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The varieties of intersections of lines and hypersurfaces in projective spaces

By

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§ 1. Introduction

We denote by X_F the hypersurface in $\mathbf{P}^n$ defined by a homogeneous polynomial $F \in \mathbf{C}[x_0, \dots, x_n]$ of degree d, and denote by $\mathbf{G}$ the set of all lines in $\mathbf{P}^n$. Let $1 \le m \le d+1$. Then the set

$$Y_{F,m} = \{(p, L) \in \mathbf{P}^n \times \mathbf{G} \mid L \text{ and } X_F \text{ intersect at } p \text{ with the multiplicity } \ge m\}$$

form a projective variety, whose defining equations are given by using the higher derivative of F (Theorem 2.1). For a general hypersurface X_F, the projective variety $Y_{F,m}$ is smooth of dimension $2n - m - 1$ (Theorem 3.2). The purpose of this research is to characterize some geometric properties of X_F by using the Hodge structure of $Y_{F,m}$. In this paper, we give a method to describe the Hodge cohomologies of $Y_{F,m}$ by the Jacobian rings, which is a generalization of the theory of Jacobian ring for a hypersurface in $\mathbf{P}^n$ by Griffiths [2]. Using this method, we study the injectivity of the infinitesimal period map for $Y_{F,m}$ (Theorem 6.2). In the case $n = d = 3$, we also yield a Torelli type theorem for the map $X_F \mapsto Y_{F,3}$ (Proposition 6.5). The full-detailed version with all proofs of this article will be appeared somewhere.

§ 2. Varieties of intersections

Let $\mathbf{P}^n$ be a complex projective space of dimension n, and let $V = H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$. We denote by $\mathbf{P} = \text{Grass}\,(n, V)$ the Grassmannian variety of all n-dimensional subspaces

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in V, and denote by $\mathcal{S}_{\mathbf{P}}$ (resp. $\mathcal{Q}_{\mathbf{P}}$) the universal sub (resp. quotient) bundle on $\mathbf{P}$. We have an exact sequence

$$0 \longrightarrow \mathcal{S}_{\mathbf{P}} \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes V \longrightarrow \mathcal{Q}_{\mathbf{P}} \longrightarrow 0.$$

Then $\mathbf{P}$ is naturally identified with $\mathbf{P}^n$, and $\mathcal{Q}_{\mathbf{P}}$ is identified with the tautological line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$. We denote by $\mathbf{G} = \operatorname{Grass}(n-1, V)$ the Grassmannian variety of all $(n-1)$-dimensional subspaces in V, and denote by $\mathcal{S}_{\mathbf{G}}$ (resp. $\mathcal{Q}_{\mathbf{G}}$) the universal sub (resp. quotient) bundle on $\mathbf{G}$. We have an exact sequence

$$0 \longrightarrow \mathcal{S}_{\mathbf{G}} \longrightarrow \mathcal{O}_{\mathbf{G}} \otimes V \longrightarrow \mathcal{Q}_{\mathbf{G}} \longrightarrow 0.$$

We remark that a point in $\mathbf{G}$ corresponds to a line in $\mathbf{P}^n$. Let $p_1 : \mathbf{P} \times \mathbf{G} \to \mathbf{P}$ and $p_2 : \mathbf{P} \times \mathbf{G} \to \mathbf{G}$ be the projections. We denote by Γ the subvariety of $\mathbf{P} \times \mathbf{G}$ defined as the zeros of the composition

$$p_2^*\mathcal{S}_{\mathbf{G}} \longrightarrow \mathcal{O}_{\mathbf{P}\times\mathbf{G}} \otimes V \longrightarrow p_1^*\mathcal{Q}_{\mathbf{P}}.$$

Then Γ is the flag variety of all pairs (p, L) of a point $p \in \mathbf{P}^n$ and a line $L \subset \mathbf{P}^n$ containing the point p. By the first projection $\phi = p_1|_\Gamma$, the subvariety Γ is considered as the $\mathbf{P}^{n-1}$-bundle

$$\phi : \Gamma = \operatorname{Grass}(n-1, \mathcal{S}_{\mathbf{P}}) = \mathbf{P}(\mathcal{S}_{\mathbf{P}}) \longrightarrow \mathbf{P}.$$

By the second projection $\pi = p_2|_\Gamma$, the subvariety Γ is considered as the $\mathbf{P}^1$-bundle

$$\pi : \Gamma = \operatorname{Grass}(1, \mathcal{Q}_{\mathbf{G}}) = \mathbf{P}(\mathcal{Q}_{\mathbf{G}}) \longrightarrow \mathbf{G}.$$

We denote by $\mathcal{Q}_\phi$ the universal quotient bundle of the Grassmannian bundle $\phi : \Gamma \to \mathbf{P}$. We have exact sequences

$$0 \longrightarrow \pi^*\mathcal{S}_{\mathbf{G}} \longrightarrow \phi^*\mathcal{S}_{\mathbf{P}} \longrightarrow \mathcal{Q}_\phi \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{Q}_\phi \longrightarrow \pi^*\mathcal{Q}_{\mathbf{G}} \longrightarrow \phi^*\mathcal{Q}_{\mathbf{P}} \longrightarrow 0.$$

Note that $\mathcal{Q}_\phi$ is an invertible sheaf. We define a decreasing filtration

$$\operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}} = \operatorname{Fil}^0 \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}} \supset \cdots \supset \operatorname{Fil}^{d+1} \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}} = 0$$

on the d-th symmetric product of $\pi^*\mathcal{Q}_{\mathbf{G}}$, as $\operatorname{Fil}^m \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}}$ being the image of the natural homomorphism

$$\operatorname{Sym}^m \mathcal{Q}_\phi \otimes \operatorname{Sym}^{d-m} \pi^*\mathcal{Q}_{\mathbf{G}} \longrightarrow \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}}.$$

Let $F \in \operatorname{Sym}^d V$. We denote by X_F the hypersurface in $\mathbf{P}$ defined as the zeros of the section $[F]_{\mathbf{P}} \in H^0(\mathbf{P}, \operatorname{Sym}^d \mathcal{Q}_{\mathbf{P}})$ which is the image of F by the natural isomorphism

$$\operatorname{Sym}^d V \simeq H^0(\mathbf{P}, \operatorname{Sym}^d \mathcal{Q}_{\mathbf{P}}).$$

We denote by $Y_{F,m}$ the subvariety in Γ defined as the zeros of the section $[F]_{\Gamma,m} \in H^0(\Gamma, \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \operatorname{Fil}^m \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}})$ which is the image of F by the natural homomorphism

$$\operatorname{Sym}^d V \simeq H^0(\Gamma, \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}) \longrightarrow H^0(\Gamma, \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \operatorname{Fil}^m \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}).$$

We denote by Z_F the subvariety in $\mathbf{G}$ defined as the zeros of the section $[F]_{\mathbf{G}} \in H^0(\mathbf{G}, \operatorname{Sym}^d \mathcal{Q}_{\mathbf{G}})$ which is the image of F by the natural isomorphism

$$\operatorname{Sym}^d V \simeq H^0(\mathbf{G}, \operatorname{Sym}^d \mathcal{Q}_{\mathbf{G}}).$$

Then a point in Z_F corresponds to a line which is contained in X_F. Let L be a line in $\mathbf{P}^n$, and let p be a point on L. The fiber of the line bundle $\mathcal{Q}_\phi$ at the point $(p, L) \in \Gamma$ is naturally identified with the kernel of the restriction

$$H^0(L, \mathcal{O}_{\mathbf{P}^n}(1)|_L) \longrightarrow H^0(p, \mathcal{O}_{\mathbf{P}^n}(1)|_p).$$

Hence, L and X_F intersect at p with the multiplicity $\geq m$ if and only if the pair (p, L) represents a point in $Y_{F,m}$. We have a diagram

$$\begin{array}{ccccccc}
 & \mathbf{P} & \xleftarrow{\phi} & \Gamma & \xrightarrow{\pi} & \mathbf{G} \\
 & \bigcup & & \bigcup & & \bigcup \\
X_F = & \phi(Y_{F,1}) & \longleftarrow & Y_{F,1} & \longrightarrow & \pi(Y_{F,1}) \\
 & \bigcup & & \bigcup & & \bigcup \\
 & \vdots & & \vdots & & \vdots \\
 & \bigcup & & \bigcup & & \bigcup \\
 & \phi(Y_{F,d}) & \longleftarrow & Y_{F,d} & \longrightarrow & \pi(Y_{F,d}) \\
 & \bigcup & & \bigcup & & \bigcup \\
 & \phi(Y_{F,d+1}) & \longleftarrow & Y_{F,d+1} & \longrightarrow & \pi(Y_{F,d+1}) = Z_F.
\end{array}$$

The morphism $\phi|_{Y_{F,1}} : Y_{F,1} \to X_F$ is the $\mathbf{P}^{n-1}$-bundle

$$\mathbf{P}(\mathcal{S}_{\mathbf{P}}|_{X_F}) = \mathbf{P}(\Omega^1_{\mathbf{P}} \otimes \mathcal{Q}_{\mathbf{P}}|_{X_F}) \longrightarrow X_F.$$

If X_F is a smooth hypersurface, then $\phi|_{Y_{F,2}} : Y_{F,2} \to X_F$ is the $\mathbf{P}^{n-2}$-bundle

$$\mathbf{P}(\Omega^1_{X_F} \otimes \mathcal{Q}_{\mathbf{P}}|_{X_F}) \longrightarrow X_F.$$

The morphism $\pi|_{Y_{F,m}} : Y_{F,m} \to \pi(Y_{F,m})$ is generically finite for $1 \le m \le d$, and the morphism $\pi|_{Y_{F,d+1}} : Y_{F,d+1} \to Z_F$ is the $\mathbf{P}^1$-bundle

$$\mathbf{P}(\mathcal{Q}_{\mathbf{G}}|_{Z_F}) \longrightarrow Z_F.$$

We remark that the isomorphism

$$\sigma : \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \operatorname{Fil}^m \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} \xrightarrow{\sim} \operatorname{Sym}^{d-m+1} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^{m-1} \pi^* \mathcal{Q}_{\mathbf{G}}$$

is induced by the homomorphism

$$\begin{aligned} \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} &\longrightarrow \operatorname{Sym}^{d-m+1} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^{m-1} \pi^* \mathcal{Q}_{\mathbf{G}}; \\ A_1 \cdots A_d &\longmapsto \tfrac{1}{d!} \textstyle\sum_{\sigma \in \mathfrak{S}_d} [A_{\sigma(1)} \cdots A_{\sigma(d-m+1)}]_{\mathbf{P}} \otimes A_{\sigma(d-m+2)} \cdots A_{\sigma(d)}, \end{aligned}$$

where $[A]_{\mathbf{P}} \in \operatorname{Sym}^j \phi^* \mathcal{Q}_{\mathbf{P}}$ denotes the image of a local section $A \in \operatorname{Sym}^j \pi^* \mathcal{Q}_{\mathbf{G}}$, and $\mathfrak{S}_d$ denotes the permutation group of the index set $\{1, \ldots, d\}$.

For $F \in \operatorname{Sym}^d V$, we define the tensor $F_k \in \operatorname{Sym}^{d-k} V \otimes \operatorname{Sym}^k V$ by

$$F_k = \frac{(d-k+1)!}{d!} \sum_{0 \le i_1, \ldots, i_{k-1} \le n} \frac{\partial^{k-1} F}{\partial x_{i_1} \cdots \partial x_{i_{k-1}}} \otimes x_{i_1} \cdots x_{i_{k-1}},$$

which does not depend on the choice of the basis $(x_0, \ldots, x_n)$ of V.

Theorem 2.1. *The subvariety $Y_{F,m}$ in Γ is defined as the zeros of the section*

$$F_m \in \operatorname{Sym}^{d-m+1} V \otimes \operatorname{Sym}^{m-1} V \simeq H^0(\Gamma, \operatorname{Sym}^{d-m+1} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^{m-1} \pi^* \mathcal{Q}_{\mathbf{G}}).$$

§ 3. Smoothness and connectedness

Since the variety $Y_{F,d+1}$ is a $\mathbf{P}^1$-bundle over Z_F, the following theorem is directly induced from the results in [1, Theorem 8] and [5, Chapter V. 4].

Theorem 3.1. *Assume $d \ge 1$.*

(1) *If $d \ge 2n-2$, then $Y_{F,d+1}$ is empty for general $F \in \operatorname{Sym}^d V \setminus \{0\}$.*

(2) *If $d \le 2n-3$, then $Y_{F,d+1}$ is non-empty for any $F \in \operatorname{Sym}^d V \setminus \{0\}$.*

(3) *If $d \le 2n-3$, then $Y_{F,d+1}$ is smooth of dimension $2n-d-2$ for general $F \in \operatorname{Sym}^d V \setminus \{0\}$.*

(4) *If $d \le 2n-4$ and $(d,n) \ne (2,3)$, then $Y_{F,d+1}$ is connected for any $F \in \operatorname{Sym}^d V \setminus \{0\}$.*

For $1 \le m \le d$, we have the following theorem.

Theorem 3.2. *Assume $1 \leq m \leq d$.*

(1) *If $m \geq 2n$, then $Y_{F,m}$ is empty for general $F \in \mathrm{Sym}^d V \setminus \{0\}$.*

(2) *If $m \leq 2n-1$, then $Y_{F,m}$ is non-empty for any $F \in \mathrm{Sym}^d V \setminus \{0\}$.*

(3) *If $m \leq 2n-1$, then $Y_{F,m}$ is smooth of dimension $2n-m-1$ for general $F \in \mathrm{Sym}^d V \setminus \{0\}$.*

(4) *If $m \leq 2n-2$, then $Y_{F,m}$ is connected for any $F \in \mathrm{Sym}^d V \setminus \{0\}$.*

In the case when X_F is a cubic hypersurface in $\mathbf{P}^n$, the variety $Y_{F,m}$ is smooth of dimension $2n-m-1$ if and only if X_F is smooth.

If $Y_{F,m}$ is smooth of dimension $2n-m-1$, then we can compute some topological invariants of $Y_{F,m}$. For example, if $m = d = 2n-1$, then $\dim Y_{F,m} = 0$, and we can compute the number of the points of $Y_{F,m}$ by Schubert calculus;

$$\begin{cases} n=1,\ m=d=1 & \Longrightarrow \sharp Y_{F,m} = 1. \\ n=2,\ m=d=3 & \Longrightarrow \sharp Y_{F,m} = 9, \\ n=3,\ m=d=5 & \Longrightarrow \sharp Y_{F,m} = 575, \\ n=4,\ m=d=7 & \Longrightarrow \sharp Y_{F,m} = 99715, \\ & \quad \cdots \end{cases}$$

for general F. Similarly, if $d = 2n-3$, then $\dim Z_F = 0$ and we have

$$\begin{cases} n=2,\ d=1 & \Longrightarrow \sharp Z_F = 1. \\ n=3,\ d=3 & \Longrightarrow \sharp Z_F = 9 \times 3 = 27, \\ n=4,\ d=5 & \Longrightarrow \sharp Z_F = 575 \times 5 = 2785, \\ n=5,\ d=7 & \Longrightarrow \sharp Z_F = 99715 \times 7 = 698005, \\ & \quad \cdots \end{cases}$$

for general F, that is known in [3]. When $\dim Y_{F,m} = 1$, we can compute the genus $\mathrm{g}(Y_{F,m})$ of $Y_{F,m}$. For example, if $m = d = 2n-2$, then $\dim Y_{F,m} = 1$ and we have

$$\begin{cases} n=2,\ m=d=2 & \Longrightarrow \mathrm{g}(Y_{F,m}) = 0. \\ n=3,\ m=d=4 & \Longrightarrow \mathrm{g}(Y_{F,m}) = 201, \\ n=4,\ m=d=6 & \Longrightarrow \mathrm{g}(Y_{F,m}) = 75601, \\ n=5,\ m=d=8 & \Longrightarrow \mathrm{g}(Y_{F,m}) = 39001985, \\ & \quad \cdots \end{cases}$$

for general F.

§ 4. Jacobian rings

We denote by

$$S = \mathbf{C}[x_0, \ldots, x_n, z_0, \ldots, z_n] = \bigoplus_{a,b \in \mathbf{Z}} S^{a,b}$$

the polynomial ring bi-graded by $\deg x_i = (1,0)$ and $\deg z_j = (0,1)$. We define homomorphisms δ and ε by

$$\delta : S^{a,b} \longrightarrow S^{a-1,b+1};\ A \mapsto \frac{1}{a} \sum_{i=0}^{n} \frac{\partial A}{\partial x_i} \cdot z_i$$

and

$$\varepsilon : S^{a,b} \longrightarrow S^{a+1,b-1};\ A \mapsto \frac{1}{b} \sum_{i=0}^{n} \frac{\partial A}{\partial z_i} \cdot x_i.$$

Let V be the $(n+1)$-dimensional vector space as in Section 2. For $F \in \operatorname{Sym}^d V$, we have a bi-homogeneous polynomial $F_1 \in S^{d,0}$ by considering $x_0, \ldots, x_n$ as a basis of V. We set the bi-homogeneous polynomial F_k by

$$F_k = \delta^{k-1}(F_1) \in S^{d-k+1,k-1}$$

for $k \geq 1$. We define the bi-graded ring $S_{F,m}$ by

$$S_{F,m} = S/(F_k;\ 1 \leq k \leq m),$$

and we define the *Jacobian ring* $R_{F,m}$ as the bi-graded ring by

$$R_{F,m} = S_{F,m-1} \Big/ \Big(\frac{\partial F_m}{\partial x_i} \cdot x_j + \frac{\partial F_m}{\partial z_i} \cdot z_j;\ 0 \leq i \leq n,\ 0 \leq j \leq n \Big)$$

for $m \geq 1$, where we set $S_{F,0} = S$. Since

$$\frac{1}{d} \sum_{i=0}^{n} \Big(\frac{\partial F_m}{\partial x_i} \cdot x_i + \frac{\partial F_m}{\partial z_i} \cdot z_i \Big) = F_m,$$

$S_{F,m-1} \to R_{F,m}$ factors through $S_{F,m}$. We set

$$S^{a,b,c}_{F,m} = \operatorname{Ker}\,(\varepsilon^c : S^{a,b}_{F,m} \longrightarrow S^{a+c,b-c}_{F,m}).$$

In the following, we describe the relation between these rings and the variety $Y_{F,m}$. We denote by T_Γ (resp. $T_{Y_{F,m}}$) the tangent bundle of Γ (resp. $Y_{F,m}$). Then we have the exact sequences

$$0 \longrightarrow \mathcal{O}_\Gamma \longrightarrow \phi^* \mathcal{S}^\vee_{\mathbf{P}} \otimes \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow T_\Gamma \longrightarrow 0,$$

where $\mathcal{S}_{\mathbf{P}}^{\vee}$ denotes the $\mathcal{O}_{\Gamma}$-dual of $\mathcal{S}_{\mathbf{P}}$. If $Y_{F,m}$ is smooth of dimension $2n-m-1$, then we define the coherent sheaf $\mathcal{N}_m$ of $\mathcal{O}_{Y_{F,m-1}}$-modules by

$$\mathcal{N}_m = \operatorname{Coker}(T_{Y_{F,m-1}}(-\log Y_{F,m}) \longrightarrow T_{\Gamma}|_{Y_{F,m-1}}).$$

Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{Y_{F,m-1}} \longrightarrow (\operatorname{Sym}^{d-m+1} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^{m-1} \pi^* \mathcal{Q}_{\mathbf{G}})|_{Y_{F,m-1}} \longrightarrow \mathcal{N}_m \longrightarrow 0.$$

Using Lemma 5.5 in the next section, we have the following proposition.

Proposition 4.1. *If $Y_{F,m}$ is smooth of dimension $2n-m-1$, then*

$$H^0(Y_{F,m-1}, \mathcal{N}_m) \simeq S_{F,m}^{d-m+1,m-1}$$

for $1 \leq m \leq n-1$.

We remark that the composition

$$V^{\vee} \otimes \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow \phi^* \mathcal{S}_{\mathbf{P}}^{\vee} \otimes \pi^* \mathcal{Q}_{\mathbf{G}} \longrightarrow T_{\Gamma} \longrightarrow \mathcal{N}_m$$

induces the homomorphism

$$\begin{array}{ccc} V^{\vee} \otimes V \simeq V^{\vee} \otimes H^0(\Gamma, \pi^* \mathcal{Q}_{\mathbf{G}}) & \longrightarrow & H^0(Y_{F,m-1}, \mathcal{N}_m) \simeq S_{F,m}^{d-m+1,m-1}; \\ x_i^{\vee} \otimes x_j & \longmapsto & \frac{\partial F_m}{\partial x_i} \cdot x_j + \frac{\partial F_m}{\partial z_i} \cdot z_j, \end{array}$$

where $x_0^{\vee}, \ldots, x_n^{\vee}$ denotes the dual basis of $x_0, \ldots, x_n$. Using Lemma 5.6 in the next section, we have the following theorem.

Theorem 4.2. *If $Y_{F,m}$ is smooth of dimension $2n-m-1$, then there is a natural injective homomorphism*

$$\rho : R_{F,m}^{d-m+1,m-1} \longrightarrow H^1(Y_{F,m-1}, T_{Y_{F,m-1}}(-\log Y_{F,m})),$$

and it is an isomorphism for $m \leq n-2$.

We set the integers $\alpha(n,m,d,q)$ and $\beta(n,m,q)$ by

$$\begin{cases} \alpha(n,m,d,q) = md - \frac{m(m-1)}{2} - n - 2 + q(d-m+1), \\ \beta(n,m,q) = \frac{m(m-1)}{2} - n + q(m-1). \end{cases}$$

Since $\Omega_{Y_{F,m-1}}^{2n-m}(Y_{F,m})$ is isomorphic to

$$(\operatorname{Sym}^{\alpha(n,m,d,0)} \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^{\beta(n,m,0)} \mathcal{Q}_{\phi})|_{Y_{F,m-1}},$$

using Lemma 5.5 in the next section, we have the following theorem.

Theorem 4.3. *If $Y_{F,m}$ is smooth of dimension $2n-m-1$, then there is a natural injective homomorphism*

$$\gamma_0 : S_{F,m-1}^{\alpha(n,m,d,0),\beta(n,m,0),1} \longrightarrow H^0(Y_{F,m-1}, \Omega^{2n-m}_{Y_{F,m-1}}(Y_{F,m})),$$

and it is an isomorphism for $m \leq n-1$.

Here we remark that $S_{F,m-1}^{\alpha(n,m,d,0),\beta(n,m,0)} = S_{F,m-1}^{\alpha(n,m,d,0),\beta(n,m,0),1}$ for $\frac{m(m-1)}{2} \leq n$.

The following theorem is proved by the similar way as Theorem 4.2, by using the exact sequence

$$0 \to \Omega^{2n-m-1}_{Y_{F,m-1}}(\log Y_{F,m}) \to T_\Gamma|_{Y_{F,m-1}} \otimes \Omega^{2n-m}_{Y_{F,m-1}}(Y_{F,m}) \to \mathcal{N}_m \otimes \Omega^{2n-m}_{Y_{F,m-1}}(Y_{F,m}) \to 0.$$

Theorem 4.4. *If $\frac{m(m-1)}{2} = n$ and $Y_{F,m}$ is smooth of dimension $2n-m-1$, then there is a natural injective homomorphism*

$$\gamma_1 : R_{F,m}^{\alpha(n,m,d,1),\beta(n,m,1)} \longrightarrow H^1(Y_{F,m-1}, \Omega^{2n-m-1}_{Y_{F,m-1}}(\log Y_{F,m})),$$

and it is an isomorphism for $m \leq n-2$.

§ 5. Computation of cohomology

In this section, we enumerate several lemmas, which is used in the proof of theorems in Section 4. For simplicity of notations, we set the invertible sheaf $\mathcal{O}_\Gamma(p,q)$ on Γ by

$$\mathcal{O}_\Gamma(p,q) = \begin{cases} \operatorname{Sym}^p \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^q \mathcal{Q}_\phi & (p \geq 0,\ q \geq 0), \\ \operatorname{Sym}^p \phi^* \mathcal{Q}_{\mathbf{P}} \otimes \operatorname{Sym}^{-q} \mathcal{Q}_\phi^\vee & (p \geq 0,\ q < 0), \\ \operatorname{Sym}^{-p} \phi^* \mathcal{Q}_{\mathbf{P}}^\vee \otimes \operatorname{Sym}^q \mathcal{Q}_\phi & (p < 0,\ q \geq 0), \\ \operatorname{Sym}^{-p} \phi^* \mathcal{Q}_{\mathbf{P}}^\vee \otimes \operatorname{Sym}^{-q} \mathcal{Q}_\phi^\vee & (p < 0,\ q < 0), \end{cases}$$

and we set $Q^r_{\mathbf{G}} = \operatorname{Sym}^r \pi^* \mathcal{Q}_{\mathbf{G}}$ for $r \geq 0$. For a sheaf $\mathcal{E}$ of $\mathcal{O}_\Gamma$-modules, we set $\mathcal{E}(p,q) = \mathcal{E} \otimes \mathcal{O}_\Gamma(p,q)$.

Lemma 5.1. *Assume $r \geq 0$.*

$$H^0(\Gamma, Q^r_{\mathbf{G}}(p,q)) = \operatorname{Ker}(\varepsilon^{r+1} : S^{p,q+r} \to S^{p+r+1,q-1}).$$

Lemma 5.2. *Assume $q \leq 0$ and $r \geq 0$.*

(1) *$H^j(\Gamma, Q^r_{\mathbf{G}}(p,q)) = 0$ for $1 \leq j \leq n-2$.*

(2) *When $n \geq 2$, if $q \geq -n+1$ or $p+r \leq -2$, then $H^{n-1}(\Gamma, Q^r_{\mathbf{G}}(p,q)) = 0$.*

Lemma 5.3. *Assume $q \leq 0$.*

(1) $H^j(\Gamma, T_\Gamma(p,q)) = 0$ *for* $1 \leq j \leq n-3$.

(2) *When* $n \geq 3$, *if* $q \geq -n+1$ *or* $p \leq -2$, *then* $H^{n-2}(\Gamma, T_\Gamma(p,q)) = 0$.

Lemma 5.4. *Assume $q \leq 0$ and $r \geq 0$.*

(1) $H^1(Y_{F,m}, Q^r_{\mathbf{G}}(p,q)|_{Y_{F,m}}) = 0$ *for* $1 \leq m \leq n-3$.

(2) *If* $q \geq \frac{n^2-7n+8}{2}$ *or* $p+r \leq (n-2)d - \frac{n^2-5n+10}{2}$, *then*

$$H^1(Y_{F,n-2}, Q^r_{\mathbf{G}}(p,q)|_{Y_{F,n-2}}) = 0.$$

Lemma 5.5. *Assume $r \geq 0$.*

(1) $H^0(Y_{F,m}, Q^r_{\mathbf{G}}(p,q)|_{Y_{F,m}}) \simeq \operatorname{Ker}(\varepsilon^{r+1} : S^{p,q+r}_{F,m} \to S^{p+r+1,q-1}_{F,m})$ *for* $1 \leq m \leq n-2$.

(2) *If* $\min\{q,0\} \geq \frac{n^2-5n+4}{2}$ *or* $p+r+\max\{q,0\} \leq (n-1)d - \frac{n^2-3n+6}{2}$, *then*

$$H^0(Y_{F,n-1}, Q^r_{\mathbf{G}}(p,q)|_{Y_{F,n-1}}) \simeq \operatorname{Ker}(\varepsilon^{r+1} : S^{p,q+r}_{F,n-1} \to S^{p+r+1,q-1}_{F,n-1}).$$

Lemma 5.6. *Assume $q \leq 0$.*

(1) $H^1(Y_{F,m}, T_\Gamma(p,q)|_{Y_{F,m}}) = 0$ *for* $1 \leq m \leq n-4$.

(2) *If* $q \geq \frac{n^2-9n+14}{2}$ *or* $p \leq (n-3)d - \frac{n^2-7n+16}{2}$, *then*

$$H^1(Y_{F,n-3}, T_\Gamma(p,q)|_{Y_{F,n-3}}) = 0.$$

§ 6. The case $n = 3$

In this section, we consider a hypersurface X_F in $\mathbf{P}^3$. Then $Y_{F,1}$ is a $\mathbf{P}^2$-bundle over X_F. If X_F is a smooth hypersurface, then $Y_{F,2}$ is a $\mathbf{P}^1$-bundle over X_F. If $d \geq 4$, then $Y_{F,4}$ is a smooth algebraic curve of genus $31d^3 - 158d^2 + 186d + 1$ for general F. If $d \geq 5$, then $\dim Y_{F,5} = 0$ and $\sharp Y_{F,5} = 5d(d-4)(7d-12)$ for general F. In the following, we study the variety $Y_{F,3}$. If $Y_{F,3}$ is smooth, then $Y_{F,3}$ is an algebraic surface of the square of the first chern class $c_1^2 = 2d(3d-8)^2$ and the second chern class $c_2 = 2d(11d^2 - 48d + 54)$.

Proposition 6.1. *If the variety $Y_{F,3}$ is smooth of dimension 2, then the morphism $\phi|_{Y_{F,3}} : Y_{F,3} \to X_F$ is the double covering branched along B_F, where B_F is the divisor on X_F defined by the equation*

$$\det\Big(\frac{\partial^2 F}{\partial x_i \partial x_j}\Big)_{0 \leq i,j \leq 3} = 0.$$

By the results in Section 4, we have natural injective homomorphisms

$$\rho : R_{F,3}^{d-2,2} \longrightarrow H^1(Y_{F,2}, T_{Y_{F,2}}(-\log Y_{F,3})),$$

$$\gamma_0 : S_{F,2}^{3d-8,0} \longrightarrow H^0(Y_{F,2}, \Omega^3_{Y_{F,2}}(Y_{F,3}))$$

and

$$\gamma_1 : R_{F,3}^{4d-10,2} \longrightarrow H^1(Y_{F,2}, \Omega^2_{Y_{F,2}}(\log Y_{F,3})).$$

By the similar way, we have a natural surjective homomorphism

$$R_{F,3}^{7d-18,2} \simeq H^1(Y_{F,2}, \Omega^3_{Y_{F,2}}(Y_{F,3}) \otimes \Omega^2_{Y_{F,2}}(\log Y_{F,3})) \longrightarrow H^1(Y_{F,2}, T_{Y_{F,2}}(-\log Y_{F,3}))^\vee.$$

Since the multiplication map

$$S_{F,2}^{3d-8,0} \otimes R_{F,3}^{4d-10,2} \longrightarrow R_{F,3}^{7d-18,2}$$

is surjective, we have the following theorem.

Theorem 6.2. *If $d \geq 3$ and $Y_{F,3}$ is smooth of dimension 2, then the homomorphism*

$$H^1(Y_{F,2}, T_{Y_{F,2}}(-\log Y_{F,3})) \longrightarrow \mathrm{Hom}_{\mathbf{C}}\left(H^0(Y_{F,2}, \Omega^3_{Y_{F,2}}(Y_{F,3})), H^1(Y_{F,2}, \Omega^2_{Y_{F,2}}(\log Y_{F,3}))\right)$$

is injective.

We consider the period map

$$\psi : M \longrightarrow W;\ [X_F] \longmapsto [H^3(Y_{F,2} \setminus Y_{F,3})],$$

where M denotes the set of isomorphism classes of hypersurfaces X_F in $\mathbf{P}^3$ such that $Y_{F,3}$ is smooth, and W denotes the set of isomorphism classes of Hodge structures of weight 2. By Theorem 6.2, the differential $\mathrm{d}\psi$ of the period map ψ at a general point in M is injective, where we remark that the sets M and W have geometric structure. Now we have a natural question of Torelli type.

Question 6.3. For smooth surfaces X_{F_1} and X_{F_2} in $\mathbf{P}^3$, if there is an isomorphism $H^3(Y_{F_1,2} \setminus Y_{F_1,3}) \simeq H^3(Y_{F_2,2} \setminus Y_{F_2,3})$ as Hodge structures, then is there an isomorphism $X_{F_1} \simeq X_{F_2}$ as algebraic varieties?

§ 6.1. The case $d = 3$

We assume that $d = 3$. If $Y_{F,3}$ is smooth, then $Y_{F,3}$ is a minimal algebraic surface with the geometric genus $p_g = 4$, the irregularity $q = 0$ and the square of the first chern class $c_1^2 = 6$. Such algebraic surfaces are classified by Horikawa, and $Y_{F,3}$ is called of type Ib in [4]. For $F \in S^{3,0}$, the cubic surface X_F is smooth if and only if $Y_{F,3}$ is a smooth surface. If X_F is a smooth cubic surface, then X_F contains 27 lines, which means that $\sharp Z_F = 27$. Hence $Y_{F,4}$ is a disjoint union of 27 rational curves, which are (-3)-curves in $Y_{F,3}$.

Proposition 6.4. *If X_F is a smooth cubic surface, then B_F has at most nodes as its singularities. A point $p \in X_F$ is a node of B_F if and only if there are three lines in X_F which contains the point p.*

Since the morphism $\phi|_{Y_{F,3}} : Y_{F,3} \to \mathbf{P}^3$ is the canonical map for $d = 3$, we have the following proposition.

Proposition 6.5. *For smooth cubic surfaces X_{F_1} and X_{F_2}, there is an isomorphism $X_{F_1} \simeq X_{F_2}$ if and only if there is an isomorphism $Y_{F_1,3} \simeq Y_{F_2,3}$.*

In the case when $d = 3$, the Hodge structure $H^2(X_F)$ is trivial, but the Hodge structure $H^3(Y_{F,2} \backslash Y_{F,3})$ is not trivial. Hence the Question 6.3 is particularly interesting in this case.

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