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Semistable objects in derived categories of K3 surfaces

By

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Abstract

For a triangulated category $\mathcal{T}$, the space of stability conditions $\mathrm{Stab}(\mathcal{T})$ is introduced by T. Bridgeland. In this article, we give a survey of the recent developments on the study of the stability conditions, and we consider moduli problems and counting invariants of semistable objects on K3 surfaces.

§ 1. Introduction

The aim of this article is to give a survey of the theory of stability conditions on triangulated categories, and introduce the results in [22], where the author studied the moduli problem of semistable objects on K3 surfaces and counting invariants of them. For a triangulated category $\mathcal{T}$, the notion of stability conditions on $\mathcal{T}$ is introduced by T. Bridgeland [4], in order to give a mathematical framework of M. Douglas' Π-stability [7], [8]. First let us introduce the stability conditions on abelian categories.

Definition 1.1. Let $\mathcal{A}$ be an abelian category. A *stability function* on $\mathcal{A}$ is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that we have

$$Z(\mathcal{A} \setminus \{0\}) \subset \mathcal{H} \cup \mathbb{R}_{<0}.$$

Here $K(\mathcal{A})$ is the Grothendieck group of $\mathcal{A}$ and $\mathcal{H} \subset \mathbb{C}$ is the upper half plane.

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Given a stability function $Z\colon K(\mathcal{A}) \to \mathbb{C}$, we can uniquely determine the *phase* $\phi(E) \in (0,1]$ for a non-zero object $E \in \mathcal{A}$ by the formula,

$$\phi(E) = \frac{1}{\pi i}\left(\log Z(E) - \log|Z(E)|\right).$$

We say $E \in \mathcal{A}$ is *Z-semistable* if for any non-zero subobject $F \subset E$ one has $\phi(F) \le \phi(E)$.

Definition 1.2. A stability function $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a *stability condition* on $\mathcal{A}$ if for any non-zero object $E \in \mathcal{A}$ there is a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that $F_i = E_i/E_{i-1}$ is Z-semistable with $\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n)$.

Now we introduce the notion of stability conditions on triangulated categories.

Definition 1.3. Let $\mathcal{T}$ be a triangulated category. A *stability condition* on $\mathcal{T}$ consists of data $\sigma = (Z, \mathcal{A})$, where $\mathcal{A} \subset \mathcal{T}$ is the heart of a bounded t-structure on $\mathcal{T}$ and Z is a stability condition on $\mathcal{A}$.

Example 1.4. (i) Let $\mathcal{T} = D(C)$ for a smooth projective curve C. Here for a variety X, we denote by $D(X)$ the bounded derived category of coherent sheaves on X. Let $Z\colon K(C) \to \mathbb{C}$ be $E \mapsto -\deg(E) + \mathrm{rk}(E)\cdot i$. Then the pair $(Z, \mathrm{Coh}(C))$ determines a stability condition on $\mathcal{T}$. In this case, an object $E \in \mathrm{Coh}(C)$ is Z-semistable if and only if it is an usual semistable sheaf.

(ii) Let A be a finite dimensional k-algebra with k a field, and $\mathcal{T} = D^b(\mathcal{A})$ where $\mathcal{A} = \mathrm{mod}\, A$ is the abelian category of finitely generated right A-modules. Then there is a finite number of simple objects $S_1, \cdots, S_N \in \mathcal{A}$ which generates $\mathcal{A}$. One can choose $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that $Z(S_i) \in \mathcal{H}$ for all $1 \le i \le N$. Then the pair $(Z, \mathcal{A})$ determines a stability condition on $\mathcal{T}$.

Remark. The original definition of stability conditions ([4, Definition 1.1]) on triangulated categories differs from Definition 1.3. Roughly speaking a stability condition in the sense of [4, Definition 1.1] is defined by a pair of a group homomorphism $Z\colon K(\mathcal{T}) \to \mathbb{C}$ and a sull subcategory $\mathcal{P}(\phi) \subset \mathcal{T}$ for each $\phi \in \mathbb{R}$ which satisfies some axiom. However [4, Proposition 4.2] shows that giving a stability condition in Definition 1.3 is equivalent to giving a stability condition in [4, Definition 1.1].

Remark. For higher dimensional varieties, the usual notion of semistable sheaves does not induce a stability condition on the derived category. For instance suppose X is a smooth projective surface and H is an ample divisor on X. One may try to construct a stability condition on $D(X)$ as follows. For the heart of a t-structure we set $\mathcal{A} = \mathrm{Coh}(X)$, and for the stability function we set $Z(E) = -c_1(E) \cdot H + \mathrm{rk}(E) \cdot i$. However the pair $(Z, \mathcal{A})$ does *not* determine a stability condition because $Z([\mathcal{O}_x]) = 0$ for closed points $x \in X$.

In the paper [4], Bridgeland showed that the set of stability conditions on $\mathcal{T}$ which satisfy some good properties has a structure of a complex manifold, denoted by $\mathrm{Stab}(\mathcal{T})$. To see how $\mathrm{Stab}(\mathcal{T})$ looks like, it is helpful to consider Example 1.4 (ii). Let $\mathrm{Stab}(\mathcal{A}) \subset \mathrm{Stab}(\mathcal{T})$ be the subset consisting of stability conditions corresponding to the fixed heart of a t-structure $\mathcal{A} \subset \mathcal{T}$. Suppose $\mathcal{A} = \mathrm{mod}\, A$ and $\mathcal{T} = D^b(\mathrm{mod}\, A)$ as in Example 1.4 (ii). Then the stability conditions constructed in Example 1.4 (ii) determine a dense open subset $U_{\mathcal{A}} \subset \mathrm{Stab}(\mathcal{A})$, isomorphic to $\mathcal{H}^N$. In some nice situations, (for example see [6],) the space $\mathrm{Stab}(\mathcal{T})$ contains the subspace having the chamber structure,

$$\bigcup_i \overline{U}_{\mathcal{A}_i} \subset \mathrm{Stab}(\mathcal{T}), \quad U_{\mathcal{A}_i} \cap U_{\mathcal{A}_j} = \emptyset \text{ for } i \neq j. \tag{1.1}$$

Here $\mathcal{A}_i$ is equivalent to $\mathrm{mod}\, A_i$ for a finite dimensional k-algebra A_i, and each $U_{\mathcal{A}_i} \cong \mathcal{H}^N$ is an open subset of $\mathrm{Stab}(\mathcal{T})$.

Remark. Obviously the group of autoequivalences $\mathrm{Auteq}(\mathcal{T})$ acts on $\mathrm{Stab}(\mathcal{T})$. In some situations (cf. [6]), the chambers $U_{\mathcal{A}_i}$ in (1.1) are obtained by the action of $\mathrm{Auteq}(\mathcal{T})$ from one of the chambers.

Remark. It is shown in [4] that the group $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, the universal cover of $\mathrm{GL}^+(2,\mathbb{R})$, also acts on $\mathrm{Stab}(\mathcal{T})$. When $\mathcal{T} = D(C)$ for an elliptic curve C, the action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is free and transitive, thus we have $\mathrm{Stab}(\mathcal{T}) \cong \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ in this case.

Suppose that X is a Calabi-Yau manifold and $\mathcal{T} = D(X)$. Then conjecturally $\mathrm{Stab}(\mathcal{T})$ is related to the so called *stringy Kähler moduli space* $\mathcal{M}_K(X)$, a subspace of the moduli space of $\mathcal{N} = 2$ super conformal field theories. Its relationship to mirror symmetry is as follows. Let $\hat{X}$ be a mirror manifold of X. According to Kontsevich's Homological mirror symmetry [18], there should exist an equivalence of triangulated categories,

$$D(X) \to D\,\mathrm{Fuk}(\hat{X}),$$

where the RHS is the derived Fukaya category on $\hat{X}$. Then $\mathcal{M}_K(X)$ should be isomorphic to $\mathcal{M}_C(\hat{X})$, the moduli space of complex structures on $\hat{X}$. More precisely it is expected that the double quotient space

$$\mathbb{C} \backslash \mathrm{Stab}(\mathcal{T}) / \mathrm{Auteq}(\mathcal{T}), \tag{1.2}$$

contains $\mathcal{M}_C(\hat{X})$. For example if $\mathcal{T} = D(C)$ for an elliptic curve C, then the space (1.2) is nothing but the modular curve $\mathcal{H}/\mathrm{SL}(2,\mathbb{Z})$. Since an elliptic curve is self mirror, we have the complete picture in this case.

§ 2. Stability conditions on K3 surfaces

In this section, we assume X is a K3 surface or an abelian surface. Let ω be an ample divisor on X. First let us recall the notion of μ_ω-stability and ω-Gieseker stability on $\mathrm{Coh}(X)$. (For the introduction, one can consult [10].)

Definition 2.1. For a torsion free sheaf $E \in \mathrm{Coh}(X)$, we set

$$\mu_\omega(E) = \frac{c_1(E)\cdot\omega}{\mathrm{rk}(E)}.$$

Then E is called μ_ω-semistable if for any non-zero subsheaf $F \subset E$, one has $\mu_\omega(F) \le \mu_\omega(E)$. Also for $E \in \mathrm{Coh}(X)$, its reduced Hilbert polynomial is defined by

$$p(E,\omega,n) = \chi(E\otimes\mathcal{O}(n\omega))/\alpha,$$

where α is the leading coefficient of the polynomial $\chi(E\otimes\mathcal{O}(n\omega)) \in \mathbb{Q}[n]$. Then E is ω-Gieseker semistable if for any non-zero subsheaf $F \subset E$ one has $p(F,\omega,n) \le p(E,\omega,n)$ for $n \gg 0$.

Remark. Obviously the notions of μ_ω-stability and ω-Gieseker stability are extended for a $\mathbb{Q}$-ample divisor ω.

Next we discuss stability conditions on $\mathcal{T} = D(X)$, studied by Bridgeland [5]. As discussed in the previous section, it is a non-trivial problem to find a stability condition on $\mathcal{T}$. Let β,ω be $\mathbb{Q}$-divisors on X with ω ample. For a torsion free sheaf $E \in \mathrm{Coh}(X)$, one has the Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E,$$

such that $F_i = E_i/E_{i+1}$ is μ_ω-semistable and $\mu_\omega(F_i) > \mu_\omega(F_{i+1})$. Then define $\mathcal{T}_{(\beta,\omega)} \subset \mathrm{Coh}(X)$ to be the subcategory consisting of sheaves whose torsion free parts have μ_ω-semistable Harder-Narasimhan factors of slope $\mu_\omega(F_i) > \beta\cdot\omega$. Also define $\mathcal{F}_{(\beta,\omega)} \subset \mathrm{Coh}(X)$ to be the subcategory consisting of torsion free sheaves whose μ_ω-semistable factors have slope $\mu_\omega(F_i) \le \beta\cdot\omega$.

Definition 2.2. We define $\mathcal{A}_{(\beta,\omega)}$ to be

$$\mathcal{A}_{(\beta,\omega)} = \left\{ E \in D(X) : \begin{array}{l} \mathcal{H}^{-1}(E) \in \mathcal{F}_{(\beta,\omega)}, \mathcal{H}^0(E) \in \mathcal{T}_{(\beta,\omega)}, \\ \text{and } \mathcal{H}^p(E) = 0 \text{ for every } p \neq -1, 0 \end{array} \right\}.$$

We define $Z_{(\beta,\omega)} \colon K(X) \to \mathbb{C}$ by the formula,

$$Z_{(\beta,\omega)}(E) = -\int e^{-(\beta+i\omega)}\,\mathrm{ch}(E)\sqrt{\mathrm{td}_X}, \tag{2.1}$$

and define $\sigma_{(\beta,\omega)}$ to be the pair $(Z_{(\beta,\omega)}, \mathcal{A}_{(\beta,\omega)})$. The following is shown in [5, Proposition 7.1].

Proposition 2.3 ([5]). *The subcategory $\mathcal{A}_{(\beta,\omega)} \subset D(X)$ is the heart of a bounded t-structure, and the pair $\sigma_{(\beta,\omega)}$ gives a stability condition on $D(X)$ if and only if for any spherical sheaf E on X, one has $Z_{(\beta,\omega)}(E) \notin \mathbb{R}_{\le 0}$. This holds whenever $\omega^2 > 2$.*

Here an object $E \in D(X)$ is called *spherical* if the following holds,

$$\operatorname{Hom}(E, E[i]) = \begin{cases} \mathbb{C} & i = 0, 2, \\ 0 & i \neq 0, 2. \end{cases}$$

Recall that the pairing

$$\chi\colon D(X) \times D(X) \ni (E, F) \mapsto \sum (-1)^i \dim \operatorname{Ext}^i(E, F) \in \mathbb{Z},$$

descends to a paring on $K(X)$. Let $\mathcal{N}(X)$ be the quotient space,

$$\mathcal{N}(X) = K(X)/\equiv,$$

where $E_1 \equiv E_2$ if and only if $\chi(E_1, F) = \chi(E_2, F)$ for any $F \in K(X)$. Note that $\mathcal{N}(X)$ is a finitely generated $\mathbb{Z}$-module.

Definition 2.4. A stability condition $\sigma = (Z, \mathcal{A})$ is called *numerical* if $Z\colon K(X) \to \mathbb{C}$ factors through the surjection $K(X) \to \mathcal{N}(X)$.

Let $\operatorname{Stab}(X)$ be the connected component of the *good* stability conditions (locally finite, numerical in the notation of [4]) which contains $\sigma_{(\beta,\omega)}$. In the paper [5], Bridgeland studies the complex manifold $\operatorname{Stab}(X)$ explicitly, and shows the following.

Theorem 2.5 ([5]). *There is an open subset in $\mathcal{N}(X)_{\mathbb{C}}$, denoted by $\mathcal{P}_0^+(X)$ in* [5]*, such that* $\operatorname{Stab}(X)$ *is a covering space over $\mathcal{P}_0^+(X)$.*

§ 3. Moduli problem of semistable objects

In this section, we discuss the moduli problem of the semistable objects in $D(X)$ for a K3 surface X. As is well-known, there are coarse moduli spaces of (μ_ω, Gieseker) semistable sheaves on projective varieties. However for $\sigma \in \operatorname{Stab}(X)$, a σ-semistable object in $D(X)$ is not necessary a sheaf, thus it is a non-trivial problem to construct the moduli space of σ-semistable objects. On the other hand, the moduli problem of objects in derived categories is addressed in [11], [19]. Here let us recall Lieblich's work [19]. We consider the following 2-functor,

$$\mathcal{M}\colon (\mathrm{Sch}/\mathbb{C}) \longrightarrow (\text{groupoid}),$$

which sends a $\mathbb{C}$-scheme S to the groupoid $\mathcal{M}(S)$ whose objects are relatively perfect objects [19, Definition 2.1.1] $\mathcal{E} \in D(X \times S)$ satisfying

$$\mathrm{Ext}^i(\mathcal{E}_s, \mathcal{E}_s) = 0, \text{ for all } i < 0 \text{ and } s \in S. \tag{3.1}$$

Lieblich [19] shows the following.

Theorem 3.1 ([19]). *The 2-functor $\mathcal{M}$ is an Artin stack of locally finite type over $\mathbb{C}$.*

One can consult [9] for the introduction of Artin stacks. Let $\sigma = (Z, \mathcal{A})$ be a stability condition on $D(X)$ and take $v \in \mathcal{N}(X)$. Note that any σ-semistable object $E \in \mathcal{A}$ satisfies (3.1). Thus we can consider the substack $\mathcal{M}^v(\sigma) \subset \mathcal{M}$, defined to be the stack of σ-semistable objects $E \in \mathcal{A}$ of numerical type v. Note that $\mathcal{M}^v(\sigma)$ is just an abstract stack, and it is not obvious that $\mathcal{M}^v(\sigma)$ is algebraic. In fact we have the following result, which is one of the main results in [22].

Theorem 3.2. *For any $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)$ and $v \in \mathcal{N}(X)$, the stack $\mathcal{M}^v(\sigma)$ is an Artin stack of finite type over $\mathbb{C}$.*

Proof. We just give the outline of the proof in [22].
Step 1. Let $M^v(\sigma)$ be the set of objects,

$$M^v(\sigma) = \{E \in D(X) \mid E \text{ is } \sigma\text{-semistable of numerical type } v\}.$$

In order to show $\mathcal{M}^v(\sigma)$ is an Artin stack of finite type, it is enough to show the following.

(i) Openness of stability: the substack $\mathcal{M}^v(\sigma) \subset \mathcal{M}$ is an open substack, i.e. given a S-valued point $\mathcal{E} \in D(X \times S)$ of $\mathcal{M}$, the locus

$$S^\circ = \{s \in S \mid \mathcal{E}_s \in M^v(\sigma)\} \subset S$$

is open in Zariski topology.

(ii) Boundedness of semistable objects: the set of objects $M^v(\sigma)$ is bounded, i.e. there exists a finite type $\mathbb{C}$-scheme Q and an object $\mathcal{F} \in D(X \times Q)$ such that any object $E \in M^v(\sigma)$ is isomorphic to $\mathcal{F}_q$ for some $q \in Q$.

In fact Theorem 3.1 and (i) ensure that $\mathcal{M}^v(\sigma)$ is an Artin stack, and it is also of finite type by (ii). Next let $\mathcal{V} \subset \mathrm{Stab}(X)$ be the subset

$$\mathcal{V} = \{\sigma_{(\beta,\omega)} \in \mathrm{Stab}(X) \mid \sigma_{(\beta,\omega)} \text{ is constructed in Proposition 2.3.}\}$$

Then it is shown in [5] that for $\sigma \in \mathrm{Stab}(X)$, there is $\Phi \in \mathrm{Auteq}\, D(X)$ and $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $\Phi \circ \sigma \circ g \in \overline{\mathcal{V}}$. Since the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ does not change the set of semistable objects, it is enough to show (i) and (ii) for $\sigma \in \overline{\mathcal{V}}$. Together with some more technical arguments, we may also assume that $\sigma = \sigma_{(\beta,\omega)} \in \mathcal{V}$. (See [22, Theorem 3.20, Step 1].)

Step 2. The conditions (i), (ii) in Step 1 are satisfied if we show (i)', (ii), where (i)' is as follows.

(i)' Generic flatness for $\mathcal{A}$: given $\mathcal{E} \in D(X \times S)$, the locus $\{s \in S \mid \mathcal{E}_s \in \mathcal{A}\}$ is open in S.

This is technically most important part, and we give the idea for the proof. For the detail, see [22, Lemma 3.13, Proposition 3.18]. We want to show that (i)' together with (ii) imply (i). Let S be a smooth quasi-projective variety and take a S-valued point of $\mathcal{M}$, $\mathcal{E} \in D(X \times S)$. Suppose that $\mathcal{E}_s \in M^v(\sigma)$ for a closed point $s \in S$. In order to show that the locus $S^\circ \subset S$ is open, it is enough to find a non-empty Zariski open subset $U \subset S$ such that $U \subset S^\circ$. (See [22, Lemma 3.6].) Since we assume (i)', we may assume that $\mathcal{E}_s \in \mathcal{A}$ for any $s \in S$.

Next we use the conditions (i)' and (ii) to show the following. There exist finite type S-schemes and objects, $(i = 1, 2,)$

$$\pi_i \colon Q_i \longrightarrow S, \quad \mathcal{F}_i \in D(X \times Q_i),$$

together with a morphism $u_1 \colon \pi_1^*\mathcal{E} \to \mathcal{F}_1$, (resp. $u_2 \colon \mathcal{F}_2 \to \pi_2^*\mathcal{E}$,) such that

- For each closed point $q \in Q_1$, the induced morphism $u_{1,q} \colon \mathcal{E}_q \to \mathcal{F}_{1,q}$ is surjective in $\mathcal{A}$ and $\phi(\mathcal{E}_q) > \phi(\mathcal{F}_{1,q})$. (resp. for each closed point $q \in Q_2$, the induced morphism $u_{2,q} \colon \mathcal{F}_2 \to \mathcal{E}_{q,2}$ is injective in $\mathcal{A}$ and $\phi(\mathcal{E}_q) < \phi(\mathcal{F}_2)$.)
- If there are $s \in S$ and a surjection $\mathcal{E}_s \to F_1$ in $\mathcal{A}$ with $\phi(\mathcal{E}_s) > \phi(F_1)$, then there exists $q \in \pi_1^{-1}(s)$ such that $F_1 \cong \mathcal{F}_{1,q}$. (resp. if there are $s \in S$ and an injection $F_2 \to \mathcal{E}_s$ in $\mathcal{A}$ with $\phi(F_2) > \phi(\mathcal{E}_s)$, then there exists $q \in \pi_2^{-1}(s)$ such that $F_2 \cong \mathcal{F}_{2,q}$.)

See [22, Proposition 3.17] for the above constructions. By the properties of π_i, an object $\mathcal{E}_s$ is an object of $M^v(\sigma)$ if and only if $s \notin (\operatorname{im}\pi_1 \cup \operatorname{im}\pi_2)$. On the other hand, the locus S° is at least dense in Zariski topology. This follows from an easy application of [1, Proposition 3.5.3], and see [22, Lemma 3.13] for the proof. We thus conclude that π_i are not dominant. Since Q_i are of finite type, we can then find an open subset

$$U \subset S \setminus (\operatorname{im}\pi_1 \cup \operatorname{im}\pi_2) = S^\circ,$$

as desired.

Step 3. By Step 1 and Step 2, it is enough to check (i)' and (ii) for $\sigma \in \mathcal{V}$. Both conditions are verified by using explicit constructions of $\mathcal{A}_{(\beta,\omega)}$. As for the generic flatness of $\mathcal{A}_{(\beta,\omega)}$, this essentially follows from the existence of relative Harder-Narasimhan filtration in μ_ω-stability [10, Theorem 2.3.2], and the proof is given in [22, Lemma 4.7]. We emphasize here that showing (i)' is much easier than showing (i) directly.

Finally we give the idea of showing (ii). Let us take a σ-semistable object $E \in \mathcal{A}_{(\beta,\omega)}$. Note that $\mathcal{H}^i(E) = 0$ unless $i = -1, 0$. Let $\mathcal{H}^0(E)_t$ be the torsion part of $\mathcal{H}^0(E)$ and set $\mathcal{H}^0(E)_f = \mathcal{H}^0(E)/\mathcal{H}^0(E)_t$. Let

$$F_1, \cdots, F_{a(E)}, \quad (\text{resp. } T_1, \cdots, T_{b(E)})$$

be the μ_ω-semistable factors of $\mathcal{H}^{-1}(E)$, (resp. $\mathcal{H}^0(E)_f$,) and $T'_1, \cdots, T'_{c(E)}$ the (β, ω)-twisted semistable factors of $\mathcal{H}^0(E)_t$. (We omit the definition of twisted stability. For the detail see [20].) Then using the σ-semistability of E, one can show that the maps

$$M^v(\sigma) \ni E \mapsto a(E), b(E), c(E) \in \mathbb{Z}$$

are bounded. Moreover one can also show that the possible numerical classes

$$\mathrm{ch}(F_i), \mathrm{ch}(T_i), \mathrm{ch}(T'_i) \in H^*(X, \mathbb{Q}),$$

are also finite. Hence the set of sheaves $\{F_i, T_i, T'_i \mid E \in M^v(\sigma)\}$ is bounded, and this implies $M^v(\sigma)$ is also bounded. (See [22, Proposition 4.11] for the detail.) □

§ 4. Counting invariants of semistable objects

In this section, we introduce the recent result of D. Joyce [15], and the related result of [22] on counting invariants of Bridgeland semistable objects in $D(X)$. D. Joyce's works [12], [13], [14], [15], [17] are attempts to introduce some structures (Frobenius structures, automorphic functions...) on $\mathrm{Stab}(\mathcal{T})$ for a triangulated category $\mathcal{T}$, using "counting invariants" of semistable objects. (However at this time, his arguments only work for $\mathrm{Stab}(\mathcal{A})$ for some abelian categories $\mathcal{A}$.)

Let $K_0(\mathrm{Var}/\mathbb{C})$ be the Grothendieck ring of varieties, i.e.

$$K_0(\mathrm{Var}/\mathbb{C}) = \bigoplus_Y \mathbb{Z}[Y]/\sim,$$

where Y is a quasi-projective variety, and the equivalence relation $\sim$ is generated by the relation,

$$[Y] \sim [Y \setminus Z] + [Z],$$

for closed subvarieties $Z \subset Y$. There is a ring structure on $K_0(\mathrm{Var}/\mathbb{C})$ given by $[Y] \cdot [Z] = [Y \times Z]$. Let Λ be a $\mathbb{Q}$-algebra. By definition, a motivic invariant is a ring homomorphism,

$$\Upsilon \colon K_0(\mathrm{Var}/\mathbb{C}) \longrightarrow \Lambda. \tag{4.1}$$

For simplicity, we write $\Upsilon([Y])$ as $\Upsilon(Y)$. We assume that $\Upsilon(Y) \in \Lambda$ is invertible for any $Y \neq \emptyset$.

Example 4.1. Let $\Lambda = \mathbb{Q}(t)$ and set Υ as

$$\Upsilon(Y) = \sum (-1)^i b_i(Y) t^i.$$

Here $b_i(Y)$ is the i-th virtual betti number of Y. If Y is smooth and projective, $b_i(Y)$ is the usual i-th betti number of Y. Then Υ satisfies the above conditions.

Let $K_0(\mathrm{St}/\mathbb{C})$ be the Grothendieck ring of Artin stacks, i.e.

$$K_0(\mathrm{St}/\mathbb{C}) = \bigoplus_{\mathcal{Y}} \mathbb{Z}[\mathcal{Y}]/\sim,$$

where $\mathcal{Y}$ is an Artin stack of finite type over $\mathbb{C}$. For some technical reasons, we assume that $\mathcal{Y}$ has affine stabilizers. The equivalence relation $\sim$ and the ring structure on $K_0(\mathrm{St}/\mathbb{C})$ are similarly defined. Under the above setting, the map Υ extends to Υ',

$$\Upsilon' \colon K_0(\mathrm{St}/\mathbb{C}) \longrightarrow \Lambda,$$

such that if G is a *special* algebraic group acting on a variety Y, then $\Upsilon'([Y/G]) = \Upsilon(Y)/\Upsilon(G)$. (See [16] for the proof.) Here an algebraic group G is called special if any principle G-bundle is Zariski locally trivial, and $[Y/G]$ is the global quotient stack.

Let $C(X) \subset \mathcal{N}(X)$ the image of $\mathrm{Coh}(X) \to \mathcal{N}(X)$. For $v \in C(X)$, let $\mathcal{M}^v(\omega)$ be the moduli stack of ω-Gieseker semistable sheaves $E \in \mathrm{Coh}(X)$ of numerical type v. As is well-known, the stack $\mathcal{M}^v(\omega)$ is an Artin stack of finite type over $\mathbb{C}$. (Actually $\mathcal{M}^v(\omega)$ is obtained as a global quotient stack of the Grothendieck Quot scheme. See [10].) We define $\hat{I}^v(\omega) = \Upsilon'(\mathcal{M}^v(\omega))$, and the weighted counting

$$\hat{J}^v(\omega) = \sum_{v_1+\cdots+v_n=v} l^{-\sum_{j>i}\chi(v_j,v_i)} \frac{(-1)^{n-1}(l-1)}{n} \prod_{i=1}^{n} \hat{I}^{v_i}(\omega) \in \Lambda. \tag{4.2}$$

Here $v_i \in C(X)$ satisfy $p(v_i,\omega,n) = p(v,\omega,n)$ and $l = \Upsilon(\mathbb{A}^1) \in \Lambda$. Joyce [15] showed that the sum (4.2) is a finite sum, and the following.

Theorem 4.2 ([15])**.** *The invariant $\hat{J}^v(\omega)$ does not depend on a choice of ω.*

Remark. The formula (4.2) is roughly speaking the logarithm of the invariants $\hat{I}^v(\omega)$, which is explained as follows. Let $A = \oplus_{v\in C(X)}\Lambda \cdot c_v$ be the algebra with multiplication given by $c_{v_1} * c_{v_2} = l^{-\chi(v_1,v_2)} c_{v_1+v_2}$. We set $\hat{\delta}^v(\omega) = \hat{I}^v(\omega)\cdot c_v \in \Lambda$. Then $\hat{J}^v(\omega)$ is the coefficient of the logarithm of $\hat{\delta}^v(\omega)$ multiplied by $(l-1)$,

$$(l-1)\cdot \sum_{v_1+\cdots+v_n=v} \frac{(-1)^{n-1}}{n} \hat{\delta}^{v_1} * \cdots * \hat{\delta}^{v_n}.$$

The algebra A is related to the Hall type algebra. (cf. [15].)

Remark. Suppose that $v \in C(X)$ is primitive and ω is in a general position of the ample cone of X. Then $\mathcal{M}^v(\omega)$ is written as $[M^v(\omega)/\mathbb{G}_m]$, where $M^v(\omega)$ is a projective symplectic variety. Then $\hat{J}^v(\omega)$ is written as $\Upsilon(M^v(\omega))$. (Note that the factor $(l-1)$ cancels out the contribution of the stabilizer group $\mathbb{G}_m$.) Now suppose that ω' is another ample divisor and $M^v(\omega')$ is birational to $M^v(\omega)$. In this case it is well known that $\Upsilon(M^v(\omega)) = \Upsilon(M^v(\omega'))$, thus Theorem 4.2 indicates this fact.

Now let us return to Bridgeland's stability conditions. The purpose of the paper [22] is to generalize Theorem 4.2 for Bridgeland's stability conditions on $D(X)$, whose precise statements are conjectured by Joyce [15]. For $\sigma \in \mathrm{Stab}(X)$ and $v \in \mathcal{N}(X)$, consider the Artin stack $\mathcal{M}^v(\sigma)$ as in the previous section. For the heart of a t-structure $\mathcal{A} \subset \mathcal{T}$, we denote $C(\mathcal{A}) = \mathrm{im}(\mathrm{ch}\colon \mathcal{A} \to \mathcal{N}(X))$.

Definition 4.3. For $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)$ and $v \in \mathcal{N}(X)$, we define $I^v(\sigma)$ as follows.

$$I^v(\sigma) = \Upsilon'(\mathcal{M}^v(\sigma)) \quad (v \in C(\mathcal{A})), \quad I^v(\sigma) = I^{-v}(\sigma) \quad (v \in -C(\mathcal{A})),$$

and if $v \notin \pm C(\mathcal{A})$, we set $I^v(\sigma) = 0$. Also we define $J^v(\sigma)$ as

$$J^v(\sigma) = \sum_{v_1 + \cdots + v_n = v} l^{-\sum_{j>i} \chi(v_j, v_i)} \frac{(-1)^{n-1}(l-1)}{n} \prod_{i=1}^{n} I^{v_i}(\sigma) \in \Lambda. \tag{4.3}$$

Here $v_i \in \mathcal{N}(X)$ satisfies $Z(v_i) \in \mathbb{R}_{>0} Z(v)$.

Note that $I^v(\sigma)$ is well defined by Theorem 3.2. One can show that (4.3) is a finite sum, thus $J^v(\sigma)$ is also well defined. The following is the analogue of Theorem 4.2 for Bridgeland's stability conditions.

Theorem 4.4 ([22]). *The invariant $J^v(\sigma)$ does not depend on a choice of $\sigma \in \mathrm{Stab}(X)$. Furthermore if $v \in C(X)$, we have $J^v(\sigma) = \hat{J}^v(\omega)$ for any ample divisor ω.*

§ 5. Future works

In this section we discuss some problems related to the work [22].

- **Stability conditions on Calabi-Yau 3-folds.**

  From the viewpoint of string theory, it is very important to study the space $\mathrm{Stab}(X)$ for a Calabi-Yau 3-fold X, and discuss the counting invariants of (semi)stable objects on X. In this case, we have the difficulty in constructing stability conditions in the sense of Bridgeland. From the discussion in [8], we guess that stability functions are given by

$$Z(E) = -\int e^{-(\beta+i\omega)}\,\mathrm{ch}(E)\sqrt{\mathrm{td}_X} + (\text{quantum corrections}),$$

in a neighborhood of the large volume limit. Note that the presence of quantum corrections is the different point from the K3 surface case. For $E \in \mathrm{Coh}(X)$, let us investigate the value $\arg Z(E)$ for $\omega \to \infty$. We have

$$\lim_{\omega\to\infty} \arg Z(E) = \begin{cases} \pi & \dim \mathrm{Supp}(E) = 0, \\ \pi/2 & \dim \mathrm{Supp}(E) = 1, \\ 0 & \dim \mathrm{Supp}(E) = 2, \\ -\pi/2 & \dim \mathrm{Supp}(E) = 3, \end{cases} \tag{5.1}$$

Let $\mathcal{T}, \mathcal{F} \subset \mathrm{Coh}(X)$ be the subcategories defined by

$$\mathcal{T} = \{E \in \mathrm{Coh}(X) \mid \dim \mathrm{Supp}(E) \le 1\},$$
$$\mathcal{F} = \{E \in \mathrm{Coh}(X) \mid \mathrm{Hom}(\mathcal{T}, E) = 0\}.$$

Then the pair $(\mathcal{T}, \mathcal{F})$ determines a torsion theory on $\mathrm{Coh}(X)$, and let $\mathcal{A} \subset D(X)$ be the corresponding tilting, i.e.

$$\mathcal{A} = \left\{ E \in D(X) : \begin{array}{c} \mathcal{H}^{-1}(E) \in \mathcal{F}, \mathcal{H}^0(E) \in \mathcal{T}, \text{ and} \\ \mathcal{H}^p(E) = 0 \text{ for every } p \neq -1, 0 \end{array} \right\}.$$

It is known that $\mathcal{A}$ is the heart of a t-structure and (5.1) implies that for a non-zero $E \in \mathcal{A}$, we have

$$Z(E) \in e^{\pi i/4} \cdot \mathcal{H},$$

for $\omega \gg 0$. Hence the phase $\phi(E) \in (1/4, 5/4)$ is well-defined for $\omega \gg 0$. (However such sufficiently big ω depends on E, so we cannot conclude that $(e^{-\pi i/4}Z, \mathcal{A})$ gives a stability condition.) From this observation, we guess that there is a heart of a t-structure $\mathcal{A}' \subset D(X)$, which is an "approximation" of $\mathcal{A}$ in some sense, such that $(Z, \mathcal{A}')$ gives a stability condition. Unfortunately we do not know how to find such $\mathcal{A}'$.

- **Counting invariants of (semi)stable objects on Calabi-Yau 3-folds and their relation to Donaldson-Thomas invariants**

For a Calabi-Yau 3-fold X, $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, let $I_n(X, \beta)$ be the moduli space of ideal sheaves $I \subset \mathcal{O}_X$ with

$$(\mathrm{ch}_0(I), \mathrm{ch}_1(I), \mathrm{ch}_2(I), \mathrm{ch}_3(I)) = (1, 0, \beta, n),$$

i.e. $I_n(X,\beta)$ is the Hilbert scheme of curves. Then the Donaldson-Thomas invariant [21] is defined by the integration over the virtual classes,

$$N_{n,\beta}=\int_{I_n(X,\beta)^{vir}} 1 \in \mathbb{Z}.$$

Note that any ideal sheaves are Gieseker stable with respect to any polarization, thus $N_{n,\beta}$ is a counting invariant of stable sheaves. As an analogy of this, we expect that for a given stability condition $\sigma\in \mathrm{Stab}(X)$, there should be the invariant $N_{n,\beta}(\sigma)$, which counts σ-stable objects as above. Furthermore the difference between $N_{n,\beta}(\sigma)$ and $N_{n,\beta}(\tau)$ should be described explicitly using the idea of Joyce [15]. However Joyce's theory does not take account of the virtual classes, and it seems a hard work to involve virtual classes in his theory.

- **Automorphic functions on** $\mathrm{Stab}(X)$ **via counting invariants.** Let X be a K3 surface and consider the invariant $J^v=J^v(\sigma)$ constructed in the previous section. (Since $J^v(\sigma)$ does not depend on σ by Theorem 4.4, we may omit σ.) Let G be the group of autoequivalences of $D(X)$, which preserves the connected component $\mathrm{Stab}(X)$. Then for $g\in G$, Theorem 4.4 implies

$$J^v=J^v(\sigma)=J^{g_*v}(g_*v)=J^{g_*v},$$

where $g_*\in \mathrm{GL}(\mathcal{N}(X))$ is the induced isomorphism. This indicates that the invariants J^v for $v\in H^*(X,\mathbb{Q})$ posses the automorphic property with respect to G. Thus it is natural to guess the existence of an automorphic form on $\mathrm{Stab}(X)$ with respect to G using J^v. For example the function (ignoring convergence),

$$\mathrm{Stab}(X)\ni\sigma=(Z,\mathcal{P})\longmapsto \sum_{v\in\mathcal{N}(X)\setminus\{0\}}\frac{J^v}{Z(v)^k}\in \Lambda\otimes_{\mathbb{Q}}\mathbb{C},$$

for $k\in\mathbb{Z}$ gives an automorphic function of weight k. Of course the above function is only one of the possibilities. It seems interesting to construct automorphic functions on $\mathrm{Stab}(X)$ via counting invariants, and compare them with a mirror side or Borcherds' automorphic functions [3]. For this purpose, the first step is to calculate the invariants J^v explicitly.

- **Moduli problems of stable objects on K3 surfaces.** Let X be a K3 surface. In [22], the author proved that the moduli stack of semistable objects in $D(X)$ is algebraic. In particular the moduli of stable objects are represented by an algebraic space of finite type. Thus it is interesting to study such moduli spaces concretely, and see how they vary under change of stability conditions. In their recent work [2], Arcara, Bertram and Lieblich study such a problem in some special situations. (They focus on the case $\mathrm{Pic}(X)=\mathbb{Z}$, and consider only special stability conditions contained in $\mathcal{V} \subset \mathrm{Stab}(X)$. They also put a certain restriction on $\mathrm{ch}(E) \in H^*(X, \mathbb{Q})$ for stable objects E.) In that situation, they prove that moduli spaces are connected by Mukai flops under change of stability conditions. It is interesting to generalize the work [2] for arbitrary K3 surfaces, numerical classes and stability conditions.

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