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On the cohomological cycle of a normal surface singularity

To the memory of Professor Eiji Horikawa

By

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Abstract

The cohomological cycles of normal surface singular points are studied by means of the chain-decomposition. It is shown that the cohomological cycle of a weakly elliptic singularity contracts to a Gorenstein singularity with the same geometric genus as the original one, and that of a weakly elliptic numerically Gorenstein singularity can be computed by Yau's elliptic sequence for the canonical cycle on the minimal resolution.

§ 1. Introduction

We shall work over an algebraically closed field k of characteristic zero. Let (V, o) be the germ of a normal surface singular point and $\pi : X \to V$ a desingularization. Since the intersection form is negative definite on $\pi^{-1}(o)$, there exists a curve D supported on $\pi^{-1}(o)$ such that $\mathcal{O}_D(-D)$ is nef. The smallest one Z among such curves exists and is called the *fundamental cycle* ([1], [2]). We have three basic genera for (V, o) (see, e.g., [9]):

- Fundamental genus $\quad p_f(V, o) := p_a(Z)$
- Arithmetic genus $\quad p_a(V, o) := \sup\{p_a(D) : 0 \prec D,\ \mathrm{Supp}(D) \subseteq \pi^{-1}(o)\}$
- Geometric genus $\quad p_g(V, o) := \dim_k R^1\pi_*\mathcal{O}_X$

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We have $p_f(V,o) \le p_a(V,o) \le p_g(V,o)$, where the inequalities are usually strict.

The geometric genus is an analytic invariant which is hard to compute, even when we know the weighted dual graph of the exceptional set. However, as shown in [7], one can associate with it a canonically determined curve as follows. If D is a sufficiently "big" curve with support in $\pi^{-1}(o)$, then we obtain an isomorphism $R^1\pi_*\mathcal{O}_X \simeq H^1(D, \mathcal{O}_D)$ from the exact sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$. In [7] (see also [8]), it is shown that there exists the smallest one among curves enjoying such a property. We denote it by Z_1 and call it the *cohomological cycle* according to [8]. Therefore, we have $h^1(D, \mathcal{O}_D) = p_g(V,o)$ when $Z_1 \preceq D$, and $h^1(D, \mathcal{O}_D) < p_g(V,o)$ when $Z_1 \not\preceq D$.

We say that (V,o) is a *numerically Gorenstein* singularity if there exists a curve Z_K such that $K_X \equiv -Z_K$ on $\pi^{-1}(o)$, where the symbol $\equiv$ means the numerical equivalence. Such a curve Z_K is called the *canonical cycle*. We have $K_X \sim -Z_K$ (linearly equivalent) if and only if (V,o) is a Gorenstein singularity, that is, $\mathcal{O}_{V,o}$ is a Gorenstein local ring. Note that $Z_K = 0$ is equivalent to saying that (V,o) is a rational double point. In [7], it is shown that $Z_1 = Z_K$ holds when (V,o) is Gorenstein (see also [8]). But our knowledge is very poor when we are in a more general situation. We do not know even whether the support of Z_1 is connected or not.

The purpose of the present note is to study the still mysterious curve Z_1 by a numerical method as a continuation of [3], where we considered the chain-decomposition of the canonical cycle among other things. After recalling from [3] some basic results for chain-connected curves in Sect. 2, we state fundamental properties of the chain-decomposition of Z_1 in Sect. 3. Then we restrict ourselves to weakly elliptic singularities in Sect. 4 in order to clarify what Z_1 is in this special case. Here, (V,o) is called weakly elliptic if $p_a(V,o) = 1$ ([9], [10]). We shall show in Theorem 4.2 that Z_1 is the canonical cycle of a weakly elliptic Gorenstein singularity with the same geometric genus as (V,o). Furthermore, Theorem 4.4 shows that the chain-decomposition of Z_1 can be realized as a subsequence of Yau's elliptic sequence [10] for the canonical cycle, when (V,o) is numerically Gorenstein. On the minimal resolution, this also follows from [6] (see, Remark after Theorem 4.4).

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§ 2. Curves on a smooth surface

In this section, we collect some results from [3] for the later use. See [3] for the full detail.

By a curve, we mean an effective (non-zero) divisor on a smooth surface X. Let D be a curve. We put $p_a(D) = 1 - \chi(\mathcal{O}_D)$ and call it the arithmetic genus of D. If D_1 is a subcurve of D, then we have an exact sequence of sheaves

$$0 \to \mathcal{O}_{D-D_1}(-D_1) \to \mathcal{O}_D \to \mathcal{O}_{D_1} \to 0,$$

which yields $p_a(D) = p_a(D_1) + p_a(D - D_1) - 1 + (D - D_1)D_1$. Since D is Gorenstein, the dualizing sheaf ω_D is invertible. We have $\omega_D = \mathcal{O}_D(K_X + D)$ by the adjunction formula and $\deg \omega_D = 2p_a(D) - 2$. A line bundle (or an invertible sheaf) L on D is called *nef* if it is of non-negative degree on any irreducible components of D.

A curve D is called *chain-connected* (s-connected in the terminology of [5]) if $\mathcal{O}_{D-\Gamma}(-\Gamma)$ is not nef for any strict subcurve $\Gamma \prec D$. It is easy to see that $h^0(D, \mathcal{O}_D) = 1$ and that a non-zero element in $H^0(D, \mathcal{O}_D)$ is nowhere vanishing, when D is chain-connected. Furthermore, the following three properties are satisfied.

Lemma 2.1. *Let L be a nef line bundle on a chain-connected curve D. Then $H^0(D, -L) \neq 0$ if and only if L is trivial.*

Lemma 2.2. *Let D_1 and D_2 be curves such that $\mathcal{O}_{D_1}(-D_2)$ is nef. If D_1 is chain-connected, then either $D_1 \preceq D_2$ or $D_1 \cap D_2 = \emptyset$.*

Lemma 2.3. *Let D be a chain-connected curve with $p_a(D) > 0$. Then there uniquely exists a subcurve $D_{\min}$ with $p_a(D_{\min}) = p_a(D)$ and $K_{D_{\min}}$ is nef. Furthermore,*

$$D_{\min} = \min_{0 \prec \Gamma \preceq D} \{p_a(\Gamma) = p_a(D)\} = \max_{0 \prec \Gamma \preceq D} \{K_\Gamma \textit{ is nef }\}.$$

The curve $D_{\min}$ as above is called the *minimal model* of D. The first half of the following can be already found in [5].

Theorem 2.4. *Let D be a curve. Then there exist a positive integer n and chain-connected subcurves $\Gamma_i \preceq D$, $1 \leq i \leq n$, such that (1) $D = \sum_{i=1}^n \Gamma_i$ and (2) $\mathcal{O}_{\Gamma_j}(-\Gamma_i)$ is nef for any $i < j$. Such an ordered decomposition is unique up to permutations of indices preserving the second property.*

The ordered decomposition as above will be referred to as the *chain-decomposition* of D. We remark that

$$h^0(D, \mathcal{O}_D) \leq n - \sum_{i<j} \Gamma_i \Gamma_j, \quad p_a(D) = \sum_{i=1}^n p_a(\Gamma_i) - (n-1) + \sum_{i<j} \Gamma_i \Gamma_j \tag{2.1}$$

hold ([3], see [5] for the first inequality). By Lemma 2.2, we have either $\Gamma_j \preceq \Gamma_i$ or $\Gamma_i \cap \Gamma_j = \emptyset$ when $i < j$. Hence, the support of every maximal curve in $\{\Gamma_i\}_{i=1}^n$ is a connected component of $\mathrm{Supp}(D)$.

There is another notion for connectedness of curves. For an integer m, a curve D is called (numerically) m-connected if $(D - D_1)D_1 \geq m$ holds for any proper subcurve $D_1 \prec D$. A nef and big curve is necessarily 1-connected by Hodge's index theorem. Every 1-connected curve is chain-connected. But the converse does not hold in general.

We sometimes need to consider a curve D with the property:

$$p_a(D') \leq 1 \text{ holds for any subcurve } D' \preceq D \tag{2.2}$$

For such curves, we have the following:

Lemma 2.5. *Let D be a curve with $p_a(D) = 1$ satisfying* (2.2). *Then D is 0-connected. If $D = \Gamma_1 + \cdots + \Gamma_n$ is the chain-decomposition, then every Γ_i is a 0-connected curve with $p_a(\Gamma_i) = 1$, $\mathcal{O}_{\Gamma_j}(-\Gamma_i)$ is numerically trivial for any $i < j$. Furthermore, $h^0(D, \mathcal{O}_D) \leq n$.*

§ 3. Cohomological cycles

From now on, we let (V, o) be the germ of a normal surface singular point with $p_g(V, o) > 0$ and $\pi : X \to V$ a resolution of (V, o). The fundamental cycle and the cohomological cycle on $\pi^{-1}(o)$ are respectively denoted by Z and Z_1. We remark that Z is chain-connected. We tacitly assume hereafter that every curve is supported in $\pi^{-1}(o)$.

§ 3.1. Some basic properties

The following lemma gives us the "dual" characterization of Z_1 that $|K_{Z_1}|$ is the common variable part of the canonical linear systems of any bigger curves.

Lemma 3.1. *The following hold.*

(1) *When $Z_1 \prec D$, every element in $H^0(D, K_D)$ vanishes identically on $D - Z_1$.*

(2) *The canonical linear system $|K_{Z_1}|$ of Z_1 has no fixed components. In particular, K_{Z_1} is nef.*

Proof. (1) We consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{Z_1}(K_{Z_1}) \to \mathcal{O}_D(K_D) \to \mathcal{O}_{D-Z_1}(K_D) \to 0.$$

The injection $H^0(Z_1, K_{Z_1}) \to H^0(D, K_D)$ is the dual map of $H^1(D, \mathcal{O}_D) \to H^1(Z_1, \mathcal{O}_{Z_1})$ which is an isomorphism, since $Z_1 \prec D$. Hence $H^0(D, K_D) \to H^0(D - Z_1, K_D)$ is the zero map. (2) Assume that there is an irreducible component C of Z_1 such that the restriction map $H^0(Z_1, K_{Z_1}) \to H^0(C, K_{Z_1})$ is zero. Then $H^0(Z_1 - C, K_{Z_1-C}) \simeq H^0(Z_1, K_{Z_1})$. By the Serre duality theorem, this gives us $H^1(Z_1, \mathcal{O}_{Z_1}) \simeq H^1(Z_1 - C, \mathcal{O}_{Z_1-C})$, which is impossible because Z_1 is the smallest curve with $h^1(Z_1, \mathcal{O}_{Z_1}) = p_g(V, o)$. □

Let $Z_1 = \Delta_1 + \cdots + \Delta_\nu$ be the chain-decomposition: each Δ_i is chain-connected, $\mathcal{O}_{\Delta_j}(-\Delta_i)$ is nef when $i < j$.

Lemma 3.2. *Any Δ_i is a subcurve of the fundamental cycle Z. If Δ_i is a minimal curve in $\{\Delta_j\}_{j=1}^{\nu}$, then K_{Δ_i} is nef and $p_a(\Delta_i) > 0$.*

Proof. $\mathcal{O}_{\Delta_i}(-Z)$ is nef. Since Δ_i is chain-connected and $\mathrm{Supp}(\Delta_i) \subseteq \pi^{-1}(o)$, we get $\Delta_i \preceq Z$ by Lemma 2.2. Let Δ_i be a minimal curve in $\{\Delta_j\}_{j=1}^{\nu}$. By a permutation of indices, we may assume that $i = \nu$. Recall that $-\sum_{j=1}^{\nu-1} \Delta_j$ is nef on Δ_ν. Since K_{Z_1} is nef and $\omega_{\Delta_\nu} = \mathcal{O}_{\Delta_\nu}(K_{Z_1} - \sum_{j=1}^{\nu-1} \Delta_j)$, we see that K_{Δ_ν} is also nef. In particular, we have $p_a(\Delta_\nu) > 0$. Then $p_a(\Delta_j) > 0$ for any j. □

The following shows that we can bound $p_g(V, o)$ by a topological data, if we could find a way to compute Z_1 from the weighted dual graph.

Lemma 3.3. $p_g(V, o) \leq \sum_{i=1}^{\nu} p_a(\Delta_i) \leq \nu \cdot p_f(V, o)$.

Proof. By (2.1), $h^0(Z_1, \mathcal{O}_{Z_1}) \leq \nu - \sum_{i<j} \Delta_i \Delta_j$ and $p_a(Z_1) = \sum_{j=1}^{\nu} p_a(\Delta_j) - (\nu - 1) + \sum_{i<j} \Delta_i \Delta_j$. Since $h^1(Z_1, \mathcal{O}_{Z_1}) = p_g(V, o)$, we get $p_g(V, o) = h^0(Z_1, \mathcal{O}_{Z_1}) - (1 - p_a(Z_1)) \leq \sum p_a(\Delta_j) + h^0(Z_1, \mathcal{O}_{Z_1}) - \nu + \sum_{i<j} \Delta_i \Delta_j \leq \sum_j p_a(\Delta_j)$. Note that we have $p_a(\Delta_j) \leq p_a(Z) = p_f(V, o)$ for each j by $\Delta_j \preceq Z$. □

§ 3.2. Numerically Gorenstein case

In this subsection, (V, o) denotes a numerically Gorenstein surface singularity with $p_g(V, o) > 0$. Let Z_K be the canonical cycle on a resolution $\pi : X \to V$. It is shown in [7] (also [8]) that $Z_1 = Z_K$ if (V, o) is Gorenstein.

Lemma 3.4 ([7]). $Z_1 \preceq Z_K$.

Proof. In order to see that $\dim R^1\pi_*\mathcal{O}_X = h^1(Z_K, \mathcal{O}_{Z_K})$, it suffices to show that the restriction $H^1(D, \mathcal{O}_D) \to H^1(Z_K, \mathcal{O}_{Z_K})$ is an isomorphism for any curve D with $Z_K \preceq D$. For this purpose, we have only to show that $H^1(D - Z_K, -Z_K) = 0$. This can be seen as follows. By duality, $H^1(D - Z_K, -Z_K)^\vee \simeq H^0(D - Z_K, K_{D-Z_K} + Z_K) = H^0(D - Z_K, D + K_X)$. Recall that K_X and $-Z_K$ are numerically equivalent. If

$H^0(D-Z_K, D+K_X)$ were not zero, since we have $\deg(D+K_X)|_{D-Z_K} = (D-Z_K)^2 < 0$, any non-zero element $s \in H^0(D-Z_K, D+K_X)$ vanishes on a component. Letting C_s be the biggest subcurve on which s vanishes identically, s induces a non-zero element s' of $H^0(D-Z_K-C_s, D+K_X-C_s)$. But we still have $\deg(D+K_X-C_s)|_{D-Z_K-C_s} = (D-Z_K-C_s)^2 < 0$ and s' should vanish on a component, which is impossible by the choice of C_s. Therefore, $H^0(D-Z_K, D+K_X) = 0$. □

Then, by the Riemann-Roch theorem and $p_a(Z_K) = 1$, we get $h^0(Z_K, \mathcal{O}_{Z_K}) = h^1(Z_K, \mathcal{O}_{Z_K}) = p_g(V, o)$.

Lemma 3.5. *Let F be the fixed part of $|K_{Z_K}|$, that is, the biggest subcurve of Z_K such that the restriction map $H^0(Z_K, K_{Z_K}) \to H^0(F, K_{Z_K})$ is zero. Then $Z_1 = Z_K - F$. In particular, $Z_1 = Z_K$ holds when (V, o) is Gorenstein.*

Proof. Since $Z_1 \preceq Z_K$, the first assertion follows from Lemma 3.1. If (V, o) is Gorenstein, then K_{Z_K} is trivial and, hence, $|K_{Z_K}|$ cannot have a base point. □

This yields the following well-known fact.

Corollary 3.6. *Let (V, o) be a Gorenstein surface singularity with $p_g(V, o) \geq 2$. Then $p_f(V, o) < p_g(V, o)$.*

Proof. We may assume that π is the minimal resolution. Then $Z \preceq Z_K$, since $K_X \sim -Z_K$ is nef. We have $h^0(Z_K, \mathcal{O}_{Z_K}) = p_g(V, o) \geq 2$, while $h^0(Z, \mathcal{O}_Z) = 1$ because Z is chain-connected. So, $Z \prec Z_K$ and we have $p_f(V, o) = h^1(Z, \mathcal{O}_Z) < h^1(Z_K, \mathcal{O}_{Z_K}) = p_g(V, o)$ by $Z_K = Z_1$. □

We let $Z_K = \Gamma_1 + \cdots + \Gamma_n$ be the chain-decomposition. It is known that $\Gamma_1 = Z$ when π is the minimal resolution (see [3]).

Lemma 3.7. *If (V, o) is a numerically Gorenstein singularity which is not Gorenstein, then $Z_1 \preceq Z_K - \Gamma_1$.*

Proof. By the assumption, K_{Z_K} is numerically trivial but not trivial. Hence, for any non-zero $s \in H^0(Z_K, K_{Z_K})$, there exists an irreducible component E_s on which s vanishes identically. Since $\mathrm{Supp}(Z_K)$ is connected, we have $\mathrm{Supp}(\Gamma_1) = \mathrm{Supp}(Z_K)$ and hence $E_s \preceq \Gamma_1$. We consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{Z_K - \Gamma_1}(K_{Z_K - \Gamma_1}) \to \mathcal{O}_{Z_K}(K_{Z_K}) \to \mathcal{O}_{\Gamma_1}(K_{Z_K}) \to 0.$$

Suppose that s restricts to a non-zero element of $H^0(\Gamma_1, K_{Z_K})$. Since Γ_1 is chain-connected and K_{Z_K} is numerically trivial, we get $\mathcal{O}_{\Gamma_1}(K_{Z_K}) \simeq \mathcal{O}_{\Gamma_1}$ by Lemma 2.1.

Note that we have $h^0(\Gamma_1, \mathcal{O}_{\Gamma_1}) = 1$ and s should be nowhere vanishing on Γ_1. This is impossible, because s vanishes on $E_s \preceq \Gamma_1$. Therefore, $H^0(Z_K, K_{Z_K}) \to H^0(\Gamma_1, K_{Z_K})$ is zero. $\square$

From the above lemmas, we get the following:

Proposition 3.8. *Let (V, o) be a numerically Gorenstein singular point. Then the following two conditions are equivalent.*

(1) *(V, o) is Gorenstein.* (2) *$Z_1 = Z_K$.*

§ 4. Weakly elliptic singularities

We say that (V, o) is a *weakly elliptic* singularity when $p_a(V, o) = 1$. It is equivalent to saying that $p_f(V, o) = 1$, as is well-known ([9], [4], see also [3]).

Lemma 4.1. *Let (V, o) be a weakly elliptic singularity. Then the cohomological cycle Z_1 is 0-connected and $p_a(Z_1) = 1$. If $Z_1 = \Delta_1 + \cdots + \Delta_\nu$ is the chain-decomposition, then $p_a(\Delta_i) = 1$ for any i, $\Delta_\nu \prec \Delta_{\nu-1} \prec \cdots \prec \Delta_1$, $\mathcal{O}_{\Delta_j}(-\Delta_i)$ is numerically trivial when $i < j$, Δ_ν is the minimal model of the fundamental cycle Z. Furthermore, $p_g(V, o) \le \nu$.*

Proof. We know from Lemma 3.1 that K_{Z_1} is nef. So $p_a(Z_1) > 0$ by $\deg K_{Z_1} = 2p_a(Z_1) - 2$. On the other hand, we have $p_a(Z_1) \le p_a(V, o) = 1$. Hence $p_a(Z_1) = 1$. Since Z_1 satisfies the property (2.2) by $p_a(V, o) = 1$, it follows from Lemma 2.5 that Z_1 is a 0-connected curve whose chain-decomposition $\Delta_1 + \cdots + \Delta_\nu$ has the properties listed there: $p_a(\Delta_i) = 1$, $\mathcal{O}_{\Delta_j}(-\Delta_i)$ is numerically trivial for $i < j$, $h^0(Z_1, \mathcal{O}_{Z_1}) \le \nu$. The last inequality shows that $p_g(V, o) \le \nu$, because $p_a(Z_1) = 1$ and $p_g(V, o) = h^1(Z_1, \mathcal{O}_{Z_1})$.

Recall that $\Delta_i \preceq Z$ and $p_a(\Delta_i) = p_a(Z) = 1$. It follows from Lemma 2.3 that each Δ_i contains the minimal model of Z as a subcurve. This is sufficient to imply that $\Delta_j \preceq \Delta_i$ when $i < j$. Note that we cannot have $\Delta_j = \Delta_i$ here, because $\Delta_i \Delta_j = 0$ but $\Delta_i^2 < 0$. Since K_{Δ_ν} is nef by Lemma 3.2 and $p_a(\Delta_\nu) = p_a(Z)$, we see that Δ_ν is nothing but the minimal model of Z. We know that K_{Δ_ν} is trivial from Lemma 2.1, because Δ_ν is chain-connected and $h^0(\Delta_\nu, K_{\Delta_\nu}) = 1$. Then it is easy to see that Δ_ν is 2-connected. $\square$

In particular, when (V, o) is weakly elliptic, we know that the support of Z_1 is connected, because it coincides with the support of the chain-connected curve Δ_1. The singular point obtained by contracting the smallest curve Δ_ν as above is a *minimally elliptic* singularity [4] (or, an elliptic Gorenstein singularity in the sense of [7]). (N.B. Δ_ν is not necessarily the fundamental cycle on its support.)

Theorem 4.2. *Let (V, o) be a weakly elliptic singularity and Z_1 the cohomological cycle. Then the singular point $(V_\flat, o_\flat)$ obtained by contracting* $\mathrm{Supp}(Z_1)$ *is a weakly elliptic Gorenstein singularity with $p_g(V_\flat, o_\flat) = p_g(V, o)$ and Z_1 is the canonical cycle of $(V_\flat, o_\flat)$.*

Proof. Recall that K_{Z_1} is nef. Since $p_a(Z_1) = 1$, we see that K_{Z_1} is numerically trivial, which is equivalent to saying that Z_1 is the canonical cycle on its support. Therefore, the singular point $(V_\flat, o_\flat)$ obtained by contracting $\mathrm{Supp}(Z_1)$ is numerically Gorenstein. We clearly have $p_g(V_\flat, o_\flat) = p_g(V, o)$. Since the canonical cycle and the cohomological cycle coincide, $(V_\flat, o_\flat)$ is a Gorenstein singularity by Proposition 3.8. □

Corollary 4.3. *If (V, o) is a normal surface singularity with $p_g(V, o) = 1$, then its cohomological cycle on the minimal resolution is the fundamental cycle of a minimally elliptic singularity.*

When (V, o) is a weakly elliptic numerically Gorenstein singularity, using Lemma 2.5 as in Lemma 4.1, one can show that the chain-decomposition $Z_K = \Gamma_1 + \cdots + \Gamma_n$ of the canonical cycle satisfies: $p_a(\Gamma_i) = 1$ for any i, $\Gamma_n \prec \Gamma_{n-1} \prec \cdots \prec \Gamma_1$, $\mathcal{O}_{\Gamma_i + \cdots + \Gamma_n}(-\Gamma_{i-1})$ is numerically trivial for $2 \le i \le n$, Γ_n is the minimal model of the fundamental cycle Z. It is shown in [3] that the sequence $\Gamma_n \prec \Gamma_{n-1} \prec \cdots \prec \Gamma_1$ is nothing more than Yau's elliptic sequence ([10]) if π is the minimal resolution. In this case, each Γ_i is the fundamental cycle on its support and $\Gamma_1 = Z$. The following in particular shows that Z_1 can be computed by using the elliptic sequence for Z_K on the minimal resolution.

Theorem 4.4. *Let (V, o) be a weakly elliptic numerically Gorenstein singularity and $\pi : X \to V$ a resolution. Let Z_K and Z_1 be the canonical cycle and the cohomological cycle on $\pi^{-1}(o)$, respectively. Then there exists a weakly elliptic Gorenstein singularity $(V_\flat, o_\flat)$ with $p_g(V_\flat, o_\flat) = p_g(V, o)$ satisfying*

(1) *$(V_\flat, o_\flat)$ is obtained by contracting the connected subset* $\mathrm{Supp}(Z_1)$ *of $\pi^{-1}(o)$ and Z_1 is the canonical cycle for $(V_\flat, o_\flat)$,*

(2) *if $Z_K = \sum_{i=1}^{n} \Gamma_i$ is the chain-decomposition of Z_K, then $Z_1 = \sum_{j=i}^{n} \Gamma_j$ is the chain-decomposition of Z_1 for some $i \in \{1, 2, \ldots, n\}$.*

In particular, $p_g(V, o) \le n - i + 1$.

Proof. By Theorem 4.2, we only have to show (2). If (V, o) itself is Gorenstein, then it suffices to take $i = 1$. Assume that (V, o) is not Gorenstein. Then $Z_1 \preceq Z_K - \Gamma_1 = \Gamma_2 + \cdots + \Gamma_n$ by Lemma 3.7. We denote by (V_1, o_1) the singularity obtained by contracting $Z_K - \Gamma_1$. Then it is a weakly elliptic numerically Gorenstein singularity with

$p_g(V_1, o_1) = p_g(V, o)$ whose canonical cycle is $Z_K - \Gamma_1$, since $-\Gamma_1$ is numerically trivial on $Z_K - \Gamma_1$. If (V_1, o_1) is Gorenstein, then we have $Z_1 = Z_K - \Gamma_1$ by Proposition 3.8 and put $i = 2$. Otherwise, we have $Z_1 \preceq Z_K - \Gamma_1 - \Gamma_2$ and let (V_2, o_2) be the weakly elliptic numerically Gorenstein singularity obtained by contracting $Z_K - \Gamma_1 - \Gamma_2$. Then $p_g(V_2, o_2) = p_g(V_1, o_1) = p_g(V, o)$ and $Z_K - \Gamma_1 - \Gamma_2$ is the canonical cycle for (V_2, o_2), since $-\Gamma_1 - \Gamma_2$ is numerically trivial on $Z_K - \Gamma_1 - \Gamma_2 = \sum_{j=3}^{n} \Gamma_j$. Now the obvious induction shows that there is an index i as in (2). Then we clearly have $p_g(V, o) \leq n - i + 1$. $\square$

Remark. As Professor T. Okuma kindly pointed out to the author, Theorem 4.4 also follows from [6] at least on the minimal resolution. In fact, since Z_1 is the canonical cycle on its support, we have $Z_1 = \Gamma_i + \cdots + \Gamma_n$ for some i by [6, Proposition 2.9 (Némethi, Tomari)]. Then it follows from [6, Lemma 2.12] that $(V_\flat, o_\flat)$ is Gorenstein.

Recall that a weakly elliptic numerically Gorenstein singularity is called *maximally elliptic*, if the geometric genus coincides with the length of the elliptic sequence (i.e., $p_g(V, o) = n$ in the above notation). Theorem 4.4 in particular implies the following result due to Yau [10].

Corollary 4.5 ([10]). *Every maximally elliptic singularity is Gorenstein.*

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