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A HOOK FORMULA FOR THE STANDARD TABLEAUX OF A GENERALIZED SHAPE

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1. Introduction

Let λ be a partition of d, Y_λ the Young (or Ferrers) diagram of shape λ, and h_v the hooklength at a cell v of Y_λ. Then the number $\#\mathrm{STab}(Y_\lambda)$ of standard tableaux of shape λ is given by the hook formula:

$$\#\mathrm{STab}(Y_\lambda) = \frac{d!}{\prod_{v\in Y_\lambda} h_v}, \tag{1.1}$$

due to J. S. Frame, G. de B. Robinson, and R. M. Thrall [2]. The purpose of this paper is to prove a hook formula:

$$\#\mathrm{STab}(\mathrm{D}(\lambda)^\vee) = \frac{d!}{\prod_{\beta\in \mathrm{D}(\lambda)} \mathrm{ht}(\beta)}, \tag{1.2}$$

for a generalized shape $\mathrm{D}(\lambda)^\vee$ in the sense of D. Peterson [1] and R. Proctor [9]. See Section 3 and 4 for unexplained notion and furthur details. In fact, the formula (1.2) is equivalent to a corollary to the main result in [6]. So, the main task of the present paper is to define the notion of standard tableaux of a generalized shape and to show the equivalence of the formula (1.2) with the one given in [6].

2. Preliminaries

Let $A = (a_{i,j})_{i,j\in I}$ be a (not necessarily symmetrizable) Cartan matrix of a Kac-Moody Lie algebra [3][5]. We denote the set of real numbers by $\mathbb{R}$. Let $\mathfrak{h}$ be an $\mathbb{R}$ -vector space and $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$ and $\langle , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{R}$ the cannonical bilinear form. We suppose the existence of linearly independent subsets $\Pi := \left\{ \alpha_i \,\middle|\, i \in I \right\} \subset \mathfrak{h}^*$ and $\Pi^\vee := \left\{ \alpha_i^\vee \,\middle|\, i \in I \right\} \subset \mathfrak{h}$ such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$. An element $\lambda \in \mathfrak{h}^*$ is said to be an *integral weight* if

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, \quad i \in I.$$

For each $i \in I$, we define the *simple reflection* $s_i \in GL(\mathfrak{h}^*)$ by:

$$s_i : \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

The group W generated by $\left\{ s_i \,\middle|\, i \in I \right\}$ is called the *Weyl group*, which acts on $\mathfrak{h}$ by:

$$\langle w(\lambda), w(h) \rangle = \langle \lambda, h \rangle, \quad w \in W, \lambda \in \mathfrak{h}^*, h \in \mathfrak{h}.$$

We define the *root system* (resp. *coroot system*) by $\Phi := W\Pi$ (resp. $\Phi^\vee := W\Pi^\vee$). We denote by Φ_+ and Φ_- the sets of positive and negative roots of Φ, respecively.
The *dual* $\beta^\vee \in \Phi^\vee$ of a root $\beta \in \Phi$ is defined so that

$$w(\beta^\vee) = w(\beta)^\vee, \quad w \in W.$$

Similarly, the *dual* $h^\vee \in \Phi$ of a coroot $h \in \Phi^\vee$ is defined so that

$$w(h^\vee) = w(h)^\vee, \quad w \in W.$$

We note that $(\beta^\vee)^\vee = \beta$ for $\beta \in \Phi$.
For each $\beta \in \Phi$, we define $s_\beta \in W$ by:

$$s_\beta(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta, \quad \lambda \in \mathfrak{h}^*,$$

or, equivalently, by

$$s_\beta(h) = h - \langle \beta, h \rangle \beta^\vee, \quad h \in \mathfrak{h}.$$

We note that $s_{\alpha_i} = s_{-\alpha_i} = s_i$.
For each $w \in W$, we define a set $\Phi(w)$ by:

$$\Phi(w) := \left\{ \beta \in \Phi_+ \mid w^{-1}(\beta) < 0 \right\}.$$

For $\beta, \gamma \in \Phi$, we have:

$$\langle \beta, \gamma^\vee \rangle = 0 \Leftrightarrow \langle \gamma, \beta^\vee \rangle = 0,$$

and

$$\langle \beta, \gamma^\vee \rangle > 0 \Leftrightarrow \langle \gamma, \beta^\vee \rangle > 0.$$

3. COLORED HOOK FORMULA FOR Path(λ)

In this section, we review main results of [6].

Definition 1. An integral weight λ is *pre-dominant* if

$$\langle \lambda, \beta^\vee \rangle \geq -1, \quad \beta \in \Phi_+.$$

The set of pre-dominant integral weights is denoted by $P_{\geq -1}$.

Definition 2. For $\lambda \in P_{\geq -1}$, the set $\mathrm{D}(\lambda)$ defined by

$$\mathrm{D}(\lambda) := \left\{ \beta \in \Phi_+ \mid \langle \lambda, \beta^\vee \rangle = -1 \right\}$$

is called the *diagram of* λ. An element of $\mathrm{D}(\lambda)$ is called a λ*-move*. An element of $\mathrm{D}(\lambda) \cap \Pi$ is called a *simple* λ*-move*. A pre-dominant integral weight λ is said to be *finite* if $\mathrm{D}(\lambda)$ is finite.

We note that $\mathrm{D}(\lambda) = \varnothing$ if and only if $\mathrm{D}(\lambda) \cap \Pi = \varnothing$. The terminology "move" is suggested by the game theoretic study of Kawanaka [4].

Lemma 3.1. *Let* $\lambda \in P_{\geq -1}$ *and* $\beta \in \mathrm{D}(\lambda)$. *Then we have:*

(1) $s_\beta(\lambda) \in P_{\geq -1}$.
(2) $\mathrm{D}(s_\beta(\lambda)) = s_\beta(\mathrm{D}(\lambda) \setminus \Phi(s_\beta))$.

Definition 3. Let $\lambda \in P_{\geq -1}$. Let l be a nonnegative integer. A sequence of positive roots $\mathcal{B} = (\beta_1, \beta_2, \cdots, \beta_l)$ is said to be a λ*-path* if

$$\beta_p \in \mathrm{D}(s_{\beta_{p-1}} \cdots s_{\beta_1}(\lambda)), \quad p = 1, \cdots, l.$$

We call l *the length of* the λ-path $\mathcal{B}$ and denote it by $\ell(\mathcal{B})$. Note that $\ell(\mathcal{B})$ may be 0. The set of λ-paths is denoted by Path(λ).

Lemma 3.2. *Let* $\lambda \in P_{\geq -1}$ *and* $\beta, \gamma \in \mathrm{D}(\lambda)$. *Then we have:*

(1) *If* $\langle \beta, \gamma^\vee \rangle = 2$, *then* $\langle \gamma, \beta^\vee \rangle = 1$ or 2
(2) *If* λ *is finite and* $\langle \beta, \gamma^\vee \rangle = \langle \gamma, \beta^\vee \rangle = 2$, *then* $\beta = \gamma$

Theorem 3.3 (Colored Hook Formula). *Let $\lambda \in P_{\geq -1}$ be finite. Then we have:*

$$\sum_{\substack{(\beta_1,\cdots,\beta_l)\in \mathrm{Path}(\lambda)\\ l\geq 0}} \frac{1}{\beta_1}\frac{1}{\beta_1+\beta_2}\cdots\frac{1}{\beta_1+\cdots+\beta_l} = \prod_{\beta\in D(\lambda)}\left(1+\frac{1}{\beta}\right). \tag{3.1}$$

where both hand sides are considered as rational functions in $\{\alpha_i \mid i \in I\} \subseteq \mathfrak{h}^*$.

We call α_i $(i \in I)$ *color variables*, when, as in Theorem 3.3, we consider them as independent variables. We note that the Weyl group W naturally acts on the rational function field $\mathbb{Q}(\alpha_i | i \in I)$ in color variables.

Let $\lambda \in P_{\geq -1}$ be finite. We denote the set of λ-paths of maximal length by MPath(λ). By Lemma 3.1, a λ-path $\mathcal{B}$ in MPath(λ) is a sequence of simple roots of length #D(λ).

Corollary 3.4. *Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)$. Then we have:*

$$\sum_{(\alpha_{i_1},\cdots,\alpha_{i_d})\in \mathrm{MPath}(\lambda)} \frac{1}{\alpha_{i_1}}\frac{1}{\alpha_{i_1}+\alpha_{i_2}}\cdots\frac{1}{\alpha_{i_1}+\cdots+\alpha_{i_d}} = \prod_{\beta\in \mathrm{D}(\lambda)}\frac{1}{\beta}. \tag{3.2}$$

Corollary 3.5. *Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)$. Then we have:*

$$\#\mathrm{MPath}(\lambda) = \frac{d!}{\prod_{\beta\in \mathrm{D}(\lambda)} \mathrm{ht}(\beta)}. \tag{3.3}$$

4. MAIN THEOREM AND REMARKS

Let d be a non-negative integer. We denote the totally oredered set $\{1, 2, \cdots, d\}$ by $[d]$.

Definition 4. Let $P = (P; \leq)$ be a finite partially ordered set. Put $d := \#P$. A bijection $T : [d] \longrightarrow P$ is said to be a *standard tableau of shape P* if the following condition holds:
(STab) If $T(j) < T(k)$, then we have $j > k$.
The set of standard tableaux of shape P is denoted by STab(P).

Definition 5. Let $\lambda \in P_{\geq -1}$. We define a set $\mathrm{D}(\lambda)^\vee$ by:

$$\mathrm{D}(\lambda)^\vee := \left\{\beta^\vee \mid \beta \in \mathrm{D}(\lambda)\right\} = \left\{\beta^\vee \in \Phi_+^\vee \mid \langle\lambda, \beta^\vee\rangle = -1\right\}.$$

We call $\mathrm{D}(\lambda)^\vee$ a *shape of* λ. We note that $\mathrm{D}(\lambda)^\vee$ is a (possibly infinite) partially ordered set with the order $\leq$ over $\Phi_+^\vee$.

We now state the main result of this paper.

Theorem 4.1. *Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)^\vee$.*

$$\#\mathrm{STab}(\mathrm{D}(\lambda)^\vee) = \frac{d!}{\prod_{\beta\in \mathrm{D}(\lambda)} \mathrm{ht}(\beta)}.$$

Through Section 5 and 6, we give a proof of Theorem 4.1. Theorem 4.1 is proved as Theorem 6.5 (2).

Remark 1. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n > 0)$ be a partition of d, and

$$Y_\lambda = \left\{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq \lambda_i\right\}$$

be the corresponding Young diagram; we consider Y_λ as a partially ordered set by:

$$(i, j) \leq (i', j') \Leftrightarrow i \geq i' \text{ and } j \geq j'.$$

Then, for a sufficiently large r, there exists some $\lambda_o \in P_{\geq -1}$ of a Lie algebra of type A_r such that Y_λ is order-isomorphic to $\mathrm{D}(\lambda_o)^\vee$. An explicit description is as follows:

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n > 0)$ be a partition of d. Put $r_o := n + \lambda_1 - 1$.
For $k = 1, \cdots, n$, put $\beta_k := \lambda_{n+1-k} + k - 1$. For $i = 1, \cdots, r_o$, define b_i as:

$$b_i := \begin{cases} -1 & \text{if } i \in \{\beta_1, \cdots, \beta_n\} \\ 0 & \text{otherwise} \end{cases}$$

For $k = 1, \cdots, n-1$, put $\gamma_k := \lambda_{n+1-k} + k$. For $i = 1, \cdots, r_o$, define c_i as:

$$c_i := \begin{cases} 1 & \text{if } i \in \{\gamma_1, \cdots, \gamma_{n-1}\} \\ 0 & \text{otherwise} \end{cases}$$

Let $A = (a_{i,j})_{i,j=1}^{r_o}$ be a Cartan matrix of type A_{r_o}, and ω_i be the i-th fundamental weight. Then, an integral weight defined by:

$$\lambda_o := \sum_{i=1}^{r_o} (b_i + c_i)\omega_i$$

is a finite pre-dominant integral weight. And, the shape $D(\lambda_o)^\vee$ is order-isomorphic to the Young diagram Y_λ. We note that the integer r_o defined above is the minimum value of r's such that the given Young diagram Y_λ is realizable in the coroot system of type A_r.

Remark 2. If a Cartan matrix A is simply-laced, then the partially ordered set $(D(\lambda); \leq)$ is order-isomorphic to $(D(\lambda)^\vee; \leq)$. In particular, we have:

$$\#\mathrm{STab}(D(\lambda)) = \frac{d!}{\prod_{\beta \in D(\lambda)} \mathrm{ht}(\beta)}.$$

Remark 3. Let $\lambda \in P_{\geq -1}$. Let $\beta \in D(\lambda)$. Put $H_\lambda(\beta) := D(\lambda) \cap \Phi(s_\beta)$ (See [6]). We call the set $H_\lambda(\beta)$ the *hook at* β, and the integer $\#H_\lambda(\beta)$ the *hooklength at* β. Then we have:

$$\mathrm{ht}(\beta) = \#H_\lambda(\beta).$$

Hence, we get:

$$\#\mathrm{STab}(D(\lambda)^\vee) = \frac{d!}{\prod_{\beta \in D(\lambda)} \#H_\lambda(\beta)}.$$

5. Proof of Theorem 4.1 (First Part)

Proposition 5.1. *Let $\lambda \in P_{\geq -1}$, $\beta \in D(\lambda)$, and $\gamma \in D(\lambda) \cap \Phi(s_\beta)$. Then we have $\langle \beta, \gamma^\vee \rangle = 1$, or 2.*

Proof. We have $s_\beta(\gamma^\vee) = \gamma^\vee - \langle \beta, \gamma^\vee \rangle \beta^\vee \leq \gamma^\vee - \beta^\vee < 0$. Since $\lambda \in P_{\geq -1}$, we have $-1 \leq \langle \lambda, -s_\beta(\gamma^\vee) \rangle = \langle \lambda, -\gamma^\vee + \langle \beta, \gamma^\vee \rangle \beta^\vee \rangle = -\langle \lambda, \gamma^\vee \rangle + \langle \beta, \gamma^\vee \rangle \langle \lambda, \beta^\vee \rangle = 1 - \langle \beta, \gamma^\vee \rangle$. Hence, we have:

$$\langle \beta, \gamma^\vee \rangle \leq 2. \tag{5.1}$$

If $\langle \beta, \gamma^\vee \rangle \leq 0$, then we have $s_\beta(\gamma^\vee) = \gamma^\vee - \langle \beta, \gamma^\vee \rangle \beta^\vee \geq \gamma^\vee > 0$. This contradicts our assumption. Hence, we have:

$$\langle \beta, \gamma^\vee \rangle \geq 1. \tag{5.2}$$

By (5.1) and (5.2), we have:

$$\langle \beta, \gamma^\vee \rangle = 1, \text{ or } 2.$$

□

Proposition 5.2. *Let $\lambda \in P_{\geq -1}$ be finite. Let $\beta, \gamma \in D(\lambda)$. Then we have:*

(1) $\langle \beta, \gamma^\vee \rangle \geq 0$.
(2) *if $\langle \beta, \gamma^\vee \rangle \geq 1$ and $\beta^\vee \neq \gamma^\vee$, then $\langle \beta, \gamma^\vee \rangle = 1$, or $\langle \gamma, \beta^\vee \rangle = 1$.*

Proof. (1) Since $s_\beta(\lambda) \in P_{\geq -1}$, we have $-1 \leq \langle s_\beta(\lambda), \gamma^\vee\rangle = -1 + \langle \beta, \gamma^\vee\rangle$. Hence, we have $\langle \beta, \gamma^\vee\rangle \geq 0$.
(2) If $\langle \beta, \gamma^\vee\rangle = 1$, then there is nothing to prove.
If $\langle \beta, \gamma^\vee\rangle = 2$, then, by Lemma 3.2 (1) and (2), we have $\langle \gamma, \beta^\vee\rangle = 1$.
Suppose $\langle \beta, \gamma^\vee\rangle \geq 3$. If $s_\gamma(\beta^\vee) > 0$, then, since $\langle s_\gamma(\lambda), s_\gamma(\beta^\vee)\rangle = \langle \lambda, \beta^\vee\rangle = -1$, we have $(\gamma, s_\gamma(\beta)) \in \mathrm{Path}(\lambda)$. Hence $s_{s_\gamma(\beta)}s_\gamma(\lambda) \in P_{\geq -1}$. Since

$$\begin{aligned} -1 \leq \langle s_{s_\gamma(\beta)}s_\gamma(\lambda), \beta^\vee\rangle &= \langle \lambda + \gamma + s_\gamma(\beta), \beta^\vee\rangle \\ &= \langle \lambda + \beta + (1 - \langle \beta, \gamma^\vee\rangle)\gamma, \beta^\vee\rangle \\ &= 1 + (1 - \langle \beta, \gamma^\vee\rangle)\langle \gamma, \beta^\vee\rangle, \end{aligned}$$

we have:

$$(\langle \beta, \gamma^\vee\rangle - 1)\langle \gamma, \beta^\vee\rangle \leq 2.$$

Hence, we have $\langle \beta, \gamma^\vee\rangle = 3$ and $\langle \gamma, \beta^\vee\rangle = 1$. If, on the other hand, $s_\gamma(\beta) < 0$, then $\beta \in \mathrm{H}_\lambda(\gamma)$. By Proposition 5.1, we have $\langle \gamma, \beta^\vee\rangle = 1$, or 2. If $\langle \gamma, \beta^\vee\rangle = 2$, then, by Lemma 3.2 (1) and (2), we have $\langle \beta, \gamma^\vee\rangle = 1$.
Thus, we always have $\langle \beta, \gamma^\vee\rangle = 1$, or $\langle \gamma, \beta^\vee\rangle = 1$. □

Proposition 5.3. *Let $\beta, \gamma \in \Phi$. Suppose $\langle \gamma, \beta^\vee\rangle = 1$, or $\langle \beta, \gamma^\vee\rangle = 1$. Then:*
(1) $\beta^\vee - \gamma^\vee \in \Phi^\vee$.
(2) *We have either $\beta^\vee < \gamma^\vee$, or $\beta^\vee > \gamma^\vee$.*

Proof. (1) If $\langle \gamma, \beta^\vee\rangle = 1$, then $\beta^\vee - \gamma^\vee = \beta^\vee - \langle \gamma, \beta^\vee\rangle\gamma^\vee = s_\gamma(\beta^\vee) \in \Phi^\vee$. If, on the other hand, $\langle \beta, \gamma^\vee\rangle = 1$, then $\beta^\vee - \gamma^\vee = -\gamma^\vee - \langle \beta, (-\gamma)^\vee\rangle\beta^\vee = s_\beta(-\gamma^\vee) \in \Phi^\vee$.
(2) Since $\beta^\vee - \gamma^\vee \in \Phi^\vee$, we have either $\beta^\vee - \gamma^\vee > 0$, or $\beta^\vee - \gamma^\vee < 0$. Hence, we have either $\beta^\vee > \gamma^\vee$, or $\beta^\vee < \gamma^\vee$. □

Let $\beta^\vee, \gamma^\vee \in \Phi^\vee$. We denote $\beta^\vee \lessdot \gamma^\vee$ if

$$\langle \gamma, \beta^\vee\rangle \geq 1,$$

and

$$\beta^\vee < \gamma^\vee.$$

Lemma 5.4. *Let $\lambda \in P_{\geq -1}$ be finite. Let $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\beta, \gamma \in \mathrm{D}(\lambda) \setminus \{\alpha_i\}$. If $\gamma^\vee \lessdot \beta^\vee$, then we have $s_i(\gamma^\vee), s_i(\beta^\vee) \in \mathrm{D}(s_{\alpha_i}(\lambda))^\vee$, and $s_i(\gamma^\vee) \lessdot s_i(\beta^\vee)$.*

Proof. By Lemma 3.1(2), we have $s_i(\gamma^\vee), s_i(\beta^\vee) \in \mathrm{D}(s_{\alpha_i}(\lambda))^\vee$.
Since $\gamma^\vee \lessdot \beta^\vee$, we have $\langle \beta, \gamma^\vee\rangle \geq 1$. By Proposition 5.2(2), we have $\langle \gamma, \beta^\vee\rangle = 1$, or $\langle \beta, \gamma^\vee\rangle = 1$. By Proposition 5.3(1), we have $\beta^\vee - \gamma^\vee \in \Phi^\vee_+$. Hence, we have either $s_i(\beta^\vee - \gamma^\vee) < 0$, or $s_i(\beta^\vee - \gamma^\vee) > 0$. Suppose $s_i(\beta^\vee - \gamma^\vee) < 0$. Then we have $\beta^\vee - \gamma^\vee = \alpha_i^\vee$. Hence, we have $\beta^\vee = \gamma^\vee + \alpha_i^\vee$. Since $\langle \lambda, \beta^\vee\rangle = \langle \lambda, \gamma^\vee + \alpha_i^\vee\rangle = (-1) + (-1) = -2$, this contradicts $\lambda \in P_{\geq -1}$. Hence, we have $s_i(\beta^\vee - \gamma^\vee) > 0$. Since $\langle s_i(\gamma), s_i(\beta^\vee)\rangle = \langle \gamma, \beta^\vee\rangle \geq 1$, we have $s_i(\gamma^\vee) \lessdot s_i(\beta^\vee)$. □

Lemma 5.5. *Let $\lambda \in P_{\geq -1}$ be finite. Let $\beta \in \mathrm{D}(\lambda)$. If β is not a simple root, then there exists $\gamma \in \mathrm{D}(\lambda) \cap \Phi\left(s_\beta\right)$ such that $\gamma^\vee < \beta^\vee$.*

Proof. Let $\alpha_i \in \Phi\left(s_\beta\right) \cap \Pi$. Since $0 > s_\beta(\alpha_i) = \alpha_i - \langle \alpha_i, \beta^\vee\rangle\beta$, we have $\langle \alpha_i, \beta^\vee\rangle \geq 1$. Since $s_i(\beta^\vee) = \beta^\vee - \langle \alpha_i, \beta^\vee\rangle\alpha_i^\vee$, we have:

$$\beta^\vee = s_i(\beta^\vee) + \langle \alpha_i, \beta^\vee\rangle\alpha_i^\vee,$$
$$0 < s_i(\beta^\vee) < \beta^\vee, \text{ and } 0 < \alpha_i^\vee < \beta^\vee.$$

Since $\lambda \in P_{\geq -1}$ and $-1 = \langle \lambda, \beta^\vee \rangle = \langle \lambda, s_i(\beta^\vee)\rangle + \langle \alpha_i, \beta^\vee \rangle \langle \lambda, \alpha_i^\vee \rangle$, we have either $\langle \lambda, \alpha_i^\vee \rangle = -1$, or $\langle \lambda, s_i(\beta^\vee)\rangle = -1$.
If $\langle \lambda, \alpha_i^\vee \rangle = -1$, then $\alpha_i \in \mathrm{D}(\lambda)$. Hence we get:

$$\alpha_i \in \mathrm{D}(\lambda) \cap \Phi\left(s_\beta\right), \text{ and } \alpha_i^\vee < \beta^\vee.$$

If $\langle \lambda, s_i(\beta^\vee)\rangle = -1$, then $\langle \lambda, \alpha_i^\vee \rangle = 0$. We have $\langle \beta, s_i(\beta^\vee)\rangle = 2 - \langle \beta, \alpha_i^\vee \rangle \langle \alpha_i, \beta^\vee \rangle$. Since β, $s_i(\beta) \in \mathrm{D}(\lambda)$, by Proposition 5.2 (1), we have $\langle \beta, \alpha_i^\vee \rangle \langle \alpha_i, \beta^\vee \rangle \leq 2$. Since $\langle \beta, \alpha_i^\vee \rangle \geq 1$ and $\langle \alpha_i, \beta^\vee \rangle \geq 1$, we have either:

$$\langle \beta, \alpha_i^\vee \rangle = 2 \text{ and } \langle \alpha_i, \beta^\vee \rangle = 1,$$

$$\langle \beta, \alpha_i^\vee \rangle = 1 \text{ and } \langle \alpha_i, \beta^\vee \rangle = 2,$$

or

$$\langle \beta, \alpha_i^\vee \rangle = 1 \text{ and } \langle \alpha_i, \beta^\vee \rangle = 1.$$

If $\langle \beta, \alpha_i^\vee \rangle = 2$ and $\langle \alpha_i, \beta^\vee \rangle = 1$, then we have $0 < 2\beta^\vee - \alpha_i^\vee = s_{s_i(\beta)}(\alpha_i^\vee) \in \Phi^\vee$. Since $\langle \lambda, s_{s_i(\beta)}(\alpha_i^\vee)\rangle = 2\langle \lambda, \beta^\vee \rangle - \langle \lambda, \alpha_i^\vee \rangle = -2$, This contradicts that $\lambda \in P_{\geq -1}$.
If $\langle \beta, \alpha_i^\vee \rangle = 1$ and $\langle \alpha_i, \beta^\vee \rangle = 2$, then we have $0 < \beta^\vee - \alpha_i^\vee = s_{s_i(\beta)}(\alpha_i^\vee) \in \Phi^\vee$. Since $s_\beta(\beta^\vee - \alpha_i^\vee) = -\beta^\vee - (\alpha_i^\vee - \beta^\vee) = -\alpha_i < 0$ and $\langle \lambda, \beta^\vee - \alpha_i^\vee \rangle = -1 - 0 = -1$, we get:

$$(\beta^\vee - \alpha_i^\vee)^\vee \in \mathrm{D}(\lambda) \cap \Phi\left(s_\beta\right), \text{ and } \beta^\vee - \alpha_i^\vee < \beta^\vee.$$

If $\langle \beta, \alpha_i^\vee \rangle = 1$ and $\langle \alpha_i, \beta^\vee \rangle = 1$, then we have $0 < \beta^\vee - \alpha_i^\vee = s_i(\beta^\vee) \in \Phi^\vee$. Since $s_\beta(\beta^\vee - \alpha_i^\vee) = -\beta^\vee - (\alpha_i^\vee - \beta^\vee) = -\alpha_i < 0$ and $\langle \lambda, \beta^\vee - \alpha_i^\vee \rangle = -1 - 0 = -1$, we get:

$$(\beta^\vee - \alpha_i^\vee)^\vee \in \mathrm{D}(\lambda) \cap \Phi\left(s_\beta\right), \text{ and } \beta^\vee - \alpha_i^\vee < \beta^\vee.$$

Thus, there always exists $\gamma \in \mathrm{D}(\lambda) \cap \Phi\left(s_\beta\right)$ such that $\gamma^\vee < \beta^\vee$. □

Definition 6. Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)^\vee$. A sequence $L = (\gamma_1^\vee, \cdots, \gamma_d^\vee)$ of elements of $\mathrm{D}(\lambda)^\vee$ of length d is said to be a *standard labelling of shape* λ if the following condition holds:
(SLab) If $\gamma_j^\vee \lhd \gamma_k^\vee$, then we have $j < k$.
The set of standard labellings of shape λ is denoted by $\mathrm{SLab}(\lambda)$.

Theorem 5.6. *Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)^\vee$.*

(1) *For a maximal λ-path $\mathcal{B} = (\alpha_{i_1}, \cdots, \alpha_{i_d})$, we put*

$$L_{\mathcal{B}} := (\gamma_1^\vee, \cdots, \gamma_d^\vee),$$

where $\gamma_k^\vee = s_{i_1} \cdots s_{i_{k-1}}(\alpha_k)^\vee$, $1 \leq k \leq d$. Then, $L_{\mathcal{B}}$ is a standard labelling of shape λ.

(2) *For a standard labelling $L = (\gamma_1^\vee, \cdots, \gamma_d^\vee)$ of shape λ, we put*

$$\mathcal{B}_L := (\beta_1, \cdots, \beta_d),$$

where $\beta_k = s_{\gamma_1} \cdots s_{\gamma_{k-1}}(\gamma_k)$, $1 \leq k \leq d$. Then, $\mathcal{B}_L$ is a maximal λ-path.

(3) *The correspondence $\mathcal{B} \mapsto L_{\mathcal{B}}$ from* $\mathrm{MPath}(\lambda)$ *to* $\mathrm{SLab}(\lambda)$ *is the inverse of the correspondence $L \mapsto \mathcal{B}_L$ from* $\mathrm{SLab}(\lambda)$ *to* $\mathrm{MPath}(\lambda)$

Proof. (1) Let $\mathcal{B} = (\alpha_{i_1}, \cdots, \alpha_{i_d}) \in \mathrm{MPath}(\lambda)$. Put $\gamma_k^\vee := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}^\vee)$, for each $1 \leq k \leq d$. Let $1 \leq k \leq d$. By Lemma 3.1 (2), we have $\mathrm{D}(s_{i_{k-1}} \cdots s_{i_1}(\lambda)) \subseteq s_{i_{k-1}} \cdots s_{i_1}(\mathrm{D}(\lambda))$. Since $\alpha_{i_k}^\vee \in \mathrm{D}(s_{i_{k-1}} \cdots s_{i_1}(\lambda))^\vee$, we have $\gamma_k^\vee = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})^\vee \in \mathrm{D}(\lambda)^\vee$. Hence, $L_{\mathcal{B}}$ is a sequence of elements of $\mathrm{D}(\lambda)^\vee$ of length d.
Let $1 \leq j, k \leq d$. Suppose $\gamma_j^\vee \lhd \gamma_k^\vee$. Suppose $j > k$. We have:

(5.3) $$s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}^\vee), s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}^\vee) \in \mathrm{D}(\lambda), \text{ and } s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}^\vee) \lhd s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}^\vee).$$

Applying Lemma 5.4 to (5.3) repeatedly, we get:

(5.4) $s_{i_k}\cdots s_{i_{j-1}}(\alpha_{i_j}^\vee),\, s_{i_k}\cdots s_{i_{k-1}}(\alpha_{i_k}^\vee) \in \mathrm{D}(s_{i_{k-1}}\cdots s_{i_1}(\lambda))$, and $s_{i_k}\cdots s_{i_{j-1}}(\alpha_{i_j}^\vee) \lessdot \alpha_{i_k}^\vee$.

This contradicts that $\alpha_{i_k}^\vee$ is a simple coroot. Hence, we have $k \le j$. Since it is trivial that $k \neq j$, we have $k < j$. Hence, we have $L_{\mathcal{B}} \in \mathrm{SLab}(\lambda)$.
(2) Let $L = (\gamma_1^\vee, \cdots, \gamma_d^\vee) \in \mathrm{SLab}(\lambda)$. Put $\beta_k := s_{\gamma_1}\cdots s_{\gamma_{k-1}}(\gamma_k)$, for each $1 \le k \le d$. Let $1 \le k \le d$. Since

$$s_{\beta_1}\cdots s_{\beta_{k-1}}(\beta_k^\vee) = \gamma_k^\vee,$$

we have $\langle s_{\beta_{k-1}}\cdots s_{\beta_1}(\lambda), \beta_k^\vee\rangle = \langle \lambda, s_{\beta_1}\cdots s_{\beta_{k-1}}(\beta_k^\vee)\rangle = \langle \lambda, \gamma_k^\vee\rangle = -1$.
Since $s_{\gamma_1}\cdots s_{\gamma_{k-1}} = s_{\beta_{k-1}}\cdots s_{\beta_1}$, we have:

$$\beta_k^\vee = s_{\gamma_1}\cdots s_{\gamma_{k-1}}(\gamma_k^\vee) = s_{\beta_{k-1}}\cdots s_{\beta_1}(\gamma_k^\vee).$$

Let $1 \le p \le k-1$. Suppose

$$s_{\beta_{p-1}}\cdots s_{\beta_1}(\gamma_k^\vee) \in \mathrm{D}(s_{\beta_{p-1}}\cdots s_{\beta_1}(\lambda))^\vee$$

and

$$\beta_p^\vee \in \mathrm{D}(s_{\beta_{p-1}}\cdots s_{\beta_1}(\lambda))^\vee \cap \Pi^\vee.$$

Then, by Lemma 3.1, we have:

$$s_{\beta_p}(s_{\beta_{p-1}}\cdots s_{\beta_1}(\gamma_k^\vee)) \in \mathrm{D}(s_{\beta_p}(s_{\beta_{p-1}}\cdots s_{\beta_1}(\lambda)))^\vee.$$

By Lemma 5.5, we have $s_{\beta_p}(s_{\beta_{p-1}}\cdots s_{\beta_1}(\gamma_k^\vee)) \in \mathrm{D}(s_{\beta_p}(s_{\beta_{p-1}}\cdots s_{\beta_1}(\lambda)))^\vee \cap \Pi^\vee$. By induction on p, we have $\beta_k^\vee \in \mathrm{D}(s_{\beta_{k-1}}\cdots s_{\beta_1}(\lambda))^\vee \cap \Pi^\vee$. Hence, we get $(\beta_1, \cdots, \beta_d) \in \mathrm{MPath}(\lambda)$.
(3) This follows from the definitions of $\mathcal{B} \mapsto L_{\mathcal{B}}$ and $L \mapsto \mathcal{B}_L$. □

By Corollary 3.5 and Theorem 5.6, we get:

$$\#\mathrm{SLab}(\lambda) = \frac{d!}{\prod_{\beta\in\mathrm{D}(\lambda)} \mathrm{ht}(\beta)}, \tag{5.5}$$

where $\mathrm{ht}(\beta)$ is the height of β.

6. Proof of Theorem 4.1 (Second Part)

Let $P = (P; \le)$ be a (possibly infinite) partially ordered set. We define a set $\min(P)$ by:

$$\min(P) := \{x \in P \mid x \text{ is a minimal element of } P\}.$$

Corollary 6.1. *Let $\lambda \in P_{\ge -1}$. Then we have:*

$$\min(\mathrm{D}(\lambda)^\vee) = \mathrm{D}(\lambda)^\vee \cap \Pi^\vee.$$

Proof. It is trivial that $\min(\mathrm{D}(\lambda)^\vee) \supseteq \mathrm{D}(\lambda)^\vee \cap \Pi^\vee$. Now, we prove the converse. Let $\beta^\vee \in \min(\mathrm{D}(\lambda)^\vee)$. Suppose $\beta^\vee \notin \mathrm{D}(\lambda)^\vee \cap \Pi^\vee$. Then, by Lemma 5.5, there exists $\gamma^\vee \in \mathrm{D}(\lambda)^\vee \cap \Phi(s_\beta)^\vee \subseteq \mathrm{D}(\lambda)^\vee$ such that $\gamma^\vee < \beta^\vee$. This contradicts that $\beta^\vee \in \min(\mathrm{D}(\lambda)^\vee)$. Hence, we have $\beta^\vee \in \mathrm{D}(\lambda)^\vee \cap \Pi^\vee$. This proves the statement. □

Lemma 6.2. *Let $P = (P; \le)$ be a finite partially ordered set. Put $d := \#P$. Let $x \in \min(P)$. Put $S := \{T \in \mathrm{STab}(P) \mid T(d) = x\}$. We define a map $\phi : \mathrm{STab}(P \setminus \{x\}) \longrightarrow S$ by:*

$$(\phi T)(k) = T(k), \quad \textit{for } k = 1, \cdots, d-1.$$

Then the map ϕ is a bijection from $\mathrm{STab}(P \setminus \{x\})$ to S.

Proof. This is straightforward to see. □

Corollary 6.3. *Let $P = (P; \leq)$ be a finite partially ordered set. Then we have:*

$$\#\mathrm{STab}(P) = \sum_{x \in \min(P)} \#\mathrm{STab}(P \setminus \{x\}).$$

Proof. This follows from Lemma 6.2. □

Lemma 6.4. *Let $\lambda \in P_{\geq -1}$ be finite. Then we have:*

$$\#\mathrm{MPath}(\lambda) = \sum_{\alpha_i \in \mathrm{D}(\lambda) \cap \Pi} \#\mathrm{MPath}(s_{\alpha_i}(\lambda)).$$

Proof. This follows from the definition of λ-paths. □

Theorem 6.5. *Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)^\vee$.*

(1) *For a standard tableau T of shape $\mathrm{D}(\lambda)^\vee$, we put*

$$L_T := (T(d), \cdots, T(1)).$$

Then, L_T is a standard labelling of shape λ.

(2) *We have:*

$$\#\mathrm{STab}(\mathrm{D}(\lambda)^\vee) = \frac{d!}{\prod_{\beta \in \mathrm{D}(\lambda)} \mathrm{ht}(\beta)}.$$

(3) *The correspondence $T \mapsto L_T$ from $\mathrm{STab}(\mathrm{D}(\lambda)^\vee)$ to $\mathrm{SLab}(\lambda)$ is a bijection.*

Proof. (1) Since whenever $\beta^\vee \lessdot \gamma^\vee$ we have $\beta^\vee < \gamma^\vee$, it is obvious that $L_T \in \mathrm{SLab}(\lambda)$.
(2) By Corollary 6.3 and Lemma 6.4, $\#\mathrm{STab}(*)$ and $\#\mathrm{MPath}(*)$ satisfy the same recursive relation. Hence, by induction on $\#\mathrm{D}(\lambda)$, we have $\#\mathrm{STab}(\mathrm{D}(\lambda)^\vee) = \#\mathrm{MPath}(\lambda)$.
(3) Since the injectivity of the correspondence $T \mapsto L_T$ is trivial, this follows from part (1), part (2), and (5.5). □

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