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DOUBLING CONSTRUCTION FOR LOCAL θ-CORRESPONDENCE OF REAL UNITARY DUAL PAIRS

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ABSTRACT. This is a survey of our paper [KK07]. We consider a unitary dual pair $(U(V), U(W))$ over $\mathbb{R}$. We first review Harris-Kudla-Sweet's doubling construction of Weil representations of $U(V)\times U(W)$ (not of their inverse image $\widetilde{U}(V)\times\widetilde{U}(W)$ in the metaplectic cover). Then we construct its Fock model and deduce the K-type correspondence under this. As a consequence, we obtain the local θ-correspondence (variant of the Howe duality correspondence under this Weil representation) between the limit of discrete series representations when $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W$. The main feature is that this correspondence is described in terms of the sign of functional equation for certain automorphic L-factors. One can view this as an archimedean analogue of the ε-dichotomy property of the local θ-correspondence of unitary dual pairs over non-archimedean local fields.

CONTENTS

1. INTRODUCTION

Let $(\mathbb{W}, \langle\langle\cdot,\cdot\rangle\rangle)$ be a symplectic space ove a local field F whose characteristic is not two. The Heisenberg group $\mathcal{H}(\mathbb{W})$ is the space $\mathbb{W}\times F$ with the multiplication law

$$(w;x)(w';x') := (w+w';x+x'+\frac{\langle\langle w,w'\rangle\rangle}{2}),\quad w,w'\in\mathbb{W},\ x,x'\in F.$$

The outer automorphism group of $\mathcal{H}(\mathbb{W})$ is the symplectic group $Sp(\mathbb{W})$ for $\mathbb{W}$ and it acts trivially on the center $F\subset\mathcal{H}(\mathbb{W})$. For a non-trivial character ψ of the additive group of F, there exists a unique isomorphism class $\rho_\psi^{\mathbb{W}} = \rho_\psi$ of irreducible unitary representations of $\mathcal{H}(\mathbb{W})$ on which the center F acts by ψ. Thus we have a (unitary) projective representation of $Sp(\mathbb{W})$ on

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the space of ρ_ψ. The image $Mp_\psi(\mathbb{W})$ of this projective representation is the *metaplectic group* of $\mathbb{W}$ and the representation $\omega_{\mathbb{W}}$ of $Mp_\psi(\mathbb{W})$ on ρ_ψ is called the *Weil representation*. Schur's lemma gives an exact sequence

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp_\psi(\mathbb{W}) \xrightarrow{p_{\mathbb{W}}} Sp(\mathbb{W}) \longrightarrow 1.$$

Take an irreducible reductive dual pair (G, G') in $Sp(\mathbb{W})$ and write $\widetilde{G}$, $\widetilde{G}'$ for their inverse images in $Mp(\mathbb{W})$, respectively. We write $\mathcal{R}(\widetilde{G}, \omega_{\mathbb{W}})$ for the set of isomorphism classes of irreducible admissible representations (Harish-Chandra modules if F is archimedean) of $\widetilde{G}$ which appear as quotients of $\omega_{\mathbb{W}}$. Then the (local) Howe duality conjecture asserts that the relation $\mathrm{Hom}_{\widetilde{G}\times\widetilde{G}'}(\omega_{\mathbb{W}}, \pi\otimes\pi') \neq 0$ determines an well-defined bijection

$$\mathcal{R}(\widetilde{G}, \omega_{\mathbb{W}}) \ni \pi \longleftrightarrow \pi' \in \mathcal{R}(\widetilde{G}', \omega_{\mathbb{W}}). \tag{1.1}$$

This is proved by R. Howe himself if F is archimedean [How89], and by Waldspurger if F is non-archimedean with odd residual characteristic [Wal90].

If F is archimedean, we have a very convenient realization of $\omega_{\mathbb{W}}$ called *Fock model*. In this realization, it is possible to describe the correspondence between the K-types of "minimal degree" in π and π' [KV78]. Using this description, the correspondence $\pi \leftrightarrow \pi'$ is explicitly determined for π in the discrete series in [Ada89], [Li90]. Further, Adams-Barbasch developed certain induction principle which avails explicit description of the whole correspondence $\mathcal{R}(\widetilde{G}, \omega_{\mathbb{W}}) \ni \pi \leftrightarrow \pi' \in \mathcal{R}(\widetilde{G}', \omega_{\mathbb{W}})$ in the cases $F = \mathbb{C}$ [AB95], of unitary groups of close degrees [Pau98], [Pau00] and of $(O^*(2n), Sp(p,q))$ [LPTZ03].

One would like to apply this correspondence to explicit construction of automorphic forms. The first obstacle for this application is that we need a correspondence between representations of G and G' in place of $\widetilde{G}$ and $\widetilde{G}'$. Namely, we seek for homomorphisms $\widetilde{\iota} : G \to \widetilde{G}, \widetilde{\iota}' : G' \to \widetilde{G}'$ satisfying $p_{\mathbb{W}} \circ \widetilde{\iota} = \mathrm{id}_G$, $p_{\mathbb{W}} \circ \widetilde{\iota}' = \mathrm{id}_{G'}$, so that we have Weil representations $\omega_{\mathbb{W}} \circ \widetilde{\iota} \times \omega_{\mathbb{W}} \circ \widetilde{\iota}'$ of $G \times G'$. As is well-known such a splitting $\widetilde{\iota}'$ of $\widetilde{G}' \twoheadrightarrow G'$ exists except if G is an orthogonal group of odd degree (so that G' is a symplectic group $Sp(W)$). In that case, we still have a homomorphism $\widetilde{\iota}' : \widetilde{Sp(W)} \hookrightarrow \widetilde{G}'$ of certain double cover $\widetilde{Sp(W)}$ of $Sp(W)$. Even if such splittings of $\widetilde{G} \twoheadrightarrow G$, $\widetilde{G}' \twoheadrightarrow G'$ exist, there may be several choices of them and these choices may affect the resulting correspondence of representations. The unitary dual pairs are typical examples of this phenomenon.

Let $(V, (\ ,\ ))$, $(W, \langle\ ,\ \rangle)$ are (non-degenerate) hermitian and skew-hermitian spaces over a separable quadratic extension E/F. We have the associated symplectic space

$$\mathbb{W} := V \otimes_E W, \quad \langle\!\langle\ ,\ \rangle\!\rangle := \frac{1}{2}\mathrm{Tr}_{E/F}((\ ,\ ) \otimes \overline{\langle\ ,\ \rangle}),$$

where $\bar{z}$ denotes the Galois conjugate of $z \in E$. Then the unitary groups G_V, G_W of V, W form a reductive dual pair in $Sp(\mathbb{W})$. As was shown in [Kud94], [HKS96], the splittings of $\widetilde{G}_V \twoheadrightarrow G_V$, $\widetilde{G}_W \twoheadrightarrow G_W$ depend on a pair $\underline{\xi} = (\xi, \xi')$ of characters of $E^\times$ satisfying the parity conditions

$$\xi|_{F^\times} = \mathrm{sgn}_{E/F}^{\dim_E W}, \quad \xi'|_{F^\times} = \mathrm{sgn}_{E/F}^{\dim_E V}.$$

Here $\mathrm{sgn}_{E/F}$ is the quadratic character of $F^\times$ associated to E/F by the local classfield theory. We denote these splittings by $\widetilde{\iota}_{W,\xi} : G_V \hookrightarrow \widetilde{G}_V$, $\widetilde{\iota}_{V,\xi'} : G_W \hookrightarrow \widetilde{G}_W$. Then we have the "Weil representation" $\omega_{W,\xi} := \omega_{\mathbb{W}} \circ \widetilde{\iota}_{W,\xi}$ and $\omega_{V,\xi'} := \omega_{\mathbb{W}} \circ \widetilde{\iota}_{V,\xi'}$ of G_V and G_W, respectively. Again we have the set $\mathcal{R}(G_V, \omega_{W,\xi})$ of irreducible admissible representations of G_V which can be realized as quotients of $\omega_{W,\xi}$, etc., and the local θ-correspondence

$$\mathcal{R}(G_V, \omega_{W,\xi}) \ni \pi_V \longleftrightarrow \pi_W \in \mathcal{R}(G_W, \omega_{V,\xi'}). \tag{1.2}$$

This dependence on $\underline{\xi}$ is important in applications, because one can replace $\underline{\xi}$ in such a way that a given irreducible representation π_V of G_V belongs to the range $\mathcal{R}(G_V, \omega_{W,\xi})$ of the local θ-correspondence. Namely, we can control the local obstruction for the non-vanishing of theta correspondence by such twisting (see *e.g.*, [Har93]).

The purpose of this note is to report the result of our recent computation of the correspondence (1.2) in the case $E/F = \mathbb{C}/\mathbb{R}$ [KK07]. The splittings $\widetilde{\iota}_{W,\xi}$, $\widetilde{\iota}_{V,\xi'}$ were defined by the doubling argument [HKS96, §1]. Namely, we consider a doubled hyperbolic hermitian space $V^{\mathbb{H}} = V \oplus -V$ where $-V$ denotes the hermitian space $(V, -(\ ,\ ))$. Its unitary group $G_{V^{\mathbb{H}}}$ contains $G_V \times G_{-V} = G_V^2$ as a diagonal subgroup. For the hyperbolic case, we have an explicit splitting $\widetilde{\iota}^{\mathbb{H}}_{W,\xi} : G_{V^{\mathbb{H}}} \to Mp_\psi(\mathbb{W}^{\mathbb{H}})$ obtained by Kudla [Kud94], where $\mathbb{W}^{\mathbb{H}} := V^{\mathbb{H}} \otimes_{\mathbb{C}} W$. Then restriction of $\widetilde{\iota}_{W,\xi}$ to the first component of $G_V \times G_V$ gives $\widetilde{\iota}_{W,\xi}$. Consequently, the associated Weil representation $\omega^{\mathbb{H}}_{W,\xi}$ of $G_{V^{\mathbb{H}}}$ restricted to $G_V \times G_V$ decomposes into a tensor products of Weil representations of G_V (see (3.1) below). Notice that simple explicit formulae for the Schrödinger model are available only for $\omega^{\mathbb{H}}_{W,\xi}$ and not for $\omega_{W,\xi}$. Thus to describe the Fock model of $\omega_{W,\xi}$, one needs an explicit description of the decomposition (3.1) for the Fock model realization. We first give an explicit system of coordinates of the Fock model of $\omega^{\mathbb{H}}_{W,\xi}$ which nicely describes this decomposition in Th.4.1. The resulting explicit formulae for the Fock model of $\omega_{W,\xi}$ looks like an weight-shift of those for the Weil representation of the covering group $\widetilde{G}_V$ (Prop.4.2). But the authors could not find neither a direct relation of $\omega_{W,\xi}$ with $\omega_{\mathbb{W}}|_{\widetilde{G}_V}$ nor any good reference for this computation.

Now we turn to the description of the correspondence (1.2) in the same size case $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W$. Recall that the analogous correspondence (1.1) for the covering groups $(\widetilde{G}_V, \widetilde{G}_W)$ has already been determined by Paul [Pau98]. Hence we have only to show how to translate her result into the present setting. The argument consists of two things. The correspondence (1.2) subordinates to the correspondence between the K-types of π_V and π_W of lowest degree in the space of joint harmonics. (See Prop.5.1 for the precise statement.) An explicit description of this correspondence (and correspondence between the infinitesimal characters of π_V, π_W) determine the Howe correspondence between the limit of discrete series representations. Secondly, the (refined) induction principle of Adams-Barbasch [AB95] allows her to reduce the correspondence for general representations to that of their Langlands data.

This latter process needs detailed analysis of K-types of irreducible representations but is of qualitative nature, so that the arguments of [Pau98] applies to our setting. Thus our task is to compute the K-type correspondence and to describe (1.2) for the limit of discrete series representations. This is done in §§5, 6. Keeping the application to theta correspondence for automorphic forms in mind, these results are stated emphasizing the dependence on ψ and the character pair $\underline{\xi}$. In particular, the local theta correspondence for the limit of discrete series representations is stated in terms of the ε-factors associated to the Langlands parameters of these representations (Th.6.1). This description should be in accordance with Prasad's conjecture on the local theta correspondence for unitary dual pairs [Pra00]. To verify this, we still need to compute the description of tempered L-packets using endoscopy. The authors hope to turn this in some near future.

2. HOWE DUALITY FOR UNITARY DUAL PAIRS

Let $(\mathbb{W}, \langle\langle\cdot,\cdot\rangle\rangle)$ be a $2N$-dimensional symplectic space over $\mathbb{R}$. We write $Sp(\mathbb{W})$ for its symplectic group. Fix a non-trivial character $\psi : \mathbb{R} \to \mathbb{C}^\times$. In this archimedean setting, we had better to modify the definition of the *Heisenberg group* $\mathcal{H}_\psi(\mathbb{W})$ to be $\mathbb{W} \times \mathbb{C}^1$ with the multiplication law

$$(w; z)(w'; z') := (w + w'; zz'\psi\Big(\frac{\langle\langle w, w'\rangle\rangle}{2}\Big)), \quad w, w' \in \mathbb{W},\ z, z' \in \mathbb{C}^1.$$

The Stone-von-Neumann theorem asserts that there exists a unique isomorphism class $\rho_\psi^{\mathbb{W}} = \rho_\psi$ of irreducible unitary representations of $\mathcal{H}_\psi(\mathbb{W})$ on which the center $\mathbb{C}^1$ acts by multiplication. The metaplectic group

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp_\psi(\mathbb{W}) \xrightarrow{p_{\mathbb{W}}} Sp(\mathbb{W}) \longrightarrow 1. \tag{2.1}$$

is the unique extension such that ρ_ψ extends to a unitary representation (also denoted by ρ_ψ) of the semidirect product $\mathcal{J}_\psi(\mathbb{W}) := \mathbb{W} \rtimes Mp_\psi(\mathbb{W})$ (the metaplectic *Jacobi group*). Here, the $Mp_\psi(\mathbb{W})$-action on $\mathbb{W}$ is the composite of the $Sp(\mathbb{W})$-action on $\mathbb{W}$ with $p_{\mathbb{W}}$. Note that $\mathcal{J}_\psi(\mathbb{W})$ fits into the extension

$$1 \longrightarrow \mathcal{H}_\psi(\mathbb{W}) \longrightarrow \mathcal{J}_\psi(\mathbb{W}) \longrightarrow Sp(\mathbb{W}) \longrightarrow 1.$$

The restriction $\omega_{\mathbb{W},\psi} = \omega_{\mathbb{W}}$ of ρ_ψ to $Mp_\psi(\mathbb{W})$ is called the *Weil representation.*

For a direct sum decomposition $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2$ of symplectic spaces, we write $\imath_{\mathbb{W}_1,\mathbb{W}_2} : Sp(\mathbb{W}_1) \times Sp(\mathbb{W}_2) \to Sp(\mathbb{W})$ for the associated (diagonal) embedding. This lifts to a homomorphism $\tilde{\imath}_{\mathbb{W}_1,\mathbb{W}_2} : Mp_\psi(\mathbb{W}_1) \times Mp_\psi(\mathbb{W}_2) \to Mp_\psi(\mathbb{W})$:

$$\begin{array}{ccc} Mp_\psi(\mathbb{W}_1) \times Mp_\psi(\mathbb{W}_2) & \xrightarrow{\tilde{\imath}_{\mathbb{W}_1,\mathbb{W}_2}} & Mp_\psi(\mathbb{W}) \\ {\scriptstyle p_{\mathbb{W}_1} \times p_{\mathbb{W}_2}}\downarrow & & \downarrow{\scriptstyle p_{\mathbb{W}}} \\ Sp(\mathbb{W}_1) \times Sp(\mathbb{W}_2) & \xrightarrow{\imath_{\mathbb{W}_1,\mathbb{W}_2}} & Sp(\mathbb{W}) \end{array}$$

and we have

$$\omega_{\mathbb{W}} \circ \tilde{\imath}_{\mathbb{W}_1,\mathbb{W}_2} \simeq \omega_{\mathbb{W}_1} \otimes \omega_{\mathbb{W}_2}. \tag{2.2}$$

If we write $-\mathbb{W}$ for the symplectic space $(\mathbb{W}, -\langle\langle\cdot,\cdot\rangle\rangle)$, then $\omega_{-\mathbb{W},\psi} = \omega_{\mathbb{W},\bar{\psi}}$ is isomorphic to the contragredient representation $\omega_{\mathbb{W},\psi}^\vee$ of $\omega_{\mathbb{W},\psi}$.

Let $(V, (\cdot,\cdot))$ and $(W, \langle\cdot,\cdot\rangle)$ be hermitian and skew-hermitian spaces over $\mathbb{C}$. Fix a square root i of -1. We can choose a basis $\boldsymbol{v}$, $\boldsymbol{w}$ of V, W, with respect to which we have

$$(v, v') = v^* I_{p,q} v', \quad \langle w, w'\rangle = iwI_{p',q'}w'^*.$$

Here, writing $\mathbf{1}_m$ for the $m \times m$ identity matrix,

$$I_{p,q} := \begin{pmatrix} \mathbf{1}_p & \\ & -\mathbf{1}_q \end{pmatrix}.$$

By abuse of terminology, we call (p', q') the *signature* of the skew-hermitian space $(W, \langle\cdot,\cdot\rangle)$. We write $n := p + q$, $n' := p' + q'$. The $\mathbb{R}$-vector space $\mathbb{W} := V \otimes_{\mathbb{C}} W$ with

$$\langle\langle v \otimes w, v' \otimes w'\rangle\rangle := \Re\Big((v, v')\overline{\langle w, w'\rangle}\Big)$$

is a symplectic space of dimension $2(N := nn')$ over $\mathbb{R}$. Here, $\Re z$ denotes the real part of $z \in \mathbb{C}$. We write G_V, G_W for the unitary groups of $(V, (\cdot,\cdot))$, $(W, \langle\cdot,\cdot\rangle)$, respectively. We have a homomorphism

$$\iota_{V,W} = \iota_W \times \iota_V : G_V \times G_W \ni (g, g') \longmapsto g^{-1} \otimes g' \in Sp(\mathbb{W}),$$

so that G_V, G_W form a reductive dual pair in $Sp(\mathbb{W})$.

If $\psi(x) = e^{iax}$ with $a > 0$, the inverse images of $\iota_W(G_V)$, $\iota_V(G_W)$ under $p_{\mathbb{W}} : Mp_\psi(\mathbb{W}) \twoheadrightarrow Sp(\mathbb{W})$ are isomorphic to

$$\widetilde{G}_V = \{(g, z) \in G_V \times \mathbb{C}^1 \,|\, z^2 = \det g^{q'-p'}\},$$
$$\widetilde{G}_W = \{(g, z) \in G_W \times \mathbb{C}^1 \,|\, z^2 = \det g^{p-q}\}.$$

We write $\mathscr{R}(\widetilde{G}_V, \omega_{\mathbb{W}})$ for the set of isomorphism classes of irreducible Harish-Chandra modules of $\widetilde{G}_V$ which appear as quotients of $\omega_{\mathbb{W}}|_{\widetilde{G}_V}$. The *Howe duality correspondence* asserts that the relation $\mathrm{Hom}_{\widetilde{G}_V \times \widetilde{G}_W}(\omega_{\mathbb{W}}, \pi_V \otimes \pi_W) \neq 0$ determines an well-defined bijection [How89]:

$$\mathscr{R}(\widetilde{G}_V, \omega_{\mathbb{W}}) \ni \begin{array}{ccc} \pi_V & \longmapsto & \theta(\pi_V, W) \\ \theta(\pi_W, V) & \longleftarrow & \pi_W \end{array} \in \mathscr{R}(\widetilde{G}_W, \omega_{\mathbb{W}}).$$

Explicit description of this correspondence in the cases $n = n'$, $n' + 1$ are obtained in [Pau98], [Pau00], respectively.

3. DOUBLING CONSTRUCTION

The Weil representation is a base of the theory of θ-correspondence of automorphic forms. In order to formulate the (local) θ-correspondence of the unitary dual pair (G_V, G_W), we need a Weil representation not of $\widetilde{G}_V \times \widetilde{G}_W$ but of $G_V \times G_W$. This is achieved by M. Harris' doubling argument. Let us briefly review the construction from [HKS96].

Writing $(V^-, (\cdot,\cdot)^-) := (V, -(\cdot,\cdot))$, we introduce a hyperbolic hermitian space $(V^{\mathbb{H}}, (\cdot,\cdot)^{\mathbb{H}}) := (V, (\cdot,\cdot)) \oplus (V^-, (\cdot,\cdot)^-)$. ${}_\Delta V := \{(v, v) \in V^{\mathbb{H}} \,|\, v \in V\}$, ${}_\nabla V := \{(v, -v) \in V^{\mathbb{H}} \,|\, v \in V\}$ are maximal isotropic subspaces of $V^{\mathbb{H}}$ dual to each other. We adopt similar notation for $(W, \langle\cdot,\cdot\rangle)$. These doublings yield the same doubled symplectic space $\mathbb{W}^{\mathbb{H}} := V^{\mathbb{H}} \otimes_{\mathbb{C}} W = V \otimes_{\mathbb{C}} W^{\mathbb{H}}$ and the same polarization

$$\mathbb{W}^{\mathbb{H}} = {}_\nabla\mathbb{W} \oplus {}_\Delta\mathbb{W}, \quad {}_\diamondsuit\mathbb{W} = {}_\diamondsuit V \otimes_{\mathbb{C}} W = V \otimes_{\mathbb{C}} {}_\diamondsuit W, \ (\diamondsuit = \nabla, \Delta).$$

Both $\iota_V^{\mathbb{H}} : G_{V^{\mathbb{H}}} \times G_W \to Sp(\mathbb{W}^{\mathbb{H}})$ and $\iota_W^{\mathbb{H}} : G_V \times G_{W^{\mathbb{H}}} \to Sp(\mathbb{W}^{\mathbb{H}})$ define reductive dual pairs. Once such a polarization is fixed, we have an explicit description $Mp_\psi(\mathbb{W}^{\mathbb{H}}) = Sp(\mathbb{W}^{\mathbb{H}}) \times \mathbb{C}^1$ where the multiplication law is given by [RR93, Th.4.1]:

$$(g_1, z_1)(g_2, z_2) = (g_1 g_2, c_{{}_\Delta\mathbb{W}}(g_1, g_2)), \quad g_i \in Sp(\mathbb{W}^{\mathbb{H}}),\ z_i \in \mathbb{C}^1.$$

The 2-cocycle $c_{{}_\Delta\mathbb{W}}(g_1, g_2)$ is the Weil constant $\gamma_\psi(L(g_1, g_2))$ of the Leray invariant of $L(g_1, g_2) = L({}_\Delta\mathbb{W}, {}_\Delta\mathbb{W}.g_2^{-1}, {}_\Delta\mathbb{W}.g_1)$.

First consider $G_{W^{\mathbb{H}}}$. We write $\underline{w}^-$ for $\underline{w}$ viewed as a basis of W^-. We choose a Witt basis for the decomposition $W = {}_\nabla W \oplus {}_\Delta W$ to be

$$\underline{\mathrm{w}}' := \frac{\underline{w} - \underline{w}^-}{\sqrt{2}}, \quad \underline{\mathrm{w}} = i I_{p',q'} \frac{\underline{w} + \underline{w}^-}{\sqrt{2}}.$$

Using this, the Siegel parabolic subgroup $P_{\Delta W} := \mathrm{Stab}({}_{\Delta}W, G_{W^{\mathbb{H}}}) = M_{\Delta W}U_{\Delta W}$ is given by

$$M_{\Delta W} = \left\{ m_{\Delta W}(a) := \begin{pmatrix} a & \mathbf{0}_{n'} \\ \mathbf{0}_{n'} & a^{*,-1} \end{pmatrix} \,\middle|\, a \in GL(n', \mathbb{C}) \right\},$$
$$U_{\Delta W} = \left\{ u_{\Delta W}(b) := \begin{pmatrix} \mathbf{1}_{n'} & b \\ \mathbf{0}_{n'} & \mathbf{1}_{n'} \end{pmatrix} \,\middle|\, b = b^* \in \mathbb{M}_{n'}(\mathbb{C}) \right\}.$$

We have the Bruhat decomposition $G_{W^{\mathbb{H}}} = \coprod_{r=1}^{n'} P_{\Delta W} \cdot w_r \cdot P_{\Delta W}$ with

$$w_r := \left(\begin{array}{cc|cc} \mathbf{0}_r & & -\mathbf{1}_r & \\ & \mathbf{1}_{n'-r} & & \\ \hline \mathbf{1}_r & & \mathbf{0}_r & \\ & & & \mathbf{1}_{n'-r} \end{array}\right).$$

For

$$g = \begin{pmatrix} a_1 & b_1 \\ & a_1^{*,-1} \end{pmatrix} w_r \begin{pmatrix} a_2 & b_2 \\ & a_2^{*,-1} \end{pmatrix} \in G_{W^{\mathbb{H}}},$$

we set $d(g) := \det(a_1a_2) \in \mathbb{C}^\times/\mathbb{R}_+^\times$, $r(g) := r = n' - \dim_{\mathbb{C}} {}_{\Delta}W.g \cap {}_{\Delta}W$. For any character ξ' of $\mathbb{C}^\times$ satisfying $\xi'|_{\mathbb{R}^\times} = \mathrm{sgn}^n$,

$$\beta^{\mathbb{H}}_{V,\xi'}(g) := \left(\gamma_\psi(1)^{2n}(-1)^q\right)^{-r(g)}\xi'(d(g)), \quad g \in G_{W^{\mathbb{H}}},$$

splits the 2-cocycle $c_V(g_1, g_2) := c_{\Delta\mathbb{W}}(\iota^{\mathbb{H}}_V(g_1), \iota^{\mathbb{H}}_V(g_2))$, $(g_1, g_2 \in G_{W^{\mathbb{H}}})$ [Kud94, Th.3.1]. That is,

$$\tilde{\iota}^{\mathbb{H}}_{V,\xi'} : G_{W^{\mathbb{H}}} \ni g \longmapsto (g, \beta^{\mathbb{H}}_{V,\xi'}(g)) \in Mp_\psi(\mathbb{W}^{\mathbb{H}})$$

is an analytic homomorphism.

As for $G_{V^{\mathbb{H}}}$, we choose a Witt basis of $V^{\mathbb{H}} = {}_{\nabla}V \oplus {}_{\Delta}V$ to be

$$\underline{\mathbf{v}}' := \frac{\underline{\mathbf{v}} - \underline{\mathbf{v}}^-}{\sqrt{2}}, \quad \underline{\mathbf{v}} := -i\frac{\underline{\mathbf{v}} + \underline{\mathbf{v}}^-}{\sqrt{2}}I_{p,q}.$$

We adopt similar definitions as in the W-side with respect to the Siegel parabolic subgroup $P_{\Delta V} := \mathrm{Stab}({}_{\Delta}V, G_{V^{\mathbb{H}}}) = M_{\Delta V}U_{\Delta V}$. Note that this is realized as

$$M_{\Delta V} = \left\{ m_{\Delta V}(a) := \begin{pmatrix} a & \mathbf{0}_n \\ \mathbf{0}_n & a^{*,-1} \end{pmatrix} \,\middle|\, a \in GL(n, \mathbb{C}) \right\},$$
$$U_{\Delta V} = \left\{ u_{\Delta V}(b) := \begin{pmatrix} \mathbf{1}_n & \mathbf{0}_n \\ b & \mathbf{1}_n \end{pmatrix} \,\middle|\, b = b^* \in \mathbb{M}_n(\mathbb{C}) \right\}.$$

with respect to $\underline{\mathbf{v}}' \cup \underline{\mathbf{v}}$. Taking a character ξ of $\mathbb{C}^\times$ satisfying $\xi|_{\mathbb{R}^\times} = \mathrm{sgn}^{n'}$,

$$\beta^{\mathbb{H}}_{W,\xi}(g) := \left(\gamma_\psi(1)^{2n'}(-1)^{p'}\right)^{-r(g)}\xi(d(g)), \quad g \in G_{V^{\mathbb{H}}}$$

splits $c_W(g_1, g_2) := c_{\Delta\mathbb{W}}(\iota^{\mathbb{H}}_W(g_1), \iota^{\mathbb{H}}_W(g_2))$, $(g_1, g_2 \in G_{V^{\mathbb{H}}})$. Hence an analytic homomorphism

$$\tilde{\iota}^{\mathbb{H}}_{W,\xi} : G_{V^{\mathbb{H}}} \ni g \longmapsto (g, \beta^{\mathbb{H}}_{W,\xi}(g)) \in Mp_\psi(\mathbb{W}^{\mathbb{H}})$$

is obtained.

Noting $G_{V^-} = G_V$, we have the (diagonal) embedding $i_V : G_V \times G_V \hookrightarrow G_{V^{\mathbb{H}}}$ and the following commutative diagram:

$$\begin{tikzcd}
G_{V^{\mathbb{H}}} \arrow[r, "\iota^{\mathbb{H}}_W"] & Sp(\mathbb{W}^{\mathbb{H}}) & G_{W^{\mathbb{H}}} \arrow[l, "\iota^{\mathbb{H}}_V"'] \\
G_V \times G_V \arrow[u, "i_V"] \arrow[r, "\iota_W \times \iota_W"] & Sp(\mathbb{W}) \times Sp(\mathbb{W}) \arrow[u, "i_{\mathbb{W}}"'] & G_W \times G_W \arrow[u, "i_W"'] \arrow[l, "\iota_V \times \iota_V"']
\end{tikzcd}$$

Fix a pair $\underline{\xi} = (\xi, \xi')$ of characters of $\mathbb{C}^\times$ such that $\xi|_{\mathbb{R}^\times} = \mathrm{sgn}^{n'}$, $\xi'|_{\mathbb{R}^\times} = \mathrm{sgn}^{n}$. We define

$$\tilde{\iota}_{W,\xi} : G_V \xrightarrow{\text{1st.}} G_V \times G_V \xrightarrow{i_V} G_{V^{\mathbb{H}}} \xrightarrow{\tilde{\iota}^{\mathbb{H}}_{W,\xi}} Mp_\psi(\mathbb{W}^{\mathbb{H}}),$$
$$\tilde{\iota}_{V,\xi'} : G_W \xrightarrow{\text{1st.}} G_W \times G_W \xrightarrow{i_W} G_{W^{\mathbb{H}}} \xrightarrow{\tilde{\iota}^{\mathbb{H}}_{V,\xi'}} Mp_\psi(\mathbb{W}^{\mathbb{H}}),$$

where the left arrows are the embeddings to the first components. These yield homomorphisms

$$\tilde{\iota}_{W,\xi} : G_V \longrightarrow Mp_\psi(\mathbb{W}), \quad \tilde{\iota}_{V,\xi'} : G_W \longrightarrow Mp_\psi(\mathbb{W}).$$

These definition show that the following diagram commutes:

$$\begin{tikzcd}
G_{V^{\mathbb{H}}} \arrow[r, "\tilde{\iota}^{\mathbb{H}}_{W,\xi}"] & Mp_\psi(\mathbb{W}^{\mathbb{H}}) & G_{W^{\mathbb{H}}} \arrow[l, "\tilde{\iota}^{\mathbb{H}}_{V,\xi'}"'] \\
G_V \times G_V \arrow[u, "i_V"] \arrow[r, "\tilde{\iota}_{W,\xi} \times \xi(\det)^{-1}\tilde{\iota}_{W,\xi}"] & Mp_\psi(\mathbb{W}) \times Mp_\psi(\mathbb{W}) \arrow[u, "\tilde{i}_{\mathbb{W}}"'] & G_W \times G_W. \arrow[u, "i_W"'] \arrow[l, "\tilde{\iota}_{V,\xi'} \times \xi'(\det)^{-1}\tilde{\iota}_{V,\xi'}"']
\end{tikzcd}$$

Here, $\tilde{i}_{\mathbb{W}} : Mp_\psi(\mathbb{W}) \times Mp_\psi(\mathbb{W}) \ni ((g,z),(g',z')) \mapsto (i_{\mathbb{W}}(g,g'), z\bar{z}') \in Mp_\psi(\mathbb{W}^{\mathbb{H}})$.
Now we can define the Weil representation $\omega_{V,W,\underline{\xi}} = \omega_{W,\xi} \times \omega_{V,\xi'}$ of $G_V \times G_W$:

$$\omega_{W,\xi} := \omega_{\mathbb{W}} \circ \tilde{\iota}_{W,\xi}, \quad \omega_{V,\xi'} := \omega_{\mathbb{W}} \circ \tilde{\iota}_{V,\xi'}.$$

Also we have the Weil representations $\omega^{\mathbb{H}}_{W,\xi} := \omega_{\mathbb{W}^{\mathbb{H}}} \circ \tilde{\iota}^{\mathbb{H}}_{W,\xi}$, $\omega^{\mathbb{H}}_{V,\xi'} := \omega_{\mathbb{W}^{\mathbb{H}}} \circ \tilde{\iota}^{\mathbb{H}}_{V,\xi'}$ of $G_{V^{\mathbb{H}}}$, $G_{W^{\mathbb{H}}}$, respectively. The above diagram combined with (2.2) show

$$\text{(3.1)} \qquad \omega^{\mathbb{H}}_{W,\xi} \circ i_V \simeq \omega_{W,\xi} \otimes \xi(\det)\omega^\vee_{W,\xi}, \quad \omega^{\mathbb{H}}_{V,\xi'} \circ i_W \simeq \omega_{V,\xi'} \otimes \xi'(\det)\omega^\vee_{V,\xi'}.$$

The explicit formula for the Schrödinger model $\mathcal{S}({}_\nabla\mathbb{W})$ for $\omega^{\mathbb{H}}_{W,\xi}$, $\omega^{\mathbb{H}}_{V,\xi'}$ is given in [Kud94, § 5].
For completeness, we adopt the following convention. If $V = 0$ (resp. $W = 0$), we set $G_V := \{1\}$ (resp. $G_W = \{1\}$) and $\omega_{V,\xi'} = \xi'_u(\det)$ (resp. $\omega_{W,\xi} = \xi_u(\det)$). Here, for a character ξ of $\mathbb{C}^\times/\mathbb{R}^\times$, we write $\xi_u : U(1,\mathbb{R}) \ni z/\bar{z} \mapsto \xi(z) \in \mathbb{C}^\times$.

4. DOUBLING CONSTRUCTION OF THE FOCK MODEL

We now study the Harish-Chandra module of $\omega_{V,W,\xi}$. Notice that $(\underline{v}\otimes\underline{w}\cup\underline{v}^-\otimes\underline{w})\cup(i\underline{v}I_{p,q}\otimes I_{p',q'}\underline{w}\cup -i\underline{v}^-I_{p,q}\otimes I_{p',q'}\underline{w})$ is a Witt basis of $\mathbb{W}^{\mathbb{H}}$. We choose another Witt basis

(4.1)

$$\begin{pmatrix} {}^{\Re}\underline{e}' \\ {}^{\Im}\underline{e}' \\ {}^{\Re}\underline{e} \\ {}^{\Im}\underline{e} \end{pmatrix} := \begin{pmatrix} \underline{\mathrm{v}}'\otimes\underline{w} \\ i\underline{\mathrm{v}}'\otimes\underline{w} \\ -\underline{\mathrm{v}}\otimes I_{p',q'}\underline{w} \\ -i\underline{\mathrm{v}}\otimes I_{p',q'}\underline{w} \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} 1_N & -1_N & 0_N & 0_N \\ 0_N & 0_N & I & I \\ \hline 0_N & 0_N & 1_N & -1_N \\ -I & -I & 0_N & 0_N \end{array}\right)\begin{pmatrix} \underline{v}\otimes\underline{w} \\ \underline{v}^-\otimes\underline{w} \\ i\underline{v}I_{p,q}\otimes I_{p',q'}\underline{w} \\ -i\underline{v}^-I_{p,q}\otimes I_{p',q'}\underline{w} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} 1_N & -1_N & 0_N & 0_N \\ 0_N & 0_N & I & I \\ \hline 0_N & 0_N & 1_N & -1_N \\ -I & -I & 0_N & 0_N \end{array}\right)\begin{pmatrix} \underline{v}\otimes\underline{w} \\ \underline{v}\otimes\underline{w}^- \\ i\underline{v}I_{p,q}\otimes I_{p',q'}\underline{w} \\ -i\underline{v}I_{p,q}\otimes I_{p',q'}\underline{w}^- \end{pmatrix} = \begin{pmatrix} \underline{v}\otimes\underline{\mathrm{w}}' \\ i\underline{v}\otimes\underline{\mathrm{w}}' \\ \underline{v}I_{p,q}\otimes\underline{\mathrm{w}} \\ i\underline{v}I_{p,q}\otimes\underline{\mathrm{w}} \end{pmatrix}$$

for $\mathbb{W}^{\mathbb{H}} = {}_{\nabla}\mathbb{W}\oplus{}_{\Delta}\mathbb{W}$, where $I := I_{p,q}\otimes I_{p',q'}$. Notice that $\underline{v}^-\otimes\underline{w}$ and $\underline{v}\otimes\underline{w}^-$ are identical in $\mathbb{W}^{\mathbb{H}}$. Let $\mathfrak{sp}(\mathbb{W}^{\mathbb{H}}) = \mathfrak{k}_{\mathbb{W}^{\mathbb{H}}}\oplus\mathfrak{p}_{\mathbb{W}^{\mathbb{H}}}$ be the Cartan decomposition of the Lie algebra of $Sp(\mathbb{W}^{\mathbb{H}})$ given by

$$\mathfrak{k}_{\mathbb{W}^{\mathbb{H}}} = \left\{\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\,\middle|\, \begin{matrix} A = -{}^tA, \\ B = {}^tB \end{matrix} \in \mathrm{M}_{2N}(\mathbb{R})\right\},$$

$$\mathfrak{p}_{\mathbb{W}} = \left\{\begin{pmatrix} A & B \\ B & -A \end{pmatrix}\,\middle|\, \begin{matrix} A = {}^tA \\ B = {}^tB \end{matrix} \in \mathrm{M}_{2N}(\mathbb{R})\right\}$$

in the realization with respect to the basis (4.1). We choose the Cartan decompositions $\mathfrak{g}_V = \mathfrak{k}_V\oplus\mathfrak{p}_V$, $\mathfrak{g}_{V^{\mathbb{H}}} = \mathfrak{k}_{V^{\mathbb{H}}}\oplus\mathfrak{p}_{V^{\mathbb{H}}}$ for the Lie algebras of G_V, $G_{V^{\mathbb{H}}}$ to be the inverse image of the above decomposition under ι_W, $\iota_{W^{\mathbb{H}}}$, respectively. We also use the similar Cartan decompositions for the W-side. Since the basis transformation matrix in (4.1) is unitary, this coincides with the usual decomposition

$$\mathfrak{k}_V := \left\{\begin{pmatrix} A & \\ & D \end{pmatrix}\in\mathfrak{g}_V\right\},\quad \mathfrak{p}_V := \left\{\begin{pmatrix} 0_p & B \\ B^* & 0_q \end{pmatrix}\in\mathfrak{g}_V\right\}$$

in the realization with respect to $\underline{v}$. We write $K_V\subset G_V$, $K_{\mathbb{W}}\subset Sp(\mathbb{W})$ for the maximal compact subgroups having the Lie algebras $\mathfrak{k}_V$, $\mathfrak{k}_{\mathbb{W}}$, respectively.

We write $\psi(x) = e^{d\psi x}$ with $d\psi\in i\mathbb{R}^\times$ and ε_ψ for the sign of $d\psi/i$. To obtain the Harish-Chandra modules of $\omega^{\mathbb{H}}_{W,\xi}$, $\omega^{\mathbb{H}}_{V,\xi'}$, we take the following totally complex polarization $\mathbb{W}^{\mathbb{H}}_{\mathbb{C}} = \mathbb{L}'\oplus\mathbb{L}$ of the complexification of $\mathbb{W}^{\mathbb{H}}$:

$$\mathbb{L}' := \mathrm{span}_{\mathbb{C}}\,({}^{\Re}\underline{e}'\cup{}^{\Im}\underline{e}'),\quad \mathbb{L} := \mathrm{span}_{\mathbb{C}}\,{}^{\Re}\underline{e}\cup{}^{\Im}\underline{e},$$

$$(4.2)\qquad {}^{\Diamond}\underline{e}' := \frac{{}^{\Diamond}\underline{e}'-\varepsilon_\psi i^{\Diamond}\underline{e}}{\sqrt{2}},\quad {}^{\Diamond}\underline{e} := \frac{{}^{\Diamond}\underline{e}-\varepsilon_\psi i^{\Diamond}\underline{e}'}{\sqrt{2}},\quad (\Diamond = \Re \text{ or } \Im).$$

The universal enveloping algebra of the complexified Lie algebra $\mathfrak{h}_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ of $\mathcal{H}_\psi(\mathbb{W}^{\mathbb{H}})$ is the quantum algebra

$$\Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) = T(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})/(w\otimes w' - w'\otimes w - d\psi\langle\langle w, w'\rangle\rangle^{\mathbb{H}}\,|\, w,\, w'\in\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$$

of $(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}, d\psi\langle\langle\cdot,\cdot\rangle\rangle^{\mathbb{H}})$. Here $T(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ is the tensor algebra of $\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}$. Thanks to the Poincaré-Birkoff-Witt theorem, we have the decomposition $\Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) = S(\mathbb{L})\oplus\Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})\mathbb{L}'$. $S(\mathbb{L})$ stands for the symmetric algebra of $\mathbb{L}$.

$\Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ carries a filtration $\Omega^0_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) = \mathbb{C} \subset \Omega^1_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) \subset \cdots \subset \Omega^n_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) \subset \dots$ induced from the grading $T(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) = \bigoplus_{n\in\mathbb{N}} T^n(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$. One can easily check that $\Omega^2_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ is isomorphic to the complexified Lie algebra $\mathfrak{j}_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ of $\mathcal{J}_\psi(\mathbb{W}^{\mathbb{H}})$. We define the representation $(r_\psi = r_\psi^{\mathbb{W}^{\mathbb{H}}}, S(\mathbb{L}))$ of $j_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ by

$$r_\psi(X)P = X.P, \quad X \in j(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}),\ P \in S(\mathbb{L}) = \Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})/\Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})\mathbb{L}'.$$

It is known that this yields the $(\mathfrak{sp}(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}), K_{\mathbb{W}^{\mathbb{H}}})$-module of the Weil representation $\omega_{\mathbb{W}}$.

We identify ${}^{\Re}\underline{\mathrm{e}} = \{{}^{\Re}\mathrm{e}_{j,k}\}$, ${}^{\Im}\underline{\mathrm{e}} = \{{}^{\Im}\mathrm{e}_{j,k}\}$ with variables $\{u_{j,k}\}$, $\{\epsilon_j\epsilon'_k v_{j,k}\}$, respectively. Then $S(\mathbb{L}^{\mathbb{H}}) = \mathbb{C}[(u_{j,k}), (v_{j,k})]$ (polynomial ring over $\mathbb{M}_{n,n'}(\mathbb{C})^2$) on which $\mathfrak{h}_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ acts by

$$r_\psi({}^{\Re}\mathrm{e}_{j,k}) = u_{j,k}, \quad r_\psi({}^{\Im}\mathrm{e}_{j,k}) = \epsilon_j\epsilon'_k v_{j,k},$$
$$r_\psi({}^{\Re}\mathrm{e}'_{j,k}) = d\psi\frac{\partial}{\partial u_{j,k}}, \quad r_\psi({}^{\Im}\mathrm{e}'_{j,k}) = \epsilon_j\epsilon'_k d\psi\frac{\partial}{\partial v_{j,k}}.$$

Here ϵ_j (resp. ϵ'_j) denotes the (j,j)-entry of $I_{p,q}$ (resp. $I_{p',q'}$). Set $w_{j,k} := \sqrt{2}^{-1}(u_{j,k} + iv_{j,k})$, $\bar{w}_{j,k} := \sqrt{2}^{-1}(u_{j,k} - iv_{j,k})$, $(1 \le j \le n, 1 \le k \le n')$. Now we can state our first result.

Theorem 4.1. *The decomposition* (3.1) *of the* $(\mathfrak{g}_{V,\mathbb{C}} \oplus \mathfrak{g}_{W,\mathbb{C}}, K_V \times K_W)$*-module* $S(\mathbb{L}^{\mathbb{H}})$ *is given by*

$$((\omega^{\mathbb{H}}_{W,\xi} \circ i_V)\times(\omega^{\mathbb{H}}_{V,\xi'} \circ i_W), \mathbb{C}[(w_{j,k}), (\bar{w}_{j,k})])$$
$$= \begin{cases} (\omega_{V,W,\underline{\xi}}, \mathbb{C}[(w_{j,k})]) \otimes (\xi(\det)\xi'(\det)\omega^\vee_{V,W,\underline{\xi}}, \mathbb{C}[(\bar{w}_{j,k})]) & \text{if } \varepsilon_\psi > 0, \\ (\omega_{V,W,\underline{\xi}}, \mathbb{C}[(\bar{w}_{j,k})]) \otimes (\xi(\det)\xi'(\det)\omega^\vee_{V,W,\underline{\xi}}, \mathbb{C}[(w_{j,k})]) & \text{if } \varepsilon_\psi < 0. \end{cases}$$

Thus the Fock model $\mathscr{P}_{V,W,\underline{\xi}}$ *for* $\omega_{V,W,\underline{\xi}}$ *is* $\mathbb{C}[(w_{j,k})]$ *if* $\varepsilon_\psi > 0$ *and* $\mathbb{C}[(\bar{w}_{j,k})]$ *otherwise.*

We also have the following explicit formulae for $(\omega_{V,W,\underline{\xi}}, \mathscr{P}_{V,W,\underline{\xi}})$. We use the basis

$$(4.3)\quad \begin{aligned} U_{j,k} &:= E_{j,k}, && (1 \le j,k \le p), & X_{j,k} &:= E_{p+j,k}, && (1 \le j \le q, 1 \le k \le p), \\ V_{j,k} &:= E_{p+j,p+k}, && (1 \le j,k \le q), & Y_{j,k} &:= E_{j,p+k}, && (1 \le j \le p, 1 \le k \le q), \\ U'_{j,k} &:= E_{j,k}, && (1 \le j,k \le p'), & X'_{j,k} &:= E_{j,p'+k}, && (1 \le j \le p', 1 \le k \le q'), \\ V'_{j,k} &:= E_{p'+j,p'+k}, && (1 \le j,k \le q'), & Y'_{j,k} &:= E_{p'+j,k}, && (1 \le j \le q', 1 \le k \le p') \end{aligned}$$

of $\mathfrak{g}_{V,\mathbb{C}}$, $\mathfrak{g}_{W,\mathbb{C}}$ realized with respect to $\underline{v}$, $\underline{w}$, respectively. Here $E_{j,k}$ denotes the (j,k)-elementary matrix. We write

$$\xi(z) = \left(\frac{z}{\bar{z}}\right)^{m/2}, \quad \xi'(z) = \left(\frac{z}{\bar{z}}\right)^{m'/2}, \quad m \equiv n',\ m' \equiv n \pmod 2, \in \mathbb{Z}.$$

Proposition 4.2. *(a) When* $\varepsilon_\psi > 0$, *we have the following explicit formulae for* $(\omega_{V,W,\underline{\xi}}, \mathscr{P}_{V,W,\underline{\xi}})$.

$$\omega_{W,\xi}(U_{j,k}) = \frac{m+q'-p'}{2}\delta_{j,k} - \sum_{\ell=1}^{p'} w_{k,\ell}\frac{\partial}{\partial w_{j,\ell}} + \sum_{\ell=p'+1}^{n'} w_{j,\ell}\frac{\partial}{\partial w_{k,\ell}},$$

$$\omega_{W,\xi}(V_{j,k}) = \frac{m+p'-q'}{2}\delta_{j,k} + \sum_{\ell=1}^{p'} w_{p+j,\ell}\frac{\partial}{\partial w_{p+k,\ell}} - \sum_{\ell=p'+1}^{n'} w_{p+k,\ell}\frac{\partial}{\partial w_{p+j,\ell}},$$

$$\omega_{W,\xi}(X_{j,k}) = -\frac{1}{|d\psi|}\sum_{\ell=1}^{p'} w_{p+j,\ell}w_{k,\ell} + |d\psi|\sum_{\ell=p'+1}^{n'} \frac{\partial^2}{\partial w_{p+j,\ell}\partial w_{k,\ell}},$$

$$\omega_{W,\xi}(Y_{j,k}) =|d\psi|\sum_{\ell=1}^{p'} \frac{\partial^2}{\partial w_{j,\ell}\partial w_{p+k,\ell}} - \frac{1}{|d\psi|}\sum_{\ell=p'+1}^{n'} w_{j,\ell}w_{p+k,\ell},$$

$$\omega_{V,\xi'}(U'_{j,k}) =\frac{m'+p-q}{2}\delta_{j,k} + \sum_{\ell=1}^{p} w_{\ell,j}\frac{\partial}{\partial w_{\ell,k}} - \sum_{\ell=p+1}^{n} w_{\ell,k}\frac{\partial}{\partial w_{\ell,j}},$$

$$\omega_{V,\xi'}(V'_{j,k}) =\frac{m'+q-p}{2}\delta_{j,k} - \sum_{\ell=1}^{p} w_{\ell,p'+k}\frac{\partial}{\partial w_{\ell,p'+j}} + \sum_{\ell=p+1}^{n} w_{\ell,p'+j}\frac{\partial}{\partial w_{\ell,p'+k}},$$

$$\omega_{V,\xi'}(X'_{j,k}) =\frac{1}{|d\psi|}\sum_{\ell=1}^{p} w_{\ell,j}w_{\ell,p'+k} - |d\psi|\sum_{\ell=p+1}^{n} \frac{\partial^2}{\partial w_{\ell,j}\partial w_{\ell,p'+k}},$$

$$\omega_{V,\xi'}(Y'_{j,k}) =-|d\psi|\sum_{\ell=1}^{p} \frac{\partial^2}{\partial w_{\ell,p'+j}\partial w_{\ell,k}} + \frac{1}{|d\psi|}\sum_{\ell=p+1}^{n} w_{\ell,p'+j}w_{\ell,k}.$$

(iii) Similarly if $\varepsilon_\psi < 0$, we have

$$\omega_{W,\xi}(U_{j,k}) =\frac{m+p'-q}{2}\delta_{j,k} + \sum_{\ell=1}^{p'} \bar{w}_{j,\ell}\frac{\partial}{\partial \bar{w}_{k,\ell}} - \sum_{\ell=p'+1}^{n'} \bar{w}_{k,\ell}\frac{\partial}{\partial \bar{w}_{j,\ell}},$$

$$\omega_{W,\xi}(V_{j,k}) =\frac{m+q'-p'}{2}\delta_{j,k} - \sum_{\ell=1}^{p'} \bar{w}_{p+k,\ell}\frac{\partial}{\partial \bar{w}_{p+j,\ell}} + \sum_{\ell=p'+1}^{n'} \bar{w}_{p+j,\ell}\frac{\partial}{\partial \bar{w}_{p+k,\ell}},$$

$$\omega_{W,\xi}(X_{j,k}) =|d\psi|\sum_{\ell=1}^{p'} \frac{\partial^2}{\partial \bar{w}_{p+j,\ell}\partial \bar{w}_{k,\ell}} - \frac{1}{|d\psi|}\sum_{\ell=p'+1}^{n'} \bar{w}_{p+j,\ell}\bar{w}_{k,\ell},$$

$$\omega_{W,\xi}(Y_{j,k}) =-\frac{1}{|d\psi|}\sum_{\ell=1}^{p'} \bar{w}_{j,\ell}\bar{w}_{p+k,\ell} + |d\psi|\sum_{\ell=p'+1}^{n'} \frac{\partial^2}{\partial \bar{w}_{j,\ell}\partial \bar{w}_{p+k,\ell}},$$

$$\omega_{V,\xi'}(U'_{j,k}) =\frac{m'+q-p}{2}\delta_{j,k} - \sum_{\ell=1}^{p} \bar{w}_{\ell,k}\frac{\partial}{\partial \bar{w}_{\ell,j}} + \sum_{\ell=p+1}^{n} \bar{w}_{\ell,j}\frac{\partial}{\partial \bar{w}_{\ell,k}},$$

$$\omega_{V,\xi'}(V'_{j,k}) =\frac{m'+p-q}{2}\delta_{j,k} + \sum_{\ell=1}^{p} \bar{w}_{\ell,p'+j}\frac{\partial}{\partial \bar{w}_{\ell,p'+k}} - \sum_{\ell=p+1}^{n} \bar{w}_{\ell,p'+k}\frac{\partial}{\partial \bar{w}_{\ell,p'+j}},$$

$$\omega_{V,\xi'}(X'_{j,k}) =-|d\psi|\sum_{\ell=1}^{p} \frac{\partial^2}{\partial \bar{w}_{\ell,j}\partial \bar{w}_{\ell,p'+k}} + \frac{1}{|d\psi|}\sum_{\ell=p+1}^{n} \bar{w}_{\ell,j}\bar{w}_{\ell,p'+k},$$

$$\omega_{V,\xi'}(Y'_{j,k}) =\frac{1}{|d\psi|}\sum_{\ell=1}^{p} \bar{w}_{\ell,p'+j}\bar{w}_{\ell,k} - |d\psi|\sum_{\ell=p+1}^{n} \frac{\partial^2}{\partial \bar{w}_{\ell,p'+j}\partial \bar{w}_{\ell,k}}.$$

Following the argument of [How89], one can deduce the local θ-correspondence under $\omega_{V,W,\underline{\xi}}$ from this. Let $\mathscr{R}(G_V,\omega_{W,\xi})$ be the set of isomorphism classes of irreducible $(\mathfrak{g}_{V,\mathbb{C}},K_V)$-modules π_V such that $\mathrm{Hom}_{(\mathfrak{g}_{V,\mathbb{C}},K_V)}(\mathscr{P}_{V,W,\underline{\xi}},\pi_V)\neq 0$. For $\pi_V\in\mathscr{R}(G_V,\omega_{W,\xi})$, let $\mathscr{N}(\pi_V)$ be the intersection of $\ker\phi$, $\phi\in\mathrm{Hom}_{(\mathfrak{g}_{V,\mathbb{C}},K_V)}(\mathscr{P}_{V,W,\underline{\xi}},\pi_V)$.

Theorem 4.3 (local θ-correspondence). *Take $\pi_V \in \mathscr{R}(G_V, \omega_{W,\xi})$.*
(i) The quotient $\mathscr{P}_{V,W,\underline{\xi}}/\mathscr{N}(\pi_V)$ is of finite length, hence has an irreducible quotient.
(ii) $\mathscr{P}_{V,W,\underline{\xi}}/\mathscr{N}(\pi_V)$ admits a unique irreducible quotient $\theta_{\underline{\xi}}(\pi_V, W)$.
(iii) This and the analogous construction in the W-side give a bijection

$$\mathscr{R}(G_V, \omega_{W,\xi}) \ni \begin{array}{ccc} \pi_V & \longmapsto & \theta_{\underline{\xi}}(\pi_V, W) \\ \theta_{\underline{\xi}}(\pi_W, V) & \longleftarrow & \pi_W \end{array} \in \mathscr{R}(G_W, \omega_{V,\xi'}).$$

5. K-TYPE CORRESPONDENCE

The next problem is to compute the bijection in Th.4.3 explicitly. We first prepare some more notation.

We write $V_+ := \mathrm{span}_{\mathbb{C}}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$, $V_- := \mathrm{span}_{\mathbb{C}}\{\boldsymbol{v}_{p+1}, \ldots, \boldsymbol{v}_n\}$ and $(\cdot,\cdot)_\pm$ for the restrictions of $(\cdot,\cdot)$ to $V_\pm$, respectively: $(V, (\cdot,\cdot)) = (V_+, (\cdot,\cdot)_+) \oplus (V_-, (\cdot,\cdot)_-)$. Then $K_V = G_{V_+} \times G_{V_-}$. We adopt the similar notation for G_W. Thus we have the seesaw dual pairs

$$\begin{tikzcd} G_V \arrow[d, dash] \arrow[dr, dash] & G_W \times G_W \arrow[d, dash] \arrow[dl, dash] \\ K_V & G_W \end{tikzcd} \qquad \begin{tikzcd} G_V \times G_V \arrow[d, dash] \arrow[dr, dash] & G_W \arrow[d, dash] \arrow[dl, dash] \\ G_V & K_W \end{tikzcd}$$

in $Sp(\mathbb{W})$.

Take any decompositions $\xi = \xi_+ \cdot \xi_-$, $\xi' = \xi'_+ \cdot \xi'_-$ such that $\xi_\pm|_{\mathbb{R}^\times} = \mathrm{sgn}^{\dim W_\pm}$, $\xi'_\pm|_{\mathbb{R}^\times} = \mathrm{sgn}^{\dim V_\pm}$, respectively. The Weil representations $\omega_{V,W_+,(\xi_+,\xi')} \otimes \omega_{V,W_-,(\xi_-,\xi')}$ of $(G_V \times G_V) \times K_W$ and $\omega_{V_+,W,(\xi,\xi'_+)} \otimes \omega_{V_-,W,(\xi,\xi'_-)}$ of $K_V \times (G_W \times G_W)$ share the same Fock model $\mathscr{P}_{V,W,\underline{\xi}}$. Using the basis (4.3), we take the Harish-Chandra decompositions $\mathfrak{g}_{V,\mathbb{C}} = \mathfrak{k}_{V,\mathbb{C}} \oplus \mathfrak{p}^+_{V,W_\pm} \oplus \mathfrak{p}^-_{V,W_\pm}$, $\mathfrak{g}_{W,\mathbb{C}} = \mathfrak{k}_{W,\mathbb{C}} \oplus \mathfrak{p}^+_{W,V_\pm} \oplus \mathfrak{p}^-_{W,V_\pm}$ as

$$\begin{aligned} \mathfrak{p}^{\varepsilon_\psi}_{V,W_+} &= \mathfrak{p}^{-\varepsilon_\psi}_{V,W_-} = \mathrm{span}_{\mathbb{C}}\{X_{j,k}\}^{1 \le k \le p}_{1 \le j \le q}, & \mathfrak{p}^{-\varepsilon_\psi}_{V,W_+} &= \mathfrak{p}^{\varepsilon_\psi}_{V,W_-} = \mathrm{span}_{\mathbb{C}}\{Y_{j,k}\}^{1 \le k \le q}_{1 \le j \le p}, \\ \mathfrak{p}^{\varepsilon_\psi}_{W,V_+} &= \mathfrak{p}^{-\varepsilon_\psi}_{W,V_-} = \mathrm{span}_{\mathbb{C}}\{X'_{j,k}\}^{1 \le k \le q'}_{1 \le j \le p'}, & \mathfrak{p}^{-\varepsilon_\psi}_{W,V_+} &= \mathfrak{p}^{\varepsilon_\psi}_{W,V_-} = \mathrm{span}_{\mathbb{C}}\{Y'_{j,k}\}^{1 \le k \le p'}_{1 \le j \le q'}. \end{aligned}$$

We define the spaces of K_W and K_V-*harmonics* to be

$$\begin{aligned} \mathscr{H}_V(K_W) &:= \{P \in \mathscr{P}_{V,W,\underline{\xi}} \,|\, \omega_{W_\pm,\xi_\pm}(\mathfrak{p}^-_{V,W_\pm})P = 0\}, \\ \mathscr{H}_W(K_V) &:= \{P \in \mathscr{P}_{V,W,\underline{\xi}} \,|\, \omega_{V_\pm,\xi'_\pm}(\mathfrak{p}^-_{W,V_\pm})P = 0\}, \end{aligned}$$

respectively. Their intersection $\mathscr{J}_{V,W,\underline{\xi}} := \mathscr{H}_V(K_W) \cap \mathscr{H}_W(K_V)$ is called the space of *joint harmonics*. Prop.4.2 applied to $(V, W_\pm)$, $(V_\pm, W)$ in place of (V, W) shows that $\mathscr{J}_{V,W,\underline{\xi}}$ consists of $P \in \mathscr{P}_{V,W,\underline{\xi}}$ killed by

$$\begin{aligned} &\sum_{\ell=1}^{p'} \frac{\partial^2}{\partial w_{j,\ell}\partial w_{p+k,\ell}}, \quad \sum_{\ell=p'+1}^{n'} \frac{\partial^2}{\partial w_{j,\ell}\partial w_{p+k,\ell}}, \quad (1 \le j \le p, 1 \le k \le q), \\ &\sum_{\ell=1}^{p} \frac{\partial^2}{\partial w_{\ell,j}\partial w_{\ell,p'+k}}, \quad \sum_{\ell=p+1}^{n} \frac{\partial^2}{\partial w_{\ell,j}\partial w_{\ell,p'+k}}, \quad (1 \le j \le p', 1 \le k \le q'), \end{aligned} \tag{5.1}$$

if $d\psi i < 0$. When $d\psi i > 0$, we have the same description with $w_{j,k}$ replaced by $\bar{w}_{j,k}$.

The following result is implicit in the proof of Th.4.3.

Proposition 5.1 (cf. [How89] §3). *(1) $\mathscr{J}_{V,W,\underline{\xi}}$ is stable under $\omega_{V,W,\underline{\xi}}(K_V \times K_W)$.*
(2) We write $\mathscr{R}(K_V, \mathscr{J}_{V,W,\underline{\xi}})$ for the set of K_V-types which appear as irreducible direct summands of $\mathscr{J}_{V,W,\underline{\xi}}$. Similarly we define $\mathscr{R}(K_W, \mathscr{J}_{V,W,\underline{\xi}})$ in the W-side. Then $\mathscr{J}_{V,W,\underline{\xi}}$ is multiplicity free as a $K_V \times K_W$-module, so that it gives a bijection

$$\mathscr{R}(K_V, \mathscr{J}_{V,W,\underline{\xi}}) \ni \begin{array}{ccc} \tau_V & \longmapsto & \theta_{\underline{\xi}}(\tau_V, K_W) \\ \theta_{\underline{\xi}}(\tau_W, K_V)) & \longleftarrow & \tau_W \end{array} \in \mathscr{R}(K_W, \mathscr{J}_{V,W,\underline{\xi}}).$$

(3) For a K_V-type τ_V, we write $\deg_{W,\xi}(\tau_V)$ for the minimum degree of polynomials in the τ_V-isotypic subspace in $\mathscr{P}_{V,W,\underline{\xi}}$. (Set $\deg_{W,\xi}(\tau_V) := \infty$ if τ_V does not appear in $\mathscr{P}_{V,W,\underline{\xi}}$.) K_V-type τ_V of $\pi_V \in \mathscr{R}(G_V, \omega_{W,\xi})$ is of minimal (W, ξ)-degree *if $\deg_{W,\xi}(\tau_V)$ is minimal among* $\det_{W,\xi}(\tau)$, *(τ runs over the set of K_V-types in π_V). Similar definition applies to the W-side.*

- (i) *Suppose τ_V is a K_V-type in $\pi_V \in \mathscr{R}(G_V, \omega_{W,\xi})$ of minimal (W, ξ)-degree. Then $\tau_V \in \mathscr{R}(K_V, \mathscr{J}_{V,W,\underline{\xi}})$.*
- (ii) *Furthermore, $\theta_{\underline{\xi}}(\tau_V, K_W)$ is a K_W-type of minimal (V, ξ')-degree in $\theta_{\underline{\xi}}(\pi_V, W)$.*

Similar assertion holds in the W-side.

Because of the assertion (3), the K-type correspondence (2) plays an important role in the explicit description of the Howe correspondence [AB95], [Pau98], [Pau00]. Following the calculation of [KV78, III.6], we obtain the following.

We write $\mathfrak{b}_V = \mathfrak{t}_{V,\mathbb{C}} \oplus \mathfrak{n}_V$, $\bar{\mathfrak{b}}_V = \mathfrak{t}_{V,\mathbb{C}} \oplus \bar{\mathfrak{n}}_V$ for the upper and lower triangular Borel subalgebras of $\mathfrak{k}_{V,\mathbb{C}}$ in the realization with respect to $\underline{v}$, respectively. Using similar notation for $\mathfrak{k}_{W,\mathbb{C}}$, we set

$$(\mathfrak{b}_{V,\psi}, \mathfrak{b}_{W,\psi}) := \begin{cases} (\bar{\mathfrak{b}}_V, \mathfrak{b}_W) & \text{if } \varepsilon_\psi = 1, \\ (\mathfrak{b}_V, \bar{\mathfrak{b}}_W) & \text{if } \varepsilon_\psi = -1. \end{cases}$$

We take a basis $\{e_1, \dots, e_p, \bar{e}_1, \dots, \bar{e}_q\}$ of $\mathfrak{t}^*_{V,\mathbb{C}}$ as

$$e_i(\mathrm{diag}(t_1, \dots, t_n)) := t_i, \quad \bar{e}_i(\mathrm{diag}(t_1, \dots, t_n)) := t_{p+i}$$

and identify $\mathfrak{t}^*_{V,\mathbb{C}}$ with $\mathbb{C}^n$ by this. We write $\{e'_1, \dots, e'_{p'}; \bar{e}'_1, \dots, \bar{e}'_{q'}\}$ for the analogous basis for $\mathfrak{t}^*_{W,\mathbb{C}}$, which gives an identification $\mathfrak{t}^*_{W,\mathbb{C}} \simeq \mathbb{C}^{n'}$.

Theorem 5.2 (K-type correspondence). *(i) Write the $\mathfrak{b}_{V,\psi}$-highest weight of a K_V-type τ_V as*

$$\begin{aligned} &\Big(\frac{m}{2}, \dots, \frac{m}{2}; \frac{m}{2}, \dots, \frac{m}{2}\Big) + \varepsilon_\psi\Big(\frac{q'-p'}{2}, \dots, \frac{q'-p'}{2}; \frac{p'-q'}{2}, \dots, \frac{p'-q'}{2}\Big) \\ &+\varepsilon_\psi(-a_1, \dots, -a_r, 0, \dots, 0, b_s, \dots, b_1; -d_1, \dots, -d_u, 0, \dots, 0, c_t, \dots, c_1), \end{aligned} \tag{5.2}$$

for some $a_1 \geq \cdots \geq a_r$, $b_1 \geq \cdots \geq b_s$, $c_1 \geq \cdots \geq c_t$, $d_1 \geq \cdots \geq d_u \in \mathbb{Z}_{>0}$. Then τ_V belongs to $\mathscr{R}(K_V, \mathscr{J}_{V,W,\underline{\xi}})$ if and only if $r + t \leq p'$, $s + u \leq q'$. In that case, the $\mathfrak{b}_{W,\psi}$-highest weight of $\theta_{\underline{\xi}}(\tau_V, K_W)$ is given by

$$\begin{aligned} &\Big(\frac{m'}{2}, \dots, \frac{m'}{2}; \frac{m'}{2}, \dots, \frac{m'}{2}\Big) + \varepsilon_\psi\Big(\frac{p-q}{2}, \dots, \frac{p-q}{2}; \frac{q-p}{2}, \dots, \frac{q-p}{2}\Big) \\ &+\varepsilon_\psi(a_1, \dots, a_r, 0, \dots, 0, -c_t, \dots, -c_1; d_1, \dots, d_u, 0, \dots, 0, -b_s, \dots, -b_1). \end{aligned} \tag{5.3}$$

(ii) Conversely, a K_W-type τ_W with the $\mathfrak{b}_{W,\psi}$-highest weight (5.3) *belongs to $\mathscr{R}(K_W, \mathscr{J}_{V,W,\underline{\xi}})$ if and only if $r + s \leq p$, $t + u \leq q$. In that case, $\theta_{\underline{\xi}}(\tau_W, K_V)$ has the $\mathfrak{b}_{V,\psi}$-highest weight* (5.2).

The same argument as in the proof of [Pau98, Prop.1.4.10] shows:

Corollary 5.3. *Let $(V, (\cdot,\cdot))$ be an n-dimensional hermitian space and τ_V be a K_V-type. Fix a pair $\underline{\xi} = (\xi, \xi')$ of characters of $\mathbb{C}^\times$ such that $\xi|_{\mathbb{R}^\times} = \xi'|_{\mathbb{R}^\times} = \mathrm{sgn}^n$. Then there exists a unique (up to isometry) n-dimensional skew-hermitian space $(W, \langle\cdot,\cdot\rangle)$ such that $\tau_V \in \mathscr{R}(K_V, \mathscr{I}_{V,W,\underline{\xi}})$.*

6. LOCAL θ-CORRESPONDENCE FOR LIMIT OF DISCRETE SERIES

As a consequence of our calculation, we compute the Howe correspondence between limit of discrete series representations of unitary dual pairs of the same size (cf. [Li90], [Pau98, 5.2]).

Consider the group $G_V = U(p,q)$ realized with respect to the basis $\underline{v}$. $T_V \subset G_V$ denotes the diagonal fundamental Cartan subgroup with the Lie algebra $\mathfrak{t}_V$. The isomorphism classes of irreducible limit of discrete series representations of G_V are classified as follows [Vog84, §2]. Up to K_V-conjugation, an *elliptic limit character* of G_V is a triple $\gamma = (\Psi, \lambda, \Lambda)$ consisting of

(LC1) Ψ is a positive system in $R(\mathfrak{g}_{V,\mathbb{C}}, \mathfrak{t}_{V,\mathbb{C}})$, the root system of $\mathfrak{t}_{V,\mathbb{C}}$ in $\mathfrak{g}_{V,\mathbb{C}}$.

(LC2) $\lambda \in \sum_{j=1}^{p}(n+1+2\mathbb{Z})e_j \oplus \sum_{j=1}^{q}(n+1+2\mathbb{Z})\bar{e}_j \subset \mathfrak{t}_{V,\mathbb{C}}^*$, satisfying the following conditions.
   (a) $\alpha^\vee(\lambda) \geq 0$ for any $\alpha \in \Psi$.
   (b) If a simple root α of Ψ satisfies $\alpha^\vee(\lambda) = 0$, it must be non-compact: $\alpha \in R(\mathfrak{p}_{V,\mathbb{C}}, \mathfrak{t}_{V,\mathbb{C}})$.

(LC3) Λ is a character of T_V such that $d\Lambda = \lambda + \rho(\Psi_{\mathrm{ncpt}}) - \rho(\Psi_{\mathrm{cpt}})$. Here $\Psi_{\mathrm{ncpt}} := \Psi \cap R(\mathfrak{p}_{V,\mathbb{C}}, \mathfrak{t}_{V,\mathbb{C}})$, $\Psi_{\mathrm{cpt}} := \Psi \cap R(\mathfrak{k}_{V,\mathbb{C}}, \mathfrak{t}_{V,\mathbb{C}})$, and $\rho(\Sigma)$ denotes the half of the sum of roots in Σ.

For such $\gamma = (\Psi, \lambda, \Lambda)$, we have an irreducible limit of discrete series representation $\pi_V(\lambda, \Psi)$ of G_V. This is characterized by its unique minimal K_V-type with the Ψ_{cpt}-highest weight Λ and the infinitesimal character λ (or its Weyl group orbit) [KV95, Ch.11]. Two such representations $\pi_V(\lambda, \Psi)$ and $\pi_V(\lambda', \Psi')$ associated to $\gamma = (\Psi, \lambda, \Lambda)$ and $\gamma' = (\Psi', \lambda', \Lambda')$ are isomorphic if and only if γ and γ' are K_V-conjugate.

Now we consider a unitary dual pair (G_V, G_W) with $n = n'$. Fix a character pair $\underline{\xi} = (\xi, \xi')$ as before:

$$\xi(z) = \left(\frac{z}{\bar{z}}\right)^{m/2}, \quad \xi'(z) = \left(\frac{z}{\bar{z}}\right)^{m'/2}, \quad m \equiv m' \equiv n \pmod 2.$$

For each odd algebraic character $\mu^a(z) := (z/\bar{z})^{a/2}$, $(a \in 2\mathbb{Z}+1)$ of $\mathbb{C}^\times$, we introduce the sign

$$\varepsilon_\psi(a) = \varepsilon_\psi(\mu^a) := \varepsilon(1/2, \mu^a, \psi_{\mathbb{C}})\mu^a(-i) = \varepsilon_\psi \cdot \mathrm{sgn}(a).$$

Here $\varepsilon(s, \mu^a, \psi_{\mathbb{C}})$ denotes the Artin ε-factor for μ^a [Tat79, (3.2)] and $\psi_{\mathbb{C}} := \psi \circ \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}$.

Let $\pi_V(\lambda, \Psi)$ be a limit of discrete series representation of G_V. Taking a suitable K_V-conjugation, we may assume

(i) λ is of the form ($[a]_k$ denotes the k-tuple $(a, \ldots, a)$.)

$$\begin{aligned}\lambda = &\Big(\overbrace{\frac{m}{2}, \ldots, \frac{m}{2}}^{p}; \overbrace{\frac{m}{2}, \ldots, \frac{m}{2}}^{q}\Big) \\ &- \frac{1}{2}\Big(\underbrace{[a_1]_{k_1}, \ldots, [a_r]_{k_r}, [b_s]_{\ell_s}, \ldots, [b_1]_{\ell_1}}_{p}; \underbrace{[a_1]_{\bar{k}_1}, \ldots, [a_r]_{\bar{k}_r}, [b_s]_{\bar{\ell}_s}, \ldots, [b_1]_{\bar{\ell}_1}}_{q}\Big),\end{aligned} \tag{6.1}$$

where $a_i, b_j \in 2\mathbb{Z}+1$ satisfy

$$\varepsilon_\psi(a_i) > 0, \quad \varepsilon_\psi(b_j) < 0, \tag{6.2}$$

$$|a_1| > |a_2| > \cdots > |a_r|, \quad |b_1| > |b_2| > \cdots > |b_s|. \tag{6.3}$$

(ii) $\Psi_{\mathrm{cpt}} = R(\mathfrak{b}_{V,\psi}, \mathfrak{t}_{V,\mathbb{C}})$.

Setting $k := \sum_{i=1}^r k_i$, $\ell := \sum_{i=1}^s \ell_i$, $\bar{k} := \sum_{i=1}^r \bar{k}_i$, $\bar{\ell} := \sum_{i=1}^r \bar{\ell}_i$, let $(W, \langle \cdot, \cdot \rangle)$ be the skew-hermitian space of signature $(p' := k + \bar{\ell}, q' := \bar{k} + \ell)$. Set

$$\begin{aligned} \lambda' := &\Bigl(\overbrace{\frac{m'}{2}, \dots, \frac{m'}{2}}^{p'}; \overbrace{\frac{m'}{2}, \dots, \frac{m'}{2}}^{q'}\Bigr) \\ &+ \frac{1}{2}\Bigl(\underbrace{[a_1]_{k_1}, \dots, [a_r]_{k_r}, [b_s]_{\bar{\ell}_s}, \dots, [b_1]_{\bar{\ell}_1}}_{p'}; \underbrace{[a_1]_{\bar{k}_1}, \dots, [a_r]_{\bar{k}_r}, [b_s]_{\ell_s}, \dots, [b_1]_{\ell_1}}_{q'}\Bigr), \end{aligned} \tag{6.4}$$

$$\Psi'_{\mathrm{cpt}} := R(\mathfrak{b}_{W,\psi}, \mathfrak{t}_{W,\mathbb{C}}), \tag{6.5}$$

$$e'_i - \bar{e}'_j \in \Psi'_{\mathrm{ncpt}} \iff \begin{cases} \text{(i)} & 1 \le i \le k, 1 \le j \le \bar{k} \text{ and } e_i - \bar{e}_j \notin \Psi, \text{ or} \\ \text{(ii)} & k < i \le p', \bar{k} < j \le q' \text{ and } \bar{e}_{i-k+\bar{k}} - e_{j-\bar{k}+k} \notin \Psi, \text{ or} \\ \text{(iii)} & (e'_i - \bar{e}'_j)^\vee(\lambda') > 0. \end{cases} \tag{6.6}$$

Using the characterization by the minimal K-type and the infinitesimal character of limit of discrete series representations, we deduce the following from Th.5.2.

Theorem 6.1. *(1) A limit of discrete series representation $\pi_V(\lambda, \Psi)$ of G_V belongs to $\mathscr{R}(G_V, \omega_{W,\xi})$ for an n-dimensional skew-hermitian space W if and only if W has the signature (p', q').*
(2) In that case, $\theta_{\underline{\xi}}(\pi_V(\lambda, \Psi), W) \simeq \pi_W(\lambda', \Psi')$.

Remark 6.2. *The signature (p', q') is determined by (p, q) and the signatures $\varepsilon_\psi(\xi^{-1}\omega_i)$, where $\underline{\omega} = (\omega_1, \dots, \omega_n)$ is the character of T_V with the differential λ. This is the archimedean analogue of the* ε-dichotomy property *of the local θ-correspondence of non-archimedean unitary dual pairs* [HKS96, Th.6.1].

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