H (\omega_k) \).

\( \forall k, 1_k \perp v_k \), so the reference vectors \( 1 \equiv \otimes_k 1_k \) and \( v \equiv \otimes_k v_k \) can not be in the same representation space. \( \mathcal{R} (1) \) contains trivial representation on \( I \), but \( \mathcal{R} (v) \) can not contain \( 1 \), and it has no trivial component. That is, \( \mathcal{R} (1) \) and \( \mathcal{R} (v) \) are not mutually equivalent.

**Definition 2-4** If a unitary representation \( \omega = \{ H, T_g \} \) satisfies the following, we call this representation of **quasi-direct product type**

\[ (\star) \quad \text{For any } j, \quad \omega = (\otimes \sim \omega [j]) \otimes \omega [j]. \]

Here \( \omega_k \) is a representation of \( G_k \) and \( \omega [j] \) is of \( G [j] \).

Of course DPR is quasi-direct type representation. But I don't know conditions under which a quasi-direct type representation is DPR.

**Lemma 2-1** For two topological groups \( H_1, H_2 \), and a unitary representation \( \omega = \{ H (\omega), T_g \} \) of \( H = H_1 \times H_2 \), if the restriction of \( \omega \) to \( H_1 \) contains some irreducible representation \( D = \{ H (D), T_g (D) \} \) as a discrete component, then \( \omega \) contains subrepresentation \( D \otimes D [2] \), where \( D [2] \) is a representation of \( H_2 \).

**Proof.** Take the maximal subspace \( H (D)^V \) of \( H (\omega) \) on which multiple of \( D \) acts. Then \( H (D)^V \) is invariant under \( \{ T_g | g \in H \} \), and any \( \{ T_g | g \in H_2 \} \) commutes with operators of \( \Sigma^\Phi D \). Since \( D \) is irreducible, so the space \( H (D)^V \) is of the form \( H (D) \otimes H (2) \), and the restriction of \( \omega \) to \( H_2 \) on \( H (D)^V \equiv H (D) \otimes H (2) \) is of the form \( 1 \otimes D [2] \). q.e.d.

Analogous result is proved.

**Lemma 2-2** For two topological groups \( H_1, H_2 \), let \( \omega = \{ H (\omega), T_g \} \) be an
irreducible unitary representation of \( H \equiv H_1 \times H_2 \), then the restriction \( \omega \big|_{H_1} \) of \( \omega \) to \( H_1 \) is a factor representation of \( H_1 \).

Moreover if \( H_1 \) is type I group, then \( \omega \) is the outer tensor product of irreducible unitary representations \( \omega_j \) of \( H_j \) (\( j = 1, 2 \)), that is, \( \omega = \omega_1 \otimes \omega_2 \).

**Proof.** If \( \omega \big|_{H_2} \) is not a factor representation, there exists a non-trivial projection \( P \) belonging to the double commutant \( (\omega \big|_{H_1})^{*} \). \( PH(\omega) \) and \((I - P)H(\omega)\) are both non-trivial \( H_1 \times H_2 \) invariant subspaces. This contradicts the assumption of irreducibility.

Next, if \( H_1 \) is of Type I, there exists an irreducible representation \( \omega_1 \) of \( H_1 \) and \( \omega \big|_{H_1} \) is a multiple of \( \omega_1 \) and the space is written as \( H(\omega) = H(\omega_1) \otimes H(\omega_2) \), the tensor product of the space of \( \omega_1 \) with some space \( H(\omega_2) \) on which operators in \((\omega \big|_{H_1})^{*} \) act, surely some representation \( \omega_2 \) of \( H_2 \). Again the irreducibility assumption of \( \omega = \omega_1 \otimes \omega_2 \) leads us to the irreducibility of \( \omega_2 \). q.e.d.

**Corollary** In our group \( G \), if all \( G_k \) are type I groups, then any irreducible unitary representation \( \omega \) of \( G \) is of quasi-direct product type.

**Proof.** For \( G(j) \equiv \Pi k \leq j G_k \), we use Lemma 2-2 repeatedly. And we conclude that every irreducible representation of \( G(j) \) is of the form \( \omega(j) = \otimes k \leq j \omega_k \) (\( 1 \leq k \leq j \)), where \( \omega_k \equiv \{H_k, T^g_k\} \) is an irreducible representation of \( G_k \).

Again we apply Lemma 2-2 to the case of \( H = G(j) \times G[j]^{*} \), where \( H_1 = G(j) \), \( H_2 = G[j]^{*} \). We get that any irreducible representation of \( G \) is of the form \( \omega[j] \otimes \omega[j] \), where \( \omega[j] \) is an irreducible representation of \( G[j]^{*} \). In other words, for arbitrary given irreducible representation \( \omega \equiv \{H, T_g\} \) of \( G \), there exist irreducible representations \( \omega_k \) of \( G_k \) determined for any \( k \leq j \), and \( \omega \) is written in the form
\[
\omega = (\otimes k \leq j \omega_k) \otimes \omega[j].
\]
q.e.d.

**[Remark]** If the assumption, "all \( G_k \) are Type I*, is omitted, then we have the following example for which the assertion of Corollary 1 fails.

**Example 2-2** Consider \( H \) the free group with two generators (Yoshizawa Group). On \( L^2(H) \), we have two groups of operators, \( KL = \{L_h; h \in H\} \) (left translations) and \( KR = \{R_h; h \in H\} \) (right translations). It is well known that both of the regular representations \([L^2(H),L_h]\) and \([L^2(H),R_h]\) of \( H \) are type II factors and so \( H \) is not type I group.

We take in our Corollary, \( G_k = H \) (\( k = 1, 2, 3, \ldots \)) and consider the representation \( \omega \) of \( G \) on the space \( H = \otimes \omega_k \) (Here \( \forall k, \omega_k = L^2(H) \) with any reference vector
Duality Theorem for Inductive Limit Group of Direct Product Type

\[ f = \otimes_k f_k \quad (\forall k, \ f_k \in H_k , \ \| f_k \| = 1) \] and the representation operators are

\[ G \ni g = (g_1, g_2, g_3, \ldots) \rightarrow T_g = L_{g_1} R_{g_2} \otimes \cdots \]

But the representation \( \omega(1, 2) \) of \( H_1 \times H_2 \) is irreducible. Apply this to the case of \( G_1 = H_1 \) and \( G_{j+1} = H_2 \), then we can assert that the representation

\[ \omega(j, j+1): G_j \times G_{j+1} \ni (g_j, g_{j+1}) \rightarrow L_{g_j} R_{g_{j+1}} \] is irreducible.

Extend this representation to \( G \), as \( \omega(j) \equiv \otimes \omega(j, j+1) \otimes (\otimes j) \), then this representation is irreducible.

Finally as the inner tensor product of representations, \( \omega = \{ \otimes \omega(j), f \} \) of \( G \) is irreducible. And this irreducible representation is not of the above form.

Proposition 2.1 For our group \( G \), any positive definite function associated to an irreducible unitary representation is a limit of a sequence of ones associated to elements in \( \Omega \) of \( G \) with Fell-topology.

Proof. By Lemma 1-1, any compact set \( C \) is contained in some \( G_j \).

In other hand, by Lemma 2-2 and Corollary, any irreducible representation \( \omega \) of \( G \) is quasi-direct product type as \( \forall m, \ \omega = (\otimes \omega_k) \otimes \omega [m] \). So, for a matrix element

\[ f(g) \equiv \langle T_{g_{v}} , v \rangle \quad (v = \otimes \omega_k \otimes \nu_k(m)) \] associated to \( \omega \), consider the DRP

(\otimes \omega_k, v_0) \rightarrow \langle T_{g_{v}} , v_0 \rangle \rightarrow \prod \langle T_{g_{v_k}} v_k, v_k \rangle \rightarrow 1 = \langle T_{g_{v_0}}, v_0 \rangle

This shows that \( f \) coincides with a matrix element \( f_0 = \langle T_{g_{v_0}}, v_0 \rangle \) on \( C \). q.e.d.

§ 3 Duality theorem

In this section, we treat our group \( G \) that is, an inductive limit group of countable direct product groups for which all component groups are type I locally compact groups.

We show a duality theorem for \( G \).

As in § 1, we put \( \Omega = \{ \omega \} \) all unitary representations of \( G \), and \( U = \{ U(\omega) \} \) a given birepresentation on \( \Omega \).

By definition, each \( U(\omega) \) is a bounded operator on the representation space of \( \omega \), and \( \{ U(\omega) \} \) satisfies the following

1. \( \omega_1, \omega_2 \in \Omega \), if \( \omega_1 \sim_A \omega_2 \), then \( U(\omega_1) = A^{-1} U(\omega_2) A \),
2. \( \forall \omega_1, \omega_2 \in \Omega, \ U(\omega_1 \oplus \omega_2) = U(\omega_1) \oplus U(\omega_2) \),
3. \( \forall \omega_1, \omega_2 \in \Omega, \ U(\omega_1 \otimes \omega_2) = U(\omega_1) \otimes U(\omega_2) \),
4. \( \forall \omega \in \Omega, \ U(\omega) = 0 \),
5. \( U(\omega) \) is weakly continuous with respect to Fell-topology.
Lemma 3-1 For a given birepresentation \( U \equiv \{ U(\omega) \} \), there exist a unique element \( g_{\omega} \in G \) such that \( U(\omega) = Tg_{\omega}(\omega) \) for any DPR \( \omega \).

Proof. Step 1. At first, for any \( k \), we consider the set \( \Omega(k) \) of index \( k \) single type representations. As we remarked, \( \Omega(k) \) is identified with the weak dual \( \Omega(G_k) \) of \( G_k \).

By restricting our birepresentation \( \{ U(\omega) \} \) to \( \Omega(k) \), we obtain a birepresentation on the weak dual \( \Omega(G_k) \) of locally compact group \( G_k \).

We can use the duality theorem for this restriction, and get unique element \( g_k \in G_k \), such that for any \( \omega_k \) in \( \Omega(k) \), \( U(\omega_k) = Tg_k(\omega_k) \).

Step 2. Next we treat FT-representation \( \omega[j] \).

Let \( \omega[j] = (\otimes \omega_k \leq j \omega_k) \otimes I(G_j) \) = \( \otimes \omega_k \leq j Tg_k(\omega_k) \) \( g = (g_1, g_2, \cdots, g_i, e, e, e, \cdots) \) \( \equiv \otimes I \). It shows the trivial representation of \( G \).

From (3) of the definition of birepresentation, \( U(\omega[j]) = \otimes \omega_k \leq j Tg_k(\omega_k) = \otimes \omega_k \leq j \mathrm{T}g(\omega[j]) \) \( (g = (g_1, g_2, \cdots, g_i, e, e, e, \cdots)) \).

It is remarkable that the above \( g_k \) depend only on the given birepresentation \( U \) and not on \( j \).

Step 3. In the case where the representation \( \omega \equiv \omega(\nu) = (\otimes \omega_k \nu_k, \nu \equiv \otimes \nu_k) \) is DPR, for any \( j \), we can write \( \omega(\nu) = \omega[j] \otimes (\omega[j])^{\nu[j]} \), where \( (\omega[j])^{\nu[j]} = I(G[j]) \otimes (\otimes \nu_k > j \nu_k) \) and \( v[j] = (\otimes 1) \otimes (\otimes \nu_k > j v_k) \).

Thus \( U(\omega(\nu)) = U(\omega[j]) \otimes U((\omega[j])^{\nu[j]}) \) \( (\otimes \omega_k \leq j Tg_k(\omega_k) \otimes U((\omega[j])^{\nu[j]})) \).

This means that birepresentation operator \( U(\omega(\nu)) \) operates on the reference vector \( \nu \equiv \otimes \nu_k \) as follows.

(*) The k-th component vector \( \nu_k \) changes to \( Tg_k(\omega_k)v_k \).

Step 4. We consider a full regular representation \( R(f) \equiv \{ \otimes \omega_k \mathfrak{R}(k), f \equiv \otimes f_k \} \). The above result means \( U(R(f)) \) must transfer the reference vector \( f \equiv \otimes f_k \) to \( \otimes Rg_k f_k \).

If for any \( j \), \( U(R(f)[j]) \neq 0 \), then \( U(R(f))f = \otimes Rg_k f_k \).

Now we assume that there exists an infinite set \( K = \{ k \} \) such that for \( \forall k \in K, g_k \neq e \). In regular representation \( R_k \) of locally compact group \( G_k \), for non-unit element \( g_k \), there exists a normalized \( L^2 \)-function \( f_k \) such that \( [f_k] \cap [Rg_k f_k] = \emptyset \), that is, \( \| f_k - Rg_k f_k \| = 2 \). Therefore the vector \( U(R(f))f = \otimes Rg_k f_k \) cannot belong to the space of \( R(f) \).

This contradicts the assumption that for birepresentation, its component \( U(\omega) \) for any \( \omega \) is a bounded operator on the representation space of \( \omega \).

Step 5. After the result in Step 4, for any given birepresentation \( U \), the element of corresponding sequence \( \{ g_k \} \) \( (g_k \in G_k) \) must be unit \( e \) except only a finite number of \( k \), in other words, \( \{ g_k \} \) is of the form \( (g_1, g_2, g_3, \ldots, g_i, e, e, \cdots) \). Therefore there exists an element \( g_0 = g_1 \times g_2 \times g_3 \times \ldots \times g_i \times e \times e \times \cdots \cdots \) in \( G \) and \( U(\omega) = \otimes \omega_k \leq j Tg_k(\omega_k) \otimes (\otimes \omega_k) = Tg(\omega(j) \otimes (\otimes \omega_k)) = Tg_0(\omega) \). q.e.d.
**Corollary** For our group \(G\), \(U(\omega) = T_{g_\omega}(\omega)\) for any irreducible unitary representation \(\omega\).

**Proof.** By Proposition 2-1, any positive definite function associated to an irreducible unitary representation of \(G\) is a limit of ones associated to an element in \(\Omega\) \(D\). And by definition, birepresentation \(\{U(\omega)\}\) is \(w\)-continuous with respect to Fell-topology.

From Lemma 3-1, on \(\Omega\) \(D\), \(<T_\omega(\omega)U(\omega)v, v> = <T_\omega(\omega)T_{g_\omega}(\omega)v, v>\), \((\forall v \in H(\omega), \forall g \in G)\). Take the limit, and we get this for any irreducible \(\omega\) too. That is, \(U(\omega) = T_{g_\omega}(\omega)\), for any irreducible representation \(\omega \in \Omega\). q.e.d.

**Theorem** Consider the inductive limit group \(G\) of countable direct product type of type I locally compact groups \(G_k\), \(k=1,2,3,\ldots\).

Then any birepresentation \(U \equiv \{U(\omega)\}\) coincides with \(T_g = \{T_g(\omega)\}\) for some \(g \in G\).

That is, the set of all birepresentations corresponds to \(G\) one to one way as a group.

**Proof** Use the notations in Lemma 3-1.

By the results of and Corollary of Lemma 3-1, \(U(\omega) = T_{g_\omega}(\omega)\) for any irreducible \(\omega\).

Now we show that \(U(\omega) = T_{g_\omega}(\omega) \quad (\forall \omega \in \Omega)\), then the proof is completed.

For any normalized positive definite function \(\varphi(\omega) \equiv <T_\omega(\omega)U(\omega)v, v(\omega)>\) associated to \(\omega\), take the function \(U(\varphi)(g) \equiv <T_\omega(\omega)U(\omega)v, v(\omega)>\).

If \(\omega\) is irreducible, \(\forall g \in G\), \(U(\varphi)(g) = T_{g_\omega}(\varphi)(g)\). Since the function \(\Omega \ni \varphi \rightarrow U(\varphi)(g) \quad (\forall g \in G)\) is continuous, using the result of Proposition 1-2, we obtain \(\forall \varphi \in \Omega\) \(p \ni \varphi \rightarrow U(\varphi)(g) \quad (\forall g \in G)\). That is, \(\forall \omega \in \Omega\), \(\forall v \in H(\omega), \forall g \in G\), \(<T_\omega(\omega)U(\omega)v, v> = <T_\omega(\omega)T_{g_\omega}(\omega)v, v>\).

So \(U(\omega)v = T_{g_\omega}(\omega)v \quad (\forall v \in H(\omega))\), i.e. \(U(\omega) = T_{g_\omega}(\omega) \quad (\forall \omega \in \Omega).\) q.e.d.

**REFERENCES**


Address:
Matsuo-chou 10-8, Nishinomiya
662-0076, Japan