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Combinatorial Aspects of $O(1)$ Loop Models

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Abstract

We review the recent developments in the study $O(1)$ loop models and related topics. Different research areas such as alternating sign matrices in combinatorics, the six-vertex model in statistical mechanics make contact through the observation by Razumov and Stroganov. Following Razumov-Stroganov (RS) conjectures, we discuss the work by Di Francesco and Zinn-Justin where one of the RS conjectures, the sum rule, was proven. As a related topic, we also discuss the loop model on a cylinder associated with the affine Hecke algebra of type A.

1 Overview

This paper is based on the talk given at the workshop "Combinatorics, Representation Theory and Related Topics," held at RIMS, Kyoto University in October 24-27, 2006.

We briefly review the recent developments in the study of the inhomogeneous $O(1)$ loop models and related topics [1, 2, 3]. Many research areas play important roles: alternating sign matrices (ASMs) in combinatorics [4, 5], the XXZ spin chain and the six-vertex model in statistical mechanics [6, 7, 8], the representation theory of the affine Temperley-Lieb (TL) algebra [9, 10] and that of the affine Hecke algebras [11].

Razumov and Stroganov proposed seven conjectures related to the eigenvector of the XXZ spin chain at the anisotropic parameter $\Delta = -1/2$ [1]. These conjectures were generalized to the $O(1)$ loop model in [12]. A typical conjecture is that the 1-sum of the ground state wavefunction of the $O(1)$ loop model with periodic boundary conditions and length $L = 2n$ is equal to the total number of $n \times n$ alternating sign matrices (Conjecture 8 in [12]). The ground state wavefunction Ψ has another remarkable property. An entry of Ψ is conjectured to be equal to the total number of fully packed loops [13]. This rule is the most popular form of the Razumov-Stroganov conjectures. Di Francesco and Zinn-Justin proved the above mentioned sum rule by introducing inhomogeneity and utilizing integrability of the six-vertex model [2]. Roughly speaking, the key is the quantum Knizhnik-Zamolodchikov (q-KZ) equation [14], which is equivalent to finding the eigenvector of the transfer matrix of the inhomogeneous $O(1)$ loop model with periodic boundary conditions. By solving the q-KZ equation at the Razumov-Stroganov point, *i.e.* $q = -\exp(\pi i/3)$, it was found that the sum of the entries of the solution is written in terms of the partition function of the six-vertex model with domain wall boundary conditions. In the homogeneous limit such that all the spectral parameters tend to 1, the sum is reduced to the total number of ASMs.

The solution of the q-KZ equation was regarded as a special polynomial representation of the affine Temperley-Lieb algebra [9]. This correspondence was also studied for the $O(1)$ loop model with cylindric boundary conditions in [10]. The special polynomial solutions of q-KZ equation for the higher-rank case of $U_q(\widehat{sl_k})$ was constructed in [11].

We also discuss the A_{k-1} generalized model on a cylinder associated with the affine Hecke algebra of type A [3]. This model is a new hybrid generalization of the $O(1)$ loop model: defined by the affine Hecke algebra instead of the affine TL algebra and with cylindric boundary conditions instead of periodic boundary conditions. The affine Hecke algebra is characterized by an extra vanishing conditions, which we call "cylindric relations."

The paper is organized as follows. In Section 2, we review basic facts about alternating sign matrices (ASMs) and the six-vertex model. These two subjects make contact through the Kuperberg's observation. The fully packed loops are also introduced. In Section 3, we move to the Razumov-Stroganov conjectures for the $O(1)$ loop model. The Temperley-Lieb algebra and the space of link patterns are introduced. The relation to the XXZ spin chain model is also stated. In Section 4, we briefly review the work by Di Francesco and Zinn-Justin. We focus on the proof of the sum rule of the RS conjectures. In Section 5, we discuss the A_{k-1} generalized model on a cylinder. Concluding remarks and some open problems are in Section 6.

2 Alternating Sign Matrix and Six-vertex Model

We briefly review how the integrability of the six-vertex model appears in the context of alternating sign matrices (ASMs) in combinatorics. For this purpose, let us recall some known facts about ASMs and six-vertex models. This section is based on [4, 7, 15] and [16].

Alternating Sign Matrix An $n\times n$ alternating sign matrix is an $n\times n$ square matrix which satisfies the following three conditions: 1) all entries are 0, 1 or -1, 2) the sum of entries in each row and in each column is one and 3) non-zero entries alternate in sign in each column and row. ASMs first appeared in the context of evaluating determinants, the Dodgson's algorithm (see [4] for interesting stories about ASMs and references therein). Mills, Robbins and Rumsey conjectured (proved by Zeilberger [5]) that the total number of $n \times n$ ASMs A_n is given by

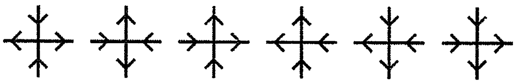
$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1, 2, 7, 42, 429, \dots. \tag{1}$$

The main idea of the proof is to construct a bijection between ASMs and plane partitions. The total number of ASMs is equal to that of descending plane partitions with largest part equal to or less than n. Furthermore, these total numbers are equal to the total number of the totally symmetric self-complement plane partitions (TSSCPPs) in a $2n \times 2n \times 2n$ box.

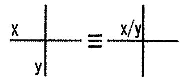
There are some classes of ASMs with a symmetry. An example is a half-turn symmetric alternating sign matrix (HTSASM). A HTSASM is invariant under the rotation by π. The total number of $2n \times 2n$ HTSASMs is

$$A_n^{HT} = A_n^2 \prod \frac{3j+2}{3j+1} = 2, 10, 140, \dots. \tag{2}$$

Six-vertex model and Fully Packed Loops Let us consider a square lattice of length n. There are four edges at a vertex in the lattice. Arrows are placed on the edges. Although there are 16 possible arrow configurations at a vertex, we consider only 6 configurations which satisfy the condition that the numbers of incoming and outgoing arrows are equal. Let us introduce the weights for configurations around a vertex with a spectral parameter x (see Figure 1).



vertex	(1)	(2)	(3)	(4)	(5)	(6)
weight	$\sigma(a^2)$	$\sigma(a^2)$	$\sigma(a/x)$	$\sigma(ax)$	$\sigma(a/x)$	$\sigma(ax)$
ASM	-1	1	0	0	0	0



$\sigma(x) = x - 1/x$
a : parameter

Figure 1: The weight of the six-vertex model

The weight of a configuration on the lattice is the product of weights at vertices. The partition function is given by the sum of the weight over all allowed arrow configurations. $2n$ spectral parameters $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ run along the vertical and horizonotal edges, respectively. The boundary arrow configuration with $2n$ outgoing vertical arrows and $2n$ incoming horizontal arrows is called the domain wall boundary conditions (DWBC) (see the middle picure in Figure 3). Korepin and Izergin showed that the partition function with DWBC is written as a determinant of the form (see [7] and references therein)

$$\begin{aligned} Z_{IK}(\mathbf{x}, \mathbf{y}; a) = & \frac{(a^2 - a^{-2})^n \prod_{i,j}(x_i y_j)^{n-2} \prod_{i,j}(a x_i^2 - a^{-1} y_j^2)(a y_j^2 - a x_i^2)}{\prod_{i<j}(x_j^2 - x_i^2)(y_i^2 - y_j^2)} \\ & \times \det \left[\frac{1}{(a x_i^2 - a^{-1} y_j^2)(a y_j^2 - a^{-1} x_i^2)} \right]_{1 \le i,j \le n}. \end{aligned} \tag{3}$$

This determinant is expressed as a Schur function when $a = \exp(\pi i/3)$. The R-matrix associated with six-vertex model is the one for $U_q(\widehat{\mathfrak{sl}_2})$. The property that the R-matrix satisfies the Yang-Baxter equation plays a central role to evaluate the partition function.

We consider the bipartite lattice on the square lattice of the six-vertex model. Vertices are classified into even and odd ones. We replace an arrow configuration on a vertex with a bold/dotted line (see Figure 2). We call a bold line configuration on the lattice fully packed loops (FPL). A FPL configuration is characterized by a link pattern (see section 3.1).

Kuperberg's observation Kuperberg [15] showed an alternative proof of the ASM theorem by the use of the integrability of the six-vertex model with DWBC. We put 0, 1 or -1 on a vertex with an arrow configuration as in Figure 1. This is a bijection between a configuration of the

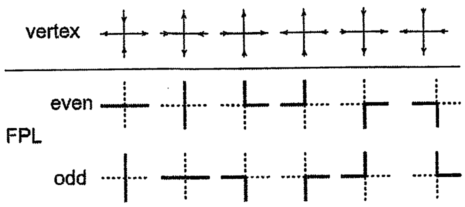


Figure 2: Six-vertex and FPL configuration

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

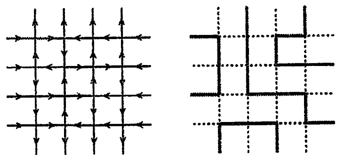


Figure 3: ASM/6V-DWBC/FPL correspondence

six-vertex model with (DWBC) and an alternating sign matrix. The total number of ASMs is equal to Z_{IK} (up to overall constant) with $a = \exp(\pi i/3)$ and all spectral parameters $x_i = y_i = 1$.

3 $O(1)$ Loop Model

Let us consider closed and non-intersecting loops on a lattice with N_T sites. We define the partition function of the $O(n)$ loop model as

$$Z_{O(n)\ loop} = \sum_{\mathcal{G}} t^{N_T - N_B} n^{N_L} \tag{4}$$

where $\mathcal{G}$ is a graph covering N_B bonds of the lattice and consisting of N_L closed, non-intersecting loops. The variable t plays a role of the temperature and n is the fugacity of a closed loop. In the zero temperature limit, the model is in the dense phase, *i.e.* every site is covered by a loop. Below, we consider the loop model a semi-infinite cylinder in the dense phase (*c.f.* Fig. 7). The state of the model is labelled by a link pattern (Section 3.1) and the Temperley-Lieb algebra (Section 3.2) plays important roles.

Starting from the definitions of a link pattern and the Temperley-Lieb algebra, we consider the eigenvector of the $O(1)$ loop model (sometimes called the TL stochastic process). Then, we summarize the Razumov-Stroganov conjectures for the $O(1)$ loop model with periodic boundary conditions. This section is based on [12, 16, 17, 18].

3.1 Link patterns

A link pattern is a diagram on a line with $L = 2n$ points in the upper-half plane, where each point is connected to another one via a non-intersecting arch (link). The number of link patterns is the Catalan number, *i.e.* $C_n = \frac{1}{n+1}\binom{2n}{n}$. Since we do not consider the direction of an arch here, only connectivities are important. When L is odd, there exists one point connected to a point at infinity.

Example: All the link patterns with $L = 3, L = 4$ and $L = 6$ are shown in Figure 4. Vertical lines in the $L = 3$ case indicate the connection to a point at infinity.

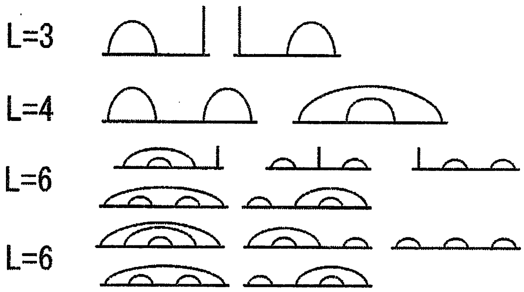


Figure 4: Link patterns

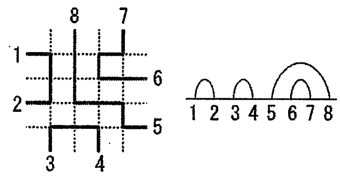


Figure 5: FPL and link pattern

FPL and link pattern: Recall that the total number of arrow configurations of the six-vertex model with DWBC on a square lattice of size $n \times n$ is that of FPLs (see Section 2). An FPL configuration consists of n lines (bold lines in left picture of Figure 5) which connect $2n$ external edges. Since these bold lines are non-intersecting, one can obtain a surjection from a FPL to a link pattern with $2n$ points. In other words, a FPL is characterized by a link pattern 5. Therefore, together with the ASM/6V-DWBC/FPL correspondence, the total number of ASMs A_n is written as

$$A_n = \sum_{\pi} A_{n,\pi}, \tag{5}$$

where $A_{n,\pi}$ is the total number of FPLs with the link pattern π and the sum is taken over all the link patterns.

Cylindric case: We may introduce the direction of a link. A link $\langle i, j\rangle$ is distinguished from a link $\langle j, i\rangle$ with $i < j$. A link pattern π is said to be a link pattern with cylindric boundary conditions (or a directed link pattern) if when π contains a link $\langle j, i\rangle$ with $i < j$, there is no link $\langle k, l\rangle$ with $k < l$ such that $k < i < l, k < j < l$ or $k < i < j < l$. In other words, a link pattern with $\langle j, i\rangle, i < j$ may contain a link $\langle k, l\rangle, k < l$, if $k < l < i < j, i < k < l < j$ or $i < j < k < l$, and a link $\langle l, k\rangle, k < l$, if $k < i < j < l$ or $i < k < l < j$. In the parenthesis notation, $\langle i, j\rangle$ (resp. $\langle j, i\rangle$) with $i < j$ is denoted by "()" (resp. ")("). For example, parenthesis ")()(" signifies $\langle 4, 1\rangle\langle 2, 3\rangle$.

A FPL configuration associated with a HTSASM is characterized by a directed link pattern.

A link pattern which contains only links $\langle i, j \rangle$ with $i < j$ is said to be undirected (the case of Figure 4). In this case, we denote a link by (i, j) where $\langle i, j \rangle$ is identified with $\langle j, i \rangle$.

Example: All the directed links with length $L = 4$ are as follows: ()(), (()), ())(,)((),)()(and))((. The first two links are the same as the $L = 4$ case in Figure 4.

3.2 Temperley-Lieb algebra

The Temperley-Lieb (TL) algebra generated by $\{e_1, \ldots, e_{L-1}\}$ satisfies the relations:

$$e_i^2 = \tau e_i, \quad \tau = -(q + q^{-1}), \qquad \text{for } 1 \leq i \leq L - 1, \tag{6a}$$

$$e_{i\pm 1} e_i e_{i\pm 1} = e_i e_{i\pm 1} e_i, \tag{6b}$$

$$e_i e_j = e_j e_i, \qquad |i - j| > 2, \tag{6c}$$

where q is an indeterminant.

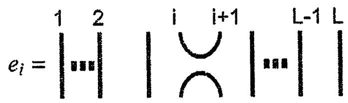
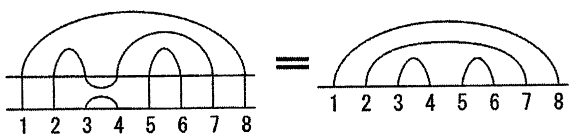


Figure 6: TL generator

A TL generator e_i is graphically depicted as in Figure 6. It is easy to show the relations (6a)-(6c) by this graphical expression. A loop in the diagram gives the weight τ due to the relation (6a). When $\tau = 1$, it is enough to remove loops from the diagram. The Temperley-Lieb algebra naturally acts on the space of link patterns locally from bottom. Recall an undirected link pattern satisfies that $L/2$ pairs of points connected by links and any link does not cross another one. We denote by (i, j) a pair of two points i and j. A link pattern with $(i, i + 1)$ is invariant under the action of e_i. After the action of e_i on a link pattern with (i, j) and $(i + 1, k)$, we obtain a link pattern with $(i, i + 1)$ and (j, k). Other pairs of points are unchanged.

We consider the additional generator e_L by imposing periodic boundary conditions in Figure 6. After the action of e_L on a link pattern with $(1, j)$ and (k, L), we obtain a link pattern with $(1, L)$ and (j, k). The relations (6a)-(6c) become cyclic with the identification $e_{L+1} \equiv e_1$.

Example: An example of the action of e_3.



e_3 naturally acts on a link pattern from bottom in the l.h.s. The obtained link pattern in the r.h.s. is topologically equivalent to the l.h.s.

3.3 Eigenvector problem

In this and the next subsections, we restrict ourselves to the case where L is even and $\tau = 1$. We consider the Hamiltonian of the $O(1)$ loop model, called the Temperley-Lieb Hamiltonian:

$$H_{TL} = \sum_{j=1}^{L}(1 - e_j), \tag{7}$$

where H_{TL} acts on the space of undirected link patterns of length L. We want to know the non-trivial ground state eigenvector satisfying

$$H_{TL}\phi_0 = 0. \tag{8}$$

The eigenvector ϕ_0 is expanded as $\phi_0 = \sum_\pi \phi_{0,\pi}|\pi\rangle$ where the sum is taken over all the link patterns.

Example: $L = 4$. The space of undirected link patters is spanned by two link patterns: ()() and (()). The Hamiltonian is given by $H_{TL} = \sum_{j=1}^{4}(1 - e_j)$. The action of H_{TL} on the two link patterns is given by

$$H_{TL}\,(()) = 2\,(()) - 2\,()(), \tag{9}$$
$$H_{TL}\,()() = 2\,()() - 2\,(()). \tag{10}$$

Therefore, the Hamiltonian is expressed as $H_{TL} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ in this basis. The non-trivial eigenvector with eigenvalue zero is $\phi_0 = (1,1)$. The sum of the entries is $\phi_{0,(())} + \phi_{0,()()} = 1 + 1 = 2$.

length L	ϕ_0	$\sum_\pi \phi_{0,\pi}$
2	(1)	1
4	$(1,1)$	2
6	$(2_2, 1_3)$	7
8	$(7_2, 3_8, 1_4)$	42
10	$(42_2, 17_{10}, 14_5, 6_{10}, 4_{10}, 1_5)$	429
$\vdots$	$\vdots$	$\vdots$

Table 1: The eigenvectors of the Hamiltonian H_{TL} up to $L = 10$.

One can obtain Table 1 by some calculations. The eigenvector ϕ_0 and the sum of its entries are in the middle and the rightmost column, respectively. When an integer i appears j-th times in the entries of ϕ_0, we denote it by i_j. The entries are written in the dominant order.

3.4 Razumov-Stroganov conjectures

There are some remarkable properties in Table 1, known as Razumov-Stroganov conjectures. All the entries of the eigenvector are positive integers when we normalized the smallest one as unity. First, a sequence of integers in the rightmost column is the same integer sequence appeared in Eqn.(1), *i.e.* the sum of the entries of the ground state of the $O(1)$ loop model with periodic boundary conditions and length $L = 2n$ is given by the total number of $n \times n$ ASMs. Second, each entry of the eigenvector has combinatorial meaning. An integer $\phi_{0,\pi}$ is the total number of FPLs characterized by the link pattern π. Therefore, the first conjecture (the sum rule) is a consequence of the second conjecture (the rule for entries). The rule for entries is the most popular form of the Razumov-Stroganov conjectures. Note that the Razumov-Stroganov conjecture gives the relation (5) in terms of the eigenvector of the $O(1)$ loop model.

3.5 The *XXZ* model

Originally, Razumov and Stroganov proposed conjectures related to the *XXZ* spin chain model at an isotropic parameter $\Delta = -1/2$ with periodic conditions [1]. In this subsection, we present the relation between the *XXZ* spin chain model and the $O(1)$ loop model [17].

Let us consider the *XXZ* model of the length L. The Hamiltonian $\mathcal{H}_{XXZ}$ is given by

$$\mathcal{H}_{XXZ} = \sum_{i=1}^{L-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z\right) + (\text{boundary terms}), \tag{11}$$

where $\vec{\sigma}$ is the Pauli matrix, Δ is the isotropic parameter and the boundary terms depend on the choice of boundary conditions. At $\Delta = -1/2$, $\mathcal{H}_{XXZ}$ is the leading term of the Hamiltonian of the $O(1)$ loop model

$$\mathcal{H}_{loop} = \sum_{i=1}^{L-1}(1 - e_i) + (\text{boundary term}), \tag{12}$$

where $e_i = \frac{1}{2}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \frac{1}{2}\sigma_i^z\sigma_{i+1}^z - \frac{\mathrm{i}\sqrt{3}}{2}(\sigma_i^z - \sigma_{i+1}^z) + \frac{1}{2})$. The operators e_i's satisfy the defining relations of the Temperley-Lieb algebra at $\tau = 1$. In particular, the boundary term of the Hamiltonian (12) is written as $\sigma^{-1}e_{L-1}\sigma$ with the cyclic operator σ for both periodic and cylindric boundary conditions. In the case of periodic boundary conditions [2, 9], $\sigma := t_{L-1}^{-1}\dots t_1^{-1}$ where $t_i = e_i + q$. The cyclic operator for cylindric boundary conditions is given in Section 5.2 (see also [3, 10]). If we have no boundary term in the Hamiltonian (12), the boundary term in (11) is $\frac{\mathrm{i}\sqrt{3}}{4}(\sigma_1^z - \sigma_L^z)$. This Hamiltonian is nothing but the $U_q(sl_2)$ invariant Hamiltonian with open boundaries [19].

Directed links $\langle i,j\rangle$ and $\langle j,i\rangle$ with $i < j$ are written in terms of spin basis as

$$|\uparrow\downarrow\rangle_{ij} + (-q)^{-1}|\downarrow\uparrow\rangle_{ij}, \qquad \text{for} \quad \langle i,j\rangle, \tag{13}$$

$$|\uparrow\downarrow\rangle_{ij} + (-q)^{-3}|\downarrow\uparrow\rangle_{ij}, \qquad \text{for} \quad \langle j,i\rangle. \tag{14}$$

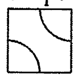
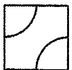
In the undirected case, every link is expressed as Eqn.(13). A link pattern is expressed in terms of spin basis as a tensor product of links (13) and (14).

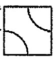
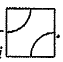
From these observations, eigenvectors observed in Section 3.3 are also eigenvectors of $\mathcal{H}_{XXZ}$. This means that we focus on the subspace in the XXZ spin chain where the total spin is zero (resp. one-half) with L even (resp. odd).

4 Inhomogeneous $O(1)$ Loop Model

Di Francesco and Zinn-Justin proved the sum rule for the inhomogeneous $O(1)$ loop model by use of the integrability of the trigonometric $\check{R}$-matrix (for the six-vertex model) [2]. The solution of the quantum Knizhnik-Zamolodchikov (q-KZ) equation is the eigenvector of the transfer matrix of the inhomogeneous $O(1)$ loop model. The q-KZ equation is considered on the space of undirected link patterns. The crucial point is that the sum of the solution of q-KZ equation at the Razumov-Stroganov point is written in terms of the partition function of the six-vertex model with DWBC. Together with the Kuperberg's observation, the sum in the homogeneous limit is equal to the total number of ASMs. This method is also applied to the $O(1)$ loop model on a cylinder (directed link patterns) and the total number of HTSASMs is given in [20]. The cylindric case is the $k = 2$ case in the section 5.

4.1 Definition of the model

The inhomogeneous $O(1)$ loop model is defined as a tiling on a semi-infinite cylinder of square lattice with the perimeter $N = 2n$, where squares on the same height are labelled in order cyclically from 1 to N. Spectral parameters z_i, $1 \le i \le N$, run along the vertical strips. We attach two kinds of unit plaquettes,  and  on a square (see Figure 7). The weight of the plaquettes in the i-th vertical strip is given by the R-matrix as

$$R(z_i,t) = \frac{qz_i - q^{-1}t}{qt - q^{-1}z_i}$$  $$+ \frac{z_i - t}{qt - q^{-1}z_i}$$  . (15)

where $q = -\exp(\pi i/3)$ and t is the horizontal spectral parameter.

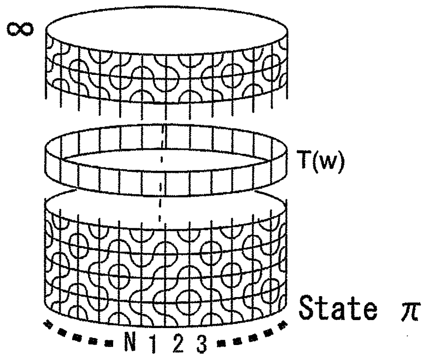


Figure 7: $O(1)$ loop model

We define a state π of the inhomogeneous $O(1)$ loop model. We mean by a site a terminal end of a curved line at the height zero. Since curved lines on a plaquette are non-intersecting, a site is connected to another one by a non-intersecting curved line. Therefore, all the N sites and curved lines form a link pattern. The space of states for the $O(1)$ loop model is the set of undirected link patterns. We denote a state by π, or $|\pi\rangle$.

The row-to-row transfer matrix of the $O(1)$ loop model is given by

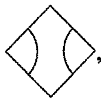
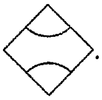
$$T(t|z_1,\dots,z_N) = \mathrm{Tr}_0(R_1(z_1,t)\dots R_N(z_N,t)), \tag{16}$$

where the trace is taken on the auxiliary quantum space. The transfer matrix naturally acts on states. We want to compute the weight distribution $\Psi(z_1,\dots,z_N) = \sum_\pi \Psi_\pi|\pi\rangle$ such that

$$T(t|z_1,\dots,z_N)\Psi(z_1,\dots,z_N) = \Psi(z_1,\dots,z_N). \tag{17}$$

4.2 Algebraic structure

The choice of the weight of plaquettes (15) is based on the R-matrix for the six-vertex model (or equivalently for the $U_q(\widehat{sl_2})$). To see why this works, we rotate the plaquette by $\pi/4$ and set two generators

$1 =$ , $\quad e_i =$ . (18)

Note that the plaquette expression of the generators is the same as Figure 6. Therefore, the generators 1 and e_i's form the Temperley-Lieb algebra (see relations (6a)-(6c)). Let us denote $\check{R} := R\mathcal{P}$ where $\mathcal{P}$ is the permutation operator (or the rotation of the plaquettes). The $\check{R}$-matrix is written as

$$\check{R}(z_i,t) = \frac{qz_i - q^{-1}t}{qt - q^{-1}z_i}1 + \frac{z_i - t}{qt - q^{-1}z_i}e_i. \tag{19}$$

With the argument in Section 3, the $\check{R}$-matrix naturally acts on the space of the states of inhomogeneous $O(1)$ loop model. The $\check{R}$-matrix is nothing but the one for the six-vertex model under the spin representation of the Temperley-Lieb algebra as in Section 3.5.

The $\check{R}$-matrix satisfies the Yang-Baxter equation

$$\check{R}_{i+1}(z_2,z_1)R_i(w,z_1)R_{i+1}(w,z_2) = R_i(w,z_2)R_{i+1}(w,z_1)\check{R}_i(z_2,z_1). \tag{20}$$

By use of the Yang-Baxter equation, one can show that

$$T(t|z_1,\dots,z_N)\check{R}_i(z_i,z_{i+1}) = \check{R}_i(z_i,z_{i+1})\tau_i T(t|z_1,\dots,z_N), \tag{21}$$

where τ_i acts on a polynomial as $\tau_i f(\dots,z_i,z_{i+1},\dots) = f(\dots,z_{i+1},z_i,\dots)$.

4.3 q-KZ equation

Instead of the eigenvector problem (17), we consider the following set of difference equations which is called the quantum Knizhnik-Zamolodchikov equation [2, 9]:

$$\check{R}_i(z_i,z_{i+1})\Psi(\mathbf{z}) = \tau_i\Psi(\mathbf{z}), \quad \text{for } 1 \le i \le N-1 \tag{22a}$$

$$\check{R}_N(z_N,sz_1)\Psi(\mathbf{z}) = \Psi(s^{-1}z_N,z_2,\dots,z_{N-1},sz_1), \tag{22b}$$

where $\Psi(\mathbf{z}) := \Psi(z_1, \ldots, z_N)$ and two parameters s and q take generic value. We want to find the minimal degree solution of the q-KZ equation on the space of link patterns and the relation between s and q. To obtain the eigenvector (17) of the transfer matrix, we restrict ourselves to $s = 1$, where the solution of the qKZ equation coincides with the eigenvector.

Let us see why this works [9, 10]. The transfer matrices with different spectral parameters commute with each other, *i.e.* $[T(t|\mathbf{z}), T(t'|\mathbf{z})] = 0$. We can define N commuting operators Y_i, $1 \le i \le N$, by $Y_i := T(z_i|\mathbf{z}) = R_{ii+1} \ldots R_{i,N} R_{i1} \ldots R_{i,i-1}$ where $R_{ij} := R_{ij}(z_j, z_i)$. Note that the simultaneous eigenvector of Y_j's is the eigenvector of the transfer matrix. On the other hand, we denote by $\hat{s}_i : z_i \to sz_i$ the shift operator. From Eqns.(22a)-(22b), we can define N commuting operators $\tilde{y}_i = R_{ii+1} \ldots R_{i,N} \hat{s}_i R_{i1} \ldots R_{i,i-1}$. The solution of the qKZ equation is the simultaneous eigenvector of $\tilde{y}_i$. Especially when $s = 1$, we have $Y_i = \tilde{y}_i$. Therefore, the solution of the qKZ equation coincides with the eigenvector of the transfer matrix only for $s = 1$.

Entries of the q-KZ equation (22a) are rewritten as

$$\frac{q^{-1}z_i - qz_{i+1}}{z_i - z_{i+1}}(\tau_i - 1)\Psi_\pi(\mathbf{z}) = \sum_{\pi' \neq \pi, e_i\pi' = \pi} \Psi_{\pi'}(\mathbf{z}), \tag{23}$$

where the sum is taken over the set $\{\pi' | \pi' \neq \pi, e_i\pi' = \pi\}$. From Eqn.(23), if there is no π' such that $e_i\pi' = \pi$, then the entry Ψ_π is factorized into $\Psi_\pi(\mathbf{z}) = (q^{-1}z_{i+1} - qz_i) \times$ (polynomial). The highest state π_0 is a state such that $\sharp\{e_i | e_i\pi_0 \neq \pi_0\}$ is smallest. Although there are $N/2$ link patterns satisfying the above condition, we define the highest state π_0 as in Figure 8.

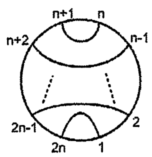


Figure 8: The highest state

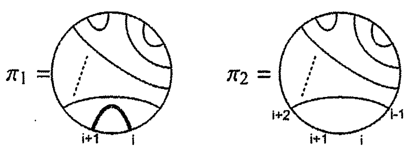


Figure 9:

The highest state π_0 is characterized by $e_{N/2}\pi_0 = e_N\pi_0 = \tau\pi_0$ and $e_i\pi_0 \neq \pi_0$ for $i \neq N/2, N$. Together with the minimal degree assumption, the entry Ψ_{π_0} is written as

$$\Psi_{\pi_0} = \prod_{1 \le i < j \le n} (qz_i - q^{-1}z_j) \prod_{n+1 \le i < j \le 2n} (q^{-1}z_j - qz_i). \tag{24}$$

Since a link pattern is obtained by acting e_i's on the highest state, the other entries are determined from Ψ_{π_0} in a triangular way through Eqn.(23). One can obtain that the relation $s = q^6$ as a compatibility condition of the q-KZ equation.

4.4 The sum rule

In this subsection, we restrict ourselves to the Razumov-Stroganov point, *i.e.* $q = -\exp(\pi i/3)$ to obtain the sum rule for the inhomogeneous $O(1)$ loop model.

We denote the sum by $W(z_1,\dots,z_N) = \sum_\pi \Psi_\pi(z_1,\dots,z_N)$. The eigen-coveter v satisfying $v\check{R}_i(z,w) = v$ for $1 \le i \le N$ is given by $v = (1,1,\dots,1)$ on the space of link patterns. Therefor, the sum is rewritten as the inner product $W(\mathbf{z}) = v \cdot \Psi(\mathbf{z})$.

There are three properties of $W(\mathbf{z})$:

(P1) W is a homogeneous polynomial with respect to $z_1,\dots,z_N$.

(P2) W is a symmetric polynomial with respect to $z_1,\dots,z_N$.

(P3) W satisfies the recurrence relation

$$W(z_1,\dots,z_N)|_{z_{i+1}=q^2z_i} = \prod_{j\neq i,i+1}(qz_i - z_j)\cdot W(\mathbf{z}\backslash\{z_i,z_{i+1}\}). \tag{25}$$

One can show the properties as follows. The highest state (24) is a homogeneous polynomial of total degree $n(n-1)$. Since the q-KZ equations (23) preserve the total degree of a polynomial solution, (P1) is satisfied. The action of τ_i on $W(\mathbf{z})$ is given by $\tau_i W(\mathbf{z}) = v\cdot\tau_i\Psi(\mathbf{z}) = v\cdot\check{R}_i\Psi(\mathbf{z}) = W(\mathbf{z})$ where we have used that $\Psi(\mathbf{z})$ is the solution of the q-KZ equation. This ensures that $W(\mathbf{z})$ is a symmetric function (P2). The property (P3) is based on the recurrence relation for entries. Let π_1 and π_2 be link patterns as in Figure 9. π_1 is a link pattern with a little arch $(i,i+1)$ and π_2 is the link patter obtained from π_1 by removing the arch $(i,i+1)$. Therefore, π_2 is a link pattern with $N-2$ points. The entry satisfies the recurrence relation

$$\Psi_{\pi_1}(\mathbf{z})|_{z_{i+1}=q^2z_i} = \prod_{j\neq i,i+1}(q^2z_i - z_j)\cdot \Psi_{\pi_2}(\mathbf{z}\backslash\{z_i,z_{i+1}\}). \tag{26}$$

Considering the total degree and partial degrees of W, we obtain the following identities:

$$\begin{aligned} W(\mathbf{z}) &= Z_{IK}(\mathbf{z};\exp(\pi i/3)) \\ &= s_{Y_n}(z_1,\dots,z_N), \end{aligned} \tag{27}$$

where we here suitably normalized the IK determinant (3) and s_{Y_n} is the Schur function with the Young diagram

$$Y_n = (n-1,n-1,n-2,n-2,\dots,2,2,1,1). \tag{28}$$

In the homogeneous limit such that all the spectral parameters z_i's tend to unity, we have

$$\begin{aligned} \lim_{z_i\to 1} W(z_1,z_2,\dots,z_N) &= 3^{n/2}A_n \\ &= 1,2,7,42,429,\dots \end{aligned} \tag{29}$$

The integer sequence (29) is the total number of ASMs (see Eqn.(1)).

5 A_{k-1} generalized model on a cylinder

In Section 4 we have discussed the $O(1)$ loop model with periodic boundary conditions. In this section, we consider the A_{k-1} generalization of the $O(1)$ loop models with cylindric boundary conditions [3] (see [21] for $k=2$).

5.1 Affine Heck algebra of type A

In what follows, we restrict ourselves to the case where $N = nk$ and $n \geq 1, k \geq 2$ are positive integers. The Hecke algebra is generated by $\{e_i : 1 \leq i \leq N-1\}$ which satisfy the defining relations

$$e_i^2 = \tau e_i, \tag{30a}$$

$$e_i e_j = e_j e_i, \qquad |i-j| > 1, \tag{30b}$$

$$e_i e_{i\pm 1} e_i - e_i = e_{i\pm 1} e_i e_{i\pm 1} - e_{i\pm 1}, \tag{30c}$$

where $\tau = -(q+q^{-1})$. We introduce the cyclic operator σ and the additional generator $e_N = \sigma^{-1} e_{N-1} \sigma$ such that $e_i = \sigma^{-1} e_{i-1} \sigma$ and the relations (30a)-(30c) become cyclic. The generators satisfy the vanishing condition $Y_k(e_i, \dots, e_{i+k-1}) = 0$ where Y_k is recursively defined by $Y_{k+1}(e_i, \dots, e_{i+k}) = Y_k(e_i, \dots, e_{i+k-1})(e_{i+k} - \mu_{k+1}) Y_k(e_i, \dots, e_{i+k-1})$where $\mu_k = U_{k-1}(\tau)/U_k(\tau)$ and $U_k(\tau)$ is the Chebyshev polynomial of the second kind. Y_k is called a q-symmetrizer. Furthermore, we consider additional vanishing conditions;

- When $N = k$, we have

$$Y_{k-1}(e_1, \cdots, e_{N-1})(e_N - \tau) Y_{k-1}(e_1, \cdots, e_{N-1}) = 0. \tag{31}$$

  Obviously, $Y_k(e_1, \cdots, e_N)$ is non-zero.

- For $N = nk$ with $n \geq 2$

$$Y_{q\text{-sym}} \cdot \prod_{i=1}^{n-1} (e_{ik} - \mu_{k-1})(e_{nk} - \tau) \cdot Y_{q\text{-sym}} = 0, \tag{32}$$

  where $Y_{q\text{-sym}} := \prod_{i=0}^{n-1} Y_{k-1}(e_{ik+1}, \cdots, e_{(i+1)k-1})$ is the product of the q-symmetrizers.

Hereafter, we call these vanishing conditions as the *cylindric relations*. We denote the affine Hecke algebra satisfying the cylindric relations by $\widehat{H}_N^{(k)}$

The Baxterization gives us the trigonometric $\check{R}$-matrix, namely

$$\check{R}_i(z, w) = \frac{qz - q^{-1}w}{qw - q^{-1}z} \mathbf{1} + \frac{z - w}{qw - q^{-1}z} e_i. \tag{33}$$

for $1 \leq i \leq N$. Note that the $\check{R}$-matrix satisfies the Yang-Baxter equation (20) with $R = \check{R}\mathcal{P}$.

5.2 Vector representation

We give the vector representation of the affine Hecke algebra $\widehat{H_N^k}(\tau)$, or equivalently, the tensor product of the vector representation [3].

Let us consider a representation $(\chi, V^{\otimes N})$ where $\chi : \widehat{H_N^{(k)}}(\tau) \to \mathrm{End}(V^{\otimes N})$ and $V \simeq \mathbb{C}^k$. We introduce two linear operators $\check{e}, \tilde{e} \in \mathrm{End}(V^2)$ as

$$\check{e} = \sum_{a,b=1}^{k} E_{ab} \otimes E_{ba} - \sum_{a,b=1}^{k} q^{\mathrm{sign}[b-a]} E_{aa} \otimes E_{bb}, \tag{34}$$

$$\tilde{e} = \sum_{a,b=1}^{k} q^{2(a-b)} E_{ab} \otimes E_{ba} - \sum_{a,b=1}^{k} q^{\mathrm{sign}[b-a]} E_{aa} \otimes E_{bb}, \tag{35}$$

where E_{ab} is a $k \times k$ matrix whose elements are $(E_{ab})_{ij} = \delta_{ai}\delta_{bj}$. Let $v = |v_1 \dots v_N\rangle$ be a base in $V^{\otimes N}$. We define the shift operator $\rho : V^{\otimes} \to V^{\otimes}, v \mapsto |v_2 \dots v_N v_1\rangle$.

A generator of the Hecke algebra is written in terms of the operator $\check{e}$:

$$\chi(e_i) = \underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{i-1} \otimes \check{e} \otimes \underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{N-i-1} \tag{36}$$

where $1 \le i \le N-1$ and $\mathbb{I}$ is a $k \times k$ identity matrix. The additional operator e_N is written in terms of $\tilde{e}$ and ρ as

$$\chi(e_N) = \rho^{-1}(\underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{N-2} \otimes \tilde{e})\rho. \tag{37}$$

The operator $\tilde{e}$ is obtained by the twist, i.e. $\tilde{e} = \Omega^{-1}\check{e}\Omega$ where $\Omega \in \mathrm{End}(V^2)$ is given by $\Omega = \mathbb{I} \otimes \mathrm{diag}(q^{-(k-1)}, q^{-(k-3)}, \dots, q^{k-1})$.

Under the above spin representation of $\widehat{H_N^{(k)}}(\tau)$, one can show that the defining relations (30a)-(30c) become cyclic and the cylindric relations (31)-(32) are satisfied.

5.3 Space of states

In the case of $k = 2$, the Temperley-Lieb algebra acts on the space of link patterns (or Dyck paths). For the case of a generic k, the affine Hecke algebra acts on the set of states labelled by a path. An unrestricted path $\pi := \pi_1\pi_2 \cdots \pi_{nk}$ of length nk is a set of nk integers satisfying $1 \le \pi_i \le k$ for $1 \le i \le nk$ and $\sharp\{i|\pi_i = j, 1 \le i \le nk\} = n$ for $1 \le j \le k$. If an unrestricted path π satisfies $\sharp\{i|\pi_i = j, 1 \le i \le l\} \ge \sharp\{i|\pi_i = j+1, 1 \le i \le l\}$ for all $1 \le j \le k$ and $1 \le l \le nk$, the path π is said to be a restricted path. The set of states labelled by restricted (resp. unrestricted) paths is considered in [21] (resp. [3]). We consider unrestricted paths. Let $|\pi\rangle$ be a state labelled by a path. A state $|\pi\rangle$ is a natural generalization of a link pattern with cylindric boundary conditions.

We consider the representation on the left ideal $\widehat{H}_N^{(k)} Y_{q\text{-sym}}$. Such a path correspoinding to $Y_{q\text{-sym}}$ is $\pi^{\Omega} = \pi_{pk+q} = q$ for $0 \le p \le n-1$ and $1 \le q \le k$. A path is identified with a line graph. All the other states can be obtained as a rhombus tiling over the path π^{Ω} shown in Fig. 10 (see [3] for the construction of states). Figure 10 is an example for $k = 3$ and $N = 12$. The bold line graph at the bottom indicates the path π^{Ω} and the line graph of top edges is a path.

We want to construct the space of states labelled by unrestricted paths satisfying the properties;

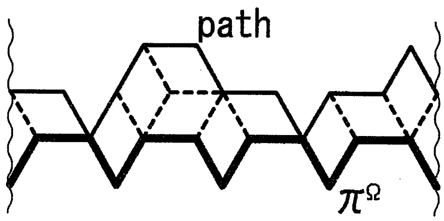


Figure 10: A rhombus tiling:

(P1) If π satisfies $\pi_i > \pi_{i+1}$, the state is invariant under the action of e_i, *i.e.* $e_i|\pi\rangle = \tau|\pi\rangle$.

(P2) If π satisfies $\pi_i < \pi_{i+1}$, the action of e_i is given by $e_i|\pi\rangle = \sum_{\pi'} C_{i,\pi,\pi'}|\pi'\rangle$. If the coefficient $C_{i,\pi,\pi'} \neq 0$, π' is obtained by adding a unit rhombus or a path below π.

(P3) If π satisfies $\pi_i = \pi_{i+1}$, the action of e_i is given by $e_i|\pi\rangle = \sum_{\pi'} C_{i,\pi,\pi'}|\pi'\rangle$. If the coefficient $C_{i,\pi,\pi'} \neq 0$, π' is a path below π.

(P4) In the properties (P2) and (P3), let us consider the case where $e_j|\pi\rangle = \tau|\pi\rangle$ for $j \neq i \pm 1$. Then, for a path π' with non-zero $C_{i,\pi,\pi'}$ it satisfies $e_j|\pi'\rangle = \tau|\pi'\rangle$.

By putting positive integers on rhombi, we obtain a state labelled by a path. When a rhombus with integer m is put between two edges π_i and π_{i+1}, the rhombus indicates $\check{L}_i(m) := e_i - \mu_{m-1}$. Therefore, a rhombus tiling with integers corresponds to a word $\prod_i \check{L}_i(m_i) \cdot Y_{q\text{-sym}}$ in the left ideal. Before giving the rule to assign integers on a rhombus tiling, we see two examples below.

Example 1: The case of $k = 3, L = 6$ (on a cylinder). A state $|231231\rangle$ is depicted and written as

$|231231\rangle =$ 

$$= e_1(e_6 - \frac{1}{\tau})e_4(e_3 - \frac{1}{\tau})Y_2(e_1, e_2)Y_2(e_4, e_5). \tag{38}$$

Example 2: The case of $k = 3, L = 3$ (on a cylinder).

We have six states:

words	e_3Y_3	$Z_{2,3}Y_3$	$Z_{1,3}Y_3$	$e_1Z_{2,3}Y_3$	$e_2Z_{1,3}Y_3$	Y_3
paths	321	312	231	132	213	123

where $Z_{i,j} = e_ie_j - 1$. One can easily show that these six states satisfy the above properties (P1)-(P4). The associated matrix representation is explicitly given as follows (up to a suitable normalization).

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \tau^2-1 & 0 & \tau & 0 & 1 & 0 \\ 0 & 1 & 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \tau^2-1 & \tau \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \tau^2-1 & \tau & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \tau & 0 \\ 1 & 0 & 0 & \tau^2-1 & 0 & \tau \end{pmatrix},$$

$$e_3 = \begin{pmatrix} \tau & 0 & 0 & 0 & 0 & 1 \\ 0 & \tau & 0 & 1 & \tau^2-1 & 0 \\ 0 & 0 & \tau & \tau^2-1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We summarize the basic ideas to construct a state of the A_{k-1} generalized model. We propose a novel graphical way, the rhombus tiling, to deal with the Yang-Baxter equation and q-symmetrizers. A rhombus represents a $\check{R}$-matrix (up to overall factor). On its face, a rhombus has a positive integer indicating the spectral parameter of the $\check{R}$-matrix. The Yang-Baxter equation, $\check{R}_{i+1}(v)\check{R}_i(u)\check{R}_{i+1}(u-v) = \check{R}_i(u-v)\check{R}_{i+1}(u)\check{R}_i(v)$, is expressed as the equivalence between two different ways of tiling of a hexagon (see Fig. 11). A q-symmetrizer is expressed as a regular $(2k+1)$-gon with integers. The q-symmetrizer Y_3 is depicted as an octagon in Fig. 12. The product of the q-symmetrizers $Y_{q\text{-sym}}$ is expressed as n regular $(2k+1)$-gons connected to each other in line. By reading the top edges of the $(2k+1)$-gons, we define the path π^{Ω} (see Fig. 10).

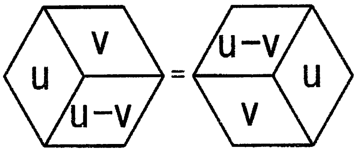


Figure 11: The Yang-Baxter equation

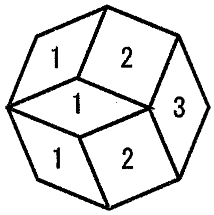


Figure 12: The q-symmetrizer Y_3

Since we consider the left ideal $\widetilde{H_N^{(k)}}(\tau)Y_{q\text{-sym}}$, the other states are obtained by piling rhombi over the $(2k+1)$-gons. All we have to do is to determine an integer on rhombi. Suppose that we have a rhombus tiling corresponding to a state. We divide the rhombus tiling into pieces of rhombus blocks. A rhombus block is a polygon where upper edges are convex. Integers on rhombus blocks are determined from top to bottom to satisfy the properties (P1)-(P4). A crucial observation is that the shape of a rhombus block is similar to that of a q-symmetrizer.

Now we consider the meaning of the cylindric relations by means of the rhombus tiling with integers. Rewrite the cylindric relation as

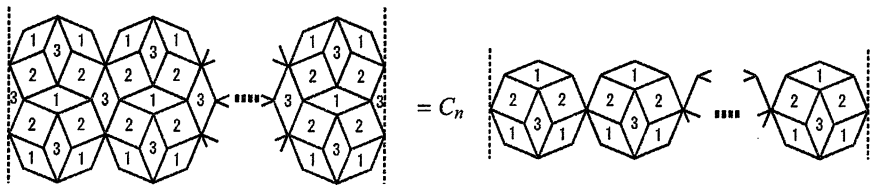
$$Y_{k-1}(e_1,\cdots,e_{k-1})\check{L}_k(k-1)Y_{k-1}(e_1,\cdots,e_{k-1}) = \mu_k^{-1}\alpha_{k-1}Y_{k-1}(e_1,\cdots,e_{k-1}) \tag{39}$$

for the case $N=k$. Here, $\check{L}_i(m) = e_i - \mu_{m-1}$ for $1 \le m \le k-1$ and $\alpha_n = \prod_{i=1}^n \mu_i^{-2^{n-i}}$. And

$$Y_{q\text{-sym}} \cdot \left(\prod_{i=1}^{n} \check{L}_{ik}(k-1)\right) \cdot Y_{q\text{-sym}} = \Delta_k^{n-1}\mu_k^{-1}\alpha_{k-1}^n Y_{q\text{-sym}} \tag{40}$$

for the case $N = nk$ with $n \ge 2$ where, $\Delta_k = \mu_k - \mu_{k-1}$. Recall that the weight of a loop surrounding the cylinder is τ in the $k=2$ case. Equations (39) and (40) mean that the weight of a "band" surrounding the cylinder is given in terms of Chebyshev polynomials of the second kind (see an example below).

Example: $k = 4, n \geq 2$. We have the following graphical representation of the cylindric relation.



where $C_n = \Delta_4^{n-1}\mu_4^{-1}\alpha_3^n$. The octagons are the q-symmetrizers Y_3. Note that all rhombi have positive integers. We choose one of the equivalent expressions of the q-symmetrizer such that two octagons share the same rhombus with the integer one. The rhombus for the operator $\check{L}_N$ is divided into two parts since the cylinder is cut along the dotted line.

5.4 q-KZ equation and the sum rule

We want the eigenvector of the transfer matrix of the A_{k-1} generalized model to obtain the eigenvector of the spin chain. We consider the system of difference equations for $\Psi(\mathbf{z}) = \Psi(z_1, \ldots, z_N)$, the quantum Knizhnik-Zamolodchikov equation (at $s = 1$ in Eqn.(23)),

$$\check{R}_i(z_i, z_{i+1})\Psi(z_1, \ldots, z_N) = \tau_i\Psi(z_1, \ldots, z_N), \tag{41}$$

for $1 \leq i \leq N$ where $\Psi(\mathbf{z}) = \sum_\pi \Psi_\pi(\mathbf{z})|\pi\rangle$ and $\tau_i f(z_i, z_{i+1}) = f(z_{i+1}, z_i)$. We have the solution of the Eqn. (41) when q is the generalized Razumov-Stroganov point, *i.e.* $q = -\exp(\pi i/(k+1))$. Hereafter, q is on the R-S point. As in Section 4, the solution is determined by the existence of the highest state and of the minimal degree solution. The highest state π_0 is

$$\Psi_{\pi_0} = \prod_{1 \leq i < j \leq N} (qz_i - q^{-1}z_j). \tag{42}$$

The other entries are determined one by one by using Eqn.(41).

Let v be the simultaneous eigen covector satisfying $ve_i = \tau v$ and $v\sigma = v$. We fix an integer m and take a special parameterization of the form

$$z_{m+j} = q^{-2j}z, \qquad 0 \leq j \leq k-1. \tag{43}$$

By using the recurrence relation for the solution of the q-KZ equation, we obtain that the weighted sum $W(\mathbf{z}) = v \cdot \Psi(\mathbf{z})$ satisfy the recurrence relation

$$W(\mathbf{z})|_{(43)} = Cz^{\frac{1}{2}k(k-1)} \left(\prod_{\substack{1 \leq j \leq N \\ j \neq m, \ldots, m+k-1}} (qz_i - q^{-1}z)^k \right) W(\mathbf{z}'), \tag{44}$$

where $\mathbf{z}' = \mathbf{z}\backslash\{z_m, \dots, z_{m+k-1}\}$ and C is some constant depending only on q and N. The sum $W(\mathbf{z})$ is written in terms of Schur functions $s_\lambda(\mathbf{z})$ as [3]:

$$W(\mathbf{z}) = (\text{const.}) \prod_{l=0}^{k-1} s_{Y^n_{k,l}}(\mathbf{z}). \tag{45}$$

Here, the Young diagrams are $Y^n_{k,l} = \delta(n^l, n-1^{k-l})$ with

$$\delta(n^l, n-1^{k-l}) = (\underbrace{n, \dots, n}_{l}, \underbrace{n-1, \dots, n-1}_{k}, \underbrace{n-2, \dots, n-2}_{k}, \dots, \underbrace{1, \dots, 1}_{k}). \tag{46}$$

Remark 1: We consider the q-KZ equation (41) only at $q = -\exp(-\pi i/(k+1))$ and $s = 1$. We want to find the eigenvector of the transfer matrix of the A_{k-1} generalized model as in Eqn.(17). The A_{k-1} generalized model has only one parameter q and Eqn.(17) corresponds to the $s = 1$ case of the q-KZ equation. Therefore, we may focus on the solution of the q-KZ equation at the Razumov-Stroganov point.

By changing the basis from the states of the A_{k-1} generalized model to the spin representation, this solution is regarded as a special solution of the q-KZ equation of the level $1 + \frac{1}{k} - k$ on the spin representation considered in [11].

Remark 2: When we take $k = 2$ and the homogeneous limit (all z_i's tend to 1), we have

$$\lim_{\mathbf{z}\to 1} W^{(k=2)}(\mathbf{z}) \propto 2, 10, 140, \dots. \tag{47}$$

The sum is equal to the total number of HTSASMs (see Eqn.(2)). The obtained sum rule contains the result in [20].

6 Conclusion

We have reviewed the recent developments related to the $O(1)$ loop models. There remain still many open problems. The meaning of the sum rule is clear for the $O(1)$ loop models in the view points of statistical mechanics; the sum is equal to the partition function of the six-vertex model with domain wall boundary conditions. The case for the A_{k-1} generalized models, however, is still not clear. We believe that the total number of a combinatorial object or the partition function of a vertex model with some boundary conditions are related to the sum rule for the A_{k-1} generalized model. Another open problem is the rule for the entries. We need more detailed analysis of the solution of the q-KZ equation on the space of link patterns. The above mentioned problems are related to the affine TL/Hecke algebras of type A. We have models with other boundary conditions [22]. It is interesting to study these models for a consistent understanding of the $O(1)$ loop models.

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