On $\log L$ and L'/L for L-functions and the associated "M-functions": Connections in optimal cases

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Introduction

0.1 – The main subject of this paper concerns the role of "M-functions" in the value-distribution theory, for logarithms and their derivatives, of L-functions $L(s,\chi)$ over a global base field K. Here, $s=\sigma+ti$ ($\sigma>1/2$) is fixed and χ runs over a certain family of Dirichlet characters on K. In the case of logarithmic derivatives $L'/L(s,\chi)$, this is a continuation of [8, 4, 6]. This paper forms a "complementary pair" with [7] where some basic results for the case of $\log L(s,\chi)$ over $K=\mathbf{Q}$ are given. Here, the main results are for both cases and over function fields over finite fields. But we also include the corresponding conditional results (i.e., under GRH, the Generalized Riemann Hypothesis) over $\mathbf{Q}$ and over imaginary quadratic fields, in order to show what one can expect in the "optimal situation". In spite of the obvious close connection in the subject itself, there are no serious logical interdependences between [7] and the present paper. The approaches also are quite different even when $K=\mathbf{Q}$.

0.2 – Let us start with explaining the conditional results over $\mathbf{Q}$. The L-function $L(s,\chi)$ associated with any non-principal Dirichlet character χ is, under GRH, holomorphic and non-vanishing on $\sigma=\mathrm{Re}(s)>1/2$; hence its logarithm on this domain can be defined in the natural manner. Thus,

(0.2.1)
$$\begin{aligned}\mathcal{L}(s,\chi):\ &=\ L'(s,\chi)/L(s,\chi) && \text{(Case 1)}\\ &=\ \log L(s,\chi) && \text{(Case 2)}\end{aligned}$$

is holomorphic on $\sigma>1/2$ in each case. But we shall fix any s in this domain, consider $\mathcal{L}(s,\chi)$ rather as a function of χ, and for any mild test function Φ on $\mathbf{C}$, study the mean value of $\Phi(\mathcal{L}(s,\chi))$ when χ runs over some natural family of characters. For this study, the basic role is played by the case where Φ is a *quasi-character* $\mathbf{C}\to\mathbf{C}^\times$ of the additive group $\mathbf{C}$. Such quasi-characters are parametrized by two complex numbers z_1,z_2, as

(0.2.2)
$$\psi_{z_1,z_2}(w)=\exp\left(\frac{i}{2}(z_1\bar{w}+z_2w)\right)$$

($i=\sqrt{-1}$), and it is a character if and only if z_1, z_2 are mutually conjugate. These are not only basic in the function space on $\mathbf{C}$, but also fit very well with this study, because each of $\psi_{z_1,z_2}(\mathcal{L}(s,\chi))$ has an Euler product expansion on $\sigma > 1$ reflecting the Euler sum decomposition of $\mathcal{L}(s,\chi)$. It should also be noticed that unless ψ_{z_1,z_2} is a character, its value at $\mathcal{L}(s,\chi)$ can be "exponentially large".

For each prime divisor [1] $\mathbf{f}$, let $\mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}}$ denote the average over all primitive Dirichlet characters χ with conductor $\mathbf{f}$. Our first main result (Theorem 3 in §1.3) asserts that

$$\lim_{\substack{\mathbf{f}\,prime \\ N(\mathbf{f})\to\infty}} \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} \psi_{z_1,z_2}(\mathcal{L}(s,\chi)) = \tilde{M}_\sigma(z_1,z_2) \qquad (\sigma > 1/2, \text{ under GRH}) \tag{0.2.3}$$

holds. In fact, the limit converges uniformly in the wider sense with respect to s, z_1, z_2. To define $\tilde{M}_\sigma(z_1,z_2)$, put

$$\mathcal{Z}(s) = \zeta'(s)/\zeta(s) \qquad \text{(Case 1)}, \tag{0.2.4}$$
$$= \log\zeta(s) \qquad \text{(Case 2)}, \tag{0.2.5}$$

$\zeta(s)$ being the Riemann zeta function, and consider the Dirichlet series

$$\exp(\frac{iz}{2}\mathcal{Z}(s)) = \sum_{n=1}^{\infty} \lambda_z(n) n^{-s} \qquad (\sigma > 1). \tag{0.2.6}$$

It is easy to see that in each of Cases 1,2, $\lambda_z(n)$ $(n = 1, 2, ...)$ are polynomials in z and multiplicative in n (see §1.2). Consider now the following Dirichlet series

$$\tilde{M}_s(z_1,z_2) = \sum_{n=1}^{\infty} \lambda_{z_1}(n)\lambda_{z_2}(n) n^{-2s} \qquad (\sigma > 1/2) \tag{0.2.7}$$

(so to speak, the "termwise product" of $\exp((iz_1/2)\mathcal{Z}(s))$ and $\exp((iz_2/2)\mathcal{Z}(s))$). This series also converges absolutely and uniformly in the wider sense on $\sigma > 1/2$, and hence defines an analytic function of 3 complex variables s, z_1, z_2 on this domain [2]. For this, GRH is unnecessary. The above function $\tilde{M}_\sigma(z_1,z_2)$ is its restriction to $s = \sigma \in \mathbf{R}$.

The above mean value theorem is an analogue of Carlson's mean value theorem on the limit average of Dirichlet series over a vertical axis inside the critical strip [1] (cf. also [14, 12, 13]). Two main differences are (i) we take averages over χ, and (ii) the Dirichlet series considered are of the type $\psi_{z_1,z_2}(\mathcal{L}(s,\chi))$.

[1] Here we use the notation system for the general case of K employed in the main text; thus, here, $\mathbf{f}$ corresponds to a prime number f, and $N(\mathbf{f}) = f$.

[2] in Case 1, it is already treated in [4].

0.3 – As for the base field K, in addition to $K = \mathbf{Q}$, we shall also include imaginary quadratic fields (also under GRH), and any algebraic function field of one variable over a finite field with *one* assigned "infinite" prime divisor $\mathfrak{p}_\infty$. In short, K is any global field with just one infinite prime. In the function field case, we shall normalize χ by the condition $\chi(\mathfrak{p}_\infty) = 1$ to kill infinitely many trivial twists not changing the conductor. The Euler factor corresponding to $\mathfrak{p}_\infty$ should be dropped from each of $L(s,\chi)$, $\mathcal{L}(s,\chi)$, $\tilde{M}_s(z_1,z_2)$. This applies also for the convolution Euler $\mathfrak{p}_\infty$-factor in the function $M_\sigma(z)$ which appears later.

0.4 – The second main result, which is closely related to the first, concerns the equality of the form

$$\lim_{\substack{\mathbf{f}\,prime \\ N(\mathbf{f})\to\infty}} \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} \Phi(\mathcal{L}(s,\chi)) = \int_{\mathbf{C}} M_\sigma(w)\Phi(w)|dw| \qquad (\sigma > 1/2), \tag{0.4.1}$$

where $|dw| = dxdy/2\pi$ for $w = x+yi$, Φ is a test function on $\mathbf{C}$, and $M_\sigma(w)$ is the density function for the distribution of values of $\{\mathcal{L}(s,\chi)\}_\chi$ constructed in [4](for Case 1), [7] (for Case 2, where it is denoted by $\mathcal{M}_\sigma(w)$); see also §5.6 of the present paper. At least in Case 2, it has a long history since Bohr and Jessen; for this, cf. the Introduction of [7].

The connection between $M_\sigma(w)$ and $\tilde{M}_s(z_1,z_2)$ is

$$\tilde{M}_\sigma(z_1,z_2) = \int_{\mathbf{C}} M_\sigma(w)\psi_{z_1,z_2}(w)|dw| \qquad (\sigma > 1/2,\ z_1,z_2 \in \mathbf{C}); \tag{0.4.2}$$

in particular, if we put $\psi_z(w) = \psi_{z,\bar{z}}(w) = e^{i\mathrm{Re}(\bar{z}w)}$ (which is a character $\mathbf{C} \to \mathbf{C}^1$), then

$$\tilde{M}_\sigma(z) := \tilde{M}_\sigma(z,\bar{z}) = \int_{\mathbf{C}} M_\sigma(w)\psi_z(w)|dw| \tag{0.4.3}$$

is the Fourier dual of $M_\sigma(z)$.

In [6], the following weaker version

$$\lim_{m\to\infty} \mathrm{Avg}_{N(\mathbf{f})\le m} \Big(\mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} \Phi(\log L(s,\chi))\Big) = \int_{\mathbf{C}} M_\sigma(w)\Phi(w)|dw| \qquad (\sigma > 1/2) \tag{0.4.4}$$

is established in Case 1 for the function field case for any Φ which is continuous and with at most a polynomial growth (which improves Theorem 7 of [4]). In [7], it is proved, among other things, that in Case 2, (0.4.4) holds for $K = \mathbf{Q}$ *unconditionally* if Φ is continuous and bounded.

In the present article, we shall prove that (0.4.1) itself holds for each of Cases 1,2, if (i) K is either any function field, or the rational number field or an imaginary quadratic field, under GRH in the latter two cases, and (ii) if Φ is any continuous function with at most *exponential* growth. This improves [6] and complements [7]. Note that the case $\Phi = \psi_{z_1,z_2}$ corresponds to the first main result (0.2.3). This special case stands at the crucial point in the proof, too.

0.5 – The above mean value theorem motivates us to study the analytic function $\tilde{M}_s(z_1, z_2)$ itself more closely. The most basic properties studied in [4] and in this paper include the following. Firstly, $\tilde{M}_s(z_1, z_2)$ has an Euler product expansion on $\sigma > 1/2$. Moreover, as can be expected, each Euler factor may be interpreted as the limit average of the corresponding Euler factor of $\psi_{z_1,z_2}(\mathcal{L}(s,\chi))$ (see (4.1.3)). Unlike the Euler product expansion of each of $\psi_{z_1,z_2}(\mathcal{L}(s,\chi))$, for which at most a conditional convergence can be expected on $1/2 < \sigma \leq 1$, the Euler product expansion for its limit average $\tilde{M}_s(z_1, z_2)$ is *absolutely* convergent on $\sigma > 1/2$.

Secondly, this function also admits an everywhere convergent *power series* expansion in z_1, z_2 with Dirichlet series coefficients (Theorem $\tilde{M}$ in §4.1). Moreover, the coefficient of $z_1^a z_2^b$ is essentially the limit average of $P^{(a,b)}(\mathcal{L}(s,\chi))$, where $P^{(a,b)}(w) = \bar{w}^a w^b$ $(a, b \geq 0)$. An amusing application shows, under GRH, that for any fixed $\sigma > 1/2$, $y > 0$, and for $N(\mathbf{f})$ sufficiently large, we have the inequalities

$$(0.5.1)\quad \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} \exp(2y\mathrm{Re}(L(s,\chi)/L'(s,\chi))) \;<\; \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} \exp(-2y\mathrm{Re}(L(s,\chi)/L'(s,\chi))),$$

$$(0.5.2)\quad \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} |L(s,\chi)|^{2y} \;>\; \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} |L(s,\chi)|^{-2y}.$$

For example, let $y = 1$. Then, in the limit $N(\mathbf{f}) \to \infty$, the latter inequality "tends to":

$$(0.5.3)\qquad \zeta(2\sigma) = \sum_n n^{-2\sigma} > \sum_{n\, squarefree} n^{-2\sigma}.$$

For these, see §4.2.

As for the zeros of $\tilde{M}_s(z_1, z_2)$, they are "merely" the collection of zeros of local Euler factors, but still, a non-trivial basic object of study. An interesting case is where $z_2 = \bar{z}_1$, and especially where $z_1 = yi$, $z_2 = -yi$ with $y \in \mathbf{R}$. This, and the study of the global "Plancherel Volume" of the density function for Case 1, can be found in [5].

Finally, the analytic property of $\tilde{M}_s(z_1, z_2)$ on a wider domain

$$(0.5.4)\qquad \{\sigma > 0\} \setminus \{1/2n, \rho/2n\}_{n\in \mathrm{N}, \zeta(\rho)=0},$$

seems also remarkable but this will be discussed in a future article.

0.6 – The main results (Theorems 1-4) are summarized in §1, except for Theorem $\tilde{M}$ related to the function $\tilde{M}_s(z_1, z_2)$ which is in §4. The remaining sections are for their proofs.

The first Theorem 1 (§1.1; the proof in §2) axiomatizes the present type of the mean-value theorem. A basic concept here is "uniformly admissible family of arithmetic functions". Theorem 2 (§1.2; the proof in §3) asserts that $\{\lambda_z\}_{|z|\leq R}$ is such a family. This is for any global field (under GRH in the number field case). The key is the estimation of $|\mathcal{L}(s,\chi)|$ on this region, and since this comes inside the exponential sign, a fairly strong estimate is required. We shall first prove a "universal" estimate on $\mathrm{Re}(s) \geq 1/2 + \epsilon$ for $|L'/L(s,\chi)|$ by using one of the "explicit formulas" (Theorem-Exp in §3.5), and then derive that for $\log L(s,\chi)$ by integration. The estimates thus obtained matches with Titchmarsh's conditional estimates of $|\log \zeta(s)|$ and $|\zeta'/\zeta(s)|$ in his book (Theorem 14.5 of [15]). It is no wonder if an appropriate L-function version, with a careful treatment of the dependence on $N(\mathbf{f}_\chi)$, already existed somewhere in an old literature. Since we could not find such, we decided to give them full proofs at the cost of the length of the paper.

From these two theorems, we obtain, directly, the next Theorem 3 (§1.3) which corresponds to the first main result mentioned above (for the base fields stated in §0.3). Then, in §4, some of the basic properties of $\tilde{M}_s(z_1, z_2)$ are stated (Theorem $\tilde{M}$) and proved. Finally, in §5, we shall give a proof of Theorem 4 (stated in §1.4) which corresponds to the second main result mentioned above. The proof is based on two key lemmas, Lemma A ("the equality (0.4.1) for some special Φ implies that for some more general Φ"), and Lemma B on the rapid decay property of $M_\sigma(z)$ whose proof contains the explanation of the explicit connections between the constructions in [9] and [7].

1 The main results

1.1 – Uniformly admissible family of arithmetic functions. Let K be a global field, i.e., either an algebraic number field of finite degree (NF) or an algebraic function field of one variable over a finite field $\mathbf{F}_q$ (FF), given together with a finite set P_∞ of prime divisors of K. We assume that P_∞ contains all the archimedean primes (NF case) and is *non-empty* also in the FF case. By an *integral divisor* we shall mean any divisor D of K having a prime factorization of the form $D = \prod_{\mathfrak{p} \notin P_\infty} \mathfrak{p}^{r_\mathfrak{p}}$ $(r_\mathfrak{p} \geq 0)$.

For an integral divisor $\mathbf{f}$, let $I_\mathbf{f}$ be the group of divisors of K coprime with $\mathbf{f}P_\infty$, and define

$G_\mathbf{f} = I_\mathbf{f}/\{(\alpha); \alpha \equiv 1(\mathrm{mod}\,\mathbf{f}), \alpha_v > 0 \text{ (all real archimedean primes } v)\}$,

where for each $\alpha \in K^\times$, (α) denotes the "prime-to-P_∞" component of the principal divisor generated by α, and α_v, the v-component.[1] Note that $G_\mathbf{f}$ is always finite (including the FF case because P_∞ is non-empty). Define

$i_\mathbf{f} : I_\mathbf{f} \to G_\mathbf{f}$: the projection,

$\hat{G}_\mathbf{f}$: the character group of $G_\mathbf{f}$, with the unit element χ_0.

For each $\chi \in \hat{G}_\mathbf{f}$ and an integral divisor D, we define $\chi(D) = \chi(i_\mathbf{f}(D))$ if $(D, \mathbf{f}) = 1$, and $\chi(D) = 0$ otherwise.

An *arithmetic function* will mean a $\mathbf{C}$-valued function $D \mapsto \lambda(D)$ on integral divisors. It will be called *admissible* if it satisfies the following three conditions (A1)-(A3):

(A1) $$\lambda(D) \ll_{\epsilon'} N(D)^{\epsilon'} \qquad \text{for any} \quad \epsilon' > 0.$$

(A2) For any integral divisor $\mathbf{f}$ and $\chi \in \hat{G}_\mathbf{f} \backslash \{\chi_0\}$, consider the Dirichlet series

$$g_\lambda(s, \chi, \mathbf{f}) = \sum_{(D,\mathbf{f})=1} \chi(D)\lambda(D)N(D)^{-s}, \tag{1.1.1}$$

where the summation is over all integral divisors coprime with $\mathbf{f}$. By (**A1**), this converges absolutely and defines a holomorphic function on $\mathrm{Re}(s) > 1$. The condition (**A2**) imposes that this extends to a holomorphic function on $\mathrm{Re}(s) > 1/2$.

(A3) In the FF case this simply imposes that

$$g_\lambda(s, \chi, \mathbf{f}) \ll_{\epsilon,\epsilon'} N(\mathbf{f})^{\epsilon'} \quad \textit{holds on } \mathrm{Re}(s) \geq 1/2 + \epsilon \tag{1.1.2}$$

for any $\epsilon, \epsilon' > 0$. In the NF cases, the condition is necessarily more complicated;

$$\mathrm{Max}(0, \log |g_\lambda(s, \chi, \mathbf{f})|) \ll_\epsilon \ell(t)\ell(\mathbf{f})^{1-2\epsilon} + \ell(t)^2 \quad \textit{on } \mathrm{Re}(s) \geq 1/2 + \epsilon \tag{1.1.3}$$

[1] The second system of conditions $\alpha_v > 0$ is optional; the results remain valid if we do not impose this (except of course for a slight difference in the formula for $|G_\mathbf{f}|$ in §2.1).

for any $0 < \epsilon < 1/2$, where $t = \mathrm{Im}(s)$ and

$$\begin{aligned} (1.1.4) \qquad \ell(\mathbf{f}) &= \log(N(\mathbf{f})+2) \qquad \textit{if } \mathbf{f} \textit{ is an integral divisor}, \\ (1.1.5) \qquad \ell(t) &= \log(|t|+2) \qquad \textit{if } t \in \mathbf{R}. \end{aligned}$$

The holomorphic functions $g_\lambda(s,\chi,\mathbf{f})$ will be called *the g-functions associated with* λ. If Λ is a family of admissible arithmetic functions such that the implicit constants in (A1) and (A3) can be chosen to be independent of $\lambda \in \Lambda$, then Λ will be called a *uniformly admissible* family of arithmetic functions. Important examples of such families will be given in Theorem 2 (§1.2).

The notion of uniformly admissible family of arithmetic function is invented because it seems to give a natural setting for the following mean value theorem. Unfortunately, at least at present, we need to assume further in this theorem that $|P_\infty| = 1$, i.e., either K is the rational number field or an imaginary quadratic field and P_∞ consists only of the unique archimedean prime, or K is a function field over a finite field and P_∞ consists of just one prime divisor (to be called $\mathfrak{p}_\infty$). The point is that in such a case the group of P_∞-units of K is finite, so that the order of $G_\mathbf{f}$ is comparable with $N(\mathbf{f})$.

By $\mathrm{Avg}_{\chi\in X} G(\chi)$, for a finite set X of characters χ and a $\mathbf{C}$-valued function $G(\chi)$ of χ, we shall mean the usual average $|X|^{-1}\sum_{\chi\in X} G(\chi)$.

Theorem 1 *Let Λ be any uniformly admissible family of arithmetic functions, and let λ, λ' run over Λ. Fix any ϵ such that $0 < \epsilon < 1/2$, and let $s = \sigma + ti$ run over the domain $\sigma \geq 1/2 + \epsilon$. In the NF case, we also fix $T > 0$ and impose additionally that $|t| \leq T$. Assume $|P_\infty| = 1$. Then:*

(i) *For any integral divisor* $\mathbf{f}$*, we have*

$$(1.1.6) \qquad \mathrm{Avg}_{\chi\in\hat{G}_\mathbf{f}\setminus\{\chi_0\}}(\overline{g_\lambda(s,\chi,\mathbf{f})}g_{\lambda'}(s,\chi,\mathbf{f})) - \sum_{(D,\mathbf{f})=1} \overline{\lambda(D)}\lambda'(D)N(D)^{-2\sigma} \ll N(\mathbf{f})^{-\epsilon/2}.$$

In particular, the quantity on the left hand side tends to 0 uniformly as $N(\mathbf{f}) \to \infty$.

(ii) *Let* $\mathbf{f}$ *run only over the prime divisors. Then*

$$(1.1.7) \qquad \lim_{N(\mathbf{f})\to\infty} \mathrm{Avg}_{\chi\in\hat{G}_\mathbf{f}\setminus\{\chi_0\}}(\overline{g_\lambda(s,\chi,\mathbf{f})}g_{\lambda'}(s,\chi,\mathbf{f})) = \sum_D \overline{\lambda(D)}\lambda'(D)N(D)^{-2\sigma},$$

and the convergence is uniform w.r.t. λ, λ', s. Moreover, the above average may be replaced by that over all χ with the conductor $\mathbf{f}_\chi = \mathbf{f}$.

The proof will be given in §2.

Remark For given $\lambda_1, ..., \lambda_k \in \Lambda$, define their $*$-product by

$$(1.1.8) \qquad (\lambda_1 * ... * \lambda_k)(D) = \sum_{D=D_1...D_k} \lambda_1(D_1)...\lambda_k(D_k).$$

Then this is also admissible, being associated with the product

$$g_{\lambda_1}...g_{\lambda_k}. \tag{1.1.9}$$

This is because if $S(D)$ denotes the number of distinct factors of D, then for any $\epsilon' > 0$ we have $S(D) \ll_{\epsilon'} N(D)^{\epsilon'}$, as is well-known in the NF case and can be proved similarly in the FF case (cf. [6]Appendix, for a unified proof). Moreover, if we fix k, then $\Lambda_k := \{\lambda_1 * ... * \lambda_k; \lambda_1, ..., \lambda_k \in \Lambda\}$ is again a uniformly admissible family of arithmetic functions. Thus, Theorem 1 remains valid if g_λ, $g_{\lambda'}$ are replaced by their k-th powers and $\lambda(D)$, $\lambda'(D)$, by their k-th $*$-powers.

1.2 – The families $\{\lambda_z\}_{|z|\le R}$ associated with L-functions. We shall consider the Dirichlet L-function associated with each $\chi \in \hat{G}_{\mathbf{f}}$, $\chi \neq \chi_0$, *without P_∞-component.* Namely, define

$$L(s, \chi, \mathbf{f}) = \prod_{\mathfrak{p} \notin P_\infty} (1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1}, \tag{1.2.1}$$

which converges absolutely on $\mathrm{Re}(s) > 1$ and extends to a holomorphic function on $\mathrm{Re}(s) > 1/2$. In the FF case, and in the NF case under GRH, it has no zeros on this domain. In these cases, $\log L(s, \chi, \mathbf{f})$ on this domain is defined as the unique holomorphic branch that vanishes at $s = +\infty$. Write:

$$\mathcal{L}(s, \chi, \mathbf{f}) = \begin{cases} \frac{L'}{L}(s, \chi, \mathbf{f}) & \text{(Case 1)}, \\ \log L(s, \chi, \mathbf{f}) & \text{(Case 2)}. \end{cases} \tag{1.2.2}$$

We shall show in the next Theorem that for any given $R > 0$, the family of functions

$$\exp\left(\frac{iz}{2}\mathcal{L}(s, \chi, \mathbf{f})\right) \tag{1.2.3}$$

parametrized by $\{z; |z| \le R\}$ forms (in each of Cases 1,2) a family of g-functions $g_{\lambda_z}(s, \chi, \mathbf{f})$ associated with a uniformly admissible family $\{\lambda_z\}_{|z|\le R}$ of arithmetic functions. To explain this, first define the polynomials $G_r(x)$, $H_r(x)$ $(r = 0, 1, 2, ...)$ of x as

$$\exp(xt/(1-t)) = \sum_{r=0}^{\infty} G_r(x) t^r, \tag{1.2.4}$$

$$\exp(-x \log(1-t)) = (1-t)^{-x} = \sum_{r=0}^{\infty} H_r(x) t^r, \tag{1.2.5}$$

by generating functions ($|t| < 1$). Explicitly, $G_0(x) = H_0(x) = 1$, and for $r \geq 1$,

$$G_r(x) = \sum_{k=1}^{r} \frac{1}{k!}\binom{r-1}{k-1}x^k, \tag{1.2.6}$$

$$H_r(x) = \sum_{k=1}^{r} \frac{1}{k!}\delta_k(r)x^k = \frac{1}{r!}x(x+1)...(x+r-1), \tag{1.2.7}$$

where

$$\delta_k(r) = \sum_{\substack{r=r_1+...+r_k \\ r_1,...,r_k \geq 1}} \frac{1}{r_1...r_k}. \tag{1.2.8}$$

Theorem 2 *For each $z \in \mathbf{C}$ and each integral divisor $D = \prod_{\mathfrak{p}} \mathfrak{p}^{r_\mathfrak{p}}$, define $\lambda_z(D)$ by*

$$\lambda_z(D) = \prod_{\mathfrak{p}|D} \lambda_z(\mathfrak{p}^{r_\mathfrak{p}}), \tag{1.2.9}$$

$$\lambda_z(\mathfrak{p}^r) = \begin{cases} G_r(-\frac{iz}{2}\log N(\mathfrak{p})) & \text{(Case 1)}, \\ H_r(\frac{iz}{2}) & \text{(Case 2)}, \end{cases} \tag{1.2.10}$$

where $i = \sqrt{-1}$. (In particular, $\lambda_z(D) = 1$ for $D = (1)$.) Then, for any K and P_∞,

(i) *the family $\{\lambda_z\}_{|z|\leq R}$ satisfies* (A1) *uniformly, i.e., with $\ll$ depending only on (ϵ' and) R.*

(ii) *Moreover, if we assume GRH in the NF case, then this also satisfies* (A2)(A3) *and is a uniformly admissible family of arithmetic functions. The associated g-function is given by*

$$g_{\lambda_z}(s, \chi, \mathbf{f}) = \exp\left(\frac{iz}{2}\mathcal{L}(s, \chi, \mathbf{f})\right). \tag{1.2.11}$$

The proof will be given in §3.

1.3 – Direct consequences of Theorems 1,2. Now consider the Dirichlet series

$$\tilde{M}_s(z_1, z_2) = \sum_D \lambda_{z_1}(D)\lambda_{z_2}(D)N(D)^{-2s} \qquad (\mathrm{Re}(s) > 1/2), \tag{1.3.1}$$

where the summation is over all integral divisors D of K. This converges absolutely and uniformly on $\mathrm{Re}(s) \geq 1/2 + \epsilon$, $|z_1|, |z_2| \leq R$ for any fixed ϵ, $R > 0$, because

$\lambda_{z_1}(D)$, $\lambda_{z_2}(D) \ll_{\epsilon'} N(D)^{\epsilon'}$ (uniformly on $|z_1|, |z_2| \leq R$) by Theorem 2; hence this is a holomorphic function of three complex variables s, z_1, z_2 on the domain $\mathrm{Re}(s) > 1/2$. Note that $\tilde{M}_s(z_1, z_2)$ is symmetric in z_1, z_2. For $z_1, z_2 \in \mathbf{C}$, let ψ_{z_1,z_2} denote the quasi-character of the additive group $\mathbf{C}$ defined by

$$\psi_{z_1,z_2}(w) = \exp\left(\frac{i}{2}(z_1\bar{w} + z_2 w)\right). \tag{1.3.2}$$

Theorem 3 *Assume $|P_\infty| = 1$, and in the NF case assume also GRH. Then*

$$\lim_{\substack{\mathbf{f}\, prime \\ N(\mathbf{f})\to\infty}} \Big(\mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} \psi_{z_1,z_2}(\mathcal{L}(s,\chi,\mathbf{f}))\Big) = \tilde{M}_\sigma(z_1, z_2) \tag{1.3.3}$$

uniformly on $|z_1|, |z_2| \leq R$ and for $s = \sigma + ti$ with $\sigma \geq 1/2 + \epsilon$, and $|t| \leq T$ in the NF case.

This is a direct consequence of Theorem 1(ii) and Theorem 2. Indeed, we have $\overline{\lambda_z(D)} = \lambda_{-\bar{z}}(D)$ and

$$\psi_{z_1,z_2}(\mathcal{L}(s,\chi,\mathbf{f})) = \overline{g_{\lambda_{-\bar{z}_1}}(s,\chi,\mathbf{f})}\, g_{\lambda_{z_2}}(s,\chi,\mathbf{f}). \tag{1.3.4}$$

Remark 1.3.5 When $\sigma > 1$, (1.3.3) holds without GRH, because we may use $\lambda_z(D)N(D)^{-1/2}$ instead of $\lambda_z(D)$.

Some basic analytic properties of $\tilde{M}_s(z_1, z_2)$ will be shown in §4. In Case 1 this is mostly a review of results of [4]§3. We only mention here that when $s = \sigma > 1/2$, we have

$$\tilde{M}_\sigma(z_1, z_2) = \int_{\mathbf{C}} M_\sigma(w)\psi_{z_1,z_2}(w)|dw|. \tag{1.3.6}$$

Here, $M_\sigma(w) = *_{\mathfrak{p}\notin P_\infty} M_{\sigma,\mathfrak{p}}(w)$ ($*$: the convolution product) is the "M"-function (here without P_∞-factors) constructed in [4](Case 1) [7](Case 2; denoted as $\mathcal{M}_\sigma$)[2]. In particular, $\tilde{M}_\sigma(z, \bar{z})$ is equal to the Fourier dual $\tilde{M}_\sigma(z)$ of $M_\sigma(z)$.

In Case 2, where $\exp(\mathcal{L}(s,\chi,\mathbf{f})) = L(s,\chi,\mathbf{f})$, Theorem 3 gives

$$\lim_{\substack{\mathbf{f}\, prime \\ N(\mathbf{f})\to\infty}} \left(\mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} \overline{L(s,\chi,\mathbf{f})}^{\frac{iz_1}{2}} L(s,\chi,\mathbf{f})^{\frac{iz_2}{2}}\right) = \tilde{M}_\sigma(z_1, z_2); \tag{1.3.7}$$

hence in particular:

[2]For a short-cut definition of M_σ relying on the classical Jessen-Wintner theory [9], see §5.6.

Corollary 1.3.8 (Case 2) *The assumptions being as in Theorem 3,*

$$\lim_{\substack{\mathbf{f}\,prime\\ N(\mathbf{f})\to\infty}} \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} |L(s,\chi,\mathbf{f})^{\frac{iz}{2}}|^2 = \tilde{M}_\sigma(-\bar{z},z); \tag{1.3.9}$$

$$\lim_{\substack{\mathbf{f}\,prime\\ N(\mathbf{f})\to\infty}} \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} (L(s,\chi,\mathbf{f})^{iz/2}/\overline{L(s,\chi,\mathbf{f})^{iz/2}}) = \tilde{M}_\sigma(\bar{z}) = \tilde{M}_\sigma(z); \tag{1.3.10}$$

in particular, for any $y \in \mathbf{R}$,

$$\lim_{\substack{\mathbf{f}\,prime\\ N(\mathbf{f})\to\infty}} \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} (L(s,\chi,\mathbf{f})/\overline{L(s,\chi,\mathbf{f})})^y = \tilde{M}_\sigma(2iy). \tag{1.3.11}$$

As is shown in [5], $\tilde{M}_\sigma(z)$ has, at least in Case 1, infinitely many purely imaginary zeros, and at most finitely many other zeros. The following Corollary will be needed later

Corollary 1.3.12 *The assumptions being as in Theorem 3, fix any* $\epsilon > 0$, $T > 0$, $a > 0$, *and let* $s = \sigma + ti$ *run over* $\sigma \geq 1/2 + \epsilon$, *and in the NF case, additionally,* $|t| \leq T$. *Then for any prime divisor* $\mathbf{f}$ *we have*

$$\mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} \exp(a|\mathcal{L}(s,\chi,\mathbf{f})|) \ll 1. \tag{1.3.13}$$

Proof Write $\ell_\chi = \mathcal{L}(s,\chi,\mathbf{f})$, $\mathrm{Avg}_\chi = \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}}$. Since $e^{a|\ell_\chi|} \leq e^{a|\mathrm{Re}(\ell_\chi)|}e^{a|\mathrm{Im}(\ell_\chi)|}$, Schwarz inequality reduces the Corollary to $\mathrm{Avg}_\chi e^{2a|\mathrm{Re}(\ell_\chi)|}$, $\mathrm{Avg}_\chi e^{2a|\mathrm{Im}(\ell_\chi)|} \ll 1$. But since $e^{2a|\mathrm{Re}(\ell_\chi)|} < |e^{a\ell_\chi}|^2 + |e^{-a\ell_\chi}|^2$, $e^{2a|\mathrm{Im}(\ell_\chi)|} < |e^{-ai\ell_\chi}|^2 + |e^{ai\ell_\chi}|^2$, and since $\mathrm{Avg}_\chi |e^{z\ell_\chi}|^2 \ll 1$ holds for $z = \pm a, \pm ai$, in view of Theorem 3, the Corollary follows. □

1.4 – Application to value distributions. As before, let $\mathcal{L}(s,\chi,\mathbf{f}) = L/L'(s,\chi,\mathbf{f})$ (Case 1), $= \log L(s,\chi,\mathbf{f})$ (Case 2), and let $M_\sigma(z)$ be the associated M-function without P_∞-component.

Theorem 4 *Let* $\sigma := \mathrm{Re}(s) > 1/2$, *assume* $|P_\infty| = 1$, *and in the NF case assume also GRH. Then*

$$\lim_{\substack{\mathbf{f}\,prime\\ N(\mathbf{f})\to\infty}} \left(\mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} \Phi(\mathcal{L}(s,\chi,\mathbf{f}))\right) = \int_{\mathbf{C}} M_\sigma(w)\Phi(w)|dw| \tag{1.4.1}$$

holds for any continuous function Φ *on* $\mathbf{C}$ *with at most exponential growth, i.e. when* $\Phi(w) \ll e^{a|w|}$ *holds with some* $a > 0$. *The equality (1.4.1) holds also when* Φ *is the characteristic function of either a compact subset of* $\mathbf{C}$ *or the complement of such a subset.*

When moreover $\sigma > 1$*, then (1.4.1) holds unconditionally for any continuous function* Φ *on* $\mathbf{C}$.

The proof will be given in §5. This Theorem for Case 1 for the FF case strengthens Theorem B of [6] (and Theorem 7 of [4]) in various sense. The condition on the test function Φ is now considerably loosened, and here, the assertion is on the limit of the average over $\mathbf{f}$, which is stronger than the previous assertions on the limit, as $m \to \infty$, of a weighted average over $N(\mathbf{f}) \leq m$. It should also be added, however, that Theorem A of [6], and the direct method for proving Theorem B as its application, may still deserve attention for independent interest.

2 Proof of Theorem 1

Throughout this section, we assume that $|P_\infty| = 1$; i.e., either K is rational or imaginary quadratic and P_∞ consists only of the unique archimedean prime (NF case), or $P_\infty = \{\mathfrak{p}_\infty\}$ for a given prime divisor $\mathfrak{p}_\infty$ (FF case).

2.1 – Preliminaries. We shall first prepare some basic materials that will be used in the sequel. Notations being as in §1.1, for each $x \geq 1$ and an integral divisor $\mathbf{f}$, let $n(c, \mathbf{f}; x)$ for each $c \in G_\mathbf{f}$ denote the number of integral divisors D of K with $N(D) \leq x$ satisfying $(D, \mathbf{f}) = 1$ and $i_\mathbf{f}(D) = c$.

Proposition 2.1.1 (i) *For any* $\mathbf{f}$ *and* x,

$$n(c, \mathbf{f}; x) \ll 1 + N(\mathbf{f})^{-1}x.$$

(ii) *There exists* $A = A_K > 0$ *such that for any* $\mathbf{f}$ *and any* $x < A \cdot N(\mathbf{f})$,

$$\text{Max}_{c \in G_\mathbf{f}}\, n(c, \mathbf{f}; x) \leq 1. \tag{2.1.2}$$

Proof First, let K be a function field over $\mathbf{F}_q$. Then, since principal divisors have norm equal to 1, two integral divisors (which means also that they are coprime with $\mathfrak{p}_\infty$) belonging to the same class c must have the equal norm. Now, Prop 3.3.16 of [6] asserts that the number of integral divisors D with the given norm $N(D) = q^m$ satisfying $(D, \mathbf{f}) = 1$ and $i_\mathbf{f}(D) = c$ cannot exceed $\text{Max}(1, q^{m+1}/N(\mathbf{f}))$. Therefore,

$$n(c, \mathbf{f}; x) \leq \text{Max}(1, qx/N(\mathbf{f})) \ll 1 + N(\mathbf{f})^{-1}x. \tag{2.1.3}$$

Moreover, if $q^m < N(\mathbf{f})$ (so that $q^{m+1} \leq N(\mathbf{f})$), there is at most one such D. Hence (ii) holds with $A_K = 1$.

When $K = \mathbf{Q}$ and $f \in \mathbf{N}$, $n(c, (f); x) \leq x/f + 1$; whence (i). Moreover, $n(c, (f); x) \leq 1$ when $x < f$; hence (ii) holds with $A_\mathbf{Q} = 1$.

Now let K be imaginary quadratic, with class number h. To prove (i), let $\mathfrak{A}_i$ ($1 \leq i \leq h$) be a set of representatives of the ideal classes in K, and for each i ($1 \leq i \leq h$) choose a fundamental domain (a parallelogram) Ω_i for the lattice $\mathfrak{A}_i$ embedded in $\mathbf{C}$. Then as a fundamental domain $\Omega_\mathbf{f}$ for any divisor $\mathbf{f} \neq (0)$, we may choose some complex scalar multiple of one of the Ω_i. For each i, the number of distinct translations of Ω_i (by an element of the lattice $\mathfrak{A}_i$) that meet the disk $\{|\xi|^2 \leq x\}$ is $\ll 1 + x$; hence the number of distinct translations of $\Omega_\mathbf{f}$ (by an element of $\mathbf{f}$) that meet $\{|\xi|^2 \leq x\}$ is $\ll 1 + x/N(\mathbf{f})$. Now (i) follows easily from the finiteness of the unit group of K. As for (ii), suppose that D, D' are distinct integral divisors belonging to c such that $N(D') \leq N(D)$. This means that $D' = (\alpha)D$ with some $\alpha \equiv 1 (\text{mod}\, \mathbf{f})$, $\alpha \neq 1, N(\alpha) \leq 1$. The last inequality gives

$N(\alpha-1) \leq 2(N(\alpha)+1) \leq 4$. On the other hand, $(\alpha-1)D$ must be integral, $\neq (0)$, and divisible by $\mathbf{f}$; hence $N(\alpha-1)N(D) \geq N(\mathbf{f})$. Therefore, $4 \geq N(\alpha-1) \geq N(D)^{-1}N(\mathbf{f})$; i.e., $N(D) \geq N(\mathbf{f})/4$. Therefore, (ii) holds with $A_K = 1/4$. □

We shall also need the formula for the cardinality of the group $G_{\mathbf{f}}$;

$$|G_{\mathbf{f}}| = \delta_K h_K \frac{N(\mathbf{f})}{w_K} \prod_{\mathfrak{p}|\mathbf{f}} (1 - N(\mathfrak{p})^{-1}), \tag{2.1.4}$$

where $\delta_K = 1$ (NF case), $= \deg \mathfrak{p}_\infty$ (FF case), h_K is the class number of K, and w_K is the number of residue classes mod $\mathbf{f}$ represented by some root of unity, except that it is 1 when $K = \mathbf{Q}$. Note that $w_K = q-1$ in the FF case over $\mathbf{F}_q$. Since

$$\prod_{N(\mathfrak{p}) \leq y} (1 - N(\mathfrak{p})^{-1}) \gg (\log y)^{-1} \tag{2.1.5}$$

(cf [6]§3.7 for a proof for the FF case), the above formula gives

$$\frac{N(\mathbf{f})}{\log N(\mathbf{f})} \ll |G_{\mathbf{f}}| \ll N(\mathbf{f}). \tag{2.1.6}$$

2.2 – The integral expression. The basic notations are as follows.

Fix ϵ such that $0 < \epsilon < 1/2$. The symbol $\ll$ will depend on ϵ but this dependence will be suppressed from the notations.

$s \in \mathbf{C}$ will always satisfy $\sigma := \mathrm{Re}(s) \geq 1/2 + \epsilon$;

$\mathbf{f}$: any integral divisor;

X: a real parameter ≥ 1.

Later, we shall choose $X = N(\mathbf{f})^\beta$, with $\beta = 1 + \epsilon/2$;

Λ: a given uniformly admissible family of arithmetic functions;

λ, $\lambda' \in \Lambda$; write $g = g_\lambda$, $g' = g_{\lambda'}$.

Proposition 2.2.1 (i) *On the space* $\mathrm{Re}(s) \geq 1/2 + \epsilon$, *one can express* $g = g(s, \chi, \mathbf{f})$ $(\chi \in \hat{G}_{\mathbf{f}} \backslash \{\chi_0\})$ *as the difference*

$$g = g_+ - g_- \tag{2.2.2}$$

of two holomorphic functions

$$g_+ = g_+(s, \chi, \mathbf{f}; X) = \frac{1}{2\pi i} \int_{\mathrm{Re}(w)=c} \Gamma(w) g(s+w, \chi, \mathbf{f}) X^w dw, \tag{2.2.3}$$

and

$$g_- = g_-(s,\chi,\mathbf{f};X) = \frac{1}{2\pi i}\int_{\mathrm{Re}(w)=\epsilon'-\epsilon} \Gamma(w) g(s+w,\chi,\mathbf{f}) X^w dw, \tag{2.2.4}$$

where c and ϵ' are any positive real numbers satisfying $c > \mathrm{Max}(0, 1-\sigma)$ and $0 < \epsilon' < \epsilon$. Each of g_+ and g_- depends on the parameter X but not on c or ϵ'.

(ii) *g_+ has a Dirichlet series expansion*

$$g_+ = \sum_{(D,\mathbf{f})=1} \chi(D)\lambda(D)\exp(-N(D)/X)N(D)^{-s}, \tag{2.2.5}$$

which is absolutely convergent for any $\chi \in \hat{G}_{\mathbf{f}}$ and any $s \in \mathbf{C}$.

Proof First, note that

$$g(s,\chi,\mathbf{f}) = \frac{1}{2\pi i}\int_B \Gamma(w) g(s+w,\chi,\mathbf{f}) X^w dw \tag{2.2.6}$$

holds, where B is the positively oriented rectangle bordering

$$\epsilon' - \epsilon \le \mathrm{Re}(w) \le c, \qquad |\mathrm{Im}(w)| \le T \tag{2.2.7}$$

($T > 0$). This is clear, because the integrand is holomorphic in w on (2.2.7) except for a simple pole at $w = 0$ with the residue $g(s,\chi,\mathbf{f})$. (In fact, since $-1 < \epsilon' - \epsilon < 0$, the only pole of $\Gamma(w)$ on (2.2.7) is a simple pole at $w = 0$ (with the residue 1), and since $\mathrm{Re}(s+w) \ge 1/2 + \epsilon' > 1/2$, $g(s+w,\chi,\mathbf{f})$ is holomorphic on (2.2.7), by (A3).)

To prove (i), let us estimate the integrand on $\epsilon' - \epsilon \le \mathrm{Re}(w) \le c$; $|\mathrm{Im}(w)| \ge T$. First, $|X^w| \le X^c$ (because $X \ge 1$); secondly, in the FF case, $g(s+w,\chi,\mathbf{f})$ is holomorphic and vertically periodic; hence bounded, and in the NF case, for each fixed $s, \mathbf{f}, \chi$,

$$|g(s+w,\chi,\mathbf{f})| \ll \exp\left(C\log^2(|\mathrm{Im}(w)|+2)\right) \tag{2.2.8}$$

with some $C = C_{s,\chi,\mathbf{f}} > 0$, by (A3). Thirdly,

$$|\Gamma(w)| \ll |\mathrm{Im}(w)|^{c-1/2}\exp(-\frac{\pi}{2}|\mathrm{Im}(w)|) \tag{2.2.9}$$

for $|\mathrm{Im}(w)| \ge 1$. Now (i) follows directly from these by letting $T \to \infty$ in (2.2.6).

(ii) Since $\sigma + c > 1$, the Dirichlet series expansion

$$g(s+w,\chi,\mathbf{f}) = \sum_{(D,\mathbf{f})=1} \chi(D)\lambda(D)N(D)^{-s-w} \tag{2.2.10}$$

is absolutely convergent on $\mathrm{Re}(w) = c$, and the convergence is uniform with respect to $\mathrm{Im}(w)$. Therefore,

$$\begin{aligned} (2.2.11) \qquad g_+ &= g_+(s, \chi, \mathbf{f}, X) \\ &= \frac{1}{2\pi i} \int_{\mathrm{Re}(w)=c} \Gamma(w) \left(\sum_{(D,\mathbf{f})=1} \chi(D)\lambda(D)N(D)^{-s-w} \right) X^w dw \\ &= \sum_D \chi(D)\lambda(D)N(D)^{-s} \left(\frac{1}{2\pi i} \int_{\mathrm{Re}(w)=c} \Gamma(w)N(D)^{-w}X^w dw \right). \end{aligned}$$

But since

$$(2.2.12) \qquad \frac{1}{2\pi i} \int_{\mathrm{Re}(u)=c} \Gamma(u)a^{-u}du = e^{-a} \qquad (a,\, c > 0),$$

we obtain the desired Dirichlet series expansion (2.2.5). Because of the exponential factor, this converges absolutely for any $s \in \mathbf{C}$ and any $\chi \in \hat{G}_{\mathbf{f}}$. This can be seen easily by noting that $\lambda(D) \ll N(D)$, and that the number of D with $N(D) = n$ is certainly $\ll n$. □

We define $g_+(s, \chi, \mathbf{f}; X)$ for any $\chi \in \hat{G}_{\mathbf{f}}$ including $\chi = \chi_0$, by (2.2.5).

Proposition 2.2.13 *Let $\sigma = \mathrm{Re}(s) \geq 1/2 + \epsilon$. Then*

(i) *For any $\epsilon' > 0$ and $\chi \in \hat{G}_{\mathbf{f}}$,*

$$(2.2.14) \qquad |g_+(s, \chi, \mathbf{f}; X)| \ll_{\epsilon'} X^{1/2+\epsilon'-\epsilon}.$$

(ii) *For any ϵ' $(0 < \epsilon' < \epsilon)$, $T > 0$, and for $|\mathrm{Im}(s)| \leq T$, $\chi \in \hat{G}_{\mathbf{f}} \backslash \{\chi_0\}$,*

$$(2.2.15) \qquad |g_-(s, \chi, \mathbf{f}; X)| \ll_{\epsilon',T} (N(\mathbf{f})X)^{\epsilon'} X^{-\epsilon}.$$

Proof (i) Since

$$(2.2.16) \qquad g_+(s, \chi, \mathbf{f}; X) = \sum_{(D,\mathbf{f})=1} \chi(D)\lambda(D) \exp(-N(D)/X)N(D)^{-s},$$

we have, by (A1),

$$\begin{aligned} (2.2.17) \qquad |g_+(s, \chi, \mathbf{f}; X)| &\ll \sum_{(D,\mathbf{f})=1} N(D)^{\epsilon'} \exp(-N(D)/X)N(D)^{-\sigma} \\ &\leq \sum_{n=1}^{\infty} a(n)n^{\epsilon'-(1/2+\epsilon)}e^{-n/X}, \end{aligned}$$

where $a(n)$ denotes the number of D with $N(D) = n$. But since $\sum_{n\le x} a(n) \ll x$ for any $x \ge 1$, we obtain, by partial summation,

$$|g_+(s,\chi,\mathbf{f};X)| \ll \int_1^\infty t|f'(t)|dt, \tag{2.2.18}$$

where $f(t) = t^{-a}e^{-t/X}$, with $a = (1/2+\epsilon) - \epsilon'$. But $f'(t)/f(t) = -(X^{-1} + at^{-1})$; hence $t|f'(t)| \ll_{\epsilon'} (X^{-1}t + 1)f(t)$; hence

$$\begin{aligned} |g_+(s,\chi,\mathbf{f};X)| &\ll_{\epsilon'} \int_1^\infty (X^{-1}t+1)t^{-a}e^{-t/X}dt = X^{1-a}\int_{1/X}^\infty (u+1)u^{-a}e^{-u}du \\ &\ll X^{1-a}(\Gamma(2-a)+\Gamma(1-a)) \ll X^{1-a} = X^{1/2+\epsilon'-\epsilon}. \end{aligned}$$

This settles (i).

(ii) By definition,

$$g_-(s,\chi,\mathbf{f};X) = \frac{1}{2\pi i}\int_{\mathrm{Re}(w)=\epsilon'-\epsilon} \Gamma(w)g(s+w,\chi,\mathbf{f})X^w dw. \tag{2.2.19}$$

Since $\mathrm{Re}(w) = \epsilon' - \epsilon$, we have $|X^w| = X^{\epsilon'-\epsilon}$, and

$$\Gamma(w) \ll \exp(-\frac{\pi}{2}|\mathrm{Im}(w)|). \tag{2.2.20}$$

In the FF case, since $\mathrm{Re}(s+w) \ge 1/2 + \epsilon'$, we have, by (A3),

$$|g(s+w,\chi,\mathbf{f})| \ll_{\varepsilon',\epsilon''} N(\mathbf{f})^{\epsilon''}; \tag{2.2.21}$$

for any $\epsilon', \epsilon'' > 0$; in particular, for $\epsilon'' = \epsilon'$; whence (2.2.15).

In the NF case, the situation is more complicated. Put $\mathrm{Im}(w) = u$, so that $\mathrm{Im}(s+w) = t + u$. Then by (1.1.3)(since $\mathrm{Re}(s+w) \ge 1/2+\epsilon'$) there exists $C = C_{\epsilon'} > 0$ such that

$$|g(s+w,\chi,\mathbf{f})| \le \exp\{C(\ell(t+u)\ell(\mathbf{f})^{1-2\epsilon'} + \ell(t+u)^2)\}. \tag{2.2.22}$$

But since $|t+u|+2 \le (|t|+2)(|u|+1)$, we may replace $\ell(t+u)$ by $\ell(t) + \log(|u|+1)$; hence also $\ell(t+u)^2$ by $2(\ell(t)^2 + \log^2(|u|+1))$. Therefore, there exists $C' = C'_{\epsilon',T} > 0$ such that when $|t| \le T$,

$$|g(s+w,\chi,\mathbf{f})| \le \exp(C'\ell(\mathbf{f})^{1-2\epsilon'})\exp\{C'(\ell(\mathbf{f})^{1-2\epsilon'}\log(|u|+1) + \log^2(|u|+1))\}, \tag{2.2.23}$$

which together with (2.2.19)(2.2.20) gives

$$|g_-(s,\chi,\mathbf{f};X)| \ll X^{\epsilon'-\epsilon}\exp(C'\ell(\mathbf{f})^{1-2\epsilon'})\int_0^\infty e^{-u}(u+1)^{C'\ell(\mathbf{f})^{1-2\epsilon'}}e^{C'\log^2(u+1)}du. \tag{2.2.24}$$

By using the Schwarz inequality

$$(2.2.25) \qquad \left(\int_0^\infty f_1(u)f_2(u)du\right)^2 \leq \left(\int_0^\infty f_1(u)^2du\right)\left(\int_0^\infty f_2(u)^2du\right)$$

for $f_1(u) = e^{-u/2}(u+1)^{C'\ell(\mathbf{f})^{1-2\epsilon'}}$, $f_2(u) = e^{-u/2}e^{C'\log^2(u+1)}$, and by noting that the integral of $f_2(u)^2du$ for this case is $\ll_{\epsilon',T} 1$, we obtain

$$|g_-(s,\chi,\mathbf{f};X)| \ll_{\epsilon',T} X^{\epsilon'-\epsilon}\exp(C'\ell(\mathbf{f})^{1-2\epsilon'})\left(\int_0^\infty e^{-u}(u+1)^{2C'\ell(\mathbf{f})^{1-2\epsilon'}}du\right)^{1/2}.$$

By putting $u+1=v$ and comparing the integral with the Γ-integral, we obtain

$$\begin{aligned} |g_-(s,\chi,\mathbf{f};X)| &\ll_{\epsilon',T} X^{\epsilon'-\epsilon}\exp(C'\ell(\mathbf{f})^{1-2\epsilon'})\Gamma(2C'\ell(\mathbf{f})^{1-2\epsilon'}+1)^{1/2} \\ &\ll X^{\epsilon'-\epsilon}\exp(C'\ell(\mathbf{f})^{1-2\epsilon'})\exp(C'\ell(\mathbf{f})^{1-2\epsilon'}\log(2C'\ell(\mathbf{f})^{1-2\epsilon'})) \\ &\ll X^{\epsilon'-\epsilon}\exp(C''\ell(\mathbf{f})^{1-2\epsilon'}\log\ell(\mathbf{f})), \end{aligned}$$

with some $C'' = C''_{\epsilon',T} > 0$. But $C''\ell(\mathbf{f})^{-2\epsilon'}\log(\ell(\mathbf{f})) < \epsilon'$ holds for $N(\mathbf{f})$ sufficiently large depending on ϵ', T. Hence this is

$$\ll_{\epsilon',T} X^{\epsilon'-\epsilon}\exp(\epsilon'\ell(\mathbf{f})) = X^{\epsilon'-\epsilon}(N(\mathbf{f})+2)^{\epsilon'} \ll X^{\epsilon'-\epsilon}N(\mathbf{f})^{\epsilon'}.$$

This settles the proof of (ii) also in the NF case. □

2.3 – Study of $\mathrm{Avg}_{\chi\in\hat{G}_\mathbf{f}}\left(\overline{g_+(\chi)}g'_+(\chi)\right)$**.** This average will give the main term of $\mathrm{Avg}_{\chi\in\hat{G}_\mathbf{f}\setminus\{\chi_0\}}(\bar{g}(\chi)g'(\chi))$, and this estimation depends only on the property (A1) of the admissible family. Here, and in what follows in this subsection, we shall suppress from the notations the dependence on $s,\mathbf{f},X$. Thus,

$$(2.3.1) \qquad g_+(\chi) = \sum_{(D,\mathbf{f})=1}\chi(D)\lambda(D)\exp(-N(D)/X)N(D)^{-s},$$

$$(2.3.2) \qquad g'_+(\chi) = \sum_{(D,\mathbf{f})=1}\chi(D)\lambda'(D)\exp(-N(D)/X)N(D)^{-s}$$

($\chi\in\hat{G}_\mathbf{f}$). The orthogonality relation for characters gives directly

$$(2.3.3) \qquad S := \mathrm{Avg}_{\chi\in\hat{G}_\mathbf{f}}(\overline{g_+(\chi)}g'_+(\chi)) = \sum_{c\in G_\mathbf{f}}\overline{T(c)}T'(c),$$

where

$$T(c) = \sum_{i_{\mathbf{f}}(D)=c} \lambda(D)\exp(-N(D)/X)N(D)^{-s}, \tag{2.3.4}$$
$$T'(c) = \sum_{i_{\mathbf{f}}(D)=c} \lambda'(D)\exp(-N(D)/X)N(D)^{-s}.$$

Now we shall make a full use of Prop 2.1.1. Let $A = A_K > 0$ be as in Prop 2.1.1(ii), and decompose as $T(c) = T_1(c) + T_2(c)$, where $T_1(c)$ (resp. $T_2(c)$) denotes the partial sum over $N(D) < AN(\mathbf{f})$ (resp. $N(D) \geq AN(\mathbf{f})$). Define $T_i'(c)$ ($i = 1, 2$) similarly. By definition, the sum for $T_1(c)$ has at most *one* term. Call $c \in G_{\mathbf{f}}$ *small* when there exists an integral divisor D such that $i_{\mathbf{f}}(D) = c$ and $N(D) < AN(\mathbf{f})$. In this case, call D_c the unique such D. Thus,

$$T_1(c) = \begin{cases} \lambda(D_c)\exp(-N(D_c)/X)N(D_c)^{-s} & (c: \ \textit{small}), \\ 0 & (\textit{otherwise}). \end{cases}$$

Since $c \mapsto D_c$ gives a bijection between small classes in $G_{\mathbf{f}}$ and integral divisors D satisfying $(D, \mathbf{f}) = 1$ and $N(D) < AN(\mathbf{f})$, we obtain

$$S_1 := \sum_{c \in G_{\mathbf{f}}} \overline{T_1(c)} T_1'(c) = \sum_{\substack{(D,\mathbf{f})=1 \\ N(D)<AN(\mathbf{f})}} \overline{\lambda(D)}\lambda'(D)\exp(-2N(D)/X)N(D)^{-2\sigma}. \tag{2.3.5}$$

Note that

$$S_1 \ll \sum_D N(D)^{\epsilon - 2\sigma} \ll \sum_D N(D)^{-1-\epsilon} \ll 1. \tag{2.3.6}$$

As for

$$T_2(c) = \sum_{\substack{i_{\mathbf{f}}(D)=c \\ N(D)\geq AN(\mathbf{f})}} \lambda(D)\exp(-N(D)/X)N(D)^{-s}, \tag{2.3.7}$$

we shall prove

$$T_2(c) \ll_{\epsilon'} N(\mathbf{f})^{-1} X^{1/2+\epsilon'-\epsilon} \tag{2.3.8}$$

for any $\epsilon' > 0$. Since $\lambda(D) \ll N(D)^{\epsilon'}$, we have

$$T_2(c) \ll \sum_{n \geq AN(\mathbf{f})} a_c(n) n^{\epsilon' - \sigma} e^{-n/X}, \tag{2.3.9}$$

where $a_c(n)$ denotes the number of D with $N(D) = n$, $i_{\mathbf{f}}(D) = c$. But since $\sum_{n\le x} a_c(n) \ll N(\mathbf{f})^{-1}x$ for $x \ge AN(\mathbf{f})$ by Proposition 2.1.1 (i), we obtain (2.3.8) exactly by the same argument as in the proof of Prop 2.2.13 (i). Therefore, by (2.1.6), $E_2 := \sum_{c\in G_{\mathbf{f}}} |T_2(c)|^2$, $E_2' := \sum_{c\in G_{\mathbf{f}}} |T_2'(c)|^2$ satisfy

$$E_2,\, E_2' \ll |G_{\mathbf{f}}|N(\mathbf{f})^{-2}X^{1-2(\epsilon-\epsilon')} \ll N(\mathbf{f})^{-1}X^{1-2(\epsilon-\epsilon')}. \tag{2.3.10}$$

Therefore, by (2.3.6) for $(T_1 = T_1')$,(2.3.10), and by the Schwarz inequality, we obtain

$$\begin{aligned} S - S_1 &= \sum_{c\in G_{\mathbf{f}}} \left(\overline{((T_1(c)+T_2(c))}(T_1'(c)+T_2'(c)) - \overline{T_1(c)}T_1'(c) \right) \\ &\ll (N(\mathbf{f})^{-1}X^{\alpha})^{1/2} + N(\mathbf{f})^{-1}X^{\alpha}, \end{aligned} \tag{2.3.11}$$

where $\alpha = 1 - 2(\epsilon - \epsilon') > 0$. We shall choose

$$X = N(\mathbf{f})^{\beta},\ \textit{with}\ 0 < \beta < \alpha^{-1}, \tag{2.3.12}$$

so that $N(\mathbf{f})^{-1}X^{\alpha}$ is a negative power of $N(\mathbf{f})$; hence

$$S - S_1 \ll (N(\mathbf{f})^{-1}X^{\alpha})^{1/2} = N(\mathbf{f})^{(-1+\alpha\beta)/2}. \tag{2.3.13}$$

We shall now treat the difference between S_1 and

$$S_0 = \sum_{(D,\mathbf{f})=1} \overline{\lambda(D)}\lambda'(D)N(D)^{-2\sigma}. \tag{2.3.14}$$

By the definitions of S_1, S_0, we have $S_0 - S_1 = E + E'$, with

$$\begin{aligned} E &= \sum_{\substack{(D,\mathbf{f})=1 \\ N(D)\ge AN(\mathbf{f})}} \overline{\lambda(D)}\lambda'(D)N(D)^{-2\sigma}, \\ E' &= \sum_{\substack{(D,\mathbf{f})=1 \\ N(D)< AN(\mathbf{f})}} \overline{\lambda(D)}\lambda'(D)\left(1 - \exp(-2N(D)/X)\right)N(D)^{-2\sigma}. \end{aligned} \tag{2.3.15}$$

As for E,

$$E \ll \sum_{N(D)\ge AN(\mathbf{f})} N(D)^{\epsilon-2\sigma} \le \sum_{N(D)\ge AN(\mathbf{f})} N(D)^{-1-\epsilon}. \tag{2.3.16}$$

But since the number of D with norm $\le x$ is $\ll x$, this gives

$$E \ll \int_{AN(\mathbf{f})}^{\infty} t|d(t^{-1-\epsilon})/dt|dt \ll N(\mathbf{f})^{-\epsilon}. \tag{2.3.17}$$

As for E', since $0 < 1 - \exp(-a) < a$ holds for any $a > 0$,
(2.3.18)
$$E' \ll \sum_{N(D)<AN(\mathbf{f})} N(D)^{\epsilon-2\sigma}(1-\exp(-2N(D)/X)) < 2AN(\mathbf{f})X^{-1}\sum_{D} N(D)^{-1-\epsilon} \ll N(\mathbf{f})X^{-1};$$

hence for the above choice of X we have $E' \ll N(\mathbf{f})^{1-\beta}$; hence

$$S_0 - S_1 \ll N(\mathbf{f})^{-\epsilon} + N(\mathbf{f})^{1-\beta}. \tag{2.3.19}$$

Therefore, combining with (2.3.13) we obtain (for the above choice of X)

$$S - S_0 \ll N(\mathbf{f})^{(-1+\alpha\beta)/2} + N(\mathbf{f})^{-\epsilon} + N(\mathbf{f})^{1-\beta}. \tag{2.3.20}$$

Now the question is how to choose $\beta > 0$ so that all the exponents of $N(\mathbf{f})$ on the right hand side of (2.3.20) are negative and the minimal of their absolute values is large enough. One of such choices is where $\epsilon' = \epsilon/4$, $\beta = 1 + \epsilon/2$, in which case $\alpha = 1 - (3/2)\epsilon$, and the three exponents are

$$(-\epsilon - (3/4)\epsilon^2)/2, \ -\epsilon, \ -\epsilon/2;$$

hence

$$S - S_0 \ll N(\mathbf{f})^{-\epsilon/2}. \tag{2.3.21}$$

(We shall see in §2.6 that this choice of β is appropriate also for the estimation of the counterpart related to $g_-(\chi)$.)

2.4 – Differences between modified averages. We now compare the averages of $\overline{g_+(\chi)}g'_+(\chi)$ over the whole group $\chi \in \hat{G}_{\mathbf{f}}$, with that over the complement of χ_0, and also when $\mathbf{f}$ is a prime divisor, with that over $\{\chi; \mathbf{f}_\chi = \mathbf{f}\}$ (Note that when the class number is greater than one, there can be non-principal characters with the conductor (1).) It is easy to see that these differences are

$$\ll \frac{1}{|G_{\mathbf{f}}|}\left(\mathrm{Max}_{\chi\in\hat{G}_{\mathbf{f}}}|g_+(\chi)|\mathrm{Max}_{\chi\in\hat{G}_{\mathbf{f}}}|g'_+(\chi)|\right). \tag{2.4.1}$$

Hence by Prop 2.2.13(i) and by (2.1.6), this is

$$\ll |G_{\mathbf{f}}|^{-1}X^\alpha \ll (\log N(\mathbf{f}))N(\mathbf{f})^{-1+\alpha\beta} \ll N(\mathbf{f})^{(-1+\alpha\beta)/2} \ll N(\mathbf{f})^{-\epsilon/2}.$$

Therefore, by combining this with the main estimation (2.3.20) of the previous subsection, we obtain

$$\mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}\setminus\{\chi_0\}}(\overline{g_+(\chi)}g'_+(\chi)) - S_0 \ll N(\mathbf{f})^{-\epsilon/2}, \tag{2.4.2}$$

together with that when $\mathbf{f}$ is a prime divisor, the average may be replaced by that over $\{\chi; \mathbf{f}_\chi = \mathbf{f}\}$.

2.5 – Final stage of the proof. It remains to estimate the difference

$$\mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}\setminus\{\chi_0\}}(\overline{g(\chi)}g'(\chi)) - \mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}\setminus\{\chi_0\}}(\overline{g_+(\chi)}g'_+(\chi)). \tag{2.5.1}$$

Recall that $g = g_+ - g_-$, $g' = g'_+ - g'_-$. But

$$\mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}\setminus\{\chi_0\}}|g_+(\chi)|^2,\ \mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}\setminus\{\chi_0\}}|g'_+(\chi)|^2 \ll 1, \tag{2.5.2}$$

because of $S \ll 1$ (which follows from (2.3.6)(2.3.13)), and because of the estimations in §3.4. On the other hand, by Prop 2.2.13(ii),

$$\mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}\setminus\{\chi_0\}}|g_-(\chi)|^2,\ \mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}\setminus\{\chi_0\}}|g'_-(\chi)|^2 \ll (N(\mathbf{f})X)^{2\epsilon''}X^{-\epsilon} \tag{2.5.3}$$

for any $\epsilon'' > 0$. Hence if we choose ϵ'' so small that $2\epsilon''(1+\beta) \leq \epsilon(\beta-1)$, which is possible since $\beta > 1$, we obtain

$$\mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}\setminus\{\chi_0\}}|g_-(\chi)|^2,\ \mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}\setminus\{\chi_0\}}|g'_-(\chi)|^2 \ll N(\mathbf{f})^{-\epsilon}. \tag{2.5.4}$$

Therefore, by the Schwarz inequality, (2.5.1) is $\ll N(\mathbf{f})^{-\epsilon/2}$. Therefore, together with (2.4.2) we obtain

$$\mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}\setminus\{\chi_0\}}(\overline{g(\chi)}g'(\chi)) - S_0 \ll N(\mathbf{f})^{-\epsilon/2}. \tag{2.5.5}$$

When $\mathbf{f}$ is a prime divisor, this average may be replaced by that over $\{\chi;\ \mathbf{f}_\chi = \mathbf{f}\}$.

Finally, in this case, it is clear that the sum for S_0, which is over all D with the condition $(D, \mathbf{f}) = 1$, and the sum over all D without this condition, differs only by a quantity $\ll N(\mathbf{f})^{\epsilon'-2\sigma} \ll N(\mathbf{f})^{-1}$. This completes the proof of Theorem 1.

3 Proof of Theorem 2

3.1 – Estimations of $\lambda_z(D)$. Let z run only over $|z| \le R$. We shall prove

$$\lambda_z(D) \ll_{R,\epsilon'} N(D)^{\epsilon'} \tag{3.1.1}$$

for any $\epsilon' > 0$, which will settle the first statement of Theorem 2. Since H_r, G_r are polynomials with positive coefficients and since $\delta_k(r) \le \binom{r-1}{k-1}$,

$$|H_r(iz/2)| \le H_r(|z|/2) \le G_r(|z|/2) \le G_r(|z|\log N(\mathfrak{p})); \tag{3.1.2}$$

hence

$$|\lambda_z(\mathfrak{p}^r)| \le G_r(|z|\log N(\mathfrak{p})) \le \exp(2\sqrt{r|z|\log N(\mathfrak{p})})$$

holds in both Cases 1,2, by [4] Sublemma 3.10.1 (and $\binom{r-1}{k-1} \le \binom{r}{k}$). Now since we may assume $\lambda_z(D) \ne 0$ in proving (3.1.1), we may take the log of $|\lambda_z(D)|$ for estimation. Denoting by $\mathrm{Supp}(D)$ the set of prime factors of D we obtain

$$\begin{aligned}\log|\lambda_z(D)| &\le 2\sqrt{|z|}\sum_{\mathfrak{p}|D}\sqrt{r_\mathfrak{p}\log N(\mathfrak{p})} \le 2\sqrt{|z|}\sqrt{|\mathrm{Supp}(D)|}\sqrt{\sum_{\mathfrak{p}|D} r_\mathfrak{p}\log N(\mathfrak{p})} \\ &\le 2\sqrt{R}\sqrt{|\mathrm{Supp}(D)|\log N(D)}.\end{aligned} \tag{3.1.3}$$

(The second inequality is by the Schwarz inequality.) On the other hand, by [4] Sublemma 3.10.5, we have

$$|\mathrm{Supp}(D)| \ll \frac{\log N(D)}{\log\log N(D)+2}. \tag{3.1.4}$$

Therefore, (3.1.3) gives

$$\log|\lambda_z(D)| \ll_R \frac{\log N(D)}{\sqrt{\log\log N(D)+2}}. \tag{3.1.5}$$

Therefore, for any $\epsilon' > 0$, $\log|\lambda_z(D)| \le \epsilon' \log N(D)$ if $N(D) \gg_{\epsilon',R} 1$. This proves (3.1.1).

3.2 – The function $g_{\lambda_z}(s,\chi,\mathbf{f})$. We shall now prove that

$$\exp\left(\frac{iz}{2}\mathcal{L}(s,\chi,\mathbf{f})\right) = \sum_{(D,\mathbf{f})=1} \chi(D)\lambda_z(D)N(D)^{-s} \tag{3.2.1}$$

holds on $\mathrm{Re}(s) > 1$ for any $\chi \in \hat{G}_\mathbf{f}\backslash\{\chi_0\}$. But on this domain, each has an absolutely convergent Euler product decomposition and the equality between their $\mathfrak{p}$-components is given by the equality (1.2.4) with $x = -\frac{iz}{2}\log N(\mathfrak{p})$, $t = \chi(\mathfrak{p})N(\mathfrak{p})^{-s}$ in Case 1, and by (1.2.5) with $x = \frac{iz}{2}$, $t = \chi(\mathfrak{p})N(\mathfrak{p})^{-s}$ in Case 2. Therefore,

$$g_{\lambda_z}(s, \chi, \mathbf{f}) = \exp(\frac{iz}{2}\mathcal{L}(s, \chi, \mathbf{f})). \tag{3.2.2}$$

It is holomorphic on $\mathrm{Re}(s) > 1/2$ in the FF case, and under GRH, also in the NF case. This settles (A2). Now we are going to prove (A3) in several steps.

3.3 – Reduction of A3 to Theorem-Est. The property (A3) will be proved as a Corollary of the following estimation Theorem.

Theorem-Est *Let $\chi \in \hat{G}_{\mathbf{f}} \backslash \{\chi_0\}$, and $s = \sigma + ti$, with $\sigma \geq 1/2 + \epsilon$ $(\epsilon > 0)$. Then*

$$\begin{aligned} \left|\frac{L'}{L}(s, \chi, \mathbf{f})\right| &\ll_\epsilon \frac{\ell(\mathbf{f})^{2-2\sigma} - 1}{1-\sigma} && (FF) \\ &\ll_\epsilon \frac{\ell(\mathbf{f})^{2-2\sigma} - 1}{1-\sigma}\left(\ell(t) + \frac{\ell(t)^2}{\ell(\mathbf{f})}\right) && (NF;\ under\ GRH). \end{aligned}$$

When $\sigma = 1$, $(\ell(\mathbf{f})^{2-2\sigma} - 1)/(1-\sigma)$ should be replaced by its limit at $\sigma = 1$; namely by $2\log\ell(\mathbf{f})$.

Corollary-Est *Let $\mathcal{L}(s, \chi, \mathbf{f})$ be either $L'/L(s, \chi, \mathbf{f})$ or $\log L(s, \chi, \mathbf{f})$, and for any $0 < \epsilon < 1/2$, let $\sigma \geq 1/2 + \epsilon$. Then*

$$\begin{aligned} |\mathcal{L}(s, \chi, \mathbf{f})| &\ll_\epsilon \ell(\mathbf{f})^{1-2\epsilon} && (FF) \\ &\ll_\epsilon \ell(\mathbf{f})^{1-2\epsilon}\left(\ell(t) + \frac{\ell(t)^2}{\ell(\mathbf{f})}\right) && (NF;\ under\ GRH). \end{aligned}$$

Reduction of Corollary-Est to Theorem-Est.

(Case 1) For each $y > 1$,

$$\frac{y^{1-\sigma} - 1}{1-\sigma} = \int_1^y u^{-\sigma} du \tag{3.3.1}$$

is monotone decreasing with σ. Therefore,

$$\frac{\ell(\mathbf{f})^{2-2\sigma} - 1}{1-\sigma} \leq \frac{\ell(\mathbf{f})^{1-2\epsilon} - 1}{1-(1/2+\epsilon)}. \tag{3.3.2}$$

But since the right hand side of (3.3.2) is $\ll_\epsilon \ell(\mathbf{f})^{1-2\epsilon}$, the Corollary for Case 1 follows immediately from Theorem-Est.

(Case 2) Put $\sigma_0 := \mathrm{Max}(\sigma, 2)$. Then

$$\log L(s,\chi,\mathbf{f}) = \int_{\sigma_0+ti}^{\sigma+ti} \frac{L'}{L}(s,\chi,\mathbf{f})\,ds + \log L(\sigma_0+ti,\chi,\mathbf{f}). \tag{3.3.3}$$

Since $|\log L(\sigma_0+ti,\chi,\mathbf{f})| \le |\log\zeta_K(2)| \ll_K 1$, where $\zeta_K(s)$ denotes the Dedekind zeta function of K, and since $|\sigma-\sigma_0| < 2-1/2 \ll 1$, the Corollary for Case 2 follows immediately from that for Case 1 by estimation of the integrand. □

From the Corollary follows directly that the present family $\{g_{\lambda_z}(s,\chi,\mathbf{f})\}_{|z|\le R}$ satisfies (A3). Thus, Theorem 2 is reduced to Theorem-Est.

3.4 – Reduction of Theorem-Est to a Key Lemma. As usual, for any integral divisor D of K, let $\Lambda(D) = \log N(\mathfrak{p})$ when $D = \mathfrak{p}^r$ for some prime divisor $\mathfrak{p}$ and $r \ge 1$, and $\Lambda(D) = 0$ otherwise. For $y > 1$ and $\chi \in \hat{G}_{\mathbf{f}}$, put

$$\psi(s,\chi,\mathbf{f};<y) = \sum_{N(D)<y} \chi(D)\Lambda(D)N(D)^{-s}, \tag{3.4.1}$$

$$\psi(s,\chi,\mathbf{f};y) = \psi(s,\chi,\mathbf{f};<y) + \frac{1}{2}\sum_{N(D)=y} \chi(D)\Lambda(D)N(D)^{-s}. \tag{3.4.2}$$

When $\mathbf{f} = \mathbf{f}_\chi$ (the conductor of χ), we shall suppress $\mathbf{f}$ from these notations and write as $\psi(s,\chi;<y)$, $\psi(s,\chi;y)$. We shall assume hereafter that y is *separated from* 1, i.e., $1-y^{-1} \gg 1$. Then we have

$$\frac{y^{1-\sigma}-1}{1-\sigma} = \int_1^y u^{-\sigma}du \ge (y-1)y^{-\sigma} \gg y^{1-\sigma}, \tag{3.4.3}$$

and also an elementary unconditional estimation (cf. [4](6.4.9)):

$$|\psi(s,\chi,\mathbf{f};y)| \le \sum_{N(\mathfrak{p})\le y} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^\sigma - 1} \ll \int_1^y u^{-\sigma}du + y^{1-\sigma} \ll \frac{y^{1-\sigma}-1}{1-\sigma}. \tag{3.4.4}$$

We shall reduce the proof of Theorem-Est to the following

Key Lemma *For $s = \sigma+ti$ with $\sigma \ge 1/2+\epsilon$ and for $y > 1$ separated from 1,*

$$\left|\frac{L'}{L}(s,\chi) + \psi(s,\chi;y)\right| \ll_\epsilon y^{1/2-\sigma}\ell(\mathbf{f}) \qquad \text{(FF)}$$

$$\ll_\epsilon y^{1/2-\sigma}(\ell(\mathbf{f})\ell(t)+\ell(t)^2) + y^{1-\sigma} \qquad \text{(NF; under GRH)}$$

(The term $y^{1-\sigma}$ can be replaced by a one which tends to 0 as $y \to \infty$ whenever $\sigma > 1/2$; but this is more complicated and not necessary for the present purpose.)

Now this reduction can be done by using the following intermediate objects and the decomposition, for a suitable choice of y.

$$\left|\frac{L'}{L}(s,\chi,\mathbf{f})\right| \leq I + II + III + IV, \tag{3.4.5}$$

where

$$I = \left|\frac{L'}{L}(s,\chi,\mathbf{f}) - \frac{L'}{L}(s,\chi)\right|, \qquad II = \left|\frac{L'}{L}(s,\chi) + \psi(s,\chi;y)\right| \tag{3.4.6}$$

$$III = |\psi(s,\chi,\mathbf{f};y) - \psi(s,\chi;y)|, \qquad IV = |\psi(s,\chi,\mathbf{f};y)|. \tag{3.4.7}$$

First, by (3.4.4), we have

$$IV \ll \frac{y^{1-\sigma}-1}{1-\sigma}. \tag{3.4.8}$$

Secondly, I and III are also minor terms obviously bounded by

$$\sum_{\mathfrak{p}|\mathbf{f}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{\sigma}-1} = I_1 + I_2, \tag{3.4.9}$$

where I_1 (resp. I_2) are the partial sums over $N(\mathfrak{p}) \leq \ell(\mathbf{f})^2$ (resp. $N(\mathfrak{p}) > \ell(\mathbf{f})^2$). By (3.4.4) we have

$$I_1 \ll \frac{\ell(\mathbf{f})^{2-2\sigma}-1}{1-\sigma}. \tag{3.4.10}$$

As for I_2, since $(\log y)/(y^{\sigma}-1)$ is monotone decreasing for $y > 1$, and since $\sum_{\mathfrak{p}|\mathbf{f}} 1 \ll \ell(\mathbf{f})/\log \ell(\mathbf{f})$ by [4]Sublemma 3.10.5, we have

$$I_2 \leq \frac{2\log \ell(\mathbf{f})}{\ell(\mathbf{f})^{2\sigma}-1} \sum_{\mathfrak{p}|\mathbf{f}} 1 \ll \ell(\mathbf{f})^{1-2\sigma} \ll \ell(\mathbf{f})^{2-2\sigma}. \tag{3.4.11}$$

Therefore,

$$I,\ III \ll \frac{\ell(\mathbf{f})^{2-2\sigma}-1}{1-\sigma}. \tag{3.4.12}$$

Now put $y = \ell(\mathbf{f})^2$ ($\geq (\log 3)^2 > 1$). Then (3.4.8) and (3.4.12) give

$$I + III + IV \ll \frac{\ell(\mathbf{f})^{2-2\sigma}-1}{1-\sigma}; \tag{3.4.13}$$

while the Key lemma gives

$$
\begin{aligned}
(3.4.14) \qquad II &\ll_\epsilon \ell(\mathbf{f})^{2-2\sigma} && \text{(FF)}\\
&\ll_\epsilon \ell(\mathbf{f})^{2-2\sigma}(\ell(t)+\ell(t)^2/\ell(\mathbf{f}))+\ell(\mathbf{f})^{2-2\sigma} && \text{(NF; under GRH)}
\end{aligned}
$$

hence by combining these with (3.4.3) we obtain Theorem-Est. Thus, Theorem-Est is reduced to the Key Lemma.

The Key Lemma in the FF case is proved in [4](6.8.4). To prove this in the NF case, we shall make use of the following "explicit formula".

3.5 – An explicit formula. Let K be any number field, let P_∞ consist only of the archimedean primes of K, and let χ be a *primitive* Dirichlet character on K, so that $L(s,\chi,\mathbf{f})$ is the usual L-function $L(s,\chi)$. Put $\delta_\chi = 1$ (resp. 0) for $\chi = \chi_0$ (resp. $\chi \neq \chi_0$).

Theorem-Exp *Let $\sigma = \mathrm{Re}(s) > 1/2$ and $y > 1$. Then:*

$$
(3.5.1) \qquad \frac{L'}{L}(s,\chi)+\psi(s,\chi;y) = -\delta_\chi\left(\frac{y^{-s}}{s}+\frac{y^{1-s}}{s-1}\right)+{\sum_\rho}' \frac{y^{\rho-s}}{s-\rho}+\ell(s,\mathrm{sign}(\chi);y),
$$

where ρ runs over all non-trivial zeros of $L(s,\chi)$, $\sum_\rho' = \lim_{T\to\infty}\sum_{|\rho|\le T}$, and

$$
\begin{aligned}
(3.5.2) \qquad \ell(s,\mathrm{sign}(\chi);y) &= \sum_{trivial\ zeros} \frac{y^{\mu-s}}{s-\mu}\\
&= (a+r_2)\sum_{i\ge 0,\, even}\frac{y^{-i-s}}{s+i} + (a'+r_2)\sum_{i\ge 1,\, odd}\frac{y^{-i-s}}{s+i}.
\end{aligned}
$$

Here, a (resp. a') denotes the number of real places of K at which χ is unramified (resp. ramified), and r_2 is the number of complex places of K.

For the proof see §3.7.

3.6 – Reduction of the Key Lemma (NF case) to Theorem-Exp.

To avoid inessential complications we shall restrict our attention to the case where P_∞ consists only of the archimedean primes. (The difference arising in the general case can be estimated easily as in the estimations of I, III in §3.4.) Recall Theorem-Exp for $\chi \neq \chi_0$, which reads as

$$\frac{L'}{L}(s,\chi) + \psi(s,\chi;y) = \sum_{\rho}{}' \frac{y^{\rho-s}}{s-\rho} + \ell(s, \mathrm{sign}(\chi); y). \tag{3.6.1}$$

As before, let $\sigma_0 = \mathrm{Max}(\sigma, 2)$. Then this gives directly

$$\frac{L'}{L}(s,\chi) + \psi(s,\chi;y) = P + Q + R, \tag{3.6.2}$$

where

$$P = y^{\sigma_0 - s}\left(\frac{L'}{L}(\sigma_0,\chi) + \psi(\sigma_0,\chi;y)\right), \tag{3.6.3}$$

$$Q = \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{\sigma_0-\rho}\right) y^{\rho-s}, \tag{3.6.4}$$

$$R = \ell(s, \mathrm{sign}(\chi); y) - y^{\sigma_0 - s}\ell(\sigma_0, \mathrm{sign}(\chi); y). \tag{3.6.5}$$

The sum over ρ in Q is, unlike that in the above explicit formula itself, absolutely convergent.

(Estimation of P) Since $\sigma_0 \geq 2 > 1$, $L'/L(\sigma_0, \chi)$ has an absolutely convergent Dirichlet series expansion

$$\frac{L'}{L}(\sigma_0,\chi) = -\sum_{D} \Lambda(D)\chi(D)N(D)^{-\sigma_0}; \tag{3.6.6}$$

hence

$$\left|\frac{L'}{L}(\sigma_0,\chi) + \psi(\sigma_0,\chi;y)\right| \leq \sum_{N(D)\geq y} \Lambda(D)N(D)^{-\sigma_0} \ll \frac{y^{1-\sigma_0}}{\sigma_0 - 1} \ll y^{1-\sigma_0}.$$

(By partial summation, using $\sum_{N(D)\leq x} \Lambda(D) \ll x$.) Hence

$$P \ll y^{1-\sigma}. \tag{3.6.7}$$

(Estimation of R) It is easy to see that $\ell(s, \mathrm{sign}(\chi); y) \ll y^{-\sigma}$; hence

$$R \ll y^{-\sigma}. \tag{3.6.8}$$

(Estimation of Q, under GRH) By definition, and by GRH,

$$|Q| \le y^{1/2-\sigma} \sum_{\rho} \frac{|\sigma_0 - s|}{|(s-\rho)(\sigma_0-\rho)|}. \tag{3.6.9}$$

Write $\rho = 1/2 + i\gamma$. The only property on the distribution of γ on the real axis that we are going to use is the standard estimation, cf. e.g. [11];

$$n_\chi(x) := \#\{\rho;\ |\gamma - x| \le 1\} \ll \log d_\chi + [K:\mathbf{Q}]\log(|x|+2) \tag{3.6.10}$$

for any $x \in \mathbf{R}$, where $d_\chi = |d_K|N(\mathbf{f}_\chi)$ (d_K: the discriminant of K). Thus, in our notations,

$$n_\chi(x) \ll \ell(\mathbf{f}) + \ell(x). \tag{3.6.11}$$

Now since

$$\begin{aligned} |\sigma_0 - s| &\le |t| + |\sigma_0 - \sigma| < |t| + 3/2 < |t| + 2, \\ \sqrt{2}|s-\rho| &\ge (\sigma - 1/2) + |t-\gamma| \ge \epsilon + |t-\gamma| \gg_\epsilon 2 + |t-\gamma|, \\ \sqrt{2}|\sigma_0 - \rho| &\ge (\sigma_0 - 1/2) + |\gamma| \ge 3/2 + |\gamma| \gg 2 + |\gamma|, \end{aligned}$$

we have, by (3.6.9),

$$\begin{aligned} |Q| &\ll y^{1/2-\sigma}(|t|+2)\sum_{\rho}\frac{1}{(|\gamma|+2)(|t-\gamma|+2)} && (3.6.12) \\ &\ll y^{1/2-\sigma}(|t|+2)(\ell(\mathbf{f})B_1(t) + B_2(t)), && (3.6.13) \end{aligned}$$

where

$$B_1(t) = \int_{-\infty}^{\infty} \frac{1}{(|x|+2)(|t-x|+2)}dx, \quad B_2(t) = \int_{-\infty}^{\infty} \frac{\log(|x|+2)}{(|x|+2)(|t-x|+2)}dx. \tag{3.6.14}$$

It is easy to see that

$$B_1(t) \ll \frac{\log(|t|+2)}{|t|+2}, \quad B_2(t) \ll \frac{\log^2(|t|+2)}{|t|+2}. \tag{3.6.15}$$

Therefore,

$$|Q| \ll y^{1/2-\sigma}(\ell(\mathbf{f})\ell(t) + \ell(t)^2). \tag{3.6.16}$$

Therefore, (3.6.7)(3.6.8)(3.6.16) combined give

$$\left|\frac{L'}{L}(s,\chi) + \psi(s,\chi;y)\right| \ll_\epsilon y^{1-\sigma} + y^{-\sigma} + y^{1/2-\sigma}(\ell(\mathbf{f})\ell(t) + \ell(t)^2). \tag{3.6.17}$$

But since $y^{-\sigma} \ll y^{1-\sigma}$, this settles the proof of the Key Lemma assuming Theorem-Exp.

3.7 – Proof of Theorem-Exp. First, Weil's explicit formula [17], applied to the function $F(x)$ on $\mathbf{R}$ defined by $F(x) = e^{(1/2-s)x}$ $(0 < x < \log y)$, $F(x) = 0$ $(x < 0$ or $x > \log y)$, $F(x) = (F(x+0) + F(x-0))/2$ (everywhere), gives directly:

$$\begin{aligned} \psi(s,\chi;y) \quad &+ \quad \delta_\chi \left(\frac{y^{-s}-1}{s} + \frac{y^{1-s}-1}{s-1} \right) \\ &= \quad \sum_\rho{}' \frac{y^{\rho-s}-1}{s-\rho} + \frac{1}{2}(\log d_\chi - N\log\pi) \\ &+ \quad \frac{a+r_2}{2} G(\frac{s}{2}) + \frac{a'+r_2}{2} G(\frac{s+1}{2}) + \ell(s, \mathrm{sign}(\chi); y), \end{aligned} \tag{3.7.1}$$

where $d_\chi = |d_K| N(\mathbf{f}_\chi)$, $N = [K : \mathbf{Q}]$ and $G(s) = \Gamma'(s)/\Gamma(s)$. (Note: $\pm \frac{N}{2}\log 2$ appears from two different terms in the Weil formula, cancelling each other.)

On the other hand, the partial fractional decomposition of $L'/L(s,\chi)$ gives

$$\begin{aligned} \frac{L'}{L}(s,\chi) + \delta_\chi \left(\frac{1}{s} + \frac{1}{s-1} \right) \quad &= \quad \sum_\rho{}' \frac{1}{s-\rho} - \frac{1}{2}(\log d_\chi - N\log\pi) \\ &- \quad \frac{a+r_2}{2} G(\frac{s}{2}) - \frac{a'+r_2}{2} G(\frac{s+1}{2}). \end{aligned} \tag{3.7.2}$$

Here, the key formula is in [11] (the formula (5.9)), but in addition, we need the (conditional) convergence of the sums $\sum_\rho{}'(1-\rho)^{-1}$ and $\sum_\rho{}' \rho^{-1}$ (cf. [8]§2), and the formula in Theorem 2 of *loc. cit.*, which asserts that the limit of the left hand side of (3.7.2) as $s \to 1$ is equal to

$$\sum_\rho{}' \frac{1}{1-\rho} - \frac{1}{2}\log d_\chi + \frac{a+r_2}{2}(\gamma + \log 4\pi) + \frac{a'+r_2}{2}(\gamma + \log\pi), \tag{3.7.3}$$

where γ denotes the usual Euler constant. By summing up (3.7.1)(3.7.2) we obtain the desired explicit formula.

This completes the proof of Theorem 2.

4 The analytic function $\tilde{M}_s(z_1, z_2)$.

4.1 – Let K and P_∞ be as at the beginning of §1.1. We shall exhibit some basic properties of the complex analytic function

$$\tilde{M}_s(z_1, z_2) = \sum_{D:\, integral} \lambda_{z_1}(D)\lambda_{z_2}(D)N(D)^{-2s} \tag{4.1.1}$$

of s, z_1, z_2 ($\mathrm{Re}(s) > 1/2$) defined in §1.3. This will supplement some results given in [4]§3.7 (Case 1) and [7] (Case 2). Its analytic property on $\mathrm{Re}(s) > 0$ will be discussed in a future article. First, note that $\lambda_z(D)$ (as well as $\chi(D)$) is multiplicative in D and hence it has an Euler product expansion, at least formally.

Theorem $\tilde{M}$ (i) *Let* $\mathfrak{p}$ *be any prime divisor of* K *not contained in* P_∞*, and* $\mathrm{Re}(s) > 0$. *Define a continuous function* $g_{s,\mathfrak{p}}(t)$ *on* $\mathbf{C}^1 = \{t \in \mathbf{C}; |t| = 1\}$ *by*

$$\begin{aligned} g_{s,\mathfrak{p}}(t) &= \frac{-(\log N(\mathfrak{p}))N(\mathfrak{p})^{-s}t}{1 - N(\mathfrak{p})^{-s}t} \qquad \text{(Case 1)} \\ &= -\log(1 - N(\mathfrak{p})^{-s}t) \qquad \text{(Case 2)} \end{aligned} \tag{4.1.2}$$

(*the principal branch of the logarithm*), *and put*

$$\tilde{M}_{s,\mathfrak{p}}(z_1, z_2) = \int_{\mathbf{C}^1} \exp\left(\frac{i}{2}(z_1 g_{s,\mathfrak{p}}(t^{-1}) + z_2 g_{s,\mathfrak{p}}(t))\right) d^\times t, \tag{4.1.3}$$

where $d^\times t$ *denotes the normalized Haar measure on* $\mathbf{C}^1$. *Then with the notations of* §*1.2,*

$$\begin{aligned} \tilde{M}_{s,\mathfrak{p}}(z_1, z_2) &= \sum_{r=0}^{\infty} \lambda_{z_1}(\mathfrak{p}^r)\lambda_{z_2}(\mathfrak{p}^r)N(\mathfrak{p})^{-2rs} \\ &= 1 + \sum_{a,b \geq 1} (\pm i/2)^{a+b} \mu_{s,\mathfrak{p}}^{(a,b)} \frac{z_1^a z_2^b}{a!b!}, \end{aligned} \tag{4.1.4}$$

where the sign is minus (*resp. plus*) *for Case 1* (*resp. Case 2*), *and*

$$\begin{aligned} \mu_{s,\mathfrak{p}}^{(a,b)} &= (\log N(\mathfrak{p}))^{a+b} \sum_{r \geq \mathrm{Max}(a,b)} \binom{r-1}{a-1}\binom{r-1}{b-1} N(\mathfrak{p})^{-2rs} \qquad \text{(Case 1)} \\ &= \sum_{r \geq \mathrm{Max}(a,b)} \delta_a(r)\delta_b(r)N(\mathfrak{p})^{-2rs} \qquad \text{(Case 2).} \end{aligned} \tag{4.1.5}$$

(ii) $\tilde{M}_s(z_1, z_2)$ *has an absolutely convergent Euler product expansion on* $\mathrm{Re}(s) > 1/2$;

$$\tilde{M}_s(z_1, z_2) = \prod_{\mathfrak{p} \notin P_\infty} \tilde{M}_{s,\mathfrak{p}}(z_1, z_2). \tag{4.1.6}$$

This convergence is uniform on $\mathrm{Re}(s) \geq 1/2+\epsilon$, $|z_1|$, $|z_2| \leq R$, *for each fixed* $\epsilon, R > 0$.

(iii) $\tilde{M}_s(z_1, z_2)$ *for each* s *with* $\mathrm{Re}(s) > 1/2$ *has an everywhere absolutely convergent power series expansion*

$$\tilde{M}_s(z_1, z_2) = 1 + \sum_{a,b\geq 1} (\pm i/2)^{a+b} \mu_s^{(a,b)} \frac{z_1^a z_2^b}{a!b!}, \tag{4.1.7}$$

with the same choice of the sign as above. Here, $\mu_s^{(a,b)}$ *denotes the following Dirichlet series which is absolutely convergent on* $\mathrm{Re}(s) > 1/2$;

$$\mu_s^{(a,b)} = \sum_D \Lambda_a(D)\Lambda_b(D)N(D)^{-2s}, \tag{4.1.8}$$

where each $\Lambda_k(D)$ *is a non-negative real number determined from the polynomial coefficients of* $\lambda_z(D)$ *by the formula*

$$\lambda_z(D) = \sum_{k=0}^{\infty} \frac{\Lambda_k(D)}{k!} (\pm iz/2)^k \tag{4.1.9}$$

(*the same choice of the sign*).

(iv) *As before, put*

$$\psi_{z_1,z_2}(w) = \exp\left(\frac{i}{2}(z_1\overline{w} + z_2 w)\right) \tag{4.1.10}$$

($z_1, z_2, w \in \mathbf{C}$), *and for* $\sigma > 1/2$, *let* $M_\sigma(z)$ *denote the "M-function" defined and studied in* [4](Case 1) [7] (Case 2). (*In the latter, it is denoted as* $\mathcal{M}_\sigma(z)$.) *Then*

$$\tilde{M}_\sigma(z_1, z_2) = \int_{\mathbf{C}} M_\sigma(w)\psi_{z_1,z_2}(w)|dw|. \tag{4.1.11}$$

In particular, $\tilde{M}_\sigma(z, \overline{z})$ *is the Fourier dual* $\tilde{M}_\sigma(z)$ *of* $M_\sigma(z)$.

Proof (i) Recall the definition (4.1.2) of $g_{s,\mathfrak{p}}(t)$, and note that $|N(\mathfrak{p})^{-s}t| < 1$. For each $k \in \mathbf{N}$, the power series expansion of the k-th power $g_{s,\mathfrak{p}}(t)^k$ in $N(\mathfrak{p})^{-s}t$ reads as

$$g_{s,\mathfrak{p}}(t)^k = \sum_{r=1}^{\infty} a_r^{(k)} (N(\mathfrak{p})^{-s}t)^r, \tag{4.1.12}$$

where $a_r^{(k)} = 0$ for $r < k$, and for $r \geq k$,

$$\begin{aligned} a_r^{(k)} &= (-\log N(\mathfrak{p}))^k \binom{r-1}{k-1} && \text{(Case 1)} \\ &= \delta_k(r) && \text{(Case 2)} \end{aligned} \tag{4.1.13}$$

(cf. §1.2). Hence by the definition of $\lambda_z(\mathfrak{p}^r)$,

$$\sum_{k=1}^{\infty} \frac{a_r^{(k)}}{k!}(iz/2)^k = \sum_{k=1}^{r} \frac{a_r^{(k)}}{k!}(iz/2)^k = \lambda_z(\mathfrak{p}^r) \tag{4.1.14}$$

for any $r \geq 1$ and $z \in \mathbf{C}$. Therefore,

$$\begin{aligned} \exp(\frac{iz}{2} g_{s,\mathfrak{p}}(t)) &= 1 + \sum_{r,\,k\geq 1} \frac{1}{k!}(iz/2)^k a_r^{(k)} (N(\mathfrak{p})^{-s}t)^r \\ &= \sum_{r=0}^{\infty} \lambda_z(\mathfrak{p}^r)(N(\mathfrak{p})^{-s}t)^r. \end{aligned} \tag{4.1.15}$$

But since $\tilde{M}_{s,\mathfrak{p}}(z_1, z_2)$ is nothing but the constant term in the Fourier expansion of

$$\exp\left(\frac{i}{2}(z_1 g_{s,\mathfrak{p}}(t^{-1}) + z_2 g_{s,\mathfrak{p}}(t))\right) \tag{4.1.16}$$

in t^n ($n \in \mathbf{Z}$), (4.1.4) follows directly.

(ii) Fix any $\epsilon, R > 0$, and let s, z_1, z_2 run over $\mathrm{Re}(s) \geq 1/2 + \epsilon$, $|z_1|, |z_2| \leq R$. It is obvious from the absolute convergence of the Dirichlet series (4.1.1) that the product $\prod_{N(\mathfrak{p})\leq y} \tilde{M}_{s,\mathfrak{p}}(z_1, z_2)$ converges to $\tilde{M}_s(z_1, z_2)$ as $y \to \infty$ uniformly. Our assertion, the absolute convergence of the infinite product, requires also that the infinite sum

$$\sum_{\mathfrak{p}\notin P_\infty} |\tilde{M}_{s,\mathfrak{p}}(z_1, z_2) - 1| \tag{4.1.17}$$

converges uniformly. But by (4.1.4) and Theorem 2 (i),

$$\begin{aligned} |\tilde{M}_{s,\mathfrak{p}}(z_1, z_2) - 1| &\leq \sum_{r=1}^{\infty} |\lambda_{z_1}(\mathfrak{p}^r)\lambda_{z_2}(\mathfrak{p}^r)| N(\mathfrak{p})^{-2r\sigma} \\ \ll_{\epsilon', R} \sum_{r=1}^{\infty} N(\mathfrak{p})^{(2\epsilon'-2\sigma)r} &\leq \sum_{r=1}^{\infty} N(\mathfrak{p})^{(-1-\epsilon)r} < 2N(\mathfrak{p})^{-1-\epsilon} \end{aligned} \tag{4.1.18}$$

(take $\epsilon' = \epsilon/2$); hence this is clear.

(iii) First, a few preliminary remarks on $\Lambda_k(D)$. When $k = 0$, we have $\Lambda_0(D) = 0$ (resp. 1) for $D \neq (1)$ (resp. $D = (1)$). When $k = 1$, $\Lambda_1(D) = 0$ unless $D = \mathfrak{p}^r$ with some $\mathfrak{p} \notin P_\infty$ and $r \geq 1$, and in this case,

$$\Lambda_1(D) = \begin{cases} \log N(\mathfrak{p}) & (\text{Case 1}), \\ 1/r & (\text{Case 2}). \end{cases} \tag{4.1.19}$$

For $k \geq 1$, Λ_k can also be expressed as the k-th iteration of Λ_1;

$$\Lambda_k(D) = \sum_{D=D_1...D_k} \Lambda_1(D_1)...\Lambda_1(D_k). \tag{4.1.20}$$

(In Case 1, this is shown in [4]. In Case 2, the proof runs as follows. Let $D = \prod_\nu \mathfrak{p}_\nu^{r_\nu}$ be the prime factorization of D, and t_ν be independent variables. Then since $\lambda_z(D)$ is multiplicative, it is equal to the coefficient of $\prod_\nu t_\nu^{r_\nu}$ in

$$\begin{aligned} \prod_\nu (\sum_{r=0}^{\infty} \lambda_z(\mathfrak{p}_\nu^r) t_\nu^r) &= \prod_\nu (\sum_{r=0}^{\infty} H_r(iz/2) t_\nu^r) \\ &= \exp(-(iz/2) \sum_\nu \log(1-t_\nu)) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (iz/2)^k (-\sum_\nu \log(1-t_\nu))^k. \end{aligned} \tag{4.1.21}$$

But since the coefficient of $\prod_\nu t_\nu^{r_\nu}$ in $(-\sum_\nu \log(1-t_\nu))^k$ is nothing but the right hand side of (4.1.20), we obtain the equality (4.1.20).)

In particular, $\mu_s^{(0,0)} = 1$, $\mu_s^{(a,0)} = \mu_s^{(0,b)} = 0$ for $ab \neq 0$. Since the Dirichlet coefficients in (4.1.8) for $a, b \geq 1$ are non-negative for all D and positive for, say, $D = \mathfrak{p}^{a+b}$, we have

$$\mu_\sigma^{(a,b)} > 0 \qquad (a, b \geq 1). \tag{4.1.22}$$

Now let us prove (iii). In Case 1, this is proved in [4](Theorem 5 in §3.7). The proof for Case 2 is almost parallel, but let us sketch this proof. Fix s with $\mathrm{Re}(s) > 1/2$, and put $w_j = iz_j/2$ $(j = 1, 2)$. Since the Dirichlet series (4.1.1) converges uniformly on $|w_1|, |w_2| \leq 1$, we obtain, first by termwise differentiation, then by putting $w_1 = w_2 = 0$, and then by (4.1.9);

$$\begin{aligned} \left(\frac{\partial^{a+b} \tilde{M}_s(z_1, z_2)}{\partial w_1^a \partial w_2^b} \right)_{(0,0)} &= \sum_D \left(\frac{\partial^a \lambda_{z_1}(D)}{\partial w_1^a} \right)_0 \left(\frac{\partial^b \lambda_{z_2}(D)}{\partial w_2^b} \right)_0 N(D)^{-2s} \\ &= \sum_D \Lambda_a(D) \Lambda_b(D) N(D)^{-2s} = \mu_s^{(a,b)}. \end{aligned} \tag{4.1.23}$$

Since $\tilde{M}_s(z_1, z_2)$ is entire, the power series (4.1.7) must converge (absolutely) everywhere.

(iv) We shall use the rapidly decreasing property $M_\sigma(w) = O(e^{-\lambda|w|^2})$ for any $\lambda > 0$, to be proved later (Lemma B §5). The integral on the right-hand side of (4.1.11) is the limit of that over $|w| \leq R$ which is holomorphic in z_1, z_2, and the convergence is uniform in a wider sense with respect to z_1, z_2. Therefore, each side of (4.1.11) is a holomorphic function of z_1, z_2. Since they are equal when $z_2 = \overline{z}_1$, as is proved in [4](Case 1) [7](Case 2), they must be equal for any $z_1, z_2 \in \mathbf{C}$.

This completes the proof of Theorem $\tilde{M}$.

4.2 – Some remarks. (I) From (4.1.7) and (4.1.11), we obtain, by partial derivation,

$$\mu_\sigma^{(a,b)} = (\pm 1)^{a+b} \int M_\sigma(w) P^{(a,b)}(w)|dw|, \tag{4.2.1}$$

where $\pm 1 = -1$ (resp. 1) for Case 1 (resp. Case 2) throughout this subsection, and $P^{(a,b)}(w) = \bar{w}^a w^b$ $(a, b \geq 0)$. Thus, by Theorem 4 (to be proved later) applied to $\Phi = P^{(a,b)}$, we also obtain

$$\mu_\sigma^{(a,b)} = (\pm 1)^{a+b} \lim_{\substack{\mathbf{f}\, prime \\ N(\mathbf{f})\to\infty}} \mathrm{Avg}_{\mathbf{f}_\chi = \mathbf{f}} P^{(a,b)}(\mathcal{L}(s,\chi)), \tag{4.2.2}$$

under the same assumption as in Theorem 4. In Case 1, this equality for $s = 1$ is proved unconditionally over $K = \mathbf{Q}$ [8], and over function fields, for any s with $\sigma > 1/2$ [4]Theorem 7(iii) [1].

(II) When $iz_1/2 = iz_2/2 = y \in \mathbf{R}$, so that $\psi_{z_1,z_2}(w) = \exp(2y\mathrm{Re}(w))$, (4.1.7) gives

$$\left(\tilde{M}_\sigma(2y/i, 2y/i) - \tilde{M}_\sigma(-2y/i, -2y/i)\right)/4y = \frac{(\pm 1)}{2} \sum_{k\geq 3,\, odd} \big(\sum_{\substack{k=a+b \\ a,b\geq 1}} \frac{\mu_\sigma^{(a,b)}}{a!b!} \big) y^{k-1}. \tag{4.2.3}$$

Note that when $y \neq 0$, this is non-zero and has the same sign as (± 1). Under the assumption of Theorem 3, this is the limit of

$$h_s(\mathbf{f}, y) = \mathrm{Avg}_{\mathbf{f}_\chi = \mathbf{f}} (e^{2y\mathrm{Re}\mathcal{L}(s,\chi)} - e^{-2y\mathrm{Re}\mathcal{L}(s,\chi)})/4y \tag{4.2.4}$$

as $N(\mathbf{f}) \to \infty$; hence for any *fixed* s and $y \neq 0$, the inequalities

$$h_s(\mathbf{f}, y) \;<\; 0 \qquad \text{(Case 1)}, \tag{4.2.5}$$

$$\qquad\qquad >\; 0 \qquad \text{(Case 2)}, \tag{4.2.6}$$

hold as long as $N(\mathbf{f})$ is *sufficiently large*. On the other hand, we have

$$h_s(\mathbf{f}, 0) = \mathrm{Avg}_{\mathbf{f}_\chi = \mathbf{f}} \mathrm{Re}(\mathcal{L}(s,\chi)), \tag{4.2.7}$$

and since (4.2.3) is $= 0$ for $y = 0$, this must tend to 0 as $N(\mathbf{f}) \to \infty$.

The sign of $h_s(\mathbf{f}, 0)$ for each *individual* $\mathbf{f}$ offers a more delicate problem. For example, let $K = \mathbf{Q}$ and $s = 1$. Then for any odd prime f, $(f-2)h_1((f), 0)$ is equal to

$$\sum_{f_\chi = f} \mathrm{Re}(L'(1,\chi)/L(1,\chi)) \;=\; \gamma_f - \gamma \qquad \text{(Case 1)}, \tag{4.2.8}$$

$$\sum_{f_\chi = f} \log|L(1,\chi)| \;=\; \log \kappa_f \qquad \text{(Case 2)}. \tag{4.2.9}$$

[1] Rigorously speaking. in [4], $\deg \mathfrak{p}_\infty = 1$ is assumed which is inessential; cf. also [6].

Here, κ_f denotes the residue of the Dedekind zeta function $\zeta_{K_f}(s)$ of the cyclotomic field $K_f = \mathbf{Q}(\mu_f)$ at $s = 1$, γ_f denotes the quotient of the constant term of its Laurent expansion at $s = 1$ divided by κ_f (the "Euler-Kronecker constant (invariant)" in the sense of [2]), and $\gamma = \gamma_1$, the usual Euler constant. The first named author considers it very likely that, in contrast to the above inequalities (4.2.5)(4.2.6),

$$\begin{aligned} &(4.2.10) && h_1((f),0) > 0 && \text{(Case 1)} \\ &(4.2.11) && \qquad\qquad < 0 && \text{(Case 2)} \end{aligned}$$

both hold [2]. Among these, the first inequality is essentially a part of the conjectures on the behaviour of γ_f raised in [3]. The second, which is equivalent with $\kappa_f < 1$, maybe new even as a conjecture. A more basic question is whether $\zeta_{K_f}(\sigma)/\zeta(\sigma) = 1 - 2^{-\sigma} - ...$ is everywhere monotone *increasing* on $\sigma > 1 - \epsilon$, as some numerical evidences suggest. In fact, both are immediate consequences of this hypothesis.

(III) In Case 2, by the second formula for $H_r(x)$ in (1.2.7), and by (4.1.4), the local factor $\tilde{M}_{s,\mathfrak{p}}(z_1, z_2)$ is nothing but the Gauss hypergeometric function

$$F(a, b, c; t) = 1 + \frac{a.b}{1.c}t + \frac{a(a+1).b(b+1)}{1.2.c(c+1)}t^2 + \cdots,$$

for

$$a = iz_1/2,\ b = iz_2/2,\ c = 1;\ \ t = N(\mathfrak{p})^{-2s}.$$

In particular, when $a = b = y = \pm 1$, they are

$$(1 - N(\mathfrak{p})^{-2s})^{-1},\ 1 + N(\mathfrak{p})^{-2s},$$

respectively; hence

$$(4.2.12) \qquad \tilde{M}_s(2/i, 2/i) = \sum_{D\, integral} N(D)^{-2s}$$

$$(4.2.13) \qquad \tilde{M}_s(-2/i, -2/i) = \sum_{\substack{D\, integral \\ squarefree}} N(D)^{-2s}.$$

Finally, as for the limit formula

$$(4.2.14) \qquad \lim_{f\to\infty} \mathrm{Avg}_{f_\chi = f}|L(s,\chi)|^2 = \zeta(2\sigma)$$

(Case 2, over $\mathbf{Q}$) also referred to as an example in the Introduction, this is known to hold unconditionally by [10] Theorem 1.

[2] These may look rather contradictory to the above inequalities (4.2.5)(4.2.6), but imagine the last moment of sunset for Case 1, and that of sunrise for Case 2. Namely, the graph of $h_1((f), y)$ for each f crosses the horizontal axis near $y = 0$ on both sides and, as $f \to \infty$, the graph tends to that of $-Cy^2 + \cdots$ (Case 1), $Cy^2 + \cdots$ (Case 2), where $C = \mu_1^{(1,2)}/2 > 0$.

5 Proof of Theorem 4

The most basic ingredient for the proof of Theorem 4 is Theorem 3, but we shall also need two other fairly basic results, Lemma A (§5.1) and Lemma B (§5.2). In §5.3 we shall give a proof of Theorem 4 assuming these two lemmas; then we shall give proofs of these lemmas in later subsections.

5.1 – Changing test functions. Let $\mathbf{R}^d = \{x = (x_1, ..., x_d);\ x_i \in \mathbf{R}\,(1 \le i \le d)\}$ be the d-dimensional Euclidean space ($d = 1, 2, ...$), and $|dx| = (dx_1...dx_d)/(2\pi)^{d/2}$ be the self-dual Haar measure with respect to the self-dual pairing $e^{i\langle x,x'\rangle}$ of $\mathbf{R}^d$, where $\langle x, x'\rangle = \sum_{i=1}^d x_i x'_i$. Write, as usual, $|x| = \langle x, x\rangle^{1/2}$. In what follows, a function will mean a $\mathbf{C}$-valued function on $\mathbf{R}^d$.

For any function f belonging to L^1, its Fourier transform $f^\wedge$ and the inverse Fourier transform $f^\vee$ are defined by

$$(5.1.1)\qquad f^\wedge(x) = \int f(y)e^{i\langle x,y\rangle}|dy|, \qquad f^\vee(x) = \int f(y)e^{-i\langle x,y\rangle}|dy|.$$

Let $\Lambda = \Lambda(\mathbf{R}^d)$ denote the space of all $f \in L^1 \cap L^\infty$ such that $f^\wedge$ also belongs to $L^1 \cap L^\infty$ and that $(f^\wedge)^\vee = f$ holds. (By definition, L^∞ consists of all continuous functions which vanish at infinity; $f \in L^1 \cap L^\infty$ implies $f \in L^p$ for all $1 \le p \le \infty$.) Let us recall here the following basic facts (cf. e.g. [16]). If $f \in L^1$, then $f \in \Lambda$ holds if and only if f is continuous and $f^\wedge$ belongs to L^1. Moreover, for any f, $g \in \Lambda$, we have

$$(5.1.2)\qquad \int \overline{f^\wedge(x)}g^\wedge(x)|dx| = \int \overline{f(x)}g(x)|dx|.$$

Call $\mathcal{S} = \mathcal{S}(\mathbf{R}^d)$ the Schwartz space, i.e., the set of all C^∞-functions f such that for any partial derivative D of any order and for any $k \ge 0$, $|x|^k D(f)$ tends to 0 as $|x| \to \infty$. Then $\mathcal{S}$ is contained in Λ, and is stable under the Fourier transform. In particular, any compactly supported C^∞-function belongs to Λ.

By a *good density function* on $\mathbf{R}^d$, we shall mean any non-negative real valued continuous function $M(x)$ on $\mathbf{R}^d$ that belongs to Λ and satisfies

$$(5.1.3)\qquad \int M(x)|dx| = 1.$$

Note that $M^\wedge$ necessarily satisfies

$$(5.1.4)\qquad |M^\wedge(x)| \le 1 \qquad \overline{M^\wedge(x)} = M^\wedge(-x).$$

Consider any *finite* measure space $X^* = (X, \omega)$ with the total measure $\omega(X) = 1$. In other words, a finite set X is equipped with a weight function $\omega(\chi) \ge 0$ ($\chi \in X$) satisfying

$\sum_\chi \omega(\chi) = 1$. For any $\mathbf{C}$-valued function ϕ on X, we define the weighted average

$$\text{(5.1.5)} \qquad \mathrm{Avg}_{X^*}\phi = \int \phi\omega = \sum_{\chi \in X} \omega(\chi)\phi(\chi).$$

Consider, now, any pair $X^{**} = (X^*, \ell)$ of an $X^* = (X, \omega)$ and a mapping $\ell : X \to \mathbf{R}^d$. We shall need some terminology related to "approximation" of the given measure space $(\mathbf{R}^d, M(x)|dx|)$ by the ℓ-images of such finite measure spaces X^*. Namely, for a sequence $\{X_n^{**}\}$ of $X_n^{**} = (X_n^*, \ell_n)$, and a test function Φ on $\mathbf{R}^d$, consider the condition

$$\text{(5.1.6)} \qquad \lim_{n\to\infty} \mathrm{Avg}_{X_n^*}(\Phi \circ \ell_n) = \int M(x)\Phi(x)|dx|.$$

(So to speak, "approximation at the level Φ".) What we shall need is to deduce, from the validity of (5.1.6) for some special classes of functions Φ to that for more general cases of funtions Φ.

Lemma A *Let $M(x)$ be any good density function on $\mathbf{R}^d$, and $\{X_n^{**}\}_{n\geq 1}$ be a sequence of pairs $X_n^{**} = (X_n^*, \ell_n)$ of a finite measure space X_n^* and a mapping $\ell_n : X_n \to \mathbf{R}^d$.*

(i) *Suppose that (5.1.6) holds for any additive characters $\Phi = \psi^{(y)} : x \to e^{i\langle x,y\rangle}$, and that the convergence is uniform in the wider sense w.r.t. the parameter $y \in \mathbf{R}^d$. Then (5.1.6) holds for any function Φ belonging to Λ. In particular, it holds for any compactly supported C^∞-function.*

(ii) *Suppose (5.1.6) holds for all compactly supported C^∞-functions Φ on $\mathbf{R}^d$. Then:*

(a) *it holds for any bounded continuous function Φ;*

(b) *it holds for any continuous function Φ satisfying*

$$\text{(5.1.7)} \qquad |\Phi(x)| \leq \phi_0(|x|),$$

if there exists a continuous monotone non-decreasing function $\phi_0(r) > 0$ of $r \geq 0$ satisfying $\lim_{r\to\infty} \phi_0(r) = \infty$ and

$$\text{(5.1.8)} \qquad \int M(x)\phi_0(|x|)|dx| \quad < \quad \infty,$$

$$\text{(5.1.9)} \qquad \mathrm{Avg}_{X_n^*}(\phi_0 \circ |\ell_n|)^2 \quad \ll \quad 1;$$

(c) *its holds when Φ is the characteristic function (the defining function) of either a compact subset of $\mathbf{R}^d$ or the complement of such a subset.*

5.2 – Rapid decay of $M_\sigma(z)$, *a la* Jessen-Wintner. We shall need the following property of $M_\sigma(z)$, which essentially goes back to Jessen-Wintner [9].

Lemma B *Fix $\sigma > 1/2$. Then in each of Cases 1,2, we have*

$$M_\sigma(z) = O(e^{-\lambda|z|^2}) \qquad \textit{any } \lambda > 0. \tag{5.2.1}$$

This proof will be given in §5.7.

5.3 – Proof of Theorem 4 assuming Lemmas A,B We shall apply Lemma A to the following situation:

$\mathbf{C}$ for $\mathbf{R}^2$ (note that $\mathrm{Re}(\overline{z}w) = \langle z, w\rangle$ (z, $w \in \mathbf{C}$)),

$M_\sigma(z)$ ($\sigma > 1/2$) for $M(x)$,

The set of prime divisors $\mathbf{f}$ ($\neq \mathfrak{p}_\infty$ in the FF case) of K, instead of $n = 1, 2, \ldots$,

The set $\hat{G}'_{\mathbf{f}} := \{\chi \in \hat{G}_{\mathbf{f}}; \mathbf{f}_\chi = \mathbf{f}\}$ for X_n, with $\omega_\chi = 1/|\hat{G}'_{\mathbf{f}}|$ for all $\chi \in \hat{G}'_{\mathbf{f}}$;

and finally,

$\mathcal{L}(s, \chi, \mathbf{f})$ for $\ell_n(\chi)$ (for each s with $\sigma = \mathrm{Re}(s) > 1/2$).

Since $\psi_{z,\bar{z}}$ ($z \in \mathbf{C}$) runs over all additive characters of $\mathbf{C}$, Theorem 3 for the case $z_2 = \bar{z_1}$ asserts that the assumption of Lemma A (i) is satisfied. Therefore, by Lemma A (i), the first common assumption of Lemma A(ii) is satisfied. Now take any $a > 0$ and put $\phi_0(r) = \exp(ar)$. It remains to show that this satisfies the assumption of (ii)(b). But (5.1.8) is obvious by Lemma B, while (5.1.9) is nothing but Corollary 1.3.12. When $\sigma > 1$, Theorem 3 holds unconditionally (Remark 1.3.5), $|\mathcal{L}(s, \chi, \mathbf{f})|$ is bounded, and $M_\sigma(w)$ is compactly supported [4](Case 1)[7](Case 2); hence the validity of (1.4.1) for any continuous function Φ is a trivial consequence of that for any compactly supported continuous function. Therefore, Theorem 4 is reduced to Lemmas A and B.

5.4 – Proof of Lemma A (i). We shall first prove (i). By assumption and by (5.1.1), we have

$$\lim_{n\to\infty} \mathrm{Avg}_{X_n^*}(\psi^{(y)} \circ \ell_n) = M^\wedge(y) \qquad (\textit{uniformly on } |y| \leq R) \tag{5.4.1}$$

for any $R > 0$, where $\psi^{(y)}(x) = e^{i\langle x, y\rangle}$. Now let Φ be any element of Λ and put

$$\Delta_n(\Phi) = \mathrm{Avg}_{X_n^*}(\Phi \circ \ell_n) - \int M(x)\Phi(x)|dx|. \tag{5.4.2}$$

(Since M, $\Phi \in \Lambda$, the above integral is finite.) Write $X_n^* = (X_n, \omega_n)$. Then since $(\Phi^\wedge)^\vee = \Phi$, $\overline{M(x)} = M(x)$ and $\overline{M^\wedge(y)} = M^\wedge(-y)$, (5.1.2) gives

$$(5.4.3) \qquad \Delta_n(\Phi) = \sum_{\chi \in X_n} \omega_n(\chi)\Phi(\ell_n(\chi)) - \int M(x)\Phi(x)|dx|$$

$$(5.4.4) \qquad = \sum_{\chi \in X_n} \omega_n(\chi) \int \Phi^\wedge(y) e^{-i\langle \ell_n(\chi), y\rangle}|dy| - \int M^\wedge(-y)\Phi^\wedge(y)|dy|$$

$$= \int \left(\sum_{\chi \in X_n} \omega_n(\chi) e^{i\langle \ell_n(\chi), -y\rangle} - M^\wedge(-y) \right) \Phi^\wedge(y)|dy|$$

$$= \int \left(\mathrm{Avg}_{X_n^*}(\psi^{(-y)} \circ \ell_n) - M^\wedge(-y) \right) \Phi^\wedge(y)|dy|.$$

But since $|\psi^{(-y)}(x)|$, $|M^\wedge(y)| \le 1$, we obtain for any $R > 0$,

$$(5.4.5) \quad |\Delta_n(\Phi)| \le \int_{|y| \le R} |\mathrm{Avg}_{X_n^*}(\psi^{(-y)} \circ \ell_n) - M^\wedge(-y)||\Phi^\wedge(y)||dy| + 2\int_{|y| \ge R} |\Phi^\wedge(y)||dy|.$$

Since $\Phi \in \Lambda$ and hence in particular $\Phi^\wedge \in L^1$, the total integral of $|\Phi^\wedge|$ is finite. Call this value I. Now, given any $\epsilon > 0$, choose R so large that the second term on the right hand side of (5.4.5) is $< \epsilon$. Then choose $\epsilon' > 0$ such that $\epsilon' I < \epsilon$. Then by (5.4.1),

$$(5.4.6) \qquad |\mathrm{Avg}_{X_n^*}(\psi^{(-y)} \circ \ell_n) - M^\wedge(-y)| < \epsilon'$$

holds on $|y| \le R$ for sufficiently large n, which implies $|\Delta_n(\Phi)| < 2\epsilon$ for such large n. This settles the proof of (i).

5.5 – Proof of Lemma A (ii) First, the validity of (5.1.6) for any compactly supported C^∞-function implies that for any compactly supported continuous function, because the latter can be approximated by the former.

As for (c), this can be proved directly by approximation of the characteristic function of a given compact set by continuous compactly supported functions, as is explained in detail in the two dimensional case in [6]§4.3.

Now to prove (a)(b), let Φ satisfy the assumptions of one of (a)(b), and put $\alpha_n = \mathrm{Avg}_{X_n^*}(\Phi \circ \ell_n)$. In each case, α_n $(n = 1, 2, ...)$ is a bounded sequence. Let α be any of its limit points. It is the limit of some subsequence α_{n_ν}. The goal is to prove

$$(5.5.1) \qquad \alpha = \int M(x)\Phi(x)|dx|.$$

To prove this, note first that for any $R > 0$ there exists a compactly supported continuous function Φ_R satisfying

$$|\Phi(x) - \Phi_R(x)| \leq (1 - \mathrm{ch}_R(x))|\Phi(x)| \tag{5.5.2}$$

where ch_R denotes the characteristic function of $\{|x| \leq R\}$. Indeed, if $E(u)$ is any compactly supported continuous function such that $0 \leq E(u) \leq 1$ everywhere and $E(u) = 1$ for $|u| \leq 1$, then $\Phi_R(x) = \Phi(x)E(x/R)$ has this property. Now choose such Φ_R for each R, and put $\alpha_{n,R} = \mathrm{Avg}_{X_n^*}(\Phi_R \circ \ell_n)$. Since Φ_R is compactly supported and continuous, we have

$$\lim_{n\to\infty} \alpha_{n,R} = \int M(x)\Phi_R(x)|dx|. \tag{5.5.3}$$

Now, (5.5.2) gives

$$\begin{aligned} \alpha_n - \alpha_{n,R} &= \mathrm{Avg}_{X_n^*}((\Phi - \Phi_R) \circ \ell_n) \\ &\ll \beta_{n,R} := \mathrm{Avg}_{X_n^*}((1 - \mathrm{ch}_R)|\Phi| \circ \ell_n), \end{aligned} \tag{5.5.4}$$

and also

$$\lim_{R\to\infty} \int M(x)\Phi_R(x)|dx| = \int M(x)\Phi(x)|dx|. \tag{5.5.5}$$

Now suppose that Φ is bounded. Then $\beta_{n,R} \ll \mathrm{Avg}_{X_n^*}((1 - \mathrm{ch}_R) \circ \ell_n)$ which tends to $\int_{|x|\geq R} M(x)|dx|$ as $n \to \infty$, because we already know that (5.1.6) holds for $1 - \mathrm{ch}_R$. Therefore, (5.5.4) for n_ν, $\nu \to \infty$ gives

$$\alpha - \int M(x)\Phi_R(x)|dx| \ll \int_{|x|\geq R} M(x)|dx|. \tag{5.5.6}$$

Therefore, by letting $R \to \infty$ we obtain (5.5.1) when Φ is bounded.

When $|\Phi(x)| \leq \phi_0(|x|)$ as in (b), (5.1.7)(5.1.9) and the Schwarz inequality give (note that $1 - \mathrm{ch}_R$ is the same as its square):

$$\begin{aligned} \beta_{n,R}^2 &\leq (\mathrm{Avg}_{X_n^*}((1 - \mathrm{ch}_R) \circ \ell_n))(\mathrm{Avg}_{X_n^*}(\phi_0 \circ |\ell_n|)^2) \\ &\ll \mathrm{Avg}_{X_n^*}((1 - \mathrm{ch}_R) \circ \ell_n). \end{aligned} \tag{5.5.7}$$

But since $\phi_0(r)$ is positive and monotone non-decreasing, we have, trivially,

$$(1 - \mathrm{ch}_R)(x)\phi_0(R)^2 \leq \phi_0(|x|)^2.$$

Therefore,

$$\mathrm{Avg}_{X_n^*}((1 - \mathrm{ch}_R) \circ \ell_n) \leq \phi_0(R)^{-2}\mathrm{Avg}_{X_n^*}(\phi_0 \circ |\ell_n|)^2 \ll \phi_0(R)^{-2}. \tag{5.5.8}$$

Therefore, by (5.5.7), we obtain $\beta_{n,R} \ll \phi_0(R)^{-1}$. Therefore, (5.5.4) for n_ν with $\nu \to \infty$ gives

$$\alpha - \int M(x)\Phi_R(x)|dx| \ll \phi_0(R)^{-1}; \tag{5.5.9}$$

hence by letting $R \to \infty$ we obtain (5.5.1) also for this case. This completes the proof of Lemma A.

5.6 – Proof of Lemma B. The general theory developed in [9], from §7 on, starts with any holomorphic function $F(z)$ on $|z| < \rho$ (in our case $\rho = 1$) satisfying $F(0) = 0$, $F'(0) \neq 0$, and from §8, also with any sequence $\{r_n\}_{n\geq 1}$ of positive real numbers satisfying $r_n \leq r$ for some $r < \rho$ and $\sum_n r_n^2 < \infty$. Then the existence of the "continuous density" $D(z)$ for the distribution on $\mathbf{C}$ of the values of

$$\sum_n F(r_n e^{2\pi i\theta_n}) \tag{5.6.1}$$

$(0 \leq \theta_n < 1)$ is established, and some basic analytic properties of $D(z)$ are proved (*loc. cit.* Theorems 14-16).

Here, (just for Case 1) we need the following (slight) generalization. Let $\{\lambda_n\}_{n\geq 1}$ be another sequence of positive real numbers satisfying $\lambda_n^{-1} \ll 1$ and $\sum_n \lambda_n^2 r_n^2 < \infty$. Consider now the distribution of

$$\sum_n \lambda_n F(r_n e^{2\pi i\theta_n}) \tag{5.6.2}$$

on $\mathbf{C}$. Then Theorems 14-16 remain valid; in particular, the existence of the density (function) $D(z)$ (Theorem 14) and the property $D(z) \ll e^{-\lambda|z|^2}$ for any $\lambda > 0$ (Theorem 16) remain valid. (Incidentally, the condition $r_n^{-1} \ll n$ in Theorem 15 need not be modified.)

Now, take

$$F(z) = \begin{cases} z/(z-1) & \text{(Case 1)}, \\ -\log(1-z) & \text{(Case 2)}, \end{cases}$$

(so that $\rho = 1$). Moreover, in Theorem 14, take $\{\mathfrak{p}; \mathfrak{p} \neq \mathfrak{p}_\infty\}$ instead of $n \in \mathbf{N}$, take $N(\mathfrak{p})^{-\sigma}$ for r_n, take $\log N(\mathfrak{p})$ (resp. 1) for λ_n in Case 1 (resp. Case 2). Then for each $\sigma > 1/2$ and for each of Cases 1,2, the above conditions are satisfied and the corresponding density function $D(z)$ is nothing but our $M_\sigma(z)$. Therefore, (the modified) Theorem 16 of [9] gives Lemma B (see also Theorem 19 for Case 2, with $N(\mathfrak{p})$ in place of p_n).

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