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On moduli spaces of quiver representations associated with dimer models

Dedicated to Professors Iku Nakamura and Eiichi Sato on their sixtieth birthdays

By

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§ 1. Introduction

Dimer models are introduced by string theorists to study four-dimensional $N = 1$ superconformal field theories. See e.g. a review by Kennaway [5] and references therein for a physical background. A dimer model is a bipartite graph on a real two-torus which encodes the information of a quiver with relations. A typical example of such a quiver is the McKay quiver determined by a finite abelian subgroup G of $SL(3, \mathbb{C})$ (see [8, 10]). In this case, the moduli space of representations of the McKay quiver (for the dimension vector $(1, 1, \ldots, 1)$) coincides with the moduli space of G-constellations considered in [1]. For a generic choice of a stability parameter θ, the moduli space of G-constellations is a crepant resolution of the quotient singularity $\mathbb{C}^3/G$ and the derived category of coherent sheaves on the moduli space is equivalent to the derived category of finitely-generated modules over the path algebra of the McKay quiver. It is expected that these kinds of statements can be generalized to the case of dimer models that are "consistent" in the physics context, which should be called "brane tilings". In this note, we discuss a slightly weaker notion of *non-degenerate dimer models*, which is strong enough to ensure that the moduli space is a crepant resolution of the three-dimensional toric singularity

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determined by the Newton polygon of the characteristic polynomial (see Theorem 6.4). We expect that one has to impose further conditions to prove the derived equivalence.

For the proof, we use a generalization of the description of a torus-fixed point on the moduli space in terms of a choice of a covering by hexagons of the fundamental region of a real 2-torus due to Nakamura [7]. Many of the arguments are similar to those in [4]. There is also a physics paper by Franco and Vegh [2] which deals with the relation between brane tilings and moduli spaces.

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§ 2. Dimer models and quivers

Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a real two-torus equipped with an orientation. A *bipartite graph* on T consists of

- a set $B \subset T$ of black vertices,
- a set $W \subset T$ of white vertices, and
- a set E of edges, consisting of embedded closed intervals e on T such that one boundary of e belongs to B and the other boundary belongs to W. We assume that two edges intersect only at the boundaries.

A bipartite graph on T is called a *dimer model* if the set of edges divide T into simply-connected polygons.

A *quiver* consists of

- a set V of vertices,
- a set A of arrows, and
- two maps $s, t : A \to V$ from A to V.

For an arrow $a \in A$, $s(a)$ and $t(a)$ are said to be the *source* and the *target* of a respectively. A *path* on a quiver is an ordered set of arrows $(a_n, a_{n-1}, \dots, a_1)$ such that $s(a_{i+1}) = t(a_i)$ for $i = 1, \dots, n-1$. We also allow for a path of length zero, starting and ending at the same vertex. The *path algebra* $\mathbb{C}Q$ of a quiver $Q = (V, A, s, t)$ is the algebra spanned by the set of paths as a vector space, and the multiplication is defined by the concatenation of paths;

$$(b_m, \dots, b_1) \cdot (a_n, \dots, a_1) = \begin{cases} (b_m, \dots, b_1, a_n, \dots, a_1) & s(b_1) = t(a_n), \\ 0 & \text{otherwise.} \end{cases}$$

A *quiver with relations* is a pair of a quiver and a two-sided ideal $\mathcal{I}$ of its path algebra. For a quiver $\Gamma = (Q, \mathcal{I})$ with relations, its path algebra $\mathbb{C}\Gamma$ is defined as the quotient algebra $\mathbb{C}Q/\mathcal{I}$.

A dimer model (B, W, E) encodes the information of a quiver $\Gamma = (V, A, s, t, \mathcal{I})$ with relations in the following way: The set V of vertices is the set of connected components of the complement $T \setminus (\bigcup_{e \in E} e)$, and the set A of arrows is the set E of edges of the graph. The directions of the arrows are determined by the colors of the vertices of the graph, so that the white vertex $w \in W$ is on the right of the arrow. In other words, the quiver is the dual graph of the dimer model equipped with an orientation given by rotating the white-to-black flow on the edges of the dimer model by minus 90 degrees.

The relations of the quiver are described as follows: For an arrow $a \in A$, there exist two paths $p_+(a)$ and $p_-(a)$ from $t(a)$ to $s(a)$, the former going around the white vertex connected to $a \in E = A$ clockwise and the latter going around the black vertex connected to a counterclockwise. Then the ideal $\mathcal{I}$ of the path algebra is generated by $p_+(a) - p_-(a)$ for all $a \in A$.

A *representation* of Γ is a module over the path algebra $\mathbb{C}\Gamma$ with relations. In other words, a representation is a collection $((V_v)_{v \in V}, (\psi(a))_{a \in A})$ of vector spaces V_v for $v \in V$ and linear maps $\psi(a) : V_{s(a)} \to V_{t(a)}$ for $a \in A$ satisfying relations in $\mathcal{I}$. The Grothendieck group of the abelian category $\operatorname{mod} \mathbb{C}\Gamma$ of finite dimensional representations of $\mathbb{C}\Gamma$ is a free abelian group generated by simple representations corresponding to the idempotents of $\mathbb{C}\Gamma$ given as the paths of length zero. A simple representation corresponding to a vertex $v \in V$ has $V_v = \mathbb{C}$, $V_w = 0$ for $w \neq v$ and $\psi(a) = 0$ for any $a \in A$. Let N be the number of vertices of Γ. Then the Grothendieck group is isomorphic to $\mathbb{Z}^N$ with respect to this basis, and the class of a module in the Grothendieck group considered as an element of $\mathbb{Z}^N$ is called its *dimension vector*. The dimension vector of a representation $((V_v)_{v \in V}, (\psi(a))_{a \in A})$ is given by $(\dim V_v)_{v \in V}$.

The *double* $\overline{Q}$ of a quiver Q is obtained from Q by adding an arrow $\overline{a}$ with $s(\overline{a}) = t(a)$ and $t(\overline{a}) = s(a)$ for each arrow a of Q. A representation $\Psi = ((V_v)_{v \in V}, (\psi(a))_{a \in A})$ of Q such that all $\psi(a)$ are linear isomorphisms determines a representation $\overline{\Psi}$ of $\overline{Q}$ by $\overline{\Psi}(a) = \Psi(a)$ and $\overline{\Psi}(\overline{a}) = \Psi(a)^{-1}$.

A *perfect matching* (or a *dimer configuration*) on a dimer model $G = (B, W, E)$ is a subset D of E such that for any vertex $v \in B \cup W$, there is a unique edge $e \in D$ connected to v. Consider the bipartite graph $\widetilde{G}$ on $\mathbb{R}^2$ obtained from G by pulling-back by the natural projection $\mathbb{R}^2 \to T$, and identify the set of perfect matchings of G with the set of periodic perfect matchings of $\widetilde{G}$. Fix a reference perfect matching D_0. Then for any perfect matching D, the union $D \cup D_0$ divides $\mathbb{R}^2$ into connected components. The height function h_{D,D_0} is a locally-constant function on $\mathbb{R}^2 \setminus (D \cup D_0)$ which increases (resp. decreases) by 1 when one crosses an edge $e \in D$ with the black (resp. white) vertex on his right or an edge $e \in D_0$ with the white (resp. black) vertex on his right. This rule determines the height function up to additions of constants. The height function may not be periodic even if D and D_0 are periodic, and the *height change* $h(D, D_0) = (h_x(D, D_0), h_y(D, D_0)) \in \mathbb{Z}^2$ of D with respect to D_0 is defined as the difference

$$h_x(D, D_0) = h_{D,D_0}(p + (1, 0)) - h_{D,D_0}(p),$$
$$h_y(D, D_0) = h_{D,D_0}(p + (0, 1)) - h_{D,D_0}(p)$$

of the height function, which does not depend on the choice of $p \in \mathbb{R}^2 \setminus (D \cup D_0)$. More invariantly, height changes can be considered as an element of $H^1(T, \mathbb{Z})$. The dependence of the height change on the choice of the reference matching is given by

$$h(D, D_1) = h(D, D_0) - h(D_1, D_0)$$

for any three perfect matchings D, D_0 and D_1. We will often suppress the dependence of the height difference on the reference matching and just write $h(D) = h(D, D_0)$.

For a fixed reference matching D_0, the characteristic polynomial of G is defined by

$$Z(x, y) = \sum_{D \in \mathrm{Perf}(G)} x^{h_x(D)} y^{h_y(D)},$$

where $\mathrm{Perf}(G)$ denotes the set of perfect matchings of G. It is a Laurent polynomial in two variables, whose Newton polygon coincides with the convex hull of the set

$$\{(h_x(D), h_y(D)) \in \mathbb{Z}^2 \mid D \in \mathrm{Perf}(G)\}$$

consisting of height changes of perfect matchings of the dimer model G.

A perfect matching can be considered as a set of walls which block some of the arrows; for a perfect matching D, let Q_D be the subquiver of Q whose set of vertices is the same as Q and whose set of arrows consists of $A \setminus D$ (recall that $A = E$). The path algebra $\mathbb{C}Q_D$ of Q_D is a subalgebra of $\mathbb{C}Q$, and the ideal $\mathcal{I}$ of $\mathbb{C}Q$ defines an ideal $\mathcal{I}_D = \mathcal{I} \cap \mathbb{C}Q_D$ of $\mathbb{C}Q_D$. A path $p \in \mathbb{C}Q$ is said to be an allowed path with respect to D if $p \in \mathbb{C}Q_D$.

As an example, consider the dimer model in Figure 1. The corresponding quiver is shown in Figure 2, whose relations are given by

$$\mathcal{I} = (dbc - cbd, dac - cad, adb - bda, acb - bca).$$

This dimer model is non-degenerate, and has four perfect matchings $D_0, \ldots, D_3$ shown in Figure 3. Their height changes with respect to D_0 are given by

$$h(D_0) = (0, 0), \quad h(D_1) = (1, 0), \quad h(D_2) = (0, 1), \quad h(D_3) = (1, 1),$$

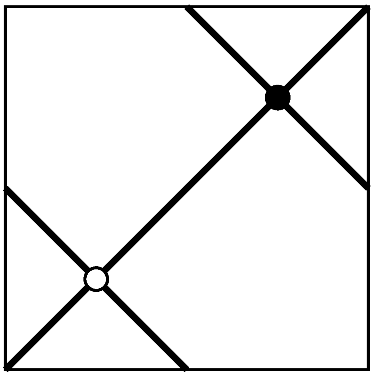


Figure 1. A dimer model

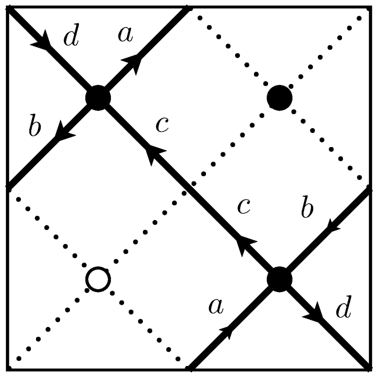
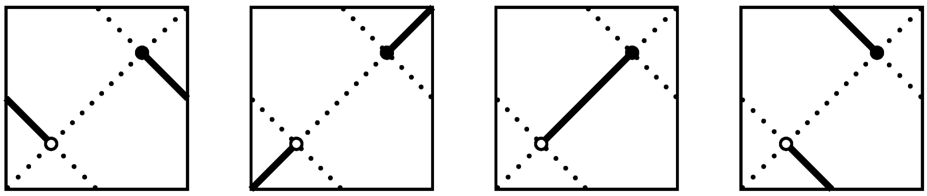


Figure 2. The corresponding quiver



D_0 D_1 D_2 D_3

Figure 3. Four perfect matchings

so that the characteristic polynomial is

$$Z(x, y) = 1 + x + y + xy.$$

§ 3. Torus actions on moduli spaces

Let $\Gamma = (V, A, s, t, \mathcal{I})$ be the quiver with relations obtained from a dimer model (B, W, E) on a real two-torus $T = \mathbb{R}^2/\mathbb{Z}^2$ and $\widetilde{\mathcal{M}}$ be the set of representations of Γ with dimension vector $(1, \dots, 1)$. In other words, $\widetilde{\mathcal{M}}$ is the subset of $\mathbb{C}^A$ consisting of linear maps $\psi(a) : \mathbb{C}s(a) \to \mathbb{C}t(a)$ for arrows $a \in A$ satisfying relations. $\widetilde{\mathcal{M}}$ has a natural scheme structure as a closed subscheme of $\mathbb{C}^A$ defined by the ideal generated by the relations. Let $\prod_v \operatorname{Aut}(V_v) \cong (\mathbb{C}^\times)^V$ be the product of the set of automorphisms of the vector spaces attached to vertices of the quiver. Two representations $((V_v)_{v\in V}, (\psi(a))_{a\in A})$ and $((W_v)_{v\in V}, (\phi(a))_{a\in A})$ will be called isomorphic if there is an element $(g_v)_{v\in V}$ such that for any $a \in A$, the following diagram commutes;

$$\begin{tikzcd}
V_{s(a)} \arrow[r, "\psi(a)"] \arrow[d, "g_{s(a)}"'] & V_{t(a)} \arrow[d, "g_{t(a)}"] \\
W_{s(a)} \arrow[r, "\phi(a)"] & W_{t(a)}.
\end{tikzcd}$$

The diagonal subgroup $\mathbb{C}^\times \subset (\mathbb{C}^\times)^V$ acts trivially on $\widetilde{\mathcal{M}}$ and the quotient $\mathcal{G} = (\mathbb{C}^\times)^V/\mathbb{C}^\times$ acts faithfully on $\widetilde{\mathcal{M}}$. The set-theoretic quotient of $\widetilde{\mathcal{M}}$ by the action of $\mathcal{G}$ will be denoted by $\mathcal{M}$. Let $\mathbb{T} \subset \mathcal{M}$ be the subset consisting of isomorphism classes $[(\psi(a))_{a\in A}]$ such that $\psi(a) \in \mathbb{C}^\times$ for any $a \in A$. It has a structure of an algebraic torus by the pointwise multiplication: For two elements $[(\psi(a))_{a\in A}]$ and $[(\phi(a))_{a\in A}]$ of $\mathbb{T}$, their composition is defined by

$$[(\psi(a))_{a\in A}] \cdot [(\phi(a))_{a\in A}] = [(\psi(a) \cdot \phi(a))_{a\in A}], \tag{3.1}$$

which gives an element of $\mathbb{T}$ again.

The set $\mathcal{M}$ of isomorphism classes does not have a good geometric structure. We use the notion of stability introduced by King [6] in order to construct moduli schemes of quiver representations.

Definition 3.1. For $\theta \in \operatorname{Hom}_{\mathbb{Z}}(K(\operatorname{mod}\mathbb{C}\Gamma), \mathbb{Z})$ such that $\theta((1,\dots,1)) = 0$, a representation $\Psi \in \widetilde{\mathcal{M}}$ is said to be *θ-stable* if for any non-trivial subrepresentation (i.e., subobject in $\operatorname{mod}\mathbb{C}\Gamma$) $S \subsetneq \Psi$, we have $\theta(S) > 0$. Ψ is *θ-semistable* if $\theta(S) \geq 0$ holds instead of $\theta(S) > 0$ above.

In this definition, θ corresponds to a character of $\mathcal{G}$. King [6] proved that there is a moduli scheme $\overline{\mathcal{M}_\theta}$ which parameterizes the S-equivalence classes of θ-semistable representations in $\widetilde{\mathcal{M}}$. It contains the moduli scheme $\mathcal{M}_\theta$ which parameterizes the isomorphism classes of θ-stable representations as an open set.

Note that if $\psi(a) \in \mathbb{C}^\times$ for any $a \in A$, then Ψ doesn't have any non-trivial subrepresentation and hence is θ-stable for any $\theta \in \operatorname{Hom}_{\mathbb{Z}}(K(\operatorname{mod}\mathbb{C}\Gamma), \mathbb{Z})$. Thus $\mathbb{T}$ is naturally contained in $\mathcal{M}_\theta$ for any θ. Moreover, there is an action of $\mathbb{T}$ on $\mathcal{M}_\theta$ defined by the pointwise multiplication just as in (3.1).

Now consider the complex

$$1 \to (\mathbb{C}^\times)^V \xrightarrow{d^1} (\mathbb{C}^\times)^A \xrightarrow{d^2} (\mathbb{C}^\times)^F \to 1.$$

Here, $F = B \cup W$ is the set of vertices of the dimer model which is in one-to-one correspondence with the set of faces of the quiver. The map d^1 is defined by

$$\begin{array}{rccc} d^1 : & (\mathbb{C}^\times)^V & \to & (\mathbb{C}^\times)^A \\ & \cup\!\!\!\!\!\!\;\; \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (g_v)_{v\in V} & \mapsto & (g_{s(a)}^{-1} \cdot g_{t(a)})_{a\in A}, \end{array}$$

and the map d^2 sends $(\psi(a))_{a\in A}$ to $(\phi_f)_{f\in F}$, where ϕ_f is the product of all $\psi(a)$ such that $a \in A = E$ is connected to $f \in B \cup W$. The above complex is the cochain complex computing the $\mathbb{C}^\times$-valued cohomologies of T with respect to the polygonal division of T determined by the quiver Γ.

Let $\mathbb{T}_0$ denote the diagonal subgroup of $(\mathbb{C}^\times)^F$ and $\widetilde{\mathbb{T}} \subset \widetilde{\mathcal{M}}$ the preimage of $\mathbb{T} \subset \mathcal{M}$. Then one has

$$\widetilde{\mathbb{T}} = (d^2)^{-1}(\mathbb{T}_0),$$

and the cohomology group in the middle of the subcomplex

$$1 \to (\mathbb{C}^\times)^V \xrightarrow{d^1} \widetilde{\mathbb{T}} \xrightarrow{d^2} \mathbb{T}_0 \to 1$$

is isomorphic to $H^1(T, \mathbb{C}^\times)$. It follows from the definition that

$$\mathbb{T} = \widetilde{\mathbb{T}} / \operatorname{Im} d^1,$$

and hence one has an exact sequence

$$1 \to H^1(T, \mathbb{C}^\times) \to \mathbb{T} \to \mathbb{T}_0. \tag{3.2}$$

This proves the following:

Lemma 3.2. *The dimension of the algebraic torus $\mathbb{T}$ is either two or three.*

§ 4. Coordinates around $\mathbb{T}$-fixed points

Suppose that a representation $\Psi = (\psi(a))_{a \in A} \in \widetilde{\mathcal{M}}$ represents a point $[\Psi] \in \mathcal{M}_\theta$, which is fixed by the action of $\mathbb{T}$. Let Γ_Ψ be the subquiver of Γ whose set of vertices is V and whose set of arrows consists of arrows $a \in A$ such that $\psi(a) \neq 0$. The stability of Ψ implies that Γ_Ψ is connected. Moreover, we have the following.

Lemma 4.1. *If $[\Psi] \in \mathcal{M}$ is fixed by the action of $H^1(T, \mathbb{C}^\times) \subset \mathbb{T}$, then Γ_Ψ can be lifted to a subquiver Γ'_Ψ of $\widetilde{\Gamma}$, which is isomorphically mapped to Γ_Ψ. Γ'_Ψ is unique up to translations by $\mathbb{Z}^2 \subset \mathbb{R}^2$.*

Proof. Fix a vertex v_0 of Γ_Ψ and lift it to a vertex $\tilde{v}_0$ of $\widetilde{\Gamma}$. For a vertex v of Γ_Ψ, take a path p of the double $\overline{\Gamma_\Psi}$ of Γ_Ψ starting from v_0 and ending at v. We can lift p to a path $\widetilde{p}$ of the double of $\widetilde{\Gamma}$ starting from $\tilde{v}_0$. We will show that the end point of $\widetilde{p}$ does not depend on the choice of p.

Assume that there are two paths p_1 and p_2 of $\overline{\Gamma_\Psi}$ starting from v_0 and ending at v such that the endpoints of their lifts $\widetilde{p}_1$ and $\widetilde{p}_2$ are different. The path $\gamma := p_2 \cdot (p_1)^{-1}$ is a loop starting from v and the assumption implies that it determines a non-trivial class $[\gamma] \in H_1(T, \mathbb{Z})$. Consider the value $\psi(\gamma)$ of Ψ at γ; we can define values of Ψ for arrows and paths of the double $\overline{\Gamma_\Psi}$ in an obvious way. Since $[\gamma]$ is a non-trivial class, there is $g \in H^1(T, \mathbb{C}^\times)$ with $(g \cdot \psi)(\gamma) \neq \psi(\gamma)$. This contradicts the assumption that $H^1(T, \mathbb{C}^\times)$ fixes $[\Psi]$. □

Let F_Ψ be the closure of the union of the connected components of $\mathbb{R}^2 \setminus (\bigcup_{e\in\widetilde{E}} e)$ corresponding to the vertices of Γ'_Ψ. It is a fundamental domain for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$.

Now recall that the set A of arrows of the quiver is identified with the set E of the edges of the dimer model. Thus for an edge $e \in E$, we write $\psi(e)$ for the value of ψ at the arrow corresponding to e. For an edge $e \in \widetilde{E}$, we also write $\psi(e)$ for the value of ψ at the corresponding edge of E. Let δ_Ψ be the union of edges $e \in \widetilde{E}$ satisfying the following:

- $\psi(e) = 0$
- e intersects with another edge e' with $\psi(e') = 0$.

The boundary of the fundamental domain F_Ψ is obviously contained in δ_Ψ. On the other hand, the relations of the quiver imply that there are no end points in δ_Ψ, and therefore the interior of F_Ψ does not intersect with δ_Ψ. Thus δ_Ψ is the union of the translations of the boundary ∂F_Ψ of F_Ψ:

$$\delta_\Psi = \bigcup_{m\in\mathbb{Z}^2} (\partial F_\Psi + m) \tag{4.1}$$

Lemma 4.2. *By replacing Ψ with a representation equivalent to Ψ, we may assume $\psi(a) = 1$ for all arrows a of Γ_Ψ.*

Proof. The assertion means that we can attach a complex number $g_v \in \mathbb{C}^\times$ to each vertex $v \in V$ such that $\psi(a) = g_{t(a)} g_{s(a)}^{-1}$ for any arrow a in Γ_Ψ. Fix a vertex $v_0 \in V$. For any $v \in V$, we take a path p in the double quiver $\overline{\Gamma_\Psi}$ starting from v_0 and ending at v and we want to put $g_v = \psi(p)$. If we show that $\psi(p)$ does not depend on the choice of p, we are done. Take two such paths p_1 and p_2. Lemma 4.1 implies that p_1 and p_2 are homotopic in T and (4.1) shows that the homotopy is generated by the relations

- $p_+(a) \sim p_-(a)$ for arrows a of Γ such that $p_+(a)$ is a path in Γ_Ψ.
- $a^{-1} \cdot a \sim e_{s(a)}$ and $a \cdot a^{-1} \sim e_{t(a)}$ for arrows a of Γ_Ψ.

Thus we obtain $\psi(p_1) = \psi(p_2)$. $\square$

From now on, we assume $\psi(a) = 1$ for all arrows a of Γ_Ψ. In other words, $\psi(a)$ is either 0 or 1. Consider the following subset U_Ψ of $\widetilde{\mathcal{M}}$:

$$U_\Psi = \{\Phi = (\phi(a))_{a\in A} \in \widetilde{\mathcal{M}} \mid \phi(a) = 1 \text{ if } \psi(a) = 1\}.$$

U_Ψ is naturally a closed subscheme of $\widetilde{\mathcal{M}}$.

Lemma 4.3. *Every point in U_Ψ is θ-stable and the natural morphism $U_\Psi \to \mathcal{M}_\theta$ is an open immersion. Thus U_Ψ can be regarded as a $\mathbb{T}$-invariant affine open neighborhood of $[\Psi]$ in $\mathcal{M}_\theta$.*

Proof. Suppose $\Phi \in U_\Psi$. Since the dimension vector of Φ is $(1, 1, \ldots, 1)$, a subrepresentation of Φ is determined by a subset V' of V. By the definition of U_Ψ, V' also determines a subrepresentation of Ψ. Thus the θ-stability for Ψ implies that for Φ; the first assertion follows. For the second assertion, put

$$\widetilde{U}_\Psi = \{\Phi = (\phi(a))_{a \in A} \in \widetilde{\mathcal{M}} \mid \phi(a) \neq 0 \text{ if } \psi(a) = 1\}.$$

This is an open subscheme of $\widetilde{\mathcal{M}}$. Then the same argument as in Lemma 4.2 shows that the morphism

$$\mathcal{G} \times U_\Psi \to \widetilde{U}_\Psi$$

induced by the action of $\mathcal{G}$ on $\widetilde{U}_\Psi$ is an isomorphism. Thus U_Ψ is a section of the morphism from $\widetilde{U}_\Psi$ to its quotient by $\mathcal{G}$. □

Lemma 4.4. *Either of the following two cases must occur:*

1. *There are four quadrivalent points of the graph δ_Ψ lying in ∂F_Ψ, and there are no points of valency three or greater than four.*

2. *There are six trivalent points of δ_Ψ lying in ∂F_Ψ, and there are no points of valency greater than three.*

Proof. Let a_n be the number of points of valency n of δ_Ψ lying in ∂F_Ψ. These points divide ∂F_Ψ into $\left(\sum_{n \geq 3} a_n\right)$ parts so that we can regard ∂F_Ψ as a polygon with $\left(\sum_{n \geq 3} a_n\right)$ edges. Since a point of valency n is contained in n translations of F_Ψ, the equation that the topological Euler number of T is zero leads to

$$1 - \frac{1}{2} \sum_{n \geq 3} a_n + \sum_{n \geq 3} \frac{a_n}{n} = 0.$$

It is easy to see from this that there are only two possibilities as stated. □

Lemma 4.5. *If* $\dim \mathbb{T} = 3$, *then it holds that* $\mathbb{T} \subset U_\Psi \cong \mathbb{C}^3$. *If* $\dim \mathbb{T} = 2$, *then* U_Ψ *is the disjoint union of* $\mathbb{T}$ *and the isolated point* $\{[\Psi]\}$.

Proof. We first consider the case 1 of Lemma 4.4. Assume that v_1, v_2, v_3, v_4 are the quadrivalent points of δ_Ψ lying on ∂F_Ψ, labeled counterclockwise. These points are mapped to a common vertex $v \in T$ of the dimer model. Since v_1 is a quadrivalent point of δ_Ψ, there are four edges e_1, e_2, e_3, e_4 of $\widetilde{E}$ that are connected to v_1 and that satisfy $\psi(e_i) = 0$. The four points v_i divides ∂F_Ψ into four parts and we may assume that e_1 is on the part between v_1 and v_2, and e_2 is between v_1 and v_4. We further assume e_1, e_2, e_3, e_4 are arranged counterclockwise around v_1.

Now take $\Phi \in U_\Psi$ and put $t_i = \phi(e_i)$. Then, the relations $p_+(a) = p_-(a)$ for arrows a corresponding to the edges in ∂F_Ψ determine the values of Φ at the edges on ∂F_Ψ:

- For an edge e of ∂F_Ψ between v_1 and v_2, $\phi(e)$ coincides with either t_1 or $t_2 t_3 t_4$, depending on the configuration of the colors of the vertices of e. In particular, since the colors of v_1 and v_2 are the same, we have $t_3 = t_2 t_3 t_4$ and $t_1 = t_4 t_1 t_2$.
- Similarly, for an edge e of ∂F_Ψ between v_1 and v_4, $\phi(e)$ coincides with either t_2 or $t_3 t_4 t_1$, and we have $t_4 = t_3 t_4 t_1$ and $t_2 = t_1 t_2 t_3$.

Moreover, since $a p_+(a)$ does not depend on an arrow a, we obtain the following:

- For an edge $e \in \widetilde{E}$ with $\psi(e) = 0$ that is not in δ_Ψ, we must have $\phi(e) = t_1 t_2 t_3 t_4$.

Thus $\Phi \in U_\Psi$ is determined by the point $(t_1, t_2, t_3, t_4) \in \mathbb{C}^4$. Conversely, for any point in $\mathbb{C}^4$ that satisfies the relations $t_3 = t_2 t_3 t_4$, $t_1 = t_4 t_1 t_2$, $t_4 = t_3 t_4 t_1$ and $t_2 = t_1 t_2 t_3$, we can find a corresponding point in U_Ψ. Solving these four equations, we obtain

$$U_\Psi \cong \{(t_1, t_2, t_3, t_4) \in \mathbb{C}^4 \mid (t_1, t_2, t_3, t_4) = 0 \text{ or } t_1 t_3 = t_2 t_4 = 1\}.$$

The two-dimensional component defined by $t_1 t_3 = t_2 t_4 = 1$ is $\mathbb{T}$-invariant and is contained in $\mathbb{T}$; hence it coincides with $\mathbb{T}$ which must be two-dimensional. The origin of $\mathbb{C}^4$ corresponds to $[\Psi]$.

Next we consider the case 2 of Lemma 4.4. Let $v_1, \dots, v_6$ denote the six trivalent points of δ_Ψ lying counterclockwise on ∂F_Ψ. In this case, v_1, v_3 and v_5 are in a single $\mathbb{Z}^2$-orbit in $\mathbb{R}^2$, and v_2, v_4 and v_6 are in another orbit. v_1 is connected to three edges e_1, e_2, e_3 of $\widetilde{E}$ that satisfy $\psi(e_i) = 0$ and v_2 is connected to e_4, e_5, e_6 similarly. We may assume that e_1 and e_4 are on the part of ∂F_Ψ cut out by v_1 and v_2 which contains no other v_i, and that e_1, e_2, e_3 and e_4, e_5, e_6 are arranged counterclockwise around v_1 and v_2 respectively. As in the case 1, Ψ is determined by $t_i := \phi(e_i) (i = 1, \dots, 6)$ and we can see

- If one of v_1 and v_2 is black and the other is white, then $(t_1, t_2, t_3) = (t_4, t_5, t_6)$.
- If the colors of v_1 and v_2 are the same, then $t_i = t_j t_k$ for $(i, j, k) = (1, 5, 6)$, $(2, 6, 4)$, $(3, 4, 5)$, $(4, 2, 3)$, $(5, 3, 1)$ and $(6, 1, 2)$.

In the first case, (t_1, t_2, t_3) gives rise to an isomorphism $U_\Psi \cong \mathbb{C}^3$ and $\mathbb{T}$ coincides with the open subset defined by $t_1 t_2 t_3 \neq 0$. In the second case, we can see

$$U_\Psi \cong \{(t_1, t_2, t_3) \in \mathbb{C}^3 \mid (t_1, t_2, t_3) = 0 \text{ or } t_1 t_2 t_3 = 1\},$$

which, as in the case 1, is the union of $\{[\Psi]\}$ and the two-dimensional tours $\mathbb{T}$. $\square$

§ 5. Moduli spaces as crepant resolutions

Our definition of dimer models in §2 contains a lot of "inconsistent" ones from a physics point of view (see Hanany–Vegh [3]). Here we introduce a condition which should be necessary (but not sufficient) for the consistency.

A dimer model is said to be *non-degenerate* if for any edge $e \in E$, there exists a perfect matching D such that $e \in D$. An *R-charge* on a dimer model $G = (B, W, E)$ is a collection of positive real numbers $R_e \in \mathbb{R}_{>0}$ indexed by edges $e \in E$, satisfying

$$\sum_{\substack{e \in E \\ e \ni v}} R_e = 2 \tag{5.1}$$

for each vertex $v \in B \cup W$. If G is non-degenerate, one can define an R-charge by averaging

$$R_e = \frac{2}{|\operatorname{Perf}(G)|} \sum_{D \in \operatorname{Perf}(G)} \chi_D(e)$$

over the set $\operatorname{Perf}(G)$ of perfect matchings. Here χ_D is the characteristic function of the subset $D \subset E$.

Remark. Alastair King pointed to us that the Birkhoff–von Neumann theorem implies that the non-degeneracy condition is in fact equivalent to the existence of an R-charge. He also remarked that Hall's marriage theorem implies that this condition is also equivalent to the following *strong marriage condition*; every proper subset of blacks is connected to strictly more whites and vice versa.

Take the parameter $0 \in \operatorname{Hom}_{\mathbb{Z}}(K(\operatorname{mod} \mathbb{C}\Gamma), \mathbb{Z})$ and consider the corresponding moduli space $\overline{\mathcal{M}_0}$. Since any representation of Γ is 0-semistable, this is the categorical quotient of $\widetilde{\mathcal{M}}$ by the action of $\mathcal{G}$. Hence $\overline{\mathcal{M}_0}$ is an affine scheme with a distinguished point $[0] \in \overline{\mathcal{M}_0}$ which is the image of $0 \in \widetilde{\mathcal{M}} \subset \mathbb{C}^A$. Moreover, for any parameter θ, we have a projective morphism $\overline{\mathcal{M}_\theta} \to \overline{\mathcal{M}_0}$.

Proposition 5.1. *Let (B, W, E) be a non-degenerate dimer model. Then we have* $\dim \mathbb{T} = 3$*, and for a generic parameter θ, the moduli space $\mathcal{M}_\theta$ is smooth and irreducible with the trivial canonical bundle $K_{\mathcal{M}_\theta}$.*

Proof. We may assume the existence of an R-charge $(R_e)_e \in (\mathbb{Q}_{>0})^E$ satisfying (5.1). Take a positive integer N such that $r_e := NR_e$ is an integer. Then $t \mapsto (t^{r_e})_{e \in E}$ is a one parameter subgroup of $\mathbb{T}$ not contained in $H^1(T, \mathbb{C}^\times)$, and hence we have $\dim \mathbb{T} = 3$.

Take an arbitrary point $[\Phi] = (\phi(a)_{a\in A})$ in $\mathcal{M}_\theta$. We will show that there is a $\mathbb{T}$-fixed point $[\Psi] \in \mathcal{M}_\theta$ such that $[\Phi] \in U_\Psi$.

Consider the morphism $\xi : \operatorname{Spec}\mathbb{C}[t] \to \overline{\mathcal{M}_0}$ defined by $t \mapsto [(t^{r_a}\phi(a))_{a\in A}]$. We have $\xi(1) = [\Phi]$ and $\xi(0) = [0]$. Moreover, for $t \neq 0$, $\xi(t)$ is θ-stable. By virtue of the valuative criterion for the projective morphism $\mathcal{M}_\theta \to \overline{\mathcal{M}_0}$, we can lift ξ to $\widetilde{\xi} : \operatorname{Spec}\mathbb{C}[t] \to \mathcal{M}_\theta$. Since U_Ψ is a $\mathbb{T}$-invariant open subset and $\widetilde{\xi}(\operatorname{Spec}\mathbb{C}[t] \setminus \{0\})$ is contained in a single $\mathbb{T}$-orbit, it suffices to show $\widetilde{\xi}(0) \in U_\Psi$ for some $\mathbb{T}$-fixed point $[\Psi]$. Since the fiber of $\mathcal{M}_\theta \to \overline{\mathcal{M}_0}$ over $[0] \in \overline{\mathcal{M}_0}$ is a $\mathbb{T}$-invariant closed subscheme projective over $\operatorname{Spec}\mathbb{C}$, we can find such Ψ as a limit point of the $\mathbb{T}$-action. Hence we have $[\Phi] \in U_\Psi$, where U_Ψ contains $\mathbb{T}$ and is isomorphic to $\mathbb{C}^3$ by Lemma 4.5. Since $[\Phi]$ is arbitrary, $\mathcal{M}_\theta$ is smooth and irreducible.

Now we prove that the canonical bundle of $\mathcal{M}_\theta$ is trivial. As in the proof of Lemma 4.5, we have a coordinate (t_1, t_2, t_3) on U_Ψ. We show that we can patch the 3-forms $dt_1 \wedge dt_2 \wedge dt_3$ on U_Ψ to obtain a global 3-form on $\mathcal{M}_\theta$. Let $[\Psi]$ and $[\Phi]$ be two $\mathbb{T}$-fixed points on $\mathcal{M}_\theta$. Then we have coordinates t_1, t_2, t_3 on U_Ψ and s_1, s_2, s_3 on U_Φ respectively. On the torus $\mathbb{T}$, we can express t_1, t_2, t_3 as Laurent monomials in s_1, s_2, s_3, and vice versa. Thus t_1, t_2, t_3 and s_1, s_2, s_3 are related by a matrix in $GL(3,\mathbb{Z})$. Moreover, we have $t_1t_2t_3 = s_1s_2s_3$. These two facts imply $dt_1 \wedge dt_2 \wedge dt_3 = \pm ds_1 \wedge ds_2 \wedge ds_3$. To determine the sign, recall that the edges e_1, e_2, e_3 that correspond to t_1, t_2, t_3 in the proof of Lemma 4.5 are arranged counterclockwise. We can make the same assumption on the choice of s_1, s_2, s_3. Then we can see that the matrix in $GL(3,\mathbb{Z})$ has the determinant one, so that $dt_1 \wedge dt_2 \wedge dt_3 = ds_1 \wedge ds_2 \wedge ds_3$. □

§ 6. Perfect matchings and toric divisors on moduli spaces

In this section, we discuss the relation between perfect matchings and $\mathbb{T}$-invariant divisors on moduli spaces. Throughout this section, we assume that $G = (B, W, E)$ is a non-degenerate dimer model.

For a generic θ and a two-dimensional $\mathbb{T}$-orbit Z in $\mathcal{M}_\theta$, pick a representation $\Psi = [(\psi(a))_{a\in A}] \in Z$ and put

$$D_Z = \{a \in A \mid \psi(a) = 0\}.$$

This does not depend on the choice of Ψ in Z.

Lemma 6.1. *If θ is generic, then D_Z is a perfect matching for any two-dimensional $\mathbb{T}$-orbit Z in $\mathcal{M}_\theta$.*

Proof. Take a $\mathbb{T}$-fixed point $[\Phi] \in \overline{Z}$ and consider the affine open neighborhood U_Φ of $[\Phi]$ appearing in §4. As in the proof of Lemma 4.5, there is a coordinate (t_1, t_2, t_3) on

U_Φ that gives rise to an isomorphism $U_\Phi \cong \mathbb{C}^3$. The action of $\mathbb{T}$ on U_Ψ is diagonalized with respect to this coordinate and hence $Z \subset U_\Phi$ is defined by $t_i = 0$ and $t_j t_k \neq 0$ where $\{i, j, k\} = \{1, 2, 3\}$. Then it follows from the proof of Lemma 4.5 that D_Z is a perfect matching. □

Suppose that D is a perfect matching. For $t \in \mathbb{C}$, we define $\Psi_t = (\psi_t(a))_{a \in A}$ by

$$\psi_t(a) = \begin{cases} t & \text{if } a \in D \subset E = A \\ 1 & \text{if } a \notin D \end{cases}. \tag{6.1}$$

Then we can see that Ψ_t satisfies the relation of the quiver and the graph Γ_{Ψ_t} is connected.

Lemma 6.2. *There is a generic parameter θ such that Ψ_0 is θ-stable. Moreover, the $\mathbb{T}$-orbit Z of $[\Psi_0]$ in $\mathcal{M}_\theta$ is two-dimensional and it satisfies $D = D_Z$.*

Proof. To find a parameter θ such that Ψ_0 is θ-stable, we can use an idea from Sardo–Infirri [9]: For an arrow $a \in A \setminus D$, take an arbitrary positive rational number ξ_a. For a vertex v of the quiver, we put

$$\theta(v) = \sum_{\substack{a \in A \setminus D \\ t(a) = v}} \xi_a - \sum_{\substack{a \in A \setminus D \\ s(a) = v}} \xi_a.$$

Then, for any non-trivial subrepresentation S of Ψ_t, we have

$$\theta(S) = \sum_{\substack{a \in A \setminus D \\ s(a) \notin S \\ t(a) \in S}} \xi_a > 0,$$

which shows that Ψ_t is θ-stable.

For the genericity of θ, it suffices to show that we can take θ so that

$$\theta(S) = \sum_{\substack{a \in A \setminus D \\ s(a) \notin S \\ t(a) \in S}} \xi_a - \sum_{\substack{a \in A \setminus D \\ s(a) \in S \\ t(a) \notin S}} \xi_a \neq 0$$

for an arbitrary non-empty subset $S \subsetneq V$. This is achieved if $(\xi_a)_{a \in A \setminus D}$ is sufficiently general.

$H^1(T, \mathbb{C}^\times)$ acts freely on Z since any element in $H_1(T, \mathbb{Z})$ can be represented by a linear combination of paths of the double $\overline{Q_D}$. This shows that Z is two-dimensional. It follows from the definition of Ψ_0 that $D = D_Z$. □

Consider the closure X' of $\mathbb{T}$ in the moduli space $\overline{\mathcal{M}_0}$ corresponding to the parameter 0. Since it is not a priori clear if X' is normal, we take the normalization X of X' which is an affine toric variety. Proposition 5.1 is saying that $\mathcal{M}_\theta$ is a crepant resolution of X for a generic θ.

Now let $\Delta \subset H^1(T, \mathbb{Z})$ be the Newton polygon of the characteristic polynomial (i.e., the convex hull of height changes) of the dimer model with respect to any fixed perfect matching. Then we have the following:

Proposition 6.3. *The affine coordinate ring of X is isomorphic to the semigroup ring $\mathbb{C}[(\mathrm{Cone}(\Delta \times \{1\}))^\circ]$ of the dual cone of the cone over $\Delta \times \{1\} \subset H^1(T, \mathbb{Z}) \times \mathbb{Z}$.*

Proof. Put $N = \mathrm{Hom}(\mathbb{C}^\times, \mathbb{T})$. Then the affine toric variety X is determined by a cone $C \subset N$. For a perfect matching D of $G = (B, W, E)$, let

$$\iota_D : \mathbb{C}^\times \to \mathbb{T}$$

be the homomorphism sending $t \in \mathbb{C}^\times$ to the representation Ψ_t defined by (6.1). Since $\iota_D(\mathbb{C}^\times)$ is the stabilizer of $[\Psi_0] \in Z \subset \mathcal{M}_\theta$ where the orbit Z of $[\Psi_0]$ is two-dimensional by Lemma 6.2, Lemmas 6.1 and 6.2 imply that the cone C is generated by the set

$$\{\iota_D \mid D \in \mathrm{Perf}(G)\} \subset N.$$

Recall that we have an exact sequence

$$1 \to H^1(T, \mathbb{C}^\times) \to \mathbb{T} \to \mathbb{C}^\times \to 1.$$

The map ι_D gives a splitting

$$\mathbb{T} \cong H^1(T, \mathbb{C}^\times) \times \iota_D(\mathbb{C}^\times)$$

of the above exact sequence. Let

$$\pi_D : \mathbb{T} \to H^1(T, \mathbb{C}^\times)$$

be the projection with respect to this splitting. Now fix a reference perfect matching D_0, and hence the splitting $\mathbb{T} \cong H^1(T, \mathbb{C}^\times) \times \mathbb{C}^\times$ given by ι_{D_0}. Then under the corresponding splitting $N \cong H^1(T, \mathbb{Z}) \times \mathbb{Z}$, we have $\iota_D \in H^1(T, \mathbb{Z}) \times \{1\}$ for any perfect matching D. Therefore it suffices to show that Δ is the convex hull of

$$\{\pi_{D_0} \circ \iota_D \mid D \in \mathrm{Perf}(G)\} \subset H^1(T, \mathbb{Z}),$$

where we have identified $\mathrm{Hom}(\mathbb{C}^\times, H^1(T, \mathbb{C}^\times))$ with $H^1(T, \mathbb{Z})$.

The projection π_D is defined as follows: For a homology class $C \in H_1(T, \mathbb{Z})$, choose an allowed path p_C with respect to D (i.e., a path which does not contain any arrow $a \in D \subset E = A$) whose homology class lies in C. Then for $\psi \in \mathbb{T}$,

$$\pi_D(\psi)(C) = \psi(p_C) \in \mathbb{C}^\times.$$

It follows that the height change $h_{D,D_0} \in H^1(T, \mathbb{Z})$ of D with respect to the reference perfect matching D_0 coincides with $\pi_{D_0} \circ \iota_D$. Since Δ is the convex hull of the set of height changes, we are done. □

By combining Proposition 5.1 with Proposition 6.3, we obtain the main theorem in this paper:

Theorem 6.4. *Let (B, W, E) be a non-degenerate dimer model. Then for a generic parameter θ, $\mathcal{M}_\theta$ is a crepant resolution of* $\operatorname{Spec} \mathbb{C}[(\operatorname{Cone}(\Delta \times \{1\}))^\circ]$.

For example, the moduli space $\mathcal{M}_\theta$ for the dimer model in Figure 1 and a generic stability parameter θ is the total space of the direct sum $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ of the tautological bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ on the projective line $\mathbb{P}^1$. There is a real-codimension one wall in the space of stability parameters and the moduli space flops as one moves from one chamber to the other.

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