The Euler-Kronecker invariants in various families of global fields

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Introduction

Let K be a global field, i.e., either an algebraic number field of finite degree (*abbreviated* NF), or an algebraic function field of one variable over a finite field (*abbreviated* FF). Let $\zeta_K(s)$ be the Dedekind zeta function of K. As in our previous article [E-K], we denote by γ_K ($\in \mathbb{R}$) the quotient, the constant term divided by the residue, in the Laurent expansion of $\zeta_K(s)$ at $s = 1$. In other words,

$$\gamma_K = \lim_{s\to 1}\left(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1}\right). \tag{0.0.1}$$

We consider γ_K as an invariant of K, and for various families $\mathbb{K}$ of global fields, shall study the behaviour of the distribution of values of γ_K for $K \in \mathbb{K}$.

As for the main motivation of this study, some basic results, and for connections with other arithmetic problems, see [E-K]. Here, we only recall that the value of γ_K becomes "very negative" when K has many primes with small norms (e.g. has many rational points in the FF-case), while it becomes (small and) positive when K has only few primes with small norms.

In this article, after some preliminaries (§1), we shall show an elementary treatment for the FF-case (§2), and in §3, shall exhibit some pictures showing the distribution of the point

$$P_K = (2\log\log\sqrt{|d_K|} + 2,\ \gamma_K + 1) \tag{0.0.2}$$

on $\mathbb{R}^2$, where K runs over some given family $\mathbb{K}$ of number fields (d_K: the discriminant of K). The set of families $\mathbb{K}$ that we shall consider includes the family of real or imaginary quadratic fields, that of real biquadratic fields, the full cyclotomic fields, their maximal real subfields, and that of the first layer K_p of the (unique) $\mathbb{Z}_p$-extension over $\mathbb{Q}$, where p runs over the odd prime numbers. We shall see how different the pictures look like depending on $\mathbb{K}$, and shall discuss some related new problems.

§1 Preliminaries

1.1. The invariant γ_K^*

Instead of γ_K itself, we prefer to use the normalized invariant

$$\begin{aligned} \gamma_K^* &= \gamma_K + 1 \qquad (NF), \\ &= \gamma_K + c_q \qquad (FF), \end{aligned} \tag{1.1.1}$$

where

$$c_q = \frac{q+1}{2(q-1)} \log q, \tag{1.1.2}$$

q being the number of elements of the constant field of K. (Note that $c_q > 1$ and that $\lim_{q\to 1} c_q = 1$.) As noted in [E-K], this makes several basic formulas simpler. For example, $\gamma_K^* = \log q$ when K is the rational function field over $\mathbb{F}_q$.

In terms of $\zeta_K(s)$, this γ_K^* can be expressed as

$$\begin{aligned} \gamma_K^* &= \lim_{s\to 1} \left(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s} + \frac{1}{s-1} \right) \qquad (NF), \\ &= \lim_{s\to 1} \left(\frac{\zeta_K'(s)}{\zeta_K(s)} + \sum_{q^\theta = 1, q} \frac{1}{s-\theta} \right) \qquad (FF), \end{aligned} \tag{1.1.3}$$

where the sum in the formula for the FF-case means the limit, as $T \to \infty$, of the sum over all poles θ of $\zeta_K(s)$ with $|\theta| < T$. As in [E-K], put

$$\begin{aligned} \alpha_K &= \log \sqrt{|d_K|} \qquad (NF), \\ &= (g_K - 1) \log q \qquad (FF), \end{aligned} \tag{1.1.4}$$

where d_K is the discriminant and g_K is the genus. Also, put

$$\begin{aligned} \beta_K &= -\frac{r_1}{2}(\gamma_{\mathbb{Q}} + \log 4\pi) - r_2(\gamma_{\mathbb{Q}} + \log 2\pi) \qquad (NF), \\ &= 0 \qquad (FF), \end{aligned} \tag{1.1.5}$$

where r_1 (resp. r_2) is the number of real (resp. complex) places of K, and let $\Lambda_K(s) = \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_K(s)$ be the "completed zeta function", where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$. Then ([E-K]§1.3-1.4)

$$\gamma_K^* + \beta_K = \lim_{s\to 1} \left(\frac{\Lambda_K'(s)}{\Lambda_K(s)} + \frac{1}{s} + \frac{1}{s-1} \right) \qquad (NF), \tag{1.1.6}$$

and in terms of the non-trivial zeros of $\zeta_K(s)$,

$$\gamma_K^* + \alpha_K + \beta_K = \sum_\rho \frac{1}{\rho} \qquad (NF \text{ and } FF), \tag{1.1.7}$$

where ρ runs over all non-trivial zeros of $\zeta_K(s)$ (counted with multiplicities), and the summation over ρ means the limit, as $T \to \infty$, of the sum of all those ρ with $|\rho| < T$. Note that this sum is positive unless $g_K = 0$; hence

$$\gamma_K^* > -(\alpha_K + \beta_K) \qquad (NF, \text{ and } FF \text{ with } g_K > 0). \tag{1.1.8}$$

Remark Why not choose, instead of γ_K^*, the quantities $\gamma_K^* + \beta_K$, or $\gamma_K^* + \alpha_K + \beta_K$? (The latter is connected directly with the symmetric form of the functional equation.) The main reason is that, in general $|\gamma_K^*|$ is much smaller than $|\beta_K|$ or $|\alpha_K|$. For example, when K is a cyclotomic field $\mathbb{Q}(\mu_m)$, the order of expected magnitude of γ_K^* is $\log m$, while $-\beta_K$, α_K are of orders m, $m \log m$, respectively. At least in studying anything related to the size of the invariant, such as upper or lower bounds, we do not want that any delicate property related to the size of γ_K^* be absorbed into that of $|\beta_K|$ etc.

Note also that in most of the known formulas for γ_K^* (cf. [E-K]), γ_K^* is expressed as the *difference* of two (larger) quantities (or as the limit of such differences). Roughly speaking, the invariant γ_K^* is not the main term but appears as the "second term". (One exception is the expression (II) for the FF-case (§2.1).)

1.2. The additivity

Let K/k be any finite Galois extension of global fields with Galois group G. For each irreducible character χ of G, let $L(\chi, s)$ denote the associated Artin L-function. Then from the multiplicative relation

$$\zeta_K(s) = \zeta_k(s) \prod_{\chi \neq \chi_0} L(\chi, s)^{\chi(1)} \tag{1.2.1}$$

among the zeta and the L-functions follows the additive relation among the constants

$$\gamma_K = \gamma_k + \sum_{\chi \neq \chi_0} \chi(1) \frac{L'(\chi, 1)}{L(\chi, 1)}. \tag{1.2.2}$$

(χ_0 denotes the trivial character.) For each subgroup H of G, let k_H denote the corresponding fixed subfield of K, and $\psi_{G/H}$ denote the character of G induced from the trivial character of H. Then it follows easily that whenever

$$\sum_H a_H \psi_{G/H} = 0 \tag{1.2.3}$$

holds for some system $(a_H)_H$ of rational numbers a_H, we have ($\sum_H a_H = 0$ and)

$$\sum_H a_H \gamma_{k_H} = 0. \tag{1.2.4}$$

In the NF-case, and in the FF-case where k and K have the same constant field, the difference $\gamma^*_{k_H} - \gamma_{k_H}$ is independent of H ; hence

$$\sum_H a_H \gamma^*_{k_H} = 0 \tag{1.2.5}$$

holds also for the γ^*-invariants.

The distribution of values of the additive factors $L'(\chi,1)/L(\chi,1)$ of (1.2.2) on the complex plane will be discussed in our future articles.

Examples (i) $G = (\mathbb{Z}/p)^2$ (p : a prime). Then

$$\gamma_K = \sum_{i=1}^{p+1} \gamma_{k_i} - p\gamma_k, \tag{1.2.6}$$

where k_i $(1 \le i \le p+1)$ are the subextensions of degree p.

In what follows, S_n will denote the symmetric group of degree n, and k_i will denote a subextension of K/k of degree i, determined uniquely up to conjugacy in each of the following cases, except for k_6 in (iii) which will mean the unique non-Galois one.

(ii) $G = S_3$

$$\gamma_K = 2\gamma_{k_3} + \gamma_{k_2} - 2\gamma_k, \tag{1.2.7}$$

(iii) $G = S_4$

$$\begin{aligned} \gamma_K &= 3\gamma_{k_6} + 3\gamma_{k_4} - \gamma_{k_3} + \gamma_{k_2} - 5\gamma_k \\ &= 3\gamma_{k_8} + 2\gamma_{k_3} - 2\gamma_{k_2} - 2\gamma_k, \end{aligned} \tag{1.2.8}$$

(iv) G cyclic. Then, no general relations of this type follow from such a group theoretic argument.

In the FF-case, there are (of course) many multiplicative relations among the zeta and the L-functions that do not follow from such a group theoretic argument. For example, if $g_K = g_k = 0$, then $\zeta_K(s) = \zeta_k(s)$ (and $L(\chi, s) = 1$ for all $\chi \neq \chi_0$). There are also some relations which hold only after specialization to $s = 1$. For example, there is a quadratic extension K of $k = \mathbf{F}_2(t)$ with $g_K = 2$ such that $\gamma^*_K = \gamma^*_k = \log 2$ (see §2.3).

§2 The function field case

2.1. Various expressions for $\gamma_K^*/\log q$

Let K be a function field of genus $g = g_K$ with exact constant field $\mathbb{F}_q$. Put $u = q^{-s}$. So,

$$\zeta_K(s) = \frac{P(u)}{(1-u)(1-qu)}, \tag{2.1.1}$$

with a polynomial $P(u)$ with coefficients in $\mathbb{Z}$ of the form

$$P(u) = \prod_{i=1}^{g}(1-\pi_i u)(1-\bar{\pi}_i u) \qquad (\pi_i\bar{\pi}_i = q \ \ (1 \le i \le g)). \tag{2.1.2}$$

We shall exhibit here (in addition to (1.1.7)) 4 different expressions (I)∼(IV) of $\gamma_K^*/\log q$. They are all essentially the same, and can be derived from one another trivially ([E-K] (1.4.3)), but one can observe from each expression some different features of the quantity $\gamma_K^*/\log q$. In what follows, $P'(u) = \frac{d}{du}P(u)$, Frob_K is the q-th power Frobenius endomorphism acting on the Jacobian J_K of the complete smooth curve C_K corresponding to K, and N_m $(m = 1, 2, \cdots)$ is the number of $\mathbb{F}_{q^m}$-rational points of C_K. Note that $P(1) = h_K = |J_K(\mathbb{F}_q)|$ is the class number of K, and recall that

$$\gamma_K^*/\log q > -(g-1) \qquad (g > 0) \tag{2.1.3}$$

by (1.1.8).

$$\text{(I)} \qquad \gamma_K^*/\log q = \left(\frac{P'(1)}{P(1)} - g\right) - (g-1)$$

$$\text{(II)} \qquad = 1 + \sum_{i=1}^{g}\left(\frac{1}{\pi_i - 1} + \frac{1}{\bar{\pi}_i - 1}\right) = 1 + \mathrm{trace}((\mathrm{Frob}_K - 1)^{-1})$$

$$\text{(III)} \qquad = (q-1)\sum_{i=1}^{g}\frac{1}{(\pi_i - 1)(\bar{\pi}_i - 1)} - (g-1)$$

$$\text{(IV)} \qquad = 1 + \sum_{m=1}^{\infty}\frac{q^m + 1 - N_m}{q^m}.$$

The first expression (I) shows that $\gamma_K^*/\log q$ is a *rational number*, and that the denominator divides the class number h_K. It also shows that

$$\gamma_K^*/\log q \equiv 1 - 2g \pmod{p} \qquad (p = char(K)), \tag{2.1.4}$$

when the p-rank of C_K is 0.

(II) shows a slightly stronger conclusion that the denominator of $\gamma_K^*/\log q$ divides the *exponent* of the finite abelian group $J_K(\mathbb{F}_q)$. This expression also shows (by the Weil's Riemann Hypothesis for curves) that $\gamma_K^*/\log q$ belongs to the closed interval

$$[1-\tfrac{2g}{\sqrt{q}+1}, 1+\tfrac{2g}{\sqrt{q}-1}].$$

This gives a good upper and a lower bound when $q \gg g$. In particular, when g is fixed and $q \to \infty$, $\gamma_K^*/\log q$ tends to 1.

(III) shows that $\gamma_K^*/\log q$ is related to the *harmonic* mean (the inverse of the arithmetic mean of the inverses) of g positive real numbers $(\pi_i - 1)(\bar{\pi}_i - 1)$ $(1 \leq i \leq g)$. In fact, their arithmetic, geometric, and harmonic means are given respectively by

$$\text{(2.1.5)} \qquad \begin{cases} a.m. = N_1 g^{-1} + (q+1)(1-g^{-1}), \\ g.m. = h_K^{(1/g)}, \\ h.m. = \frac{(q-1)g}{(\gamma_K^*/\log q)+g-1} \end{cases}$$

([E-K](1.4.5)). Thus, in a sense, $\gamma_K^*/\log q$ is the "third daughter" having N_1 and h_K as "elder sisters". The well-known general inequalities assert that $h.m. \leq g.m. \leq a.m.$. Note here that the denominator in the above formula for $h.m.$ is always positive and hence $h.m.$ can become as large as possible only when $\gamma_K^*/\log q$ is as close as possible to $1-g$ (hence in particular, negative). It is a new invariant; it cannot be expressed only by $(q,g), N_1$ and h_K, unless $g \leq 2$. There have been a lot of work done by many authors to find K having large N_1, and also towards the other direction, to find K having small h_K. Here, we shall be interested in finding K having negative minimal and positive maximal γ_K^* for some given (q,g), at the moment mainly for curiosity, but also keeping in mind the possibility of further interesting comparison with the NF-case.

Remark A word on a negative aspect. A characteristic property of harmonic means is that if one of the members is very close to 0, then the harmonic mean will also be close to 0, no matter how large all other members are. In our case, however, each of $(\pi_i - 1)(\bar{\pi}_i - 1)$ is separated from 0 by at least $(\sqrt{q}-1)^2$. So, the present environment is not so suitable for "her" to make full use of this general property.

Examples (i) $K = \mathbb{F}_p(x,y)$; $y^p - y = x^2$, with $p \equiv 1 \pmod 4$. Then

$$g = (p-1)/2,\ P(u) = (1-pu^2)^g,$$
$$N_1 = p+1,\ h = (p-1)^g,\ \gamma_K^*/\log p = 2;$$
$$a.m. = p+1,\ g.m. = p-1,\ h.m. = (p-1)^2/(p+1);$$

(ii) $K = \mathbb{F}_8(x,y)$; $x^7 + y^7 = 1$. Then $g = 15$, and

$$N_1 = 21,\ h = 2^6.7^{15},\ \gamma_K^*/\log 8 = -2;$$
$$a.m. = 9.8,\ g.m. = 9.236,\ h.m. = 8.75.$$

The fourth expression (IV) is an infinite one, but this shows first that $\gamma_K^*/\log q$ tends to be *negative* when N_m for small m's (esp. N_1) are big. See [E-K] (and [Ts$_2$]) for the extreme negative case. Secondly, from (IV), we easily obtain a nice upper bound for $\gamma_K^*/\log q$ when $g \gg q$. To see this, let $g \geq 1$ and denote by M the smallest positive integer satisfying

$$q^{M/2} + q^{-M/2} \geq 2g, \tag{2.1.6}$$

or equivalently,

$$M \geq 2\log(g + \sqrt{g^2 - 1})/\log q. \tag{2.1.7}$$

Proposition 1

$$\gamma_K^*/\log q \leq M + \frac{1 - q^{1-M}}{q-1} + 2gq^{-M/2}(1 - q^{-1/2})^{-1} \tag{2.1.8}$$

$$\leq M + \frac{1 - q^{1-M}}{q-1} + \frac{1 + q^{-M}}{1 - q^{-1/2}}. \tag{2.1.9}$$

The second bound is weaker but its approximate size (below) is more apparent.

$$2\log g/\log q \ + \ O(1) \ \ (O\text{: absolute}).$$

Proof The simplest combination of the obvious inequalities $N_m \geq 0$ and the Weil's Riemann Hypothesis for curves, in (IV). Namely, use

$$q^m + 1 - N_m \leq q^m + 1 \tag{2.1.10}$$

for $m < M$ and

$$q^m + 1 - N_m \leq 2gq^{m/2} \tag{2.1.11}$$

for $m \geq M$. □

Remarks (i) If we use $N_m \geq N_1$ instead of $N_m \geq 0$, then we obtain, similarly,

$$\gamma_K^*/\log q \leq M' + (1 - N_1)\frac{1 - q^{1-M'}}{q-1} + 2gq^{-M'/2}(1 - q^{-1/2})^{-1}, \tag{2.1.12}$$

where M' is the smallest positive integer satisfying

$$M' \geq 2\log(g + \sqrt{g^2 + N_1 - 1})/\log q. \tag{2.1.13}$$

This is useful when N_1 is large.

(ii) Let us compare the upper bound given by [E-K] (Th.1(FF)) and the above Proposition 1. First, we note that by changing the proof of the former slightly (instead of using c_q just as a number > 1, use this as the difference between γ_K^* and γ_K), we obtain

$$\gamma_K^*/\log q < \left(\left(\frac{\alpha_K+1}{\alpha_K-1}\right)(2\log\alpha_K + 1 + \log q) + 1\right)/\log q, \tag{2.1.14}$$

under the restriction $g > 2$, or $g = 2$ and $q > 2$. Call $UB0$ (resp. $UB1$) the right hand side of (2.1.14) (resp. (2.1.8)). Then they are both $2\log g/\log q + O(1)$, but for $q \geq 7$, $UB1$ is slightly smaller and hence gives a better bound. (On the other hand, for smaller q , $UB0$ is better when g is large enough; e.g., $UB0 < UB1$ holds for $q = 2, g > 7$.)

Thus, for each fixed q, $\limsup((\gamma_K^*/\log q)/\log g) \leq 2$. The author has not succeeded in deciding whether $\limsup(\gamma_K^*/\log q) = \infty$ or not. As for the lower bound, we know ([E-K]) that $\liminf((\gamma_K^*/\log q)/(g-1))$ is (finite and) negative for each q and is equal to $-(\sqrt{q}+1)^{-1}$ when q is a square.

For each $m \geq 1$, denote by B_m the number of prime divisors of K with degree m, so that

$$N_m = \sum_{d|m} dB_d. \tag{2.1.15}$$

Then (IV) can be rewritten in terms of B_m as

$$\gamma_K^*/\log q = 1 + \textstyle\sum_{m=1}^{\infty} \frac{m(B_m^0 - B_m)}{q^m - 1}, \tag{V}$$

where B_m^0 is "B_m for the genus 0 case", i.e., the number of conjugacy classes of elements of degree m over $\mathbb{F}_q$ for $m > 1$, and this added by one when $m = 1$. If one tries to use this to improve Proposition 1 (because $B_m \geq 0$ is slightly stronger than $N_m \geq 0$), some complications arise in the evaluations of $m(B_m^0 - B_m)$ using the Weil's Riemann Hypothesis.

We end this § by giving a general formula for $\gamma_K^*/\log q$ in terms of $(B_1, ..., B_g)$, for some small g.

$$(g = 0) \qquad \gamma_K^*/\log q = 1,$$

$$(g = 1) \qquad = \tfrac{q-1}{B_1},$$

$$(g = 2) \qquad = -1 + 2(q-1)\tfrac{B_1+q+1}{B_1^2+B_1+2B_2-2q},$$

$$(g = 3) \qquad = -2 + 3(q-1)\tfrac{B_1^2+(2q+3)B_1+2(B_2+q^2+1)}{B_1^3+3B_1^2+6B_3+B_1(6B_2-6q+2)},$$

and so on. In each case of $g > 0$, the denominator is $g!\ h_K$.

2.2. QuasiCurve data (QC data)

Fix a prime power q and a non-negative integer g. Let $P(u)$ be any polynomial of degree $2g$ with rational integral coefficients. Consider the following conditions to be imposed upon $P(u)$;

(O) $$P(0) = 1,$$

(FE) $$q^g u^{2g} P((qu)^{-1}) = P(u).$$

The polynomials $P(u) = \sum_{i=0}^{2g} a_i u^i$ $(a_i \in \mathbb{Z}\ (0 \le i \le 2g))$ satisfying (O) and (FE) (i.e., $a_0 = 1, a_{2g-i} = q^{g-i} a_i\ (0 \le i \le g-1)$), and the ordered sets $BT = (B_1, \cdots B_g)$ of g integers $B_1, \cdots B_g \in \mathbb{Z}$, are in a one-to-one correspondence with each other via the congruence

(2.2.1) $$\frac{P(u)}{(1-u)(1-qu)} \equiv \prod_{i=1}^{g} (1-u^i)^{-B_i} \qquad (\mathrm{mod}\ u^{g+1}).$$

Extend BT to an infinite sequence $(B_m)_{m=1}^{\infty}$ of integers B_m by the identity

(2.2.2) $$\frac{P(u)}{(1-u)(1-qu)} = \prod_{m=1}^{\infty} (1-u^m)^{-B_m}$$

in the formal power series algebra over $\mathbb{Z}$. Note that the comparison of coefficients of $u, u^2, \cdots$ inductively determines $B_1, B_2, \cdots$.

In addition to (O) and (FE), consider also the following "Weil Riemann Hypothesis";

(RH) *all reciprocal roots of $P(u)$ have (complex) absolute values $q^{1/2}$.*

Note that $P(u)$ satisfies all these conditions (O),(FE) and (RH) if and only if it is of the form

(2.2.3) $$P(u) = \prod_{i=1}^{g} (1-\pi_i u)(1-\bar{\pi}_i u) \qquad (\pi_i \bar{\pi}_i = q \quad (1 \le i \le g)).$$

By Honda-Tate theorem, such $P(u)$ correspond bijectively with the $\mathbb{F}_q$-isogeny classes of g-dimensional abelian varieties over $\mathbb{F}_q$ (the characteristic polynomial of the Frobenius action on the Tate modules). When BT corresponds to such $P(u)$, we call BT an *abelian* datum.

When (2.2.2) is equal to the zeta function of some function field K over $\mathbb{F}_q$, then B_m is the number of prime divisors of K of degree m; hence, necessarily,

(Non-Neg) $\qquad B_m \geq 0 \qquad (m \geq 1).$

When BT satisfies (Non-Neg) (in addition to (O),(FE) and (RH)), we call BT a *quasi-curve* datum (*abbrev.* QC-datum). And when BT does correspond to an actual curve, it will be called a *curve* datum (*abbrev.* C-datum).

Note that when $g = 1$, *abelian* implies *curve* and hence *quasi-curve*. In general, the necessary condition (NonNeg) for an abelian datum BT to correspond to a curve is non-trivial but far from being sufficient. For example, when $(q, g) = (2, 2)$, the numbers of abelian,quasi-curve,curve data are 35,23,20, respectively.

Remark The following consequence of the Riemann-Roch theorem satisfied by every C-datum BT is also satisfied by any datum BT (corresponding to conditions (O) and (FE) for $P(u)$). Define, for each BT, the sequence of integers $\{D_m\}$ $(m \geq 0)$ by the equality

$$\frac{P(u)}{(1-u)(1-qu)} = \prod_{m=1}^{\infty}(1-u^m)^{-B_m} = \sum_{m=0}^{\infty} D_m u^m. \tag{2.2.4}$$

When B_m is non-negative for all m, D_m is also, and when BT corresponds to a curve, D_m is the number of effective divisors of degree m. The Riemann-Roch theorem, averaged over the ideal classes, gives (cf. [Ts$_1$])

$$D_m = q^{m-g+1}D_{2g-2-m} + P(1)\frac{q^{m+1-g}-1}{q-1} \qquad (\mathrm{m} \in \mathbb{Z}), \tag{2.2.5}$$

where we put $D_m = 0$ when $m < 0$. As is well-known, the functional equation is a consequence of the Riemann-Roch theorem, but also conversely, the Riemann-Roch theorem in this form is a consequence of (FE). Indeed, assume (FE) and let a_m denote the coefficient of u^m in $P(u)$. Then $a_m = D_m - (q+1)D_{m-1} + qD_{m-2}$ and $a_{2g-m} = q^{g-m}a_m \quad (m \in \mathbb{Z})$. So if we put $\delta_m = D_m - q^{m-g+1}D_{2g-2-m}$, then $\delta_{g-1} = 0$ and $\delta_m - (q+1)\delta_{m-1} + q\delta_{m-2} = 0 \quad (m \in \mathbb{Z})$. Therefore, $\delta_m = C\frac{q^{m-g+1}-1}{q-1}$ with some constant C. But $P(1) = \sum_{m=0}^{2g} a_m = D_{2g} - qD_{2g-1} = \delta_{2g} - q\delta_{2g-1}$; hence $P(1) = C$.

Reduction of the condition (Non-Neg) to finitely many m

Proposition 2 (Lemma 2.1(i) of [EHKPWZ]) *Let $BT = (B_1, \cdots B_g)$ be an abelian datum over $\mathbb{F}_q$ with $g \geq 2$. If m is so large that $q^{m/2} \geq 6g + 3$, then $B_m \geq 0$.*

In fact, the proof in [EHKPWZ] uses only the formula $\sum_{d|m} dB_d = q^m + 1 - \sum_{i=1}^{g}(\pi_i^m + \bar{\pi}_i^m)$ and (RH).

Corollary 1 *At least when $g \geq g_q$ is satisfied, then the non-negativity of B_m for $m \leq g$ implies that for all m. Here, g_q is the smallest integer g such that $q^{(g+1)/2} \geq 6g + 3$; explicitly, $g_q = 2$ for all $q \geq 7$, and $g_2 = 12, g_3 = 6, g_4 = 4, g_5 = 3$. In particular, if either $g \geq 12$ or $q \geq 7$, then the non-negativity of B_m for all $m \leq g$ implies that for all m.*

Let us define the invariant γ^*_{BT} for any QC-datum $BT = (B_1, \cdots B_g)$ by the same formula §2.1 (I). Then, as the proofs show, the upper bounds for γ^*_K, given by Theorem 1

of [E-K], and that given by Proposition 1 above, are both valid for γ^*_{BT} . The asymptotic lower bound, Theorem 2 of [E-K], holds also for γ^*_{BT}. On the other hand, Theorem 3 of [E-K] uses the gonalty of curves, and cannot be applied directly to γ^*_{BT}.

2.3. Examples for $q = 2$

(i) $(q, g) = (2, 2)$. There are 23 QC-data BT, among which 20 correspond to curves and 3 not. In fact, there are exactly 20 isomorphism classes over $\mathbb{F}_2$ of hyperelliptic curves of genus 2, and they give 20 distinct BT's (their Jacobians are not isogenous to each other over $\mathbb{F}_2$). The 3 exceptional BT are $(0, 4), (1, 5), (3, 4)$, all with reducible $P(u)$ given respectively by

$$(1 - u + 2u^2)(1 - 2u + 2u^2), \ \ (1 - u + 2u^2)^2, \ (1 - u + 2u^2)(1 + u + 2u^2).$$

The maximal and the minimal values of $\gamma^*_{BT}/\log q$ among these 23 BT are attained by the case of curves, and they are, respectively,

$$\begin{aligned} \gamma^*_{BT}/\log 2 = 3 \qquad & BT = (1, 2), \\ = -10/19 \qquad & BT = (6, 0). \end{aligned}$$

(Affine equations : $y^2 + y = x^5 + x^3 + 1$, resp. $y^2 + y = (x^2 + x)/(x^3 + x + 1)$.)
The integers $0, 1$ and 2 also appear as $\gamma^*_K/\log 2$-values. For example, $\gamma^*_K/\log 2 = 1$ for $BT = (1, 3)$ (defined by $y^2 + y = (x^3 + x + 1)/(x^2 + x + 1)^{-1}$ and $P(u) = 1 - 2u + 3u^2 - 4u^3 + 4u^4$).

(ii) $(q, g) = (2, 3)$. In this case, there are 147 QC-data. The maximal and the minimal values of $\gamma^*_{BT}/\log q$ are

$$\begin{aligned} \gamma^*_{BT}/\log 2 = 4 \qquad & BT = (0, 1, 1), (1, 3, 0) \\ = -88/71 \qquad & BT = (7, 0, 1), \end{aligned}$$

respectively. Each of these 3 corresponds to a curve;

$(0, 1, 1)$: plane quartic: $X^4 + Y^4 + XY^3 + X^3Z + XY^2Z + Y^3Z + Z^4 = 0$,
$(1, 3, 0)$: hyperelliptic: $y^2 + y = (x^5 + x^2 + 1)/(x^2 + x + 1)$,
$(7, 0, 1)$: plane quartic: $X^2Y^2 + X^3Y + X^3Z + Y^3Z + Y^2Z^2 + XZ^3 = 0$.

The last plane quartic is the one that passes through all 7 rational points of the projective 2-space.

The author is not sure about the exact number of BT that correspond to curves. The number of BT that correspond to some hyperelliptic curve is, according to his calculation, 59. That corresponding to some plane quartic seems to be around 57 (from 78 isomorphism

classes). Here, the author is indebted to a precious table made in 1975 by Kazuhisa Kato (Master's thesis, Univ. Tokyo), but unfortunately, it is not mistake-free. There are non-isomophic curves having the same BT, even one hyperelliptic and the other not. It seems that the total number of BT corresponding to some curves is etwa 90-100. The famous $PGL(3,2)$-stable plane quartic (the Klein curve)

$$(X+Y+Z)^4+(XY+YZ+ZX)^2+XYZ(X+Y+Z)=0$$

corresponds to $BT=(0,7,8), P(u)=(1-u+2u^2)^3$ and $\gamma_K^*/\log 2=-1/2$. (Incidentally, the more "famous (0,0,7)" is QC but does not correspond to any curve.) To $BT=(1,2,3)$ correspond two non-isomorphic curves,

$$\begin{cases} Y^4+XY^3+Y^2Z^2+YZ^3+X^4+X^3Z+X^2Z^2=0, \\ y^2+y=(x^4+x+1)/(x^3+x+1). \end{cases} \tag{2.3.1}$$

(iii) $(q,g)=(2,4)$ There are 1035 QC-data. The maximal and the minimal values of $\gamma_{BT}^*/\log q$ are, respectively,

$$\begin{aligned} \gamma_{BT}^*/\log 2 &= 6 && BT=(0,0,0,1), \\ &= -260/133 && BT=(8,0,1,0). \end{aligned}$$

But neither of them corresponds to any curve, as M.Tsfasman and R.Schoof kindly let me know in response to my questions. First, as for the datum attaining the maximal value 6, [LMQ] contains a proof that $(0,0,0,1)$ does not correspond to any curve. The second largest value for $\gamma_{BT}^*/\log q$ among all 1035 QC-data is considerably smaller, i.e., $9/2$, attained by three distinct BT; (0,0,6,2),(0,1,3,3),(0,2,0,3). I do not know at present whether at least one of these corresponds to a curve. As for the minimal value, Schoof has shown me how to prove the non-existence of a curve corresponding to $(8,0,1,0)$. This uses the decomposition of $P(u)$ as a function of $u+2/u$ over Z, and the decomposition of the corresponding Jacobian variety as a polarized abelian variety (the argument used by Serre). On the other hand, since 8 is the maximal number of rational points of a curve of genus 4 over $\mathbb{F}_2$, and since $BT=(8,0,0,2)$ is the only other QC-datum with $B_1=8$, $(8,0,0,2)$ must be a C-datum. This $(8,0,0,2)$ gives the second minimal value $-503/260$ for $\gamma_{BT}^*/\log 2$; hence the minimal value for curves.

If we restrict ourselves to hyperelliptic curves of genus 4, then the maximal (resp. minimal) value for $\gamma_{BT}^*/\log 2$ is $15/4$ (resp. $-239/139$), each being attained by a unique isomorphism class of hyperelliptic curves. Their BT are $(1,2,1,3)$ (resp.$(6,2,2,2)$).

§3 Pictures of "pebble streams" of the Euler-Kronecker invariants for various families of number fields

3.1. Plotting points P_K, the GRH-bounds

For each number field K with $N = [K : \mathbb{Q}] > 1$, we plot the point

$$P_K = (x_K, y_K) \in \mathbb{R}^2$$

on the 2-dimensional Euclidean space $\mathbb{R}^2$, where

$$\begin{cases} x_K = 2\log \alpha_K + 2 = 2\log\log \sqrt{|d_K|} + 2, \\ y_K = \gamma_K^* = \gamma_K + 1. \end{cases} \tag{3.1.1}$$

This coordinate system is chosen in view of Theorems 1 and 3 of [E-K] giving, respectively, an upper and a lower bound for γ_K under the Generalized Riemann Hypothesis (*abbreviated* GRH). Let K run over some given family of number fields, and let us see how the stream of these "pebbles" P_K looks like.

Theorem 1 of [E-K] asserts, under (GRH), that P_K with $\alpha_K > 1.16$ (i.e., $x_K > 2.297$) must lie *below* the curve

$$y = u(x) = \frac{e^{\frac{x}{2}-1}+1}{e^{\frac{x}{2}-1}-1}x - \frac{2}{e^{\frac{x}{2}-1}-1}, \tag{3.1.2}$$

which has the asymptote line $u_\infty(x) = x$. Theorem 3 (*loc.cit*) asserts, under (GRH), that when N is fixed, P_K with $\alpha_K > N - 1$ must lie *above* the curve

$$y = l(N, x) = -\left(\frac{e^{\frac{x}{2}-1} - N + 1}{e^{\frac{x}{2}-1} + N - 1}\right)(N-1)(x - 2\log(N-1)), \tag{3.1.3}$$

which has the asymptote line $l_\infty(N, x) = -(N-1)(x - 2\log(N-1))$.

Method for computations As in [E-K], let

$$\Phi_K(t) = \frac{1}{t-1}\sum_{N(P)^k < t}\left(\frac{t}{N(P)^k} - 1\right)\log N(P) \qquad (t > 1), \tag{3.1.4}$$

where (P, k) runs over the pairs of a non-archimedean prime P of K and a positive integer k such that $N(P)^k < t$. Put

$$A_K^*(t) = \log t - \Phi_K(t) \qquad (t > 1). \tag{3.1.5}$$

Then

$$\gamma_K^* = \lim_{t\to\infty} A_K^*(t) \qquad \text{(unconditionally)}, \tag{3.1.6}$$

(3.1.7) $\quad |\gamma_K^* - A_K^*(t)| < 2\ \dfrac{\alpha_K+1}{\alpha_K-1}(\alpha_K + 2\log\alpha_K)\dfrac{1}{\sqrt{t}-1} + N\ \dfrac{\log t+1}{t-1}$ (under GRH)

when $\alpha_K > 1.16$ (cf. [E-K]; the arguments in §1.5-1.6). Roughly speaking, the error in (3.1.7) is about $2\alpha_K/\sqrt{t}$. We shall compute the "t-approximation" $A_K^*(t)$ of γ_K^* for each K for a suitable choice of t depending on K.

3.2. Quadratic Fields

Figure 1 (resp. Figure 2) plots P_K for all real (resp. imaginary) quadratic fields K with $|d_K| < 5 \times 10^3$, computed using the t-approximation $A_K^*(t)$ of γ_K^* for $t = 10^4$. The right vertical lines merely indicate the limit of the range of our present calculations. Observe that while the points going up are rather sporadic, the points going down towards right draw such a clean curve in each case. More careful examinations show that these down-slope curves in the imaginary and the real cases are similar but different, and the difference will not disappear under any vertical translations (i.e., they will not coincide even if we use e.g. $\gamma_K^* + \beta_K$ instead of γ_K^* for the y-coordinate.)

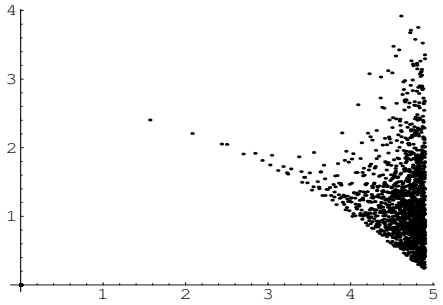

Figure 1: Real Quadratic Fields

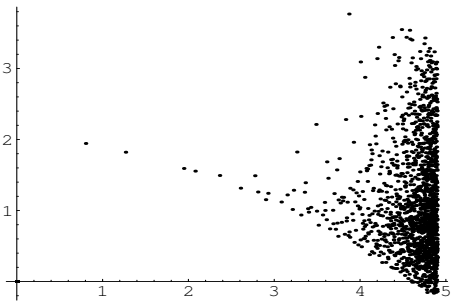

Figure 2: Imaginary Quadratic Fields

Let us now restrict ourselves to the imaginary quadratic case and try to construct point sequences near the lower and the higher actual boundaries of Figure 2.

(Construction of a point sequence going down)
For each $M < 18$, we take the imaginary quadratic field K with minimal $|d_K|$ in which the first M primes starting from 2 decompose completely. Figure 3 is the set of points P_K for these fields K. Their discriminant starts with -23, and ends with -2155919, -6077111.

(Construction of a point sequence flying up)
We simply replace "decompose completely" by "remain prime" in the above construction. Their discriminants include $-19, -43, -67$, notably -163 (the most conspicuous one with the coordinates (3.870.., 3.767..)), and $-1333963, -2404147$. (Figure 4.)

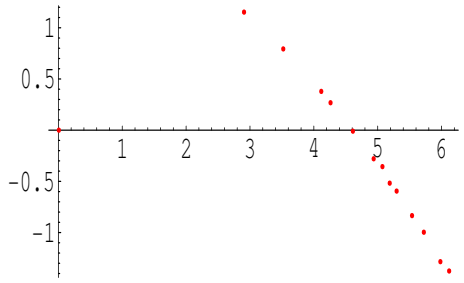

Figure 3: Low Points

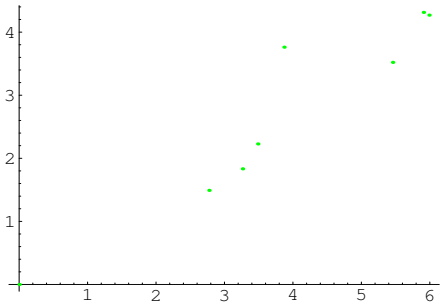

Figure 4: High Points

(Joint graph with all other points)
The low point sequence going down constructed above fits very well with the actual lower boundary of plotted points; see Figure 5. On the other hand, the high points constructed above do not seem to constitute the highest flying up sequence.

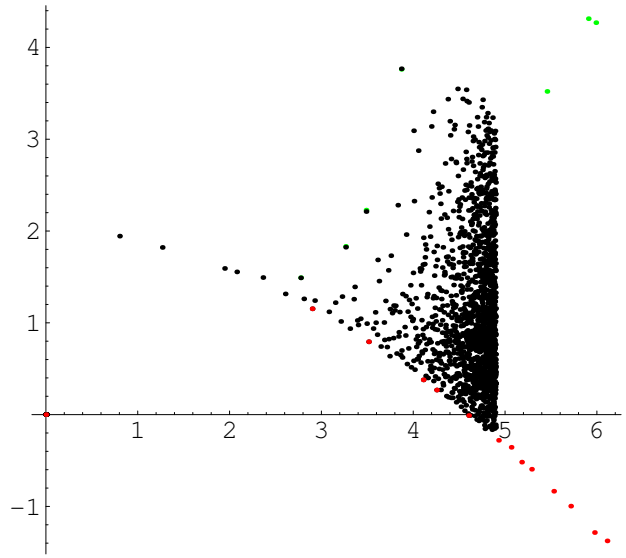

Figure 5: Joint graph with other points

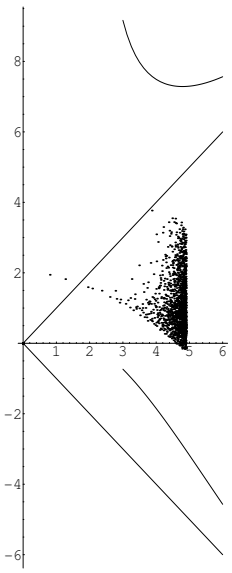

Figure 6: Together with GRH-bounds and Asymptotes

Figure 6 shows the joint graph with our GRH-upper (resp. lower) bound $y = u(x)$

(resp. $y = l(2,x)$) and their asymptote lines $y = x$ (resp. $y = -x$). *Are these GRH-embarkments really safe ?*

3.3. Real BiQuadratic Fields

We consider biquadratic fields $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where d_1,d_2 are distinct discriminants of quadratic fields. Put $d_3^\star = (d_1 d_2)/(gcd(d_1, d_2))^2$. Then the discriminant of the third quadratic subfield is $d_3 = d_3^\star \times \epsilon$, where $\epsilon = 1$ when $d_3^\star$ is a discriminant of a quadratic field and $\epsilon = 4$ otherwise. The discriminant of K is $d_1 d_2 d_3$. The invariant γ_K^* is given by the formula (1.2.6) for $p = 2$, i.e., the sum of $\gamma_{k_i}^*$ for three quadratic subfields k_i $(i = 1, 2, 3)$ *minus* twice the $\gamma_{\mathbb{Q}}^*$. Figure 7 plots P_K for *real* biquadratic fields of discriminant up to $400^3 = 6.4 \times 10^7$ (there are 2729 such K), using the t-approximation for each quadratic subfield at $t = 4 \times 10^3$. Figure 8 is the joint graph with our GRH-upper (resp. lower) bound $y = u(x)$ (resp. $y = l(4,x)$) and their asymptote lines $y = x$ (resp. $y = -3(x - 2\log 3)$).

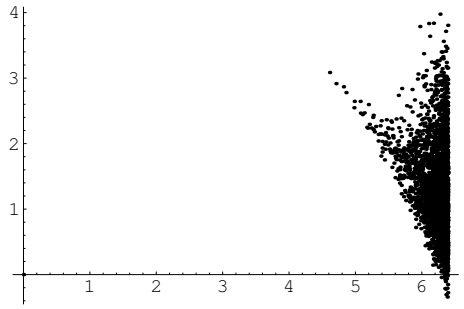

Figure 7: Real BiQuadratic Fields

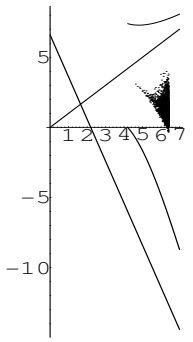

Figure 8: Together with the GRH-bounds and their Asymptote lines

Note that the actual lower boundary curve seems to have the slope tending towards that of $l_\infty(x)$, i.e., -3, giving an evidence that our GRH-lower bound, Theorem 3 of [E-K], for a fixed N, is quite sharp.

3.4. Full Cyclotomic Fields and their Maximal Real Subfields

We consider the full cyclotomic field $K_m = \mathbb{Q}(\mu_m)$ and its maximal real subfield $K_m^+ = \mathbb{Q}(\cos(2\pi/m))$. We may and shall assume that m is either odd or divisible by 4. Recall that $\alpha_K = \log\sqrt{|d_K|}$ for these K are given by

$$\alpha_{K_m} = \tfrac{\phi(m)}{2}(\log m - \textstyle\sum_{p|m}(\log p)/(p-1)),$$
$$\alpha_{K_m^+} = \tfrac{\alpha_{K_m}}{2} - \epsilon(m),$$

where $\phi(m)$ is the Euler function, and $\epsilon(m) = 0, (\log p)/4, (\log 2)/2$, according to whether m is not a prime power, a power of an odd prime p, or a power of 2, respectively.

Figure 9 plots the 10^6-approximation of P_{K_m} for all such m that $\alpha_{K_m} < 1909$. (As for the range of calculations, the limit is chosen in terms of x-coordinate instead of m. But we have first chosen the limit 600 for m, and then appended all other points whose x-coordinates lie within the range. This contains all $m \le 600$, no prime $m > 600$, and the maximal value of m included is $2520(= 7!/2)$.)

Figure 10 plots 10^6-approximations of $P_{K_m^+}$ for all such m that $\alpha_{K_m^+} < 952.9$. (The procedure starting with $m \le 600$ is the same.)

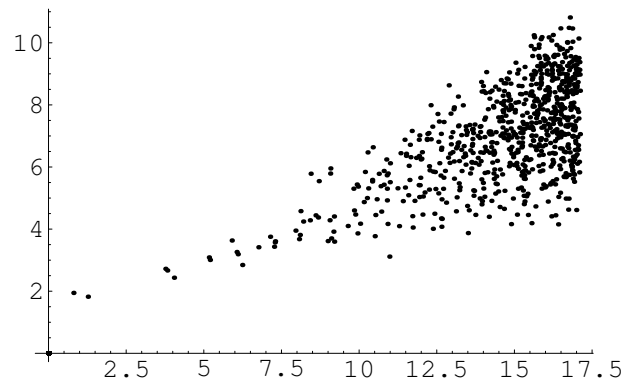

Figure 9: Full Cyclotomic Fields

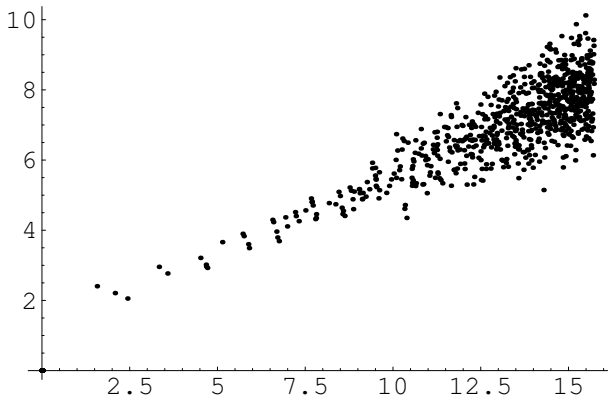

Figure 10: Maximal Real Subfields

We have three things to discuss; (I) positivity and growing tendency of $\gamma^*_{K_m}$ and $\gamma^*_{K_m^+}$, (II) signs of the relative invariant $\gamma^*_{K_m} - \gamma^*_{K_m^+}$, and (III) range and accuracy of computations.

(I) Positivity and Growing tendency of $\gamma^*_{K_m}$ and $\gamma^*_{K_m^+}$.
The fields K_m and K_m^+ have *only few* primes with small norms. In particular, when m is a prime number, K_m has *no* primes with norm $< m$. In other words, the m-approximation of their γ^*-invariant is $\log m$. So, it is natural to expect that the γ^*-invariants of these

fields are "more positive" than in other previous cases. Theoretically, the question of positivity is directly related to the non-existence of too many primes with norm $< m^2$ or $m^{2+\epsilon}$. The sieve method gives us some estimate from below, but still not the positivity. Experimentally, though limited both in range and accuracy, our numerical tests suggest that one may expect the positivity for all γ_{K_m} and $\gamma_{K_m^+}$. Among them, the case of K_m^+ looks more convincing. The numerical tests include Mahoro Shimura's extended computations of γ_{K_m} (for $m \leq 3 \times 10^4$, though t is not really large enough), and special cares for some "dangerous low points" (see (III) below). Being supported by these evidences, I raise :

Conjecture 1 (i) *γ_{K_m} and $\gamma_{K_m^+}$ are positive (even for the γ-invariants),*
(ii) *there exist positive constants c_1, c_2, c_1^+, c_2^+, all ≤ 2, such that for any $\varepsilon > 0$,*

$$(c_1 - \varepsilon)\log m < \gamma^*_{K_m} < (c_2 + \varepsilon)\log m, \tag{3.4.1}$$

$$(c_1^+ - \varepsilon)\log m < \gamma^*_{K_m^+} < (c_2^+ + \varepsilon)\log m, \tag{3.4.2}$$

hold for all sufficiently large m;
(iii) *when m is restricted to primes, one can choose $c_1 = 1/2$, $c_1^+ = 1$ and $c_2 = c_2^+ = 3/2$.*
(Since $x_K = 2\log \alpha_K + 2 \sim 2\log m$, the slopes in Figures 9,10 correspond to $1/2$ of these.)

The reason for $c_2, c_2^+ \leq 2$ is Theorem 1 of [E-K], which, under (GRH), gives

$$\gamma^*_{K_m}, \gamma^*_{K_m^+} < (2 + \varepsilon)\log m \tag{3.4.3}$$

for all sufficiently large m.

As for (iii), take, for example, all 50 primes m between 701 and 1039. Then the maximal values of $\gamma^*_{K_m}/\log m$ (resp. $\gamma^*_{K_m^+}/\log m$) for these m are 1.533 (resp. 1.512), while the minimal values are 0.589 (resp. 0.899).

There is a close connection between Conjecture 1 (iii) and "uniformity" of distribution, mod $(2\pi/\log m)$, of the imaginary part of the non-trivial zeros of $\zeta_{K_m}(s)$. Assume (GRH) for K_m, and for each m, consider the "weighted average"

(3.4.4)
$$c(m) = \left(\sum_\rho \frac{m^{\rho-1/2}}{\rho(1-\rho)}\right) / \left(\sum_\rho \frac{1}{\rho(1-\rho)}\right) = \left(\sum_\rho \frac{\cos(\gamma \log m)}{1/4+\gamma^2}\right) / \left(\sum_\rho \frac{1}{1/4+\gamma^2}\right)$$

of $\cos(\gamma \log m)$, where $\rho = 1/2 + \gamma i$ runs over all non-trivial zeros of $\zeta_{K_m}(s)$ counted with multiplicities. Note that $|c(m)| \leq 1$. Now, since

$$\left(\int_{-\infty}^{\infty} \frac{\cos(t\log m)}{1/4+t^2}dt\right) / \left(\int_{-\infty}^{\infty} \frac{1}{1/4+t^2}dt\right) = \frac{1}{\sqrt{m}} \qquad (m > 1), \tag{3.4.5}$$

the quantity

$$\sqrt{m}c(m) - 1 \tag{3.4.6}$$

in a sense measures how *non-uniform* the distribution of γ mod $(2\pi/\log m)$ for small $|\gamma|$ is. Experimentally, for large m, (3.4.6) remains small in absolute value, and it seems that they (gradually) get into the interval $(1-\epsilon,\ \epsilon)$. Define $c^+(m)$ similarly for K_m^+. The following assertion may explain why the lower boundary slope for the maximal real subfield case is steeper.

Proposition 3 (Under GRH) *Let M be any given infinite set of prime numbers, and let m run over M. Then, $|\gamma^*_{K_m}/\log m|$ (resp. $|\gamma^*_{K_m^+}/\log m|$) is bounded if and only if $|\sqrt{m}c(m)-1|$ (resp. $|\sqrt{m}c^+(m)-1|$) is so, and when these conditions are satisfied, we have*

$$\gamma^*_{K_m}/\log m = \frac{3}{2} + (\sqrt{m}c(m)-1) + O\left(\frac{1}{\log m}\right) \qquad (m \in M), \tag{3.4.7}$$

$$\gamma^*_{K_m^+}/\log m = \frac{3}{2} + \frac{1}{2}(\sqrt{m}c^+(m)-1) + O\left(\frac{1}{\log m}\right) \qquad (m \in M), \tag{3.4.8}$$

respectively. In particular, the above Conjecture 1 (iii) is equivalent to that $\sqrt{m}c(m)-1$ (resp. $\sqrt{m}c^+(m)-1$) belongs to the interval $(-1-\varepsilon,\ \varepsilon)$ when m is sufficiently large.

(Proof) By [E-K](1.2.1)(1.4.1), we have, for any number field K and $t>1$,

$$\gamma^*_K = \log t - \Phi_K(t) + \frac{1}{t-1}\sum_\rho \frac{t^\rho - 1}{\rho(1-\rho)} + \frac{r_1}{2}\left(\log\frac{t+1}{t-1} + \frac{2}{t-1}\log\frac{t+1}{2}\right) + r_2\left(\log\frac{t}{t-1} + \frac{\log t}{t-1}\right), \tag{3.4.9}$$

$$\frac{1}{2}\sum_\rho \frac{1}{\rho(1-\rho)} = \gamma^*_K + \alpha_K + \beta_K, \tag{3.4.10}$$

where ρ runs over all non-trivial zeros of $\zeta_K(s)$ (counted with multiplicities). Now let $K = K_m$, with m a prime, and put $t = m$. Then since $\Phi_K(m) = 0$, we obtain directly from (3.4.9) that

$$\gamma^*_{K_m} = \frac{3}{2}\log m + \frac{\sqrt{m}c(m)-1}{m-1}\sum_\rho \frac{1}{\rho(1-\rho)} + \frac{m-1}{2}\log\frac{m}{m-1}. \tag{3.4.11}$$

(The last term on the right hand side of (3.4.9) gives $(\log m)/2$.) Since $\alpha_K = ((m-2)/2)\log m$ and $\beta_K = -(m-1)(\gamma_{\mathbb{Q}} + \log 2\pi)/2$, we obtain from (3.4.10),(3.4.11),

$$\left(1 - 2\frac{\sqrt{m}c(m) - 1}{m-1}\right)\frac{\gamma^*_{K_m}}{\log m} = \frac{3}{2} + (\sqrt{m}c(m) - 1)(1 - a_m) + b_m, \tag{3.4.12}$$

where

$$a_m = \frac{1}{m-1} + \frac{\gamma_{\mathbb{Q}} + \log 2\pi}{\log m}, \tag{3.4.13}$$

$$b_m = \frac{\log((1 + \frac{1}{m-1})^{m-1})}{2\log m}. \tag{3.4.14}$$

Note that a_m, b_m are $O(1/\log m)$. But since $\gamma^*_{K_m} = O(m)$ (see below), our assertions related to K_m follow directly. The case of the maximal real subfield is almost similar. The coefficient $1/2$ of $\sqrt{m}c^+(m) - 1$ comes from the difference between $\alpha_{K_m^+}$ and α_{K_m}. □

Remark As for $\gamma^*_{K_m}$ and $\gamma^*_{K_m^+}$, we have a (GRH)-upper bound (3.4.3), and Conjecture 1 for their positivity. But the (GRH)-*lower* bounds that we have been able to establish so far are (unfortunately) much weaker; namely, $|\gamma^*_{K_m}|$ and $|\gamma^*_{K_m^+}|$ are (i) $O(m\log m)$ (by (1.1.8)), (ii) $O(m\log\log m)$ (by Theorem 3 [E-K]), (iii) $O(\sqrt{m}\log m)$ (by [Ih-Sh]§4), and (iv) $O((\log m)^2)$ (by a sieve method). Among them, (i) is unconditional.

(II) Signs of the relative invariants $\gamma^*_{K_m} - \gamma^*_{K_m^+}$.
This relative invariant seems to take both signs. Here, I only present a table for comparison of $\gamma^*_{K_m}$ and $\gamma^*_{K_m^+}$ for some small ranges of m (10^7-approximations).

m	$\gamma^*_{K_m}/\log m$	$\gamma^*_{K_m^+}/\log m$
67	1.497	1.545
68	1.274	1.079
69	1.077	1.221
71	1.467	1.413
72	1.030	1.018
600	0.782	0.937
601	1.084	1.056
603	1.402	1.303
604	1.296	1.140
605	1.312	1.382
607	1.348	1.166

(III) Range and accuracy of computations.
In order to widen the range of m, or that of the x-coordinate, we must take t larger and larger to make the t-approximation $A^*_{K_m}(t)$ of $\gamma^*_{K_m}$ and $\gamma^*_{K_m^+}$ sufficiently accurate. First of all, if we fix $t = t_0$ and let m grow, then the graph will be stopped by the "roof" $y = \log t_0$. The GRH error estimate (3.1.7) suggests that a reasonable choice of t is such that $\alpha_{K_m}/\sqrt{t}$ is acceptably small. But this estimate does not take into account any cancellations among the ρ-terms (in the explicit formula for $\Phi_K(t)$ (cf. [E-K] §1.1)), and in fact, in actuality, the convergence of $A_K(t)$ to γ^*_K seems faster. For the above range of computations (Figures 9, 10), the choice $t = 10^6$ seems fairly appropriate. The pictures do not visibly change when we replace 10^6 by e.g. 3×10^6 or even by 10^5, except that for the latter, the slope of the upper border near the right summit is slightly more flat (an alarm that t is not big enough!). Also, for some individual m, I computed further approximations up to 10^8, but the errors seem to be within an acceptable range.

(For example, let us take a "low point" P_{K_m} for $m = 112$, which is conspicuous in Figure 9. Then the x-coordinate is 10.97.., and the y-coordinate for $t = 10^6, 10^7, 10^8$ are 3.114, 3.1162, 3.1166, respectively. For $m = 443$, $x = 16.406..$, and the y-coordinate for these t are 4.153, 4.118, 4.146 respectively. Outside the above range, $m = 7439 = 43 \times 173$ gives a low point.)

3.5. Other Families of Fields

Cubic Fields Figure 11 plots P_K for *cyclic* cubic extensions $K/\mathbb{Q}$ ramified at one prime, i.e., those cubic extensions having discriminant m^2, where m is either 9 or a prime $\equiv 1 (\mathrm{mod}\ 3)$. The range is $m < 2000$, and $t = 5 \times 10^4$. Note that the graph is quite similar to quadratic or biquadratic cases. The lower boundary slope here seems to tend to $-(N-1) = -2$. The most conspicuous high point (5.19.., 4.62..) corresponds to $m = 139$. It is interesting to observe that in this case, too, there appears some point-sequence which rises up steeply and parabolically, as if going to violate the GRH, and also that it seems to stop and start again from a lower base. Can this be explained?

All fields treated so far are abelian. For non-Galois fields, our computations have been rather fragmentary. Among them is a (small) family of totally real non-Galois cubic fields listed in the table of [D-F] (copied in [C-R]). It consists of 28 fields with discriminant up to 1257 (the point corresponding to the last one has the x-coordinate 4.544). The plot graph of these 28 points P_K (also $t = 5 \times 10^4$) fits in very well with this part (5 points on the extreme left) of Figure 11. This is shown in Figure 12. The shape of the lower boundary curve is affected by the data (r_1, r_2), but does not seem to be affected by Galois theoretic structures (at least within the range of small discriminants).

We also treat another family of cubic fields defined by the equation of the form

$$X^3 - X + 24m = 0. \tag{3.5.1}$$

This equation has discriminant $4(1 - 3888m^2)$, but 2 is unramified. In fact, primes 2 and

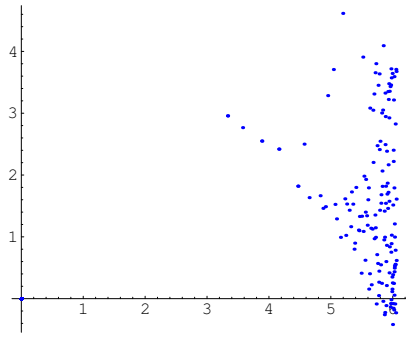

Figure 11: Cubic Cyclic; one-point-ramified

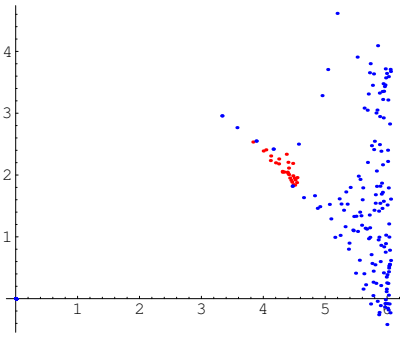

Figure 12: Few non-Galois appended

3 split completely, pulling down the y-coordinates to some extent. Note that $r_1 = r_2 = 1$. The graphs are shown in Figures 13, 14, where the latter is together with the GRH-bounds and their asymptotes. Here, m runs over those positive integers up to 200 such that (3.5.1) is irreducible and that $3888m^2 - 1$ is square free, and 2×10^4 is chosen for t.

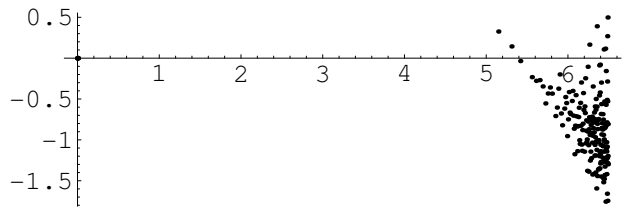

Figure 13: Some Special Cubics

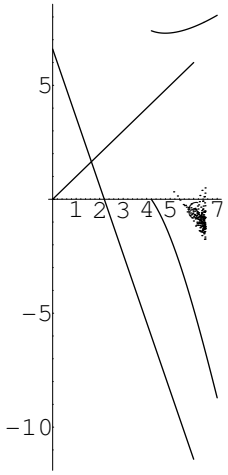

Figure 14: Joint with GRH-bounds and Asymptotes

Some other Quartic Fields

In [C-R], there is also a table of totally real quartic fields with small discriminant whose Galois closure has the symmetric Galois group S_4. This consists of 9 fields and the maximal discriminant is 8069. The plot graph of the corresponding 9 points also fits

very well with this part of Figure 7 for real biquadratic fields, i.e., with the sequence of 6 leftmost points in Figure 7.

The Fields K_p

Finally, let us take up the fields K_p studied in [E-K], i.e., for each odd prime p, K_p is the unique cyclic extension of degree p contained in the field of p^2-th roots of unity. It is totally real, and the discriminant is p^{2p-2}. A rational prime l splits in K_p completely if and only if

$$l^{p-1} \equiv 1 \pmod{p^2}. \tag{3.5.2}$$

Thus, when there exists small l satisfying this congruence, K_p contains many primes with small norms, and accordingly, $\gamma^*_{K_p}$ can become very negative. Figure 15 plots P_{K_p} for $p \leq 1223$, where $t = 10^5$ (sometimes 10^6, depending on necessity), together with the upper and the lower bounds given by [E-K](Theorems 1, resp. 3), *except* for two points "far too low" to be included; they are

$$(19.34, -174.13), \quad (19.88, -747.14), \tag{3.5.3}$$

which correspond to $p = 863,\ 1093$, respectively. The smallest splitting primes in these cases are $l = 13,\ l = 2$, respectively.

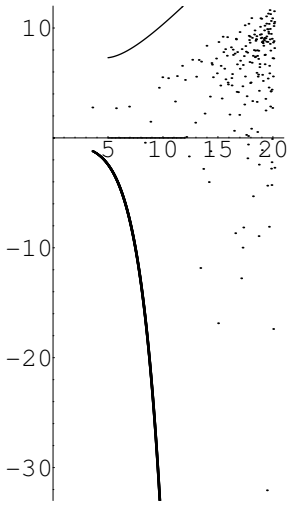

Figure 15: K_p without two very low points (3.5.3)

A crucial quantity for the study of extreme negative values of $\gamma^*_{K_p}$ is

$$\liminf \frac{\gamma_{K_p}}{p} \quad (= \liminf \frac{\gamma^*_{K_p}}{p}). \tag{3.5.4}$$

As is shown in [E-K](§2.3 Cor 3), whether (3.5.4) is 0, finite negative, or $-\infty$, is closely related to how small or large the set of primes p satisfying the congruence (3.5.2) (for fixed l) is. Consider now the decomposition (1.2.2) for this case, i.e.,

$$\gamma_{K_p} = \gamma_{\mathbb{Q}} + \sum_{\chi} \frac{L'(\chi,1)}{L(\chi,1)}, \tag{3.5.5}$$

where χ runs over all non-trivial characters mod p^2 satisfying $\chi^p = 1$. Put

$$S_p = \frac{\gamma_{K_p}}{p} = \frac{1}{p}\left(\gamma_{\mathbb{Q}} + \sum_{\chi} \frac{L'(\chi,1)}{L(\chi,1)}\right), \tag{3.5.6}$$

$$T_p = \frac{1}{p}\left((\gamma_{\mathbb{Q}})^2 + \sum_{\chi} |\frac{L'(\chi,1)}{L(\chi,1)}|^2\right), \tag{3.5.7}$$

so that $S_p^2 \leq T_p$. Some intuitive argument, and the following table (which contains two extreme negative cases $p = 1093, 3511$ for $\gamma^*_{K_p}$) suggest that the fluctuation of values of T_p is much smaller than that for S_p, and even that T_p might have a limit

$$\tau = \lim T_p = \sum_{n=1}^{\infty} \Lambda(n)^2/n^2 = \sum_{p} (\log p)^2/(p^2-1) = 0.80521... \tag{3.5.8}$$

This would imply that (3.5.4) must lie in the interval $[-\sqrt{\tau}, 0]$, and hence (under GRH; cf. [E-K] §2.3) that if l_i, $(1 \leq i \leq N)$ are distinct primes such that $\sum_i (\log l_i)/(l_i - 1) > \sqrt{\tau}$, (e.g. $l_1 = 2, l_2 = 3$), then the simultaneous congruence (3.5.2) for $l = l_1, \cdots, l_N$ can have at most finitely many solutions p.

p	S_p	T_p		p	S_p	T_p
67	0.09	0.81		1091	−0.003	0.82
71	−0.18	0.70		1093	−0.68	1.11
73	0.08	0.74		1097	0.003	0.79
79	−0.05	0.84		3499	0.003	0.79
83	0.02	0.83		3511	−0.69	1.12
89	0.09	0.90		3517	0.003	0.79

Remark If we replace $|\quad|^2$ by $(\mathrm{Re}(\quad))^2$ in the definition of T_p, then $S_p^2 \leq T_p$ remains

valid, but the fluctuation for this seems too big, and its convergence (e.g. to $\tau/2$) is less convincing.

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