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# Characterization of certain spaces of $C^\infty$-vectors of irreducible representations of solvable Lie groups

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**Abstract**

Let $\pi$ be an irreducible unitary representation of an exponential solvable Lie group $G$. Realizing $\pi$ on $L^2(G/H, \chi_l)$ as an induced representation from a unitary character $\chi_l$ of a subgroup $H$ of $G$, we are concerned with certain subspaces of $C^\infty$-vectors. We describe the subspace $\mathcal{SE}$ of vectors with a certain property of rapidly decreasing at infinity as the space of $C^\infty$-vectors of an irreducible unitary representation of an exponential solvable Lie group $F$ containing $G$. Furthermore, the space $\mathcal{ASE}$ introduced by Ludwig in [8] is expressed by our space $\mathcal{SE}$. Here we shall announce some results in [5], and we shall give brief discussions on fundamental examples.

## 1 Introduction

Let $G$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, and $\pi$ be an irreducible unitary representation of $G$. By the orbit method, which associates $\pi$ with a coadjoint orbit, we realize $\pi$ as an induced representation from a unitary character of a subgroup as follows: There exists a linear form $l \in \mathfrak{g}^*$ and a real polarization $\mathfrak{h}$ at $l$ such that $\pi \simeq \mathrm{ind}_H^G \chi_l$, where $H = \exp \mathfrak{h}$ is the connected and simply connected subgroup corresponding to $\mathfrak{h}$, and $\chi_l$ is the unitary character of $H$ defined by $\chi_l(X) = e^{il(X)}$ $(X \in \mathfrak{h})$.

We give the standard construction of the induced representation $\pi = \pi_{l,H} = \mathrm{ind}_H^G \chi_l$: Let $\mathcal{K}(G/H)$ be the space of continuous functions $f$ on $G$ with compact support modulo $H$ such that $f(gh) = \Delta_{H,G}(h) f(g)$ for all $g \in G$, $h \in H$, where $\Delta_G$ and $\Delta_H$ are the

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modular functions of $G$ and $H$, respectively, and $\Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$. Then there exists a positive left invariant linear functional

$$f \mapsto \mu(f) = \oint_{G/H} f(g) d\mu_{G/H}(g) \tag{1.1}$$

uniquely up to a constant factor (see [3]). Let $C(G/H, \chi_l)$ be the space of continuous functions $\phi$ on $G$ with compact support modulo $H$ such that

$$\phi(gh) = \chi_l(h)^{-1} \Delta_{H,G}(h)^{1/2} \phi(g), \quad \forall g \in G, h \in H,$$

and let $\mathcal{H}_\pi = L^2(G/H, \chi_l)$ be the completion of the space $C(G/H, \chi_l)$ with respect to the norm

$$\|\phi\|_\pi := \left( \oint_{G/H} |\phi(g)|^2 d\mu_{G/H}(g) \right)^{1/2}.$$

Then we define the action of $g \in G$ in $\mathcal{H}_\pi$ by

$$\pi_{l,H}(g)\phi(x) = \phi(g^{-1}x), \quad \phi \in L^2(G/H, \chi_l), \quad g, x \in G.$$

Let us briefly recall some well-known facts of the case of nilpotent Lie groups. Suppose $G$ is nilpotent, and taking a supplementary Malcev basis for $\mathfrak{h}$ in $\mathfrak{g}$, identify $G/H$ with $\mathbb{R}^k$, where $k = \dim(G/H)$, and realize $\pi$ on $L^2(\mathbb{R}^k)$. Then by results of Kirillov [6] and Corwin-Greenleaf-Penney [4], the actions of the enveloping algebra $\mathfrak{u}(\mathfrak{g})$ form the algebra of differential operators on $\mathbb{R}^k$ with polynomial coefficients. Thus the space of $C^\infty$vectors $\mathcal{H}_\pi^\infty$ coincides with the space of Schwartz functions $\mathcal{S}(\mathbb{R}^k)$ on $\mathbb{R}^k$ as a Fréchet space.

We next observe an example of exponential groups which are not nilpotent, where the specific descriptions of $C^\infty$vectors are different from those of nilpotent groups.

**Example 1.1.** ($ax+b$ group) Let $\mathfrak{g}$ be a two-dimensional Lie algebra with basis $\{X, Y\}$ whose bracket relation is $[X, Y] = Y$, and let $G = \exp \mathfrak{g}$. Then with the dual basis $\{X^*, Y^*\}$ in $\mathfrak{g}^*$, the coadjoint orbits of $G$ are described as follows:

$$\mathcal{O}_+ := \{l \in \mathfrak{g}^*;\ l(Y) > 0\}, \qquad \mathcal{O}_- := \{l \in \mathfrak{g}^*;\ l(Y) < 0\},$$
$$\{\xi X^*\}, \quad \xi \in \mathbb{R}.$$

Let $l_\varepsilon := \varepsilon Y^*$ ($\varepsilon = \pm 1$) and $\mathfrak{h} := \mathbb{R}Y$, $H := \exp \mathfrak{h}$. Then $\mathfrak{h}$ is a polarization at $l_\varepsilon$ and $\pi_\varepsilon := \mathrm{ind}_H^G \chi_{l_\varepsilon}$ is an irreducible representation of $G$. We realize $\pi_\varepsilon$ on $L^2(\mathbb{R})$ identifying $\mathbb{R}$ with $G/H$ by $\mathbb{R} \ni x \mapsto \exp(xX)H$, as follows:

$$\pi(\exp aX)\phi(x) = \phi(x-a)$$
$$\pi(\exp bY)\phi(x) = e^{i\varepsilon b e^{-x}} \phi(x), \quad \phi \in L^2(\mathbb{R}), \quad a, b \in \mathbb{R}.$$

Then the actions of $\mathfrak{g}$ are described by

$$d\pi(X)\phi(x) = -\frac{d\phi}{dx}$$
$$d\pi(Y)\phi(x) = i\varepsilon e^{-x}\phi(x).$$

It shows that if $\phi$ is a $C^\infty$vector, then $\phi$ decreases rapidly at $x \to -\infty$, but it does not necessarily decrease so rapidly at $x \to +\infty$ as at $x \to -\infty$.

Here we shall announce some results in [5], giving brief discussions on fundamental examples. For an exponential solvable group $G$ and an irreducible unitary representation $\pi$ of $G$, we construct $\pi$ in $L^2(G/H, \chi_l)$ by taking $l \in \mathfrak{g}^*$ and a suitable polarization $\mathfrak{h}$. Then we shall define a subspace $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ of vectors with some property of rapidly decreasing at infinity and show that it can be identified with the space of $C^\infty$vectors of an irreducible representation of an exponential solvable group $F$ containing $G$. Next, using our $\mathcal{SE}$ space, we shall describe the space $\mathcal{ASE}$ introduced by Ludwig in [8].

## 2 The space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$

In the sequel, let $G$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$. For example, we can take the nilradical of $\mathfrak{g}$ as $\mathfrak{n}$, or we can also take $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. Let $l \in \mathfrak{g}^*$ as above, and let

$$\mathfrak{n}^l := \{X \in \mathfrak{g};\ l([X, \mathfrak{n}]) = \{0\}\}.$$

**Definition 2.1.** (See [9].) We say that a polarization $\mathfrak{h}$ at $l$ is adapted to $\mathfrak{n}$ if it satisfies (1) and (2).

(1) The subalgebra $\mathfrak{h} \cap \mathfrak{n}$ is a polarization at $l|_\mathfrak{n}$ in $\mathfrak{n}$.

(2) $[\mathfrak{n}^l, \mathfrak{h} \cap \mathfrak{n}] \subset \mathfrak{h} \cap \mathfrak{n}$.

**Remark 2.2.** (1) Suppose that a polarization $\mathfrak{h}$ at $l$ is adapted to $\mathfrak{n}$. Then there exists a polarization $\mathfrak{h}_0 \subset \mathfrak{n}^l$ at $l|_{\mathfrak{n}^l}$ such that $\mathfrak{h} = \mathfrak{h}_0 + (\mathfrak{h} \cap \mathfrak{n})$ and $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{n}^l$.

Furthermore, it satisfies the Pukanszky condition

$$\mathrm{Ad}^*(H)l = \mathfrak{h}^\perp + l,$$

where $\mathfrak{h}^\perp := \{f \in \mathfrak{g}^*;\ f(\mathfrak{h}) = \{0\}\}$, and thus we obtain a realization of the irreducible representation corresponding to the orbit $\mathrm{Ad}^*(G)l$ by $\mathrm{ind}_H^G \chi_l$.

(2) For any $l$ and $\mathfrak{n}$, there exists a polarization $\mathfrak{h}$ at $l$ adapted to $\mathfrak{n}$. For example, a Vergne polarization associated with a refinement of the sequence of ideals $\{0\} \subset \mathfrak{n} \subset \mathfrak{g}$ satisfies the condition (1) and (2) of Definition 2.1 above.

Starting from $\mathfrak{n}$, $l$ and a polarization $\mathfrak{h}$ at $l$ adapted to $\mathfrak{n}$, we realize the irreducible representation $\pi = \pi_{l,H} = \mathrm{ind}_H^G \chi_l$ in $L^2(\mathbb{R}^n)$ ($n = \dim(G/H)$) as follows.

Let $\{T_1, \cdots, T_m, R_1, \cdots, R_v\}$ be a coexponential basis for $\mathfrak{h}$ in $\mathfrak{g}$ such that

$$G = \exp \mathbb{R}T_1 \cdots \exp \mathbb{R}T_m \cdot NH,$$
$$NH = \exp \mathbb{R}R_1 \cdots \exp \mathbb{R}R_v \cdot H,$$

and identify

$$\mathbb{R}^m \times \mathbb{R}^v \simeq (G/NH) \times (NH/H) \simeq G/H$$

by

$$\begin{aligned}\mathbb{R}^m \times \mathbb{R}^v \ni &(t,r) = (t_1, \cdots, t_m, r_1, \cdots, r_v) \\ &\mapsto E(t,r) := \exp t_1 T_1 \cdots \exp t_m T_m \exp r_1 R_1 \cdots \exp r_v R_v \text{ (modulo } H).\end{aligned}$$

Then the left invariant functional (1.1) is described by

$$\mu(f) = \int_{\mathbb{R}^{m+v}} f(E(t,r))dtdr, \quad f \in \mathcal{K}(G/H)$$

(see [7]), and we have $L^2(G/H, \chi_l) \simeq L^2(\mathbb{R}^{m+v})$.

Denoting by $\mathfrak{D}_{t,r}$ the algebra of differential operators on $\mathbb{R}^{m+v}$ with polynomial coefficients, we define the space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ as follows:

**Definition 2.3.** Let $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ be the space of vectors $\phi \in \mathcal{H}_{\pi_{l,H}} = L^2(G/H, \chi_l)$ such that

(1) $\phi$ is a $C^\infty$ function.

(2)
$$\|\phi\|^2_{a,D} := \int_{\mathbb{R}^{m+v}} e^{a\|t\|} |D(\phi \circ E)(t,r)|^2 dtdr < \infty, \quad \forall a \in \mathbb{R}_+, \forall D \in \mathfrak{D}_{t,r},$$
where $\|t\|$ denotes a norm on $\mathbb{R}^m$.

Let us remark that the space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ is independent of the choice of coexponential basis.

In [5], we obtained the following result. There exists an exponential solvable Lie group $F$ containing $G$, and an irreducible representation $\pi_0$ of $F$ such that $\pi_0|_G \simeq \pi$ and the space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ is naturally identified with the space of $C^\infty$vectors of $\pi_0$. More specifically, we can construct an exponential solvable Lie algebra $\mathfrak{f}$ which has the properties (i), (ii) and (iii):

(i) $\mathfrak{f}$ is described as $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$, where $\mathfrak{a}$ is an abelian ideal of $\mathfrak{f}$ satisfying
$$[\mathfrak{a}, \mathfrak{n} + \mathfrak{h}] = \{0\}.$$

(ii) $\dim \mathfrak{a} = 2\dim(\mathfrak{g}/(\mathfrak{n}+\mathfrak{h})) = 2m$, and there exist a coexponential basis $\{X_1, \cdots, X_m\}$ for $\mathfrak{n} + \mathfrak{h}$ in $\mathfrak{g}$ and a basis $\{A_1, \cdots, A_m, B_1, \cdots, B_m\}$ of $\mathfrak{a}$ such that
$$[X_j, A_k] = \delta_{jk} A_k, \quad [X_j, B_k] = -\delta_{jk} B_k, \quad 1 \le j,k \le m.$$

(iii) For all extension $l_1 \in \mathfrak{f}^*$ of $l \in \mathfrak{g}^*$, we have that

$$\dim(\mathfrak{f}(l_1)) = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a}),$$

where $\mathfrak{f}(l_1) := \{X \in \mathfrak{f};\ l_1([X,\mathfrak{f}]) = \{0\}\}$, $\mathfrak{g}(l) := \{X \in \mathfrak{g};\ l([X,\mathfrak{g}]) = \{0\}\}$. Thus the subalgebra $\mathfrak{p} := \mathfrak{h} + \mathfrak{a}$ is a polarization at $l_1$ adapted to the nilpotent ideal $\mathfrak{n}+\mathfrak{a}$ of $\mathfrak{f}$.

Let $F = \exp \mathfrak{f}$, $P = \exp \mathfrak{p}$, and $\chi_{l_1}$ be the unitary character of $F$ defined by $\chi_{l_1}(\exp X) = e^{il_1(X)}$ for $X \in \mathfrak{p}$. Then by (iii) above, the induced representation $\pi_{l_1,P} := \mathrm{ind}_P^G \chi_{l_1}$ is irreducible and $\pi_{l_1,P}|_G \simeq \pi$. In fact, the intertwining operator

$$\mathcal{R}_{l_1} : \mathcal{H}_{\pi_{l_1,P}} = L^2(F/P, \chi_{l_1}) \to \mathcal{H}_{\pi_{l,H}} = L^2(G/H, \chi_l)$$

is defined by

$$\mathcal{R}_{l_1}\psi = \psi|_G, \qquad \psi \in L^2(F/P, \chi_{l_1}),$$

and the inverse $\mathcal{S}_{l_1} := \mathcal{R}_{l_1}^{-1}$ is

$$\mathcal{S}_{l_1}\phi(g \exp Y) := e^{-il_1(Y)}\phi(g), \quad \phi \in L^2(G/H, \chi_l), \quad g \in G,\ Y \in \mathfrak{a}.$$

It can be seen easily that

$$\mathcal{S}_{l_1}(\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})) \subset \mathcal{H}^\infty_{\pi_{l_1,P}}.$$

Now we define another family of seminorms $\{\|\cdot\|_{l_1,U}\}$ on $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$:

$$\|\phi\|_{l_1,U} := \|d\pi_{l_1,P}(U)\mathcal{S}_{l_1}\phi\|_{\pi_{l_1,P}}, \quad U \in \mathfrak{u}(\mathfrak{f}).$$

**Theorem 2.4.** ([5]) *Let $G, \mathfrak{n}, l, \mathfrak{h}$ be as above. Then there exists an exponential solvable Lie algebra $\mathfrak{f}$ having the property (i), (ii), (iii) above and satisfying the following:*

*(iv) There exists an extension $l_0 \in \mathfrak{f}^*$ of $l$ such that the family of seminorms $\{\|\cdot\|_{l_0,U}, U \in \mathfrak{u}(\mathfrak{f})\}$ is equivalent to the family of seminorms $\{\|\cdot\|_{a,D}, a \in \mathbb{R}_+, D \in \mathfrak{D}_{t,r}\}$; and thus we have*

$$\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}) = \mathcal{R}_{l_0}(\mathcal{H}^\infty_{\pi_{l_0,P}}).$$

**Example 2.5.** ($ax + b$ group) Let $\mathfrak{g} = \mathbb{R}X + \mathbb{R}Y$ and $\mathfrak{h} = \mathbb{R}Y$ be as in Example 1.1, and let $l := Y^*$ and $\mathfrak{n} = \mathbb{R}Y$, which is the nilradical of $\mathfrak{g}$. Then the polarization $\mathfrak{h}$ is obviously adapted to $\mathfrak{n}$. We construct $\pi_l$ in $L^2(\mathbb{R})$ as in Example 1.1. Then a square integrable smooth function $\phi$ belongs to $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ if and only if

$$\int_{\mathbb{R}} e^{a|x|}|D\phi(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad \forall D \in \mathfrak{D}_x,$$

where $\mathfrak{D}_x$ is the algebra of differential operators on $\mathbb{R}$ with polynomial coefficients. Applying Theorem 2.4 above, we have

$$\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}, \quad \mathfrak{a} = \mathbb{R}A + \mathbb{R}B$$
$$[X, A] = A, \quad [X, B] = -B, \quad [Y, A] = [Y, B] = 0.$$

Let $l_0 \in \mathfrak{f}^*$ be an extension of $l$ such that $l_0(B) \neq 0$. Then we have

$$\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}) = \mathcal{R}_{l_0}(\mathcal{H}^\infty_{\pi_{l_0,P}}). \tag{2.2}$$

In fact, realizing $\pi_{l_0,P} = \mathrm{ind}_P^F \chi_{l_0}$ in $L^2(F/P, \chi_{l_0}) = \mathcal{S}_{l_0}(L^2(G/P, \chi_l)) \simeq L^2(\mathbb{R})$, we have that $\mathfrak{f}$ acts by

$$d\pi_{l_0,P}(X)\phi(x) = -\frac{d\phi}{dx}, \tag{2.3}$$

$$d\pi_{l_0,P}(Y)\phi(x) = ie^{-x}\phi(x), \tag{2.4}$$

$$d\pi_{l_0,P}(A)\phi(x) = il_0(A)e^{-x}\phi(x), \tag{2.5}$$

$$d\pi_{l_0,P}(B)\phi(x) = il_0(B)e^{x}\phi(x). \tag{2.6}$$

Thus we can directly verify the equality (2.2).

**Remark 2.6.** In Example 2.5, replacing $F$ with a subgroup $F'$ of $F$, we can also identify the space $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ with the space of $C^\infty$vectors of an extension of $\pi_l$. Let $\mathfrak{a}' := \mathbb{R}B$, $\mathfrak{f}' := \mathfrak{g} \ltimes \mathfrak{a}'$, $\mathfrak{p}' := \mathfrak{h} + \mathfrak{a}'$, $F' := \exp \mathfrak{f}'$ and $P' := \exp \mathfrak{p}'$. Then $\mathfrak{p}'$ is a polarization at any extension $l_1' \in \mathfrak{f}'^*$ of $l$ and $\pi_{l_1',P'}|_G \simeq \pi_l$, where $\pi_{l_1',P'} = \mathrm{ind}_{P'}^{F'} \chi_{l_1'}$. We denote the intertwining operator by $\mathcal{R}_{l_0'} : L^2(F'/P', \chi_{l_0'}) \to L^2(G/H, \chi_l)$ as above. Suppose that an extension $l_0' \in \mathfrak{f}'^*$ of $l$ satisfies $l_0'(B) \neq 0$. Then we also obtain that

$$\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}) = \mathcal{R}_{l_0'}(\mathcal{H}^\infty_{\pi_{l_0',P'}}). \tag{2.7}$$

In fact, letting $l_0 \in \mathfrak{f}^*$ be any extension of $l_0'$, we have $\pi_{l_0,P}|_{F'} \simeq \pi_{l_0',P'}$, and using the descriptions (2.3), (2.4) and (2.6), we obtain the equality (2.7).

**Example 2.7.** (Heisenberg group) Taking $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ instead of the nilradical, we observe an example of nilpotent groups. Let $\mathfrak{g} = \mathbb{R}\text{-span}\{X, Y, Z\}$ be the 3-dimensional Lie algebra whose non-trivial bracket relation is $[X, Y] = Z$. Let $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] = \mathbb{R}Z$, $l = Z^*$ and $\mathfrak{h} = \mathbb{R}Y + \mathbb{R}Z$. Then $\mathfrak{h}$ is a polarization adapted to $\mathfrak{n}$. We realize the representation $\pi_{l,H}$ in $L^2(\mathbb{R})$ by the coexponential basis $\{X\}$ for $\mathfrak{h}$ in $\mathfrak{g}$. Then we have

$$d\pi_{l,H}(X)\phi(x) = -\frac{d\phi}{dx}, \quad d\pi_{l,H}(Y)\phi(x) = -ix\phi(x), \quad d\pi_{l,H}(Z) = i. \tag{2.8}$$

We have that a smooth function $\phi(x) \in L^2(\mathbb{R})$ belongs to $\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$ if and only if

$$\int_{\mathbb{R}} e^{a|x|}|D\phi(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad \forall D \in \mathfrak{D}_x,$$

Applying Theorem 2.4, we have

$$\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}, \quad \mathfrak{a} = \mathbb{R}A + \mathbb{R}B,$$
$$[X, A] = A, \quad [X, B] = -B, \quad [Y, A] = [Y, B] = [Z, A] = [Z, B] = 0.$$

Let $l_0 \in \mathfrak{f}^*$ be an extension of $l$ such that $l_0(A) \neq 0$ and $l_0(B) \neq 0$. Then we have

$$\mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h}) = \mathcal{R}_{l_0}(\mathcal{H}^\infty_{\pi_l, P}).$$

In fact, we have

$$d\pi_{l_0,P}(A)\phi(x) = il_0(A)e^{-x}\phi(x), \tag{2.9}$$

$$d\pi_{l_0,P}(B)\phi(x) = il_0(B)e^{x}\phi(x). \tag{2.10}$$

By the actions (2.8), (2.9) and (2.10), we can obtain the conclusion.

# 3 The space $\mathcal{ASE}$ and the space $\mathcal{SE}^\infty$

Let $G, \mathfrak{n}, l, \mathfrak{h}$ be as above. As we mentioned in Remark 2.2, we have that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{n}^l) + (\mathfrak{h} \cap \mathfrak{n})$, so we have $\mathfrak{h} \subset \mathfrak{n} + \mathfrak{n}^l$. We choose a coexponential basis $\{T_1, \cdots, T_\nu, S_1, \cdots, S_u\}$ for $\mathfrak{h} + \mathfrak{n}$ in $\mathfrak{g}$ along with the sequence $\mathfrak{g} \supset \mathfrak{n} + \mathfrak{n}^l \supset \mathfrak{n} + \mathfrak{h}$, where $\nu + u = m$, so that

$$G = \exp \mathbb{R}T_1 \cdots \exp \mathbb{R}T_\nu \cdot N^l N$$
$$N^l N = \exp \mathbb{R}S_1 \cdots \exp \mathbb{R}S_u \cdot NH.$$

(Here we write $N^l := \exp \mathfrak{n}^l$.) In the sequel, we identify

$$\mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^v \simeq (G/N^l N) \times (N^l N/NH) \times (NH/H) \simeq G/H$$

by

$$\begin{aligned}\mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^v \ni (t, s, r) =& (t_1, \cdots, t_\nu, s_1, \cdots, s_u, r_1, \cdots, r_v) \\ & \mapsto E(t, s, r) = E(t)E(s)E(r) \text{ (modulo } H),\end{aligned}$$

where

$$E(t) = \exp t_1 T_1 \cdots \exp t_\nu T_\nu, \quad E(s) = \exp s_1 S_1 \cdots \exp s_u S_u,$$
$$E(r) = \exp r_1 R_1 \cdots \exp r_v R_v, \quad (t, s, r) \in \mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^v.$$

For $\phi \in \mathcal{SE}(G, \mathfrak{n}, l, \mathfrak{h})$, let $\hat{\phi}_s(t, s, r)$ be the partial Fourier transform of $\phi$ in $s$:

$$\hat{\phi}_s(t, s, r) := \int_{\mathbb{R}^u} \phi(E(t, x, r))e^{i\langle x, s\rangle} dx,$$

where $\langle x, s\rangle$ is the standard inner product of $\mathbb{R}^u$. Denoting by $\mathfrak{D}_{t,s,r}$ the algebra of differential operators on $\mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^v$ with polynomial coefficients, we define the space $\mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h})$ introduced in [8], where this space has been denoted by $\mathcal{ES}$.

**Definition 3.1.** (See [8].) Let $\mathcal{ASE}(G, \mathfrak{n}, l, \mathfrak{h})$ be the space of functions $\phi \in L^2(G/H, \chi_l)$ such that

(1) $\phi \in \mathcal{SE}(G,\mathfrak{n},l,\mathfrak{h})$,

(2)

$$\|\hat{\phi}_s(t,s,r)\|^2_{a,b,D} := \int_{\mathbb{R}^{\nu+u+v}} e^{a\|t\|}e^{b\|s\|}|D\hat{\phi}_s(t,s,r)|^2 dtdsdr < \infty,$$
$$\forall (a,b) \in \mathbb{R}^2_+, \quad \forall D \in \mathfrak{D}_{t,s,r}.$$

**Remark 3.2.** The space $\mathcal{ASE}$ is independent of the choice of coexponential bases. We write the letter $\mathcal{A}$ in front to indicate that the functions $\phi \in \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$ are analytic in the direction $s$. It has been shown in [8] and [1] that for $\phi$ and $\psi$ in $\mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$ there exists a function $f \in L^1(G)$, more precisely in the subalgebra $\mathcal{ES}(G)$ (see [8]) such that

$$\pi_{l,H} f(\xi) = \langle \xi, \psi \rangle \phi, \quad \xi \in \mathcal{H}_{\pi_{l,H}}.$$

Let $\mathcal{P}(\mathfrak{h})$ be the set of polarizations $\tilde{\mathfrak{h}}$ at $l$ adapted to $\mathfrak{n}$ and satisfying $\tilde{\mathfrak{h}} \cap \mathfrak{n} = \mathfrak{h} \cap \mathfrak{n}$. For $\tilde{\mathfrak{h}} \in \mathcal{P}(\mathfrak{h})$, we have $\mathrm{ind}_{\tilde{H}}^G \chi_l \simeq \mathrm{ind}_H^G \chi_l$, where $\tilde{H} = \exp \tilde{\mathfrak{h}}$. We denote the intertwining operator by $I_{\mathfrak{h},\tilde{\mathfrak{h}}} : L^2(G/\tilde{H},\chi_l) \to L^2(G/H,\chi_l)$ (see [2].)

**Definition 3.3.** We define

$$\mathcal{SE}^\infty(G,\mathfrak{n},l,\mathfrak{h}) := \cap_{\tilde{\mathfrak{h}}\in\mathcal{P}(\mathfrak{h})} I_{\mathfrak{h},\tilde{\mathfrak{h}}}(\mathcal{SE}(G,\mathfrak{n},l,\tilde{\mathfrak{h}})).$$

Then we have the following result:

**Theorem 3.4.** ([5])

$$\mathcal{SE}^\infty(G,\mathfrak{n},l,\mathfrak{h}) = \mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h}).$$

**Example 3.5.** Let $\mathfrak{g} = \mathbb{R}\text{-span}\{X,Y,Z\}$, $\mathfrak{n}$, $l$, $\mathfrak{h}$ be as in Example 2.7. Concerning Theorem 3.4, we have $\mathfrak{n}^l = \mathfrak{g}$ and a smooth function $\phi \in L^2(\mathbb{R})$ belongs to $\mathcal{ASE}(G,\mathfrak{n},l,\mathfrak{h})$ if and only if

$$\int_{\mathbb{R}} e^{a|x|}|D\phi(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad D \in \mathfrak{D}_x, \tag{3.11}$$

$$\int_{\mathbb{R}} e^{a|x|}|D\hat{\phi}(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad D \in \mathfrak{D}_x, \tag{3.12}$$

where

$$\hat{\phi}(x) = \int_{\mathbb{R}} e^{ixs}\phi(s)ds.$$

Since $\mathfrak{n} = \mathbb{R}Z$ is the center of $\mathfrak{g}$, any polarization $\tilde{\mathfrak{h}}$ at $l$ belongs to $\mathcal{P}(\mathfrak{h})$. Thus by Theorem 3.4, we have that the intersection of $I_{\mathfrak{h},\tilde{\mathfrak{h}}}(\mathcal{SE}(G,\mathfrak{n},l,\tilde{\mathfrak{h}}))$ for all polarizations $\tilde{\mathfrak{h}}$ at $l$ consists of analytic functions $\phi$ satisfying (3.11) and (3.12).

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