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Binary codes and the classification of holomorphic framed vertex operator algebras

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Abstract

In this paper we report a general theory of a framed vertex operator algebra (VOA). We show that there exists a strong duality between the structure codes (C, D) of a framed VOA V. As a consequence, we see that every framed VOA is a simple current extension of its code sub VOA. We also list up the necessary and sufficient conditions for the structure codes of a holomorphic framed VOA and discuss the classification problem of holomorphic framed VOAs. It is also proved that the moonshine VOA can be characterized as the holomorphic framed VOA of central charge 24 with trivial weight one subspace.

1 Introduction

A framed vertex operator algebra (VOA) is a simple vertex operator algebra having a full sub VOA called an *Ising frame* which is isomorphic to a tensor product of copies of 2D Ising model $L(1/2, 0)$. The study of framed VOAs originates in [DMZ], where it is verified

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that the moonshine VOA [FLM] has an Ising frame and the authors discussed certain structural theory of the moonshine VOA by using an Ising frame. After [DMZ], Dong et al. [DGH] and Miyamoto [M1, M2, M3] independently developed a general theory of framed VOAs to a great amount and it is shown in [DGH, M2] that we can attach a pair of binary linear codes called *structure codes* to a framed VOA and the structure theory can be built upon the structure codes. Given a framed VOA V, one can always define its code sub VOA and V can be thought as an extension of the code sub VOA by its irreducible modules. This extension affords a control given by the theory of so-called quantum Galois theory [DM1, DLM1] and one can also use Miyamoto involutions defined in [M1]. A code VOA is completely determined by its structure code and the representation theory of a code VOA is almost completed in [M2]. Combining these results, framed VOAs form a class of well-understood VOAs so far and many important progresses, including a reconstruction of the moonshine VOA as a framed VOA in [M3], have been done.

This article is an exposition and extension of our recent results in [LaY1, LaY2] concerning framed VOAs and their automorphism groups. In [LaY1], the authors proceeded with developing the theory and they revealed a strong duality on structure codes (cf. Theorem 5.1). This duality turns out to show that every framed VOA forms a *simple current extension* [DM2, La, M2, Y1, Y2] of its code sub VOA (cf. Theorem 5.2). Since we have a uniqueness property of simple current extension shown in [DM2], this result suggests that it is within our reach to think of classification of framed VOAs, especially holomorphic ones of small central charge. In [LaY1], the authors converted this problem into a coding theoretical one and they gave a necessary and sufficient conditions for a pair of codes to be a structure codes of a holomorphic framed VOA (cf. Theorem 5.6). In addition, the authors succeeded to characterize the moonshine VOA as the unique holomorphic framed VOA of central charge 24 with trivial weight one subspace in [LaY2]. We shall discuss these problems in this article. In [LaY1], automorphisms keeping given Ising frame is also discussed and it is shown that the group shape of pointwise frame stabilizer is determined in terms of the associated structure codes (cf. Theorem 6.3). It is not easy to describe automorphisms of a VOA in general, but for framed VOA case, one can accomplish enough informations for certain automorphisms. As an example, we shall present an explicit subgroup of the Monster simple group, which is the full automorphism group of the moonshine VOA [FLM], by using our theory.

The organization of this paper is as follows. In Section 2, we will review definitions of vertex operator superalgebras within two approaches. We will also present an explicit construction of the vertex operator superalgebra $L(1/2,0) \oplus L(1/2,1/2)$ which is a minimal piece to construct a framed VOA. In Section 3 we will review the theory of quantum Galois theory and discuss how we can use this theory to introduce so-called Miyamoto involutions. In Section 4 we will introduce the notions of framed vertex operator algebras and structure codes. Preliminary results will be presented based on the arguments in Section 3. In Section 5 we will show the duality on structure codes and the consequences. A classification problem of holomorphic framed vertex operator algebras and related topics will be discussed. In Section 6 we will discuss on the structure of point-wise frame stabilizers. As an application, we will also present a subgroup of the Monster in terms of binary linear codes. Section 7 is an appendix and devoted to give a brief sketch of the argument in [LaY2] where a characterization of the moonshine VOA by means of Ising frames is established.

Notation and Terminology. In this article, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the set of non-negative integers, integers, and the complex numbers, respectively. Every vertex operator algebra (VOA for short) is defined over the complex number field $\mathbb{C}$ unless otherwise stated.

Let A be a vector space. We define

$$A[\![z]\!] := \{\textstyle\sum_{n\geq 0} a_n z^n \mid a_n \in A\}, \qquad A[\![z,z^{-1}]\!] := \{\textstyle\sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in A\},$$
$$A(\!(z)\!) := \{\textstyle\sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in A, \quad a_n = 0 \ \text{ for } \ n \ll 0\}.$$

For $n \in \mathbb{Z}$, we define

$$(z+w)^n := \sum_{i\geq 0} \binom{n}{i} z^{n-i} w^i \in \mathbb{C}[\![z,z^{-1}]\!][\![w]\!].$$

Note that $(z+w)^n \neq (w+z)^n$ if n is negative. For $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in A[\![z,z^{-1}]\!]$, we define the formal residue of $f(z)$ by $\mathrm{Res}_z f(z) := a_{-1}$, which takes the coefficient of z^{-1} in $f(z)$.

We denote the ring $\mathbb{Z}/p\mathbb{Z}$ with $p \in \mathbb{Z}$ by $\mathbb{Z}_p$ and often identify the integers $0, 1, \ldots, p-1$ with their images in $\mathbb{Z}_p$. An additive subgroup C of $\mathbb{Z}_2^n$ together with the standard $\mathbb{Z}_2$-bilinear form is called a linear code. For a codeword $\alpha = (\alpha_1, \ldots, \alpha_n) \in C$, we define the support of α by $\mathrm{supp}(\alpha) := \{i \mid \alpha_i = 1\}$ and the weight by $\mathrm{wt}(\alpha) := |\mathrm{supp}(\alpha)|$. For a subset A of C, we define $\mathrm{supp}(A) := \cup_{\alpha\in A}\mathrm{supp}(\alpha)$. For a binary codeword $\gamma \in \mathbb{Z}_2^n$ and for any linear code $C \subset \mathbb{Z}_2^n$, we denote $C_\gamma := \{\alpha \in C \mid \mathrm{supp}(\alpha) \subset \mathrm{supp}(\gamma)\}$ and $C^{\perp_\gamma} := \{\beta \in C^\perp \mid \mathrm{supp}(\beta) \subset \mathrm{supp}(\gamma)\}$. The all one vector is a codeword $(11\ldots1) \in \mathbb{Z}_2^n$. Seen $\mathbb{Z}_2^n$ as a direct sum of rings, we denote the product by $\alpha \cdot \beta := (\alpha_1\beta_1, \ldots, \alpha_n\beta_n)$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_2^n$.

2 Vertex operator superalgebras

We begin by giving the definition of vertex operator superalgebras (SVOAs) (cf. [B, FHL, FLM, K, MN]).

Definition 2.1. A *vertex operator superalgebra* is a quadruple $(V, Y(\cdot, z), \mathbb{1}, \omega)$ subject to the following conditions:

(1) $V = \oplus_{n\in\mathbb{N}} V_{n/2} = V^{(0)} \oplus V^{(1)}$ is a $\mathbb{C}$-linear space bi-graded by $\mathbb{N} \cup (\mathbb{N}+1/2)$ and $\mathbb{Z}_2$ such that $V^{(i)} = \oplus_{n\in\mathbb{N}} V_{n+i/2}$ and each homogeneous subspace $V_{n+i/2}$ is finite dimensional.

(2) $Y(\cdot, z) : V \to \mathrm{End}(V)[\![z, z^{-1}]\!]$, $a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, is a linear map called the *vertex operator map* on V such that $Y(a, z)b \in V^{(i+j)}((z))$ for $\mathbb{Z}_2$-homogeneous $a \in V^{(i)}$ and $b \in V^{(j)}$.

(3) For $\mathbb{Z}_2$-homogeneous $a \in V^{(i)}$ and $b \in V^{(j)}$, there exists an integer $N = N(a, b)$ such that the *locality* $(z_1 - z_2)^N Y(a, z_1)Y(b, z_2) = (-1)^{ij}(-z_2 + z_1)^N Y(b, z_2)Y(a, z_1)$ holds.

(4) $\mathbb{1} \in V_0$ is the *vacuum vector* such that $Y(\mathbb{1}, z) = \mathrm{id}_V$ and $Y(a, z)\mathbb{1} \in a + V[\![z]\!]z$ for $a \in V$.

(5) $\omega \in V_2$ is the *conformal vector* such that the associated vertex operator $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ generates the Virasoro algebra

$$[L(m), L(n)] = (m - n)L(m + n) + \delta_{m+n,0}\frac{m^3 - m}{12}c, \tag{2.1}$$

where $c \in \mathbb{C}$ is called the *central charge* or *rank* of V.

(6) $L(0) = \omega_{(1)}$ acts on the homogeneous subspace V_n by n and $L(-1) = \omega_{(0)}$ satisfies $Y(\omega_{(0)}a, z) = [L(-1), Y(a, z)] = \partial_z Y(a, z)$.

A vertex operator superalgebra $(V, Y(\cdot, z), \mathbb{1}, \omega)$ is often denoted simply by V. If the odd part of V is trivial, V is referred to as a *vertex operator algebra* (VOA). We list basic properties of a vertex operator superalgebra (cf. [FHL, K, MN]).

Proposition 2.2. *Let V be an SVOA and $a \in V^{(i)}$, $b \in V^{(j)}$ be $\mathbb{Z}_2$-homogeneous.*

(1) $Y(\cdot, z) : V \ni x \mapsto Y(x, z) \in \mathrm{End}(V)[\![z, z^{-1}]\!]$ is injective.

(2) $Y(a, z)b = (-1)^{ij} e^{zL(-1)} Y(b, -z)a$.

(3) $Y(a_{(n)}b, z) = \mathrm{Res}_x\{(x - z)^n Y(a, x)Y(b, z) - (-z + x)^n Y(b, z)Y(a, x)\}$.

By (2) of the proposition above, we see that there is no distinguishment between right ideals and left ideals of V. A vertex operator algebra is called *simple* if it has no non-trivial ideals. By (3) of the proposition, $Y(a_{(n)}b, z)$ is uniquely determined by $Y(a, z)$ and

$Y(b,z)$. Actually, based on the locality in (3) of Definition 2.1 and (3) of Proposition 2.2, one can reformulate the concept of a vertex operator superalgebra as follows.

Let $M = M^0 \oplus M^1$ be a $\mathbb{Z}_2$-graded vector space. A series $a(z) \in \mathrm{End}(M)[\![z,z^{-1}]\!]$ is called a *field* on M if $a(z)v \in M(\!(z)\!)$ holds for all $v \in M$. We denote by $\mathcal{F}(M)$ the space of fields on M. Set $\mathrm{End}(M)^0 := \mathrm{End}(M^0) \oplus \mathrm{End}(M^1)$ and $\mathrm{End}(M)^1 := \mathrm{Hom}(M^0,M^1) \oplus \mathrm{Hom}(M^1,M^0)$. Then $\mathrm{End}(M) = \mathrm{End}(M)^0 \oplus \mathrm{End}(M)^1$ also attains a $\mathbb{Z}_2$-grading. A field $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1} \in \mathcal{F}(M)$ is said to $\mathbb{Z}_2$-homogeneous if all $a_{(n)} \in \mathrm{End}(M)$, $n \in \mathbb{Z}$, are $\mathbb{Z}_2$-homogeneous and have the same $\mathbb{Z}_2$-parity. We also set $\mathcal{F}(M)^i$ to be the subspace of $\mathbb{Z}_2$-homogeneous fields with $\mathbb{Z}_2$-parity $i \in \{0,1\}$. Let $a(z) \in \mathcal{F}(M)^i$ and $b(z) \in \mathcal{F}(M)^j$ be $\mathbb{Z}_2$-homogeneous fields. We define the *n-th normal product* $a(z)_{(n)}b(z)$ by

$$a(z)_{(n)}b(z) := \mathrm{Res}_x\{(x-z)^n a(x)b(z) - (-1)^{ij}(-z+x)b(z)a(x)\} \tag{2.2}$$

and extend to $\mathcal{F}(M)^0 \oplus \mathcal{F}(M)^1$ linearly for both $a(z)$ and $b(z)$. Then one can verify that $a(z)_{(n)}b(z)$ is again a field on M. We say $a(z)$ and $b(z)$ are *local* if there exists $N \in \mathbb{Z}$ such that

$$(z_1-z_2)^N a(z_1)b(z_2) = (-1)^{ij}(-z_2+z_1)^N b(z_2)a(z_1). \tag{2.3}$$

Fields which are not necessarily $\mathbb{Z}_2$-homogeneous are said to local if they are sums of mutually local $\mathbb{Z}_2$-homogeneous fields.

Proposition 2.3. *(Dong's Lemma [K, Li]) If $a(z)$, $b(z)$, $c(z)$ are mutually local fields on M, then $a(z)_{(n)}b(z)$ and $c(z)$ are also local for all $n \in \mathbb{Z}$.*

By the proposition above, mutually local fields generate a subspace closed under the normal products. This idea leads to a notion of vertex superalgebras.

Definition 2.4. A *vertex superalgebra* is a $\mathbb{Z}_2$-graded subspace $\mathfrak{A} = \mathfrak{A}^0 \oplus \mathfrak{A}^1$ of $\mathcal{F}(M)^0 \oplus \mathcal{F}(M)^1$ satisfying:

(1) any two fields in $\mathfrak{A}$ are local.

(2) $\mathfrak{A}$ is closed under the normal products, i.e., $\mathfrak{A}_{(n)}\mathfrak{A} \subset \mathfrak{A}$ for any $n \in \mathbb{Z}$.

(3) $\mathfrak{A}$ contains the identity field $\mathbb{1}_{\mathfrak{A}}(z) := \mathrm{id}_M$.

The even part $\mathfrak{A}^0$ forms a vertex subalgebra and $\mathfrak{A}$ is called a *vertex algebra* if $\mathfrak{A}^1 = 0$.

Remark 2.5. It follows that $a(z)_{(-1)}\mathbb{1}_{\mathfrak{A}}(z) = a(z)$ and $a(z)_{(-2)}\mathbb{1}_{\mathfrak{A}}(z) = \partial_z a(z)$ for $a(z) \in \mathfrak{A}$.

Now, let V be a vertex operator superalgebra in the sense of Definition 2.1. By Proposition 2.2, the space $\mathfrak{A}(V) = \{Y(a,z) \mid a \in V\} \subset \mathcal{F}(V)$ forms a vertex superalgebra in the sense of Definition 2.4. Conversely, given a vertex superalgebra $\mathfrak{A}$, we define $Y_{\mathfrak{A}}(a(z),x)b(z) := \sum_{n\in\mathbb{Z}} a(z)_{(n)}b(z)x^{-n-1}$ for $a(z), b(z) \in \mathfrak{A}$ and we obtain a vertex operator superalgebra $(\mathfrak{A}, Y_{\mathfrak{A}}(\cdot,x), \mathbb{1}_{\mathfrak{A}}(z))$ without conformal vector. Therefore, a vertex operator superalgebra can be defined as a vertex superalgebra with the Virasoro algebra symmetry.

The notion of modules over a vertex operator superalgebra can be naturally defined in terms of the formulation of fields.

Definition 2.6. A *module*[1] over a vertex operator superalgebra V is a $\mathbb{Z}_2$-graded vector space $M = M^0 \oplus M^1$ equipped with a vertex algebra homomorphism $Y_M(\cdot, z) : V \to \mathcal{F}(M)$ such that $Y_M(V^{(i)}, z) \subset \mathcal{F}(M)^i$, $Y_M(\mathbb{1}, z) = \mathbb{1}_M(z)$ and $Y_M(a_{(n)}b, z) = Y_M(a,z)_{(n)}Y_M(b,z)$ holds for $a, b \in V$ and $n \in \mathbb{Z}$. Submodules, quotient modules, irreducibility are defined as usual.

Example: the Virasoro vertex algebra. Let $\mathrm{Vir} = \oplus_{n\in\mathbb{Z}}\mathbb{C}L(n) \oplus \mathbb{C}c$ be the Virasoro algebra with Lie bracket relation (2.1) and $[\mathrm{Vir}, c] = 0$. Let $M(c,h)$ be the Verma module over Vir with central charge c and highest weight h. On $M(c,h)$, we consider the field $\omega(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ generated by the action of Vir. It follows from (2.1) that we have the locality:

$$(z_1 - z_2)^4\omega(z_1)\omega(z_2) = (z_1 - z_2)^4\omega(z_2)\omega(z_1).$$

Therefore, the Virasoro field $\omega(z)$ together with the identity field generate a vertex algebra inside $\mathcal{F}(M(c,h))$. It is not hard to see that this vertex algebra is actually a vertex operator algebra.

Example: free fermionic vertex superalgebra. Let A be the algebra over $\mathbb{C}$ with generators $\{\psi_{n+1/2} \mid n \in \mathbb{Z}\}$ and relations $\psi_r\psi_s + \psi_s\psi_r = \delta_{r+s,0}$. Let A^+ be the subalgebra generated by $\{\psi_{n+1/2} \mid n \in \mathbb{N}\}$ and $\mathbb{C}\mathbb{1}$ a trivial A^+-module. We consider an induced module $M := A \otimes_{A^+} \mathbb{C}\mathbb{1}$. M has a natural basis $\{\psi_{-r_1}\cdots\psi_{-r_k}\mathbb{1} \mid r_1 > \cdots > r_k,\ k \in \mathbb{N}\}$ and we have a standard $\mathbb{Z}_2$-grading $M = M^0 \oplus M^1$ with $M^i = \mathrm{Span}_{\mathbb{C}}\{\psi_{-r_1}\cdots\psi_{-r_k}\mathbb{1} \mid k \equiv i \bmod 2\}$. Define a generating series $\psi(z) := \sum_{n\in\mathbb{Z}} \psi_{n+1/2}z^{-n-1}$. It is easy to see that $\psi(z)$ defines an odd field on M and local to itself:

$$(z_1 - z_2)\psi(z_1)\psi(z_2) = (z_2 - z_1)\psi(z_2)\psi(z_1).$$

Thus $\psi(z)$ together with the identity field $\mathrm{id}_M(z)$ generate a vertex superalgebra inside $\mathcal{F}(M)$. Now we define the vertex operator map $Y(\cdot, z) : M \to \mathrm{End}(M)[[z, z^{-1}]]$ by

[1]There exist several notions of modules; weak modules, admissible modules, and ordinary modules, etc [DLM2]. These notions are distinguished by assumptions on the actions of the Virasoro algebra generated by the vertex operator $Y_M(\omega, z)$ associated to the conformal vector ω of V.

$Y(\mathbb{1},z) := \mathrm{id}_M(z)$, $Y(\psi_{-1/2}\mathbb{1},z) := \psi(z)$ and inductively $Y(\psi_{-r}a,z) := \psi(z)_{(-r-1/2)}Y(a,z)$ for $a \in M$. Then by Dong's Lemma, $Y(a,z)$, $a \in M$, are mutually local fields on M. Let $\omega := (1/2)\cdot\psi_{-3/2}\psi_{-1/2}\mathbb{1}$. Then it follows that ω is the conformal vector of central charge $1/2$ and the structure $(M,Y(\cdot,z),\mathbb{1},\omega)$ forms a vertex operator superalgebra. In fact, M can be uniquely characterized as a simple vertex superalgebra with $Y(\mathbb{1},z) = \mathrm{id}_M$ and $Y(\psi_{-1/2}\mathbb{1},z) = \psi(z)$ (cf. [FFR, K]). It is shown in [KR] that $M^0 \simeq L(1/2,0)$ and $M^1 \simeq L(1/2,1/2)$ as modules over the Virasoro algebra, where $L(c,h)$ denotes the irreducible quotient of the Verma module over the Virasoro algebra with central charge c and highest weight h.

3 Automorphisms of a vertex operator algebra

We begin by giving a definition of automorphisms of a vertex operator algebra.

Definition 3.1. Let $(V,Y(\cdot,z),\mathbb{1},\omega)$ be a vertex operator algebra. An *automorphism* of V is a linear isomorphism f of V such that $f\mathbb{1} = \mathbb{1}$, $f\omega = \omega$ and $fY(a,z)b = Y(fa,z)fb$ holds for all $a,b \in V$. The group of automorphisms of V is denoted by $\mathrm{Aut}(V)$.

Let G be a finite automorphism group of a simple VOA V. The fixed point subspace V^G of V by G forms a subalgebra called *G-orbifold* of V. We denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G and by M_χ an irreducible G-module affording character $\chi \in \mathrm{Irr}(G)$. Then as a $\mathbb{C}[G]$-module one has the isotypical decomposition

$$V = \bigoplus_{\chi\in\mathrm{Irr}(G)} M_\chi \otimes \mathrm{Hom}_{\mathbb{C}[G]}(M_\chi, V). \tag{3.1}$$

Set $V_\chi := \mathrm{Hom}_{\mathbb{C}[G]}(M_\chi, V)$. Since the actions of V^G and $\mathbb{C}[G]$ commute, V^G naturally acts on V_χ so that each V_χ affords a structure of module over V^G. The following Schur-Weyl type duality has been shown in [DM1, DLM1].

Theorem 3.2. *Let V be a simple VOA and G a finite subgroup of* $\mathrm{Aut}(V)$.
(1) $V_\chi \neq 0$ for all $\chi \in \mathrm{Irr}(G)$.
(2) V_χ is irreducible over V^G. In particular, V^G is simple.
(3) $V_\chi \simeq V_\mu \iff \chi = \mu$.

Let us apply Theorem 3.2 to the simplest case that $G = \langle g\rangle$ is a finite cyclic group. Set $V^i := \{a \in V \mid ga = e^{2\pi\sqrt{-1}i/|g|}a\}$ for $0 \le i \le |g|-1$. Then $V^0 = V^G$ is a simple sub VOA of V and all V^i, $0 \le i \le |g|-1$, are inequivalent irreducible V^0-submodules of V. Therefore, the irreducible decomposition

$$V = V^0 \oplus V^1 \oplus \cdots \oplus V^{|g|-1}$$

as a V^0-module coincides with the isotypical decomposition of V as a G-module. From this observation we see that every finite automorphism of a simple vertex operator algebra is realized by an irreducible decomposition with respect to certain simple subalgebras. Now we have a question: "Which sub VOA realizes an automorphism of a given VOA?". One answer has been proposed by Miyamoto [M1] and we can use representations of the free fermionic SVOA.

Miyamoto involutions. Let V be a VOA and we consider the situation that V contains a simple Virasoro VOA $L(1/2, 0)$ as a sub VOA. Since $L(1/2, 0)$ is generated by its conformal vector, one can find a Virasoro vector e of V of central charge $1/2$ such that e generates $L(1/2, 0)$. For simplicity we call such a vector an *Ising vector* in this article. We shall often write $\mathrm{Vir}(e)$ to denote the Virasoro sub VOA generated by an Ising vector e.

It is shown in [DMZ] that $L(1/2, 0)$ has three inequivalent irreducible modules $L(1/2, h)$, $h \in \{0, 1/2, 1/16\}$ and every module over $L(1/2, 0)$ is semisimple. Therefore, one can obtain the isotypical decomposition

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16) \tag{3.2}$$

of V as a $\mathrm{Vir}(e) \simeq L(1/2, 0)$-module, where $V_e(h)$ denotes the sum of all irreducible $\mathrm{Vir}(e)$-submodules of V isomorphic to $L(1/2, h)$. The component $V_e(0)$ is known to form a subalgebra and Miyamoto found that one can use the decomposition (3.2) to define involutive automorphisms of vertex operator algebras.

Theorem 3.3. *([M1]) Let e be an Ising vector of V. The linear automorphism τ_e which acts by identity on $V_e(0) \oplus V_e(1/2)$ and by -1 on $V_e(1/16)$ defines an automorphism of V. On the fixed point subalgebra $V^{\langle \tau_e \rangle} = V_e(0) \oplus V_e(1/2)$, the linear automorphism σ_e which acts by identity on $V_e(0)$ and by -1 on $V_e(1/2)$ defines an automorphism of $V^{\langle \tau_e \rangle}$.*

By this theorem, one can always associate involutive automorphisms to an Ising vector. The automorphisms $\tau_e \in \mathrm{Aut}(V)$ and $\sigma_e \in \mathrm{Aut}(V^{\langle \tau_e \rangle})$ are called *Miyamoto involutions*. An Ising vector $e \in V$ is called *of σ-type* on V if $\tau_e = 1$ on V.

4 Framed vertex operator algebras

We will introduce a class of vertex operator algebras which are controlled by Ising vectors.

Definition 4.1. A simple vertex operator algebra $(V, Y(\cdot, z), \mathbb{1}, \omega)$ is called *framed* if there exists a set $\{e^i \mid 1 \le i \le n\}$ of Ising vectors of V such that $\omega = e^1 + \cdots + e^n$ and $[Y(e^i, z_1), Y(e^j, z_2)] = 0$ for distinct i and j.

In the definition above, we also call $\{e^i \mid 1 \le i \le n\}$ an *Ising frame* or *Virasoro frame*. An Ising frame of a framed VOA is not unique in general.

Examples. Let L be a positive definite even lattice having a *4-frame*, that is, L contains a set $\{\pm x^i \mid 1 \le i \le \mathrm{rank}(L)\}$ of vectors such that $\langle x^i, x^j\rangle = 4\delta_{i,j}$. Then the vertex operator algebra V_L associated to L is a framed VOA (cf. [FLM, DMZ]). In particular, lattice VOAs associated to the E_8-lattice, the Niemeier lattices and the Leech lattice are framed. The most interesting example would be the moonshine VOA $V^\natural$ which is acted by the Monster simple group [FLM]. It is also shown in [DMZ] that $V^\natural$ is a framed VOA.

As mentioned above, there are many examples of framed VOAs having larger symmetry. In order to present a structure theory of a framed VOA, we recall a general theory of vertex operator algebras.

Lemma 4.2. *([FHL]) Let $(V^i, Y^i(\cdot,z), \mathbb{1}^i, \omega^i)$, $i = 1,2$, be vertex operator algebras. Then the tensor product $(V^1\otimes V^2, Y^1(\cdot,z)\otimes Y^2(\cdot,z), \mathbb{1}^1\otimes\mathbb{1}^2, \omega^1\otimes\mathbb{1}^2+\mathbb{1}^1\otimes\omega^2)$ over $\mathbb{C}$ forms a vertex operator algebra[2]. $V^1\otimes V^2$ is simple if and only if both V^1 and V^2 are simple.*

Proposition 4.3. *([FHL]) Let V^1 and V^2 be vertex operator algebras.*
(1) Let $(M^i, Y_{M^i}(\cdot,z))$ be V^i-modules for $i=1,2$. Then $(M^1\otimes M^2, Y^1(\cdot,z)\otimes Y^2(\cdot,z))$ is a $V^1\otimes V^2$-module.
(2) An irreducible $V^1\otimes V^2$-module is isomorphic to a tensor product of irreducible V^i-modules.
(3) If every V^i-module is semisimple, then every $V^1\otimes V^2$-module is also semisimple.

Let V be a framed VOA with Ising frame $\omega = e^1+\cdots+e^n$. By definition, the sub VOA $\mathrm{Vir}(e^i)$ generated by e^i is isomorphic to the simple Virasoro VOA $L(1/2,0)$. Subalgebras $\mathrm{Vir}(e^i)$, $1\le i\le n$, are mutually commutative so that the sub VOA F generated by e^i, $1\le i\le n$, is isomorphic to a tensor product $L(1/2,0)^{\otimes n}$. By abuse of notation, we also refer the subalgebra $F = \mathrm{Vir}(e^1)\otimes\cdots\otimes\mathrm{Vir}(e^n)$ generated by $\{e^i \mid 1\le i\le n\}$ to as an *Ising frame* of V. By Lemma 4.2, F is simple, and by Proposition 4.3, an irreducible F-module is isomorphic to a tensor product

$$L(1/2,h_1)\otimes\cdots\otimes L(1/2,h_n),\quad h_i\in\{0,1/2,1/16\}, \tag{4.1}$$

of irreducible $L(1/2,0)$-modules. Since V is semisimple as an F-module, V is a finite direct sum of irreducible F-submodules:

$$V = \bigoplus_{h_i\in\{0,1/2,1/16\}} m_{h_1,\ldots,h_n} L(1/2,h_1)\otimes\cdots\otimes L(1/2,h_n), \tag{4.2}$$

[2]The central charge of $V^1\otimes V^2$ is a sum of those of V^i, $i=1,2$.

where $m_{h_1,\dots,h_n} \in \mathbb{N}$ denotes the multiplicity. We simplify the decomposition (4.2) with the help of Miyamoto involutions. By Theorem 3.3, each e^i defines the τ-involution τ_{e^i} on V which preserves the decomposition (4.2). To obtain the isotypical decomposition shown in (3.1), we define the associated binary codeword, called the *1/16-word*, $\tau(X)$ of an irreducible F-module X by

$$\tau(L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)) := (16h_1, \dots, 16h_n) \in \mathbb{Z}_2^n. \tag{4.3}$$

For each $\alpha \in \mathbb{Z}_2^n$ we denote by V^α the sum of irreducible F-submodules of V having the 1/16-word α and set $D := \{\alpha \in \mathbb{Z}_2^n \mid V^\alpha \neq 0\}$. Then the decomposition (4.2) becomes as follows.

$$V = \bigoplus_{\alpha \in D} V^\alpha. \tag{4.4}$$

We call the decomposition above the *1/16-word decomposition.* Define $\tau : \mathbb{Z}_2^n \to \mathrm{Aut}(V)$ by

$$\tau : \mathbb{Z}_2^n \ni \beta = (\beta_1, \dots, \beta_n) \longmapsto \tau_\beta := \prod_{i \in \mathrm{supp}(\beta)} \tau_{e^i} = \tau_{e^1}^{\beta_1} \cdots \tau_{e^n}^{\beta_n} \in \mathrm{Aut}(V). \tag{4.5}$$

It follows from definition that τ_β acts on V^α by a scalar $(-1)^{\langle \alpha, \beta \rangle}$ so that $\ker \tau = D^\perp := \{\beta \in \mathbb{Z}_2^n \mid \langle \beta, D \rangle = 0\}$. The decomposition (4.4) coincides with that given in Theorem 3.2 for $G = \tau(\mathbb{Z}_2^n)$ and hence $V^G = V^0$ is a simple sub VOA and each V^α, $\alpha \in D$, forms an irreducible V^0-submodule. We also find that for $x^\alpha \in V^\alpha$ and $x^\beta \in V^\beta$, one has $Y(x^\alpha, x)x^\beta \in V^{\alpha+\beta}((z))$ so that (4.4) gives a D-graded structure on V. The decomposition (4.4) which describes V as an extension of V^0 is much more convenient to deal with than the decomposition (4.2) which describes V as an extension of F.

Let us deal with the structure of the subalgebra V^0. By definition, V^0 has the following F-module shape:

$$V^0 = \bigoplus_{h_i \in \{0, 1/2\}} m_{h_1,\dots,h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n). \tag{4.6}$$

By Theorem 3.3, one can define σ-involutions

$$\sigma : \mathbb{Z}_2^n \ni \delta = (\delta_1, \dots, \delta_n) \longmapsto \sigma_\delta := \prod_{i \in \mathrm{supp}(\delta)} \sigma_{e^i} = \sigma_{e^1}^{\delta_1} \cdots \sigma_{e^n}^{\delta_n} \in \mathrm{Aut}(V^0). \tag{4.7}$$

Set $H = \sigma(\mathbb{Z}_2^n)$. Then the fixed point subalgebra $(V^0)^H$ is simple by Theorem 3.2, and from this we have $(V^0)^H = F$. Moreover, each isotypical component in (4.6) forms an irreducible $(V^0)^H = F$-submodule so that we have $m_{h_1,\dots,h_n} \in \{0, 1\}$ if all $h_i \in \{0, 1/2\}$. Setting

$$C := \{\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_2^n \mid m_{\beta_1/2,\dots,\beta_n/2} \neq 0\}, \tag{4.8}$$

the decomposition (4.6) can be expressed by

$$V^0 = \bigoplus_{\beta=(\beta_1,\dots,\beta_n)\in C} L(1/2,\beta_1/2)\otimes\cdots\otimes L(1/2,\beta_n/2). \tag{4.9}$$

It is easy to see that the decomposition above gives a C-grading on V^0 as an extension of F. Summarizing, given a framed VOA V with an Ising frame $F=\mathrm{Vir}(e^1)\otimes\cdots\otimes\mathrm{Vir}(e^n)$, one can associate a pair of binary codes (C,D) such that there exists extensions $F\subset V^0\subset V$ which are graded by C and D:

$$V=\bigoplus_{\alpha\in D} V^\alpha,\quad V^0 = \bigoplus_{\beta=(\beta_1,\dots,\beta_n)\in C} L(1/2,\beta_1/2)\otimes\cdots\otimes L(1/2,\beta_n/2). \tag{4.10}$$

We shall call (C,D) the *structure codes* of V with respect to an Ising frame F. We recall the basic facts on the structure codes.

Proposition 4.4. *([DGH, M2]) Let (C,D) be a structure code of a framed VOA.*
(1) C and D are even linear binary codes.
(2) For any $\alpha\in D$, its weight $\mathrm{wt}(\alpha)$ is divisible by 8.
(3) C and D are mutually orthogonal.

We have seen that every framed VOA contains a subalgebra of the shape (4.9) which is graded by a linear code C. It is known that the code C completely determines the VOA structures on (4.9).

Theorem 4.5. *([M2]) Given an even linear binary code $C\subset\mathbb{Z}_2^n$, there exists a unique simple (framed) vertex operator algebra structure on the $L(1/2,0)^{\otimes n}$-module*

$$V_C = \bigoplus_{\beta=(\beta_1,\dots,\beta_n)\in C} L(1/2,\beta_1/2)\otimes\cdots\otimes L(1/2,\beta_n/2).$$

By the theorem above, the simple VOA V_C is uniquely determined by C so that we refer V_C to as the *code vertex operator algebra* associated to C. The representation theory of V_C is well-developed in [M2] and it is controlled by finite dimensional semisimple algebras.

Let $C\subset\mathbb{Z}_2^n$ be an even linear code. Denote $\nu^1=(10\dots0),\nu^2=(01\dots0),\dots,\nu^n=(00\dots1)$. We define $\epsilon:\mathbb{Z}_2^n\to\{\pm1\}\subset\mathbb{C}^*$ by $\epsilon(\nu^i,\nu^j)=-1$ if $i>j$ and $\epsilon(\nu^i,\nu^j)=1$ otherwise, and we extend to $\mathbb{Z}_2^n$ linearly. Then ϵ defines a 2-cocycle in $Z^2(\mathbb{Z}_2^n,\mathbb{C}^*)$ such that $\epsilon(\alpha,\beta)=(-1)^{\langle\alpha,\beta\rangle+\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}\epsilon(\beta,\alpha)$. Let $\mathbb{C}^\epsilon[C]$ be the twisted group algebra of C associated to ϵ, that is, $\mathbb{C}^\epsilon[C]=\oplus_{\alpha\in C}e^\alpha$ as a linear space and carries the bilinear product $e^\alpha e^\beta=\epsilon(\alpha,\beta)e^{\alpha+\beta}$. It is well-known that $\mathbb{C}^\epsilon[C]$ forms a semisimple associative algebra. It is shown in [M2] that every V_C-module affords an action of a subalgebra of $\mathbb{C}^\epsilon[C]$.

Let M be a V_C-module and take an irreducible $F = L(1/2, 0)^{\otimes n}$-submodule X, which is possible since every $L(1/2, 0)^{\otimes n}$-module is semisimple by (3) of Proposition 4.3. Then it is shown in [M2] that $\mathrm{Hom}_F(X, M)$ affords an action of the twisted group algebra $\mathbb{C}^{\epsilon}[C_{\tau(X)}]$ associated to the subcode $C_{\tau(W)} := \{\beta \in C \mid \mathrm{supp}(\beta) \subset \mathrm{supp}(\tau(X))\}$ of C, where $\tau(X) \in \mathbb{Z}_2^n$ is defined by (4.3). Since $\mathbb{C}^{\epsilon}[C_{\tau(X)}]$ is semisimple, one can decompose the isotypical component $X \otimes \mathrm{Hom}_F(X, M) \subset M$ into irreducible $\mathbb{C}^{\epsilon}[C_{\tau(X)}]$-submodules, and this irreducible decomposition induces an irreducible decomposition of the V_C-submodule $V_C \cdot X$ generated by X as a V_C-module. As a result, one sees that every V_C-module is semisimple and irreducible V_C-modules are parameterized by those of subalgebras of $\mathbb{C}^{\epsilon}[C]$.

Theorem 4.6. *([M2]) Let $C \subset \mathbb{Z}_2^n$ be an even linear binary code and V_C the associated code VOA.*
(1) Every V_C-module is semisimple.
(2) Given an irreducible $L(1/2, 0)^{\otimes n}$-module W, there exists an (untwisted or twisted) irreducible V_C-module containing W as a $L(1/2, 0)^{\otimes n}$-submodule.

That V_C-modules are classified by finite dimensional semisimple algebras is due to the fact that V_C forms a *simple current extension* of a rational[3] VOA $L(1/2, 0)^{\otimes n}$. We just refer the definition of simple current extension and related results to [Y2]. The main point is that simple current extensions form a quite nice class among other extensions and one can deal with the details for this class. For example, it is generally shown in [La, Y1] that the representation theory of a simple current extension of a rational VOA is controlled by certain finite dimensional semisimple associative algebras.

5 Duality on structure codes

As we have seen in the previous section, every framed VOA is an extension of a code sub VOA by its irreducible modules. We can use the representation theory of code VOAs given in Theorem 4.6 to study the structure of a framed VOA, and the following strong duality on the structure codes is obtained in [LaY1].

Theorem 5.1. *([LaY1]) Let (C, D) be the structure code of a framed VOA. Then for any $\alpha \in D$, the subcode $\{\beta \in C \mid \mathrm{supp}(\beta) \subset \mathrm{supp}(\alpha)\}$ of C contains a doubly even self dual subcode with respect to the support of α.*

The duality above provides quite strong restrictions on the structure codes. As an immediate consequence, we have inclusions $D \subset C \subset D^{\perp}$. Using the duality shown

[3]A VOA V is call *rational* if every $\mathbb{N}$-gradable V-module is semisimple (cf. [DLM2]).

above, we can also prove that a framed VOA is always a simple current extension of its code sub VOA.

Theorem 5.2. *([LaY1]) Let V be a framed VOA with an Ising frame F and (C, D) the associated structure codes. Then in the series of extensions $F \subset V_C \subset V$, V is a D-graded simple current extension of V_C.*

A general theory on simple current extensions is well-developed (cf. [DM2, La, Y1]), and we have the following results as corollaries.

Corollary 5.3. *([LaY1]) The number of isomorphism classes of framed VOAs with the fixed central charge is finite.*

Corollary 5.4. *Given $C, D \subset \mathbb{Z}_2^n$, the number of framed VOAs having (C, D) as a structure code is bounded by the order of* $\mathrm{Hom}(D, \mathbb{Z}_2^n/C)$.

By Theorem 5.1, 5.2 and its corollaries, it is possible to think about the classification of framed VOAs with small central charges. The most important objects in the VOA theory would be holomorphic VOAs. A *holomorphic* VOA is a simple rational VOA V such that V itself is the unique irreducible V-module. It is shown in [Z] that the character of a holomorphic VOA is a modular function[4] and it follows that the central charge of a holomorphic VOA is divisible by 8. For a framed VOA, one can easily determine whether it is holomorphic or not by considering its structure codes.

Theorem 5.5. *([DGH, M3]) A framed VOA with structure codes (C, D) is holomorphic if and only if $C = D^{\perp}$.*

By Proposition 4.4, Theorem 5.1 and 5.5, structure codes (C, D) of a holomorphic framed VOA necessarily satisfy the following conditions:

(a) $C, D \subset \mathbb{Z}_2^n$ with n divisible by 16.
(b) $C = D^{\perp}$.
(c) C is even linear.
(d) $8|\mathrm{wt}(\alpha)$ for any $\alpha \in D$.
(e) For any $\alpha \in D$, C_α contains a doubly even self dual subcode with support α.

It is shown in [LaY1] that the conditions above are also sufficient.

Theorem 5.6. *([LaY1]) There exists a holomorphic framed VOA with structure codes (C, D) if and only if the pair (C, D) satisfies the conditions (a)–(e) above.*

The conditions (a)–(e) seems a little bit complicated, but one can replace them by the following simple ones.

[4]To be precise, one needs to assume one more condition, the C_2-*finiteness* [Z, DLM3].

Lemma 5.7. *There exists a holomorphic framed VOA with structure codes* $(D^{\perp}, D)$ *if and only if* $D \subset \mathbb{Z}_2^{16k}$ *satisfies (d) and* $(11\ldots1) \in D$.

It would be an interesting problem to classify the code D satisfying the conditions in Lemma 5.7 and then try to classify holomorphic framed VOAs with small central charge. Holomorphic VOAs with central charge 8 and 16 are classified in [DM3] and it turns out that all these holomorphic VOAs are framed. Namely, the holomorphic VOA of central charge $c = 8$ is isomorphic to the lattice VOA associated to the E_8-lattice, which is the unique even unimodular positive definite lattice of rank 8, and in the $c = 16$ case there exist two holomorphic VOAs, the lattice VOAs associated to the $E_8 \oplus E_8$-lattice and D_{16}^{+}-lattice, where these two lattices exhaust all the even unimodular positive definite lattices of rank 16 up to isometry. These three lattices possess 4-frames so that associated lattice VOAs are framed.

The $c = 24$ case is open and it is conjectured in [S] that there would be 71 inequivalent holomorphic VOAs including the moonshine VOA. This problem involves a famous conjecture of Frenkel-Lepowsky-Meurman [FLM] on the uniqueness of the moonshine VOA $V^{\natural}$. That is, if $V = \oplus_{n\in\mathbb{N}} V_n$ is a holomorphic VOA of central charge 24 with trivial weight one subspace $V_1 = 0$, then V is expected to be isomorphic to $V^{\natural}$ constructed in [FLM] (see also [DGL]). If the weight one subspace V_1 of a holomorphic VOA V is non-trivial, then there exists a (finite-dimensional) Lie algebra structure on V_1, and it is discussed in [DM2, DM3] that this Lie algebra is reductive and obeys many restrictions. Schellekens [S] also used the Lie algebra structure on V_1 together with other assumptions to obtain his list of 70 holomorphic VOAs with non-trivial Lie algebras. The arguments in [DM3, S] are based on the non-trivial Lie algebra structure on V_1 so that their method is not applicable to the FLM conjecture. However, Dong-Mason [DM3] showed a characterization of the lattice VOA associated to the Leech lattice that if the Lie algebra structure on the weight one subspace V_1 of a holomorphic VOA V is abelian, then V_1 is 24-dimensional and V is isomorphic to the Leech lattice VOA. Based on this fact, we can solve a weaker version of the FLM conjecture, that is, if we add one extra condition that V is also *framed*, then V is isomorphic to the moonshine VOA.

Theorem 5.8. *([LaY2]) Let* V *be a holomorphic framed VOA of central charge 24 and suppose that the weight one subspace* V_1 *of* V *is trivial. Then* V *is isomorphic to the moonshine VOA* $V^{\natural}$ *constructed in [FLM].*

We shall give a brief sketch of the proof in appendix.

6 Point-wise frame stabilizer

Let V be a simple VOA and G a finite subgroup of $\mathrm{Aut}(V)$. As shown in Theorem 3.2, the V^G-module structure on V is closely related to the G-module structure and one can classify irreducible V^G-submodules involved in V with the support of G. It is generally difficult to classify all the irreducible V^G-modules not involved in V. However, if V is framed and also so is the fixed point subalgebra V^G, it becomes much easier to complete the representation theory of V^G. In this section we discuss the structure of the subgroups G of $\mathrm{Aut}(V)$ for a framed VOA V such that V^G is again framed.

Let V be a framed VOA and $F = \mathrm{Vir}(e^1) \otimes \cdots \otimes \mathrm{Vir}(e^n)$ an Ising frame. Then it is easily verified that for a subgroup G of $\mathrm{Aut}(V)$, the fixed point subalgebra V^G is framed if and only if any $g \in G$ fixes F point-wisely. So we introduce the following subgroup:

$$\mathrm{Stab}_V^{\mathrm{pt}}(F) := \{g \in \mathrm{Aut}(V) \mid gF = F \text{ and } g|_F = \mathrm{id}_F\}. \tag{6.1}$$

It is easy to see that the subgroup $\tau(\mathbb{Z}_2^n)$ generated by τ-involutions τ_{e^i}, $1 \le i \le n$, is a normal subgroup of $\mathrm{Stab}_V^{\mathrm{pt}}(F)$. Let (C, D) be the structure codes of V with respect to F. Then $V = \oplus_{\alpha\in D} V^\alpha$ and $V^0 \simeq V_C$. Since V and V^0 are extensions of $V^0 = V^{\tau(\mathbb{Z}_2^n)}$ and $F = (V^0)^{\sigma(\mathbb{Z}_2^n)}$ by their irreducible modules, respectively, one have the following by Schur's lemma.

Lemma 6.1. *([LaY1]) Let $V = \oplus_{\alpha\in D} V^\alpha$ be the 1/16-word decomposition of a framed VOA V.*
(1) Let $g \in \mathrm{Aut}(V)$ such that $g|_{V^0} = \mathrm{id}_{V^0}$. Then $g \in \langle \tau_{e^1}, \dots, \tau_{e^n}\rangle$.
(2) Let $h \in \mathrm{Aut}(V^0)$ such that $h|_F = \mathrm{id}_F$. Then $h \in \langle \sigma_{e^1}, \dots, \sigma_{e^n}\rangle$.

Let $\theta \in \mathrm{Stab}_V^{\mathrm{pt}}(F)$. By definition, θ keeps the isotypical decomposition (4.2) invariant, and so one sees that θ acts on V^0. Since V^0 is a C-graded extension of F, θ acts by each irreducible F-submodules of V^0 by a scalar, which is ± 1 by the C-grading. Therefore, $\theta|_{V^0}$ is realized by σ-involutions and this also implies that θ^2 acts by identity on V^0. Summarizing, we have:

Proposition 6.2. *([LaY1]) Let $\theta \in \mathrm{Stab}_V^{\mathrm{pt}}(F)$. Then there exist $\eta, \xi \in \mathbb{Z}_2^n$ such that $\theta^2 = \tau_\eta$ and $\theta|_{V^0} = \sigma_\xi$. In particular, $\theta^4 = 1$.*

If ξ above is in $C^\perp$, then θ is actually a product of τ-involutions. Since we know $\tau(\mathbb{Z}_2^n) \subset \mathrm{Stab}_V^{\mathrm{pt}}(F)$ explicitly, we are interested in the case $\theta|_{V^0}$ is non-trivial, that is, we want to know for which $\xi \notin C^\perp$ we can extend $\sigma_\xi \in \mathrm{Aut}(V^0)$ to $\mathrm{Aut}(V)$. The answer is given by the following theorem.

Theorem 6.3. *([LaY1]) For a codeword $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Z}_2^n \setminus C^\perp$, there exists $\theta \in \mathrm{Stab}_V^{\mathrm{pt}}(F)$ such that $\theta|_{V^0} = \sigma_\xi$ if and only if $\alpha \cdot \xi \in C$ for all $\alpha = (\alpha_1, \dots, \alpha_n) \in D$, where $\alpha \cdot \xi = (\alpha_1\xi_1, \dots, \alpha_n\xi_n) \in \mathbb{Z}_2^n$ denotes the bilinear product in the ring $\mathbb{Z}_2^n$. Moreover, $|\theta| = 2$ if and only if $\mathrm{wt}(\alpha \cdot \xi) \equiv 0 \mod 4$ for all $\alpha \in D$, and otherwise $|\theta| = 4$.*

By the theorem above, we define

$$P := \{\xi \in \mathbb{Z}_2^n \mid \xi \cdot D \subset C\}. \tag{6.2}$$

Note that P forms a linear subcode of $\mathbb{Z}_2^n$.

Lemma 6.4. *(1) $C^\perp \subset P$.*
(2) Let $\theta_1, \theta_2 \in \mathrm{Stab}_V^{\mathrm{pt}}(F)$ such that $\theta_1|_{V^0} = \theta_2|_{V^0}$. Then $\theta_1 = \theta_2 \tau_\eta$ for some $\eta \in \mathbb{Z}_2^n$.

By the lemma above, the point-wise frame stabilizer satisfies the following exact sequence:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tau(\mathbb{Z}_2^n) & \longrightarrow & \mathrm{Stab}_V^{\mathrm{pt}}(F) & \longrightarrow & \mathrm{Stab}_V^{\mathrm{pt}}(F)/\tau(\mathbb{Z}_2^n) & \longrightarrow & 1 \\
 & & \downarrow\wr & & \| & & \downarrow\wr & & \\
1 & \longrightarrow & \mathbb{Z}_2^n/D^\perp & \longrightarrow & \mathrm{Stab}_V^{\mathrm{pt}}(F) & \longrightarrow & P/C^\perp & \longrightarrow & 1
\end{array} \tag{6.3}$$

To determine the extension above, one can use the following result.

Proposition 6.5. *Let $\xi^1, \xi^2 \in P$ and θ_{ξ^i}, $i = 1, 2$, be extensions of σ_{ξ^i} to $\mathrm{Stab}_V(F)$. Then $[\theta_{\xi^1}, \theta_{\xi^2}] = 1$ if and only if $\langle \alpha \cdot \xi^1, \alpha \cdot \xi^2 \rangle = 0$ for all $\alpha \in D$.*

By these results, the point-wise frame stabilizer $\mathrm{Stab}_V^{\mathrm{pt}}(F)$ can be described in terms of the structure codes (C, D) which is computable if one is given a framed VOA and its frame F.

Example: 4A-element of the Monster. Let $V^\natural$ be the moonshine VOA [FLM]. It is shown in [DMZ] that $V^\natural$ is a framed VOA and a pair of structure codes for certain Ising frame is computed in [DGH, M3]. Let $F = \mathrm{Vir}(e^1) \otimes \cdots \otimes \mathrm{Vir}(e^{48})$ be the Ising frame discussed in [DMZ, DGH, M3]. Then the structure codes $(\mathcal{C}, \mathcal{D})$ for this frame is presented as

$$\mathcal{C} = \mathcal{D}^\perp, \quad \mathcal{D} = \mathrm{Span}_{\mathbb{Z}_2}\{(1^{16}0^{32}), (0^{32}1^{16}), (\alpha, \alpha, \alpha) \mid \alpha \in \mathrm{RM}(1,4)\},$$

where $\mathrm{RM}(1,4) \subset \mathbb{Z}_2^{16}$ is the first order Reed-Muller code defined by the generator matrix

$$\begin{bmatrix}
1111 & 1111 & 1111 & 1111 \\
1111 & 1111 & 0000 & 0000 \\
1111 & 0000 & 1111 & 0000 \\
1100 & 1100 & 1100 & 1100 \\
1010 & 1010 & 1010 & 1010
\end{bmatrix}.$$

Note that $\dim_{\mathbb{Z}_2} \mathcal{D} = 7$ and $\dim_{\mathbb{Z}_2} \mathcal{C} = 41$. The code $\mathcal{C}$ can be expressed as

$$\mathcal{C} = \{(\alpha, \beta, \gamma) \in \mathbb{Z}_2^{48} \mid \alpha, \beta, \gamma \in \mathbb{Z}_2^{16} \text{ are even and } \alpha + \beta + \gamma \in \mathrm{RM}(2,4)\},$$

where $\mathrm{RM}(2,4) := \mathrm{RM}(1,4)^{\perp} \subset \mathbb{Z}_2^{16}$ is the second order Reed-Muller code of length 16. Set $\mathcal{P} := \{\xi \in \mathbb{Z}_2^{48} \mid \xi \cdot \mathcal{D} \subset \mathcal{C}\}$.

Lemma 6.6. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{P}$ be defined as above. Then*

$$\mathcal{P} = \{(\alpha, \beta, \gamma) \in \mathbb{Z}_2^{48} \mid \alpha, \beta, \gamma \in \mathrm{RM}(2,4) \text{ and } \alpha + \beta + \gamma \in \mathrm{RM}(1,4)\}.$$

By the lemma, we see that $\dim_{\mathbb{Z}_2} \mathcal{P} = 27$. Therefore, the point-wise frame stabilizer $\mathrm{Stab}^{\mathrm{pt}}_{V^\natural}(F)$ has a shape $2^7.2^{20}$ (see [ATLAS] for the notation). Set

$$\xi = (1100\ 0000\ 1100\ 0000\ 0110\ 0000\ 0110\ 0000\ 1010\ 0000\ 1010\ 0000) \in \mathbb{Z}_2^{48}.$$

Then one can directly check that $\xi \in \mathcal{P}$ so that by Theorem 6.3 there exists $\theta_\xi \in \mathrm{Stab}^{\mathrm{pt}}_{V^\natural}(F)$ such that the restriction of θ_ξ on the code sub VOA of $V^\natural$ coincides with a non-trivial involution σ_ξ. Therefore, θ_ξ has an order 4, and it is shown in [LaY1] that θ_ξ actually belongs to the 4A conjugacy class of the Monster simple group $\mathbb{M} = \mathrm{Aut}(V^\natural)$ (cf. [ATLAS]).

7 Appendix: proof of Theorem 5.8

In this appendix we will present a sketch of proof of Theorem 5.8. We refer the reader to [LaY2] and references therein for details and undefined terms. Let V be a holomorphic framed VOA with central charge 24 and F an Ising frame of V. We suppose $V_1 = 0$. Let (C, D) be the structure codes and $V = \oplus_{\alpha\in D} V^\alpha$ the 1/16-word decomposition with $V^0 \simeq V_C$. Since V is holomorphic, $C = D^\perp$ by Theorem 5.5, and since V_1 is trivial, C contains no weight two codeword. Let $\delta = (110^{46}) \in \mathbb{Z}_2^{48}$. Then $\delta \notin C = D^\perp$ and hence $\tau_\delta \in \mathrm{Aut}(V)$ is a non-trivial involution. Set $\tilde{C} = C \sqcup (C + \delta)$ which forms an even linear code. The code VOA $V_{\tilde{C}}$ associated to $\tilde{C}$ contains V_C as a full sub VOA and there is an irreducible V_C-submodule $V_{C+\delta}$ of $V_{\tilde{C}}$ such that $V_{\tilde{C}} = V_C \oplus V_{C+\delta}$ as a V_C-module. Set $D^0 := \{\alpha \in D \mid \langle \alpha, \delta\rangle = 0\}$. Then $V^{\langle \tau_\delta\rangle} = \oplus_{\alpha\in D^0} V^\alpha$ is a simple sub VOA of V. It is shown in [LaY1, M3, Y1] that we can uniquely extend $V_{C+\delta}$ to an (untwisted) irreducible $V^{\langle\tau_\delta\rangle}$-module which we denote by $\mathrm{Ind}_{V_C}^{V^{\langle\tau_\delta\rangle}} V_{C+\delta}$. It is also shown in [LaY1, M3, Y2] that one can extend $V^{\langle\tau_\delta\rangle}$ to a holomorphic framed VOA

$$V(\tau_\delta) := V^{\langle\tau_\delta\rangle} \oplus \mathrm{Ind}_{V_C}^{V^{\langle\tau_\delta\rangle}} V_{C+\delta} \tag{7.1}$$

on which one can define an involution θ acting by identity on $V^{\langle\tau_\delta\rangle}$ and by -1 on $\mathrm{Ind}_{V_C}^{V^{\langle\tau_\delta\rangle}} V_{C+\delta}$. Since $V(\tau_\delta)$ involves $V_{C+\delta}$ as a V_C-submodule, the weight one subspace

$V(\tau_\delta)_1$ of $V(\tau_\delta)$ is non-trivial. Moreover, since the weight one subspace of $V^{\langle\tau_\delta\rangle}$ is trivial, the involution θ acts on the weight one subspace by -1. Consider the Lie algebra structure on $V(\tau_\delta)_1$. As we have explained, θ acts on $V(\tau_\delta)_1$ by -1 so that θ acts on the derived subalgebra $[V(\tau_\delta)_1, V(\tau_\delta)_1]$ by identity. Thus $[V(\tau_\delta)_1, V(\tau_\delta)_1] \subset V(\tau_\delta)^{\langle\theta\rangle} = V^{\langle\tau_\delta\rangle}$ and since the weight one subspace of $V^{\langle\tau_\delta\rangle}$ is trivial, we conclude that the Lie algebra structure on $V(\tau_\delta)_1$ is abelian. It is shown in [DM3] that if the Lie algebra structure on the weight one subspace of a holomorphic VOA with central charge 24 is abelian, then it is isomorphic to the lattice VOA V_Λ associated to the Leech lattice Λ. Therefore, $V(\tau_\delta) \simeq V_\Lambda$. Now consider the induced action of θ on V_Λ. We have already seen that θ acts on the weight one subspace of V_Λ by -1. It is well-known (cf. [DGH, LaY1]) that such an involution is unique up to conjugacy in $\mathrm{Aut}(V_\Lambda)$. Therefore, $V^{\langle\tau_\delta\rangle}$ is isomorphic to the fixed point subalgebra V_Λ^+ of V_Λ by the lift of -1 isometry on Λ (cf. [FLM]). Since the given holomorphic VOA V is a (simple current) extension of $V^{\langle\tau_\delta\rangle} \simeq V_\Lambda^+$ with trivial weight one subspace, V is isomorphic to the moonshine VOA $V^\natural$ constructed in [FLM] by the classification of the irreducible V_Λ^+-modules obtained in [D] and the uniqueness of simple current extensions established in [DM2]. ∎

References

[ATLAS] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, ATLAS of finite groups, Clarendon Press, Oxford, 1985.

[B] R.E. Borcherds, Vertex algebras, Kac-Moody algebras and the Monster, *Proc. Nat. Acad. Sci. USA* **83** (1986), 3068–3071.

[D] C. Dong, Representations of the moonshine module vertex operator algebra, Contemporary Math. **175**, Amer. Math. Soc., Providence, 1994, 27–36.

[DGH] C. Dong, R.L. Griess and G. Höhn, Framed vertex operator algebras, codes and the moonshine module, *Comm. Math. Phys.* **193** (1998), 407–448.

[DGL] C. Dong, R.L. Griess and C.H. Lam, On the uniqueness of the moonshine vertex operator algebra, *Amer. J. Math.* **139**(2), 583–609 (2007); `math.QA/0506321`.

[DLM1] C. Dong, H. Li and G. Mason, Compact automorphism groups of vertex operator algebras, *Internat. Math. Res. Notices* **18** (1996), 913–921.

[DLM2] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571–600.

[DLM3] C. Dong, H. Li and G. Mason, Modular-invariance of trace functions in orbifold theory and generalized moonshine, *Comm. Math. Phys.* **214** (2000), 1–56.

[DM1] C. Dong and G. Mason, On quantum Galois theory, *Duke Math. J.* **86** (1997), 305–321.

[DM2] C. Dong and G. Mason, Rational vertex operator algebras and the effective central charge, *Internat. Math. Res. Notices* **56** (2004), 2989–3008.

[DM3] C. Dong and G. Mason, Holomorphic vertex operator algebras of small central charge, *Pacific J. Math.* **213** (2004), 253–266.

[DMZ] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, Proc. Symp. Pure. Math., American Math. Soc. **56** II (1994), 295–316.

[FFR] A. J. Feingold, I. B. Frenkel and J. F.X. Ries, Spinor construction of vertex operator algebras, triality, and $E_8^{(1)}$, *Contemp. Math.* **121** (1991).

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs Amer. Math. Soc.* **104**, 1993.

[FLM] I.B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Academic Press, New York, 1988.

[KR] V. G. Kac and A. K. Raina, "Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie algebras," World Scientific, Singapore, 1987.

[K] V. G. Kac, Vertex algebras for beginners, second edition, Cambridge University Press, Cambridge, 1990.

[La] C.H. Lam, Induced modules for orbifold vertex operator algebras, *J. Math. Soc. Japan* **53** (2001), 541–557.

[LaY1] C.H. Lam and H. Yamauchi, On the structure of framed vertex operator algebras and their pointwise frame stabilizers, to appear in *Comm. Math. Phys.*; `math.QA/0605176`.

[LaY2] C.H. Lam and H. Yamauchi, A characterization of the moonshine vertex operator algebra by means of Virasoro frames, *Internat. Math. Res. Notices* 2007 Article ID:mm003, p. 9.; `math.QA/0609718`.

[Li] H. Li, Local systems of vertex operators, vertex superalgebras and modules, *J. Pure Appl. Algebra* **109** (1996), 143–195.

[MN] A. Matsuo and K. Nagatomo, Axiom for vertex operator algebra and the locality of quantum fields, MSJ-Memoirs **4**, Mathematical Society of Japan, 1999.

[M1] M. Miyamoto, Griess algebras and conformal vectors in vertex operator algebras, *J. Algebra* **179** (1996), 528–548.

[M2] M. Miyamoto, Representation theory of code vertex operator algebras, *J. Algebra* **201** (1998), 115-150.

[M3] M. Miyamoto, A new construction of the moonshine vertex operator algebra over the real number field, *Ann. of Math* **159** (2004), 535–596.

[S] A.N. Schellekens, Meromorphic $c = 24$ conformal field theories, *Comm. Math. Phys.*, **153** (1993),159–185.

[Y1] H. Yamauchi, Module categories of simple current extensions of vertex operator algebras, *J. Pure Appl. Algebra* **189** (2004), 315–328.

[Y2] H. Yamauchi, A theory of simple current extensions of vertex operator algebras and applications to the moonshine vertex operator algebra, Ph.D. thesis, University of Tsukuba, 2004.; available on the author's web site: `http://auemath.aichi-edu.ac.jp/~yamauchi/math/phd/myphd.pdf`

[Z] Y.Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–302.