# Some density functions and their Fourier transforms arising from number theory

**Yasutaka Ihara** [1]

## 1 Introduction

**1.1** – The object of our study is a certain density function $M(z)$ on the complex plane $\mathbf{C}$ and its Fourier dual $\tilde{M}(z)$ arising from number theory. These functions have been constructed and studied in the previous article [5], but here (after briefly reviewing the definitions in §2), we shall focus on the following two topics not treated there; namely,

(i) the zeros of the Fourier dual $\tilde{M}(z)$,

(ii) the "Plancherel Volume" (see below).

The author's main motivation for having chosen this subject in this interdisciplinary workshop was, mainly, to ask colleagues in probability theory about the probabilistic meaning (and possible applications) of results related to (i)(ii).

The density function $M(z)$ on $\mathbf{C}$ to be treated here was introduced in order to study the distribution of values of the *logarithmic derivative* of $L$-functions $L(\chi, s)$ on the complex plane. Here, the complex "variable" $s$ is *fixed* while the character $\chi$ runs over some nice family of characters on the base field. The distribution density is then expected to depend only on the base field and on the real part $\sigma = \mathrm{Re}(s)(> 1/2)$ of $s$. Thus, $M(z)$ and $\tilde{M}(z)$ depend on the base field and on the real parameter $\sigma > 1/2$. In general, the connection with $L$-function is still conjectural; only in some special cases the actual connection was proved ([5]§4,6).

Here, the readers may regard this as an example of distribution of arithmetic objects on the complex plane which, from the point of view of (i), is very much different from the Gaussian distribution but on the other hand, from that of (ii), is very close to the Gaussian.

The function $M(z)$ is (being the density function) real non-negative valued, and satisfies

$$(1.1.1) \qquad \int M(z)|dz| = 1 \qquad (|dz| = dxdy/2\pi;\ z = x + yi)$$

[1] P.E., RIMS Kyoto University; email:ihara@kurims.kyoto-u.ac.jp

(The integral is always over the whole complex plane.) It belongs to class $C^\infty$ and is rapidly decreasing. Moreover, the center of gravity is the origin, i.e.,

$$\int M(z)x|dz| = \int M(z)y|dz| = 0. \tag{1.1.2}$$

It is, though, *not* rotation-invariant. Write

$$\psi_z(w) = \psi_w(z) = e^{i\mathrm{Re}(\bar{z}w)}, \tag{1.1.3}$$

and let

$$\tilde{M}(w) = \int M(z)\psi_w(z)|dz| \tag{1.1.4}$$

be the Fourier dual. Then the inversion formula

$$M(z) = \int \tilde{M}(w)\psi_z(-w)|dz| \tag{1.1.5}$$

holds, and we have

$$\nu := \int M(z)^2|dz| = \int |\tilde{M}(w)|^2|dw|, \tag{1.1.6}$$

the Plancherel formula. Let us call this quantity the "Plancherel Volume".

As for the topic (i), we shall state (with basic indications for proofs but without details) that $\tilde{M}(z)$ has infinity of zeros on the imaginary axis $\mathrm{Re}(z) = 0$, and, in addition, depending on the basic field and $\sigma$, finitely many zeros off the imaginary axis. (What does this mean?)

As for (ii) (the Plancherel Volume $\nu$), first of all, intuitively, this should be a basic invariant and should deserve enough attention. After some trials to understand what this is, the author's attention was directed to its comparison with the multiplicative inverse $\mu^{-1}$ of the variance

$$\mu := \int M(z)|z|^2|dz|, \tag{1.1.7}$$

because $\mu\nu = 1$ holds for the two-dimensional Gaussian distribution with center $O$. We shall show by some examples that the product $\mu\nu$ for our density function is often quite close to 1; in particular, we have noticed that when the base field is the rational function field over the finite field $\mathbf{F}_q$ and when $q \mapsto \infty$, then this product actually tends to 1. In this sense, our distribution is close to the Gaussian.

In §2, we briefly recall the constructions of $M, \tilde{M}$, and in §3 (resp. §4) we deal with the zeros of $\tilde{M}(z)$ (resp. the Plancherel Volume). The main results are Theorems 1A,1B(§3) and Theorems 2,3 (§4). Further details, if desired, will be published later.

## 2 Brief review of $M_\sigma(z)$ and $\tilde{M}_\sigma(z)$.

**2.1** – Let $K$ be a global field, i.e., either an algebraic number field, or an algebraic function field of one variable over a finite field. Let $\sigma$ be a real parameter, with $\sigma > 1/2$. Associated with such $K$ and $\sigma$, we have contructed and studied mutually Fourier-dual functions $M_\sigma(z)$, $\tilde{M}_\sigma(z)$ of $z \in \mathbf{C}$, in connection with the distribution of values of the logarithmic derivative of $L$-functions $L(\chi, s)$ [5], where $\chi$ varies and $s$ is fixed ($\sigma = \mathrm{Re}(s)$). ($K$ is usually suppressed from the notations.) Here, we briefly recall the constructions. For those readers only interested in the case of the rational number field $K = \mathbf{Q}$, we add that "non-archimedean prime divisor $\wp$" simply means prime number $p$, and its norm $N(\wp)$ is just $p$ itself.

Let $\mathbf{C}^1$ denote the multiplicative group of complex numbers with norm 1, and for each $\sigma > 0$ and a non-archimedean prime divisor $\wp$ of $K$, consider the rational function

$$g_{\sigma,\wp}(t) = -(\log N(\wp))\frac{t}{N(\wp)^\sigma - t} \tag{2.1.1}$$

of $t \in \mathbf{C}^1$. Its image is another circle, with center $c_{\sigma,\wp}$ and radius $r_{\sigma,\wp}$ given by

$$c_{\sigma,\wp} = -\frac{\log N(\wp)}{N(\wp)^{2\sigma} - 1}, \qquad r_{\sigma,\wp} = \frac{(\log N(\wp))N(\wp)^\sigma}{N(\wp)^{2\sigma} - 1}. \tag{2.1.2}$$

For each real number $y > 1$, consider the distribution on $\mathbf{C}$ of points

$$g_\sigma^{(y)}(t) = \sum_{N(\wp)\le y} g_{\sigma,\wp}(t_\wp), \tag{2.1.3}$$

where $\wp$ runs over all non-archimedean prime divisors of $K$ whose norm does not exceed $y$, and $t_\wp$ runs independently over the points of $\mathbf{C}^1$. Then the density measure $M_\sigma^{(y)}(z)|dz|$ for this distribution exists and is explicitly given as the convolution product

$$M_\sigma^{(y)}(z) = *_{N(\wp)\le y} M_{\sigma,\wp}(z) \tag{2.1.4}$$

of local functions $M_{\sigma,\wp}(z)$, which, in polar coordinates with center $c_{\sigma,\wp}$, is given by

$$M_{\sigma,\wp}(c_{\sigma,\wp} + re^{i\theta}) = \frac{N(\wp)^{2\sigma} - 1}{|N(\wp)^\sigma - e^{i\theta}|^2} \cdot \frac{\delta(r - r_{\sigma,\wp})}{r} \tag{2.1.5}$$

($\delta(r)$ : the Dirac delta function). Each local factor is a Schwarz distribution, but the convolution product of more than one factor is a usual function, which gets smoother as the number of factors increases.

Now when $\sigma > 1/2$, $M_\sigma(z) = \lim_{y\mapsto\infty} M_\sigma^{(y)}(z)$ exists. The convergence is uniform on $\mathbf{C}$. The limit function belongs to class $C^\infty$, and is sufficiently rapidly decreasing (see below). The center of gravity may, at the first glance, look like the point $c_\sigma = \sum_\wp c_{\sigma,\wp}$

(which is finite for any $\sigma > 1/2$), but it is actually the origin $O$, i.e., (1.1.2) holds. Now since $M_\sigma(z)$ is the limit of the convolution product over $\wp$, its Fourier dual $\tilde{M}_\sigma(z)$ has a (uniformly and absolutely) convergent Euler product expansion

$$\tilde{M}_\sigma(z) = \prod_{\wp} \tilde{M}_{\sigma,\wp}(z), \tag{2.1.6}$$

and each Euler factor $\tilde{M}_{\sigma,\wp}(z)$ , which is the Fourier dual of $M_{\sigma,\wp}(z)$, can be expressed explicitly in terms of Bessel functions;

$$\tilde{M}_{\sigma,\wp}(z) = e^{ic_{\sigma,\wp}\mathrm{Re}(z)} \left( \sum_{n=0}^{\infty} \epsilon_n \left(\frac{i}{N(\wp)^\sigma}\right)^n \cos(n\vartheta) J_n(r_{\sigma,\wp}|z|) \right), \tag{2.1.7}$$

where $\epsilon_n = 1$ $(n = 0)$,$= 2$ $(n \geq 1)$, $\vartheta = \mathrm{Arg}(z)$, and $J_n(x)$ is the Bessel function of order $n$.

For the main properties of $M_\sigma(z)$, $\tilde{M}_\sigma(z)$, see [5]§2,3, and for the connection with the $L'/L$-values, see *ibid* §4,6.

We add here the following statements on the orders of their decay as $|z| \mapsto \infty$, and on the zeros of $M_\sigma(z)$, that were not treated in [5]. (As for the former, in [5], we have only shown that they are both $O(|z|^{-N})$ for any $N \geq 1$, which were enough for the main purpose of that paper.)

Fix the base field $K$ and any $\sigma > 1/2$. Then

$$M_\sigma(z) = O(e^{-\lambda|z|^2}) \qquad \textit{for any } \lambda > 0. \tag{2.1.8}$$

This can be proved by a slight modification of the arguments used in [7] (proof of Theorem 16). As a reflection, the Fourier dual cannot have the same order of decay;

$$\tilde{M}_\sigma(z) \neq O(e^{-\lambda|z|^2}) \qquad \textit{for any } \lambda > 0. \tag{2.1.9}$$

What we can prove is:

$$\tilde{M}_\sigma(z) = O(e^{-|z|^{1/(\sigma+\epsilon)}}) \qquad \textit{for any } \epsilon > 0. \tag{2.1.10}$$

For this, we use an estimation of $|\tilde{M}_\sigma(z)|^2$ given later in §4.3.

As for the zeros of $M_\sigma(z)$, when $\sigma > 1$, the sum $r_\sigma = \sum_\wp r_{\sigma,\wp}$ converges, and we know that $M_\sigma(z)$ vanishes outside the disk with center $c_\sigma$ and radius $r_\sigma$. When $\sigma \leq 1$, we can prove, also by a slight modification of [7] (proof of Theorem 14), that $M_\sigma(z) > 0$ everywhere. We shall now go on to the study of zeros of the Fourier dual.

## 3 Zeros of $\tilde{M}_\sigma(z)$

**3.1 – Global Zeros** We first state our theorem on the set of zeros $z$ of

$$\tilde{M}_\sigma(z) = \prod_{\wp} \tilde{M}_{\sigma,\wp}(z). \tag{3.1.1}$$

Note that $\tilde{M}_\sigma(z)$ is real-analytic but not complex analytic; in particular, the discreteness of zeros is not guaranteed. And since $\tilde{M}_\sigma(-\bar{z}) = \tilde{M}_\sigma(-z) = \overline{\tilde{M}_\sigma(z)}$, it is real-valued on the imaginary axis.

**Theorem** 1*A* (i) *Each zero of* $\tilde{M}_\sigma(z)$ *comes from a zero of some local factor.*
(ii) *The set of zeros of* $\tilde{M}_\sigma(z)$ *is infinite and discrete.*
(iii) *All but finitely many zeros lie on the imaginary axis.*

To be more precise, (i) means that for any given compact subset $\mathfrak{K}$ of $\mathbf{C}$ there exists a finite set $S_{\mathfrak{K}}$ of primes $\wp$ such that $\tilde{M}_{\sigma,\wp}(z) \neq 0$ on $\mathfrak{K}$ for all $\wp \notin S_{\mathfrak{K}}$ and moreover that the product of these local factors converges absolutely to a nowhere vanishing function on $\mathfrak{K}$ (cf. [5] §3). Thus, (ii) will follow from the corresponding statement on the zeros of each local factor.

**3.2 – Local Zeros** To study the zeros of each local factor, let us consider the function $h_q(z)$ of $z$ parametrized by $q > 1$, defined by

$$\begin{aligned} h_q(z) &= \frac{q^2-1}{2\pi} \int_0^{2\pi} \frac{\exp(i|z|\cos(\theta - \vartheta))}{|q - e^{i\theta}|^2} d\theta \\ &= \sum_{n=0}^{\infty} \epsilon_n (i/q)^n \cos(n\vartheta) J_n(|z|), \end{aligned} \tag{3.2.1}$$

where, as before, $\vartheta = \mathrm{Arg}(z)$. Then by (2.1.7) we have

$$\tilde{M}_{\sigma,\wp}(z) = e^{ic_{\sigma,\wp}\mathrm{Re}(z)} \cdot h_{N(\wp)^\sigma}(r_{\sigma,\wp} z). \tag{3.2.2}$$

Note that $r_{\sigma,\wp} \mapsto 0$ as $N(\wp) \mapsto \infty$. Note also that $h_q(-\bar{z}) = h_q(-z) = \overline{h_q(z)}$ ; hence $h_q(z)$ is real-valued on the imaginary axis. We shall call a zero $z$ of $h_q(z)$ *regular* if $\mathrm{Re}(z) = 0$, and *irregular* otherwise.

Thus, (ii)(iii) of Theorem 1A are reduced to

**Theorem** 1*B* *Fix any* $q > 1$. *Then*
(i) $h_q(z)$ *has an infinite discrete set of regular zeros.*

(ii) *Each* $h_q(z)$ *has at most finitely many irregular zeros, and there exists an absolute constant* $C$ *such that when* $q > C$, $h_q(z)$ *has no irregular zeros at all.*

As for Theorem 1B (i), we can prove a much stronger statement (analogous to the corresponding well-known result on the zeros of $J_n(x)$):

**Proposition 3.2.3** *Let* $q > 1$. *Then there exists* $M_q \geq 0$ *such that if* $m \in \mathbf{Z}$, $|m| \geq M_q$
(I) *there is no zero* $y$ *of* $h_q(yi)$ *in the closed interval* $[m\pi, (m+\frac{1}{2})\pi]$, *and*
(II) *exactly one simple zero in* $[(m+\frac{1}{2})\pi, (m+1)\pi]$.
*Moreover, when* $q$ *is sufficiently large,* $M_q = 0$, *i.e.,* (I)(II) *hold for all* $m \in \mathbf{Z}$.

The proof is based on the formulas both induced from (3.2.1):

(3.2.4) $$h_q(yi) = \sum_{m=0}^{\infty} \epsilon_m q^{-2m} J_{2m}(y),$$

(3.2.5) $$y^{-1} \cdot d(y \cdot h_q(yi))/dy = \sum_{m=0}^{\infty} \epsilon_m q^{-2m} J_{2m-1}(y).$$

(Recall that $J_{-1}(y) = -J_1(y)$), and an approximation formula

(3.2.6) $$y^{1/2} J_n(y) - (2/\pi)^{1/2} \cos(y - \pi/4 - n\pi/2) = O_{abs}((n^3+1)y^{-1}) \qquad (y > 0).$$

(For each fixed $n$, this is well-known; the above statement can be derived from lemma 3.3.4 of [5].)

As for Theorem 1B (ii), it consists of three separate statements;
(a) For each $q > 1$, the irregular zeros have bounded absolute values;
(b) When $q$ is sufficiently large, $h_q(z)$ has no irregular zeros;
(c) The set of irregular zeros of $h_q(z)$ is discrete.

Among them, (a)(b) can be derived by using (3.2.6), as follows.
Decompose $h_q(z)$ into the real and the imaginary parts and write

(3.2.7) $$h_q(z) = h_q^{(1)}(z) + i \cdot (\cos\vartheta) h_q^{(2)}(z).$$

The irregular zeros of $h_q(z)$ are the common zeros of $h_q^{(1)}(z)$ and $h_q^{(2)}(z)$ off the imaginary axis. We have

(3.2.8) $$h_q^{(1)}(z) = J_0(|z|) + 2\sum_{m=1}^{\infty} \left(\frac{-1}{q^2}\right)^m \cos(2m\vartheta) J_{2m}(|z|),$$

$$h_q^{(2)}(z) = \frac{2}{q} \sum_{m=0}^{\infty} \left(\frac{-1}{q^2}\right)^m \frac{\cos(2m+1)\vartheta}{\cos\vartheta} J_{2m+1}(|z|).$$

From these and (3.2.6) we obtain easily, for each fixed $q_0 > 1$ and for any $q \geq q_0$,

$$(3.2.9) \qquad |z|^{1/2} h_q^{(1)}(z) - \left(\frac{2}{\pi}\right)^{1/2} \frac{q^4 - 1}{|q^2 - e^{2i\vartheta}|^2} \cos(|z| - \pi/4) = O_{q_0}(|z|^{-1}),$$
$$\frac{q}{2}|z|^{1/2} h_q^{(2)}(z) - \left(\frac{2}{\pi}\right)^{1/2} \frac{q^2(q^2 - 1)}{|q^2 - e^{2i\vartheta}|^2} \sin(|z| - \pi/4) = O_{q_0}(|z|^{-1});$$

hence if $z_0$ is an irregular zero of $h_q(z)$, then $|z_0|^{-1} \gg_{q_0} 1$; whence (a). On the other hand, (3.2.8) also gives

$$(3.2.10) \qquad h_q^{(1)}(z) - J_0(|z|) = O_{abs}(q^{-2}),$$
$$\frac{q}{2} h_q^{(2)}(z) - J_1(|z|) = O_{abs}(q^{-2});$$

hence $\mathrm{Max}(|J_0(|z_0|)|, |J_1(|z_0|)|) = O_{abs}(q^{-2})$ which is impossible for $q$ large and $|z_0|$ bounded, whence (b).

As for the actual bound for $q$ in (b), $q \geq 5$ is enough, and numerical computations suggest that the bound is probably $q = 1.91148...$ (see below).

As for (c), we need to look, not only at the first partial derivatives but also the second partial derivatives of the transformation $z \mapsto h_q(z)$ of the complex plane $\mathbf{C}$, and here we use the partial differential equations with respect to the $(x, y)$ coordinates of $z = x + yi$:

$$(3.2.11) \qquad \frac{\partial h_q(z)}{\partial x} = \frac{i}{2q}\left((q^2 + 1)h_q(z) - (q^2 - 1)J_0(|z|)\right),$$

$$(3.2.12) \qquad \triangle h_q(z) = -h_q(z),$$

where $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Incidentally, these two equations, together with the vanishing at $|z| = \infty$ characterize $h_q(z)$. By using these equations we can show that each irregular zero $z_0$ with $J_0(|z_0|) \neq 0$ must be isolated. The rest of the proof of (c) is easier.

We add here the following remark which will be needed later. If we call $h_q^*(y)$ the right hand side of (3.2.4) considered as a (complex valued) holomorphic function of a complex variable $y$, it is entire with order 1, satisfying $h_q^*(0) = 1$, $(dh_q^*/dy)(0) = 0$; hence it possesses a factorization

$$(3.2.13) \qquad h_q^*(y) = \prod_\eta \left(1 - \frac{y}{\eta}\right) e^{y/\eta},$$

where $\eta$ runs over the zeros of $h_q^*(y)$. From this we obtain easily that

$$(3.2.14) \qquad \sum_\eta \eta^{-2} = -(y^{-1} d \log h_q^*(y)/dy)_0 = \frac{1}{2}(1 - q^{-2}).$$

Whether all zeros $\eta$ are real depends on $q$. It seems that this is true if $q > q_1 = 1.54007....$ ($q_1$ is the largest $q$ such that $h_q^*(y)$ has a multiple zero on the real axis. When $q_1$ moves towards left on the real axis, it produces a pair of non-real zeros nearby). Thus, for such $q$, (3.2.14) gives directly

$$\sum_{\rho} (\mathrm{Im}\,\rho)^{-2} = \frac{1}{2}(1 - q^{-2}), \tag{3.2.15}$$

where $\rho$ runs over all *regular* zeros of $h_q(z)$ (which are simple for $q > q_1$).

**3.3 – Finding irregular zeros** The largest $q$ for which an irregular zero of $h_q(z)$ was found is $q = 1.91148....$ As stated above, the irregular zeros of $h_q(z)$ are the common zeros of $h_q^{(1)}(z)$ and $h_q^{(2)}(z)$ off the imaginary axis. But we first try to find their common zeros *on* the imaginary axis and then change $q$, $z$ slightly. By the differential equation (3.2.11), we see that when $h_q^{(1)}(yi) = 0$, the vanishings of $h_q^{(2)}(yi)$ and of $J_0(y)$ are equivalent. So we start with a positive zero $\gamma$ of $J_0(y)$ and try to find $q$ such that $h_q^{(1)}(\gamma i) = 0$. This is possible only when $\gamma = \gamma_{2n}$, where $0 < \gamma_1 < \gamma_2 < ...$ denotes the increasing sequence of positive zeros of $J_0(y)$. In fact, for each $n \geq 1$ there exists $q = q_n > 1$ such that $h_{q_n}^{(1)}(\gamma_{2n} i) = 0$. We have

$$\gamma_2 = 5.520078 < \gamma_4 = 11.791534 < ..., \tag{3.3.1}$$

$$q_1 = 1.91148 > q_2 = 1.51775 > .... \tag{3.3.2}$$

When we change $q_n$ slightly to a smaller $q_n'$, we can actually find a pair of *irregular* zeros of $h_{q_n'}(z)$ near $z = \gamma_{2n} i$. For example ($n = 1$), $q_1' = 1.9108$ gives such zeros $z$ with $\mathrm{Re}(z) = \pm 0.1$ and $\mathrm{Im}(z) = 5.5198$ (close to $\gamma_2$). For $n = 2$, $q_2' = 1.5$ gives $z$ whose real (resp. imaginary) part is approximately $\pm 1$ (resp. $\gamma_4$). This is how they have been found.

**3.4 – Global zeros (-continued)** By using Proposition 3.2.3 , we can obtain some basic information on the number $N_\sigma(T)$ of zeros $z$ of $\tilde{M}_\sigma(z)$ with $|\mathrm{Im}(z)| \leq T$. Here, the counted multiplicity is, by definition, the number of distinct primes $\wp$ such that $\tilde{M}_{\sigma,\wp}(z) = 0$.

**Proposition 3.4.1**

$$N_\sigma(T) \approx \begin{cases} T^{\sigma^{-1}} (\log T)^{\sigma^{-1}-1} & \cdots \frac{1}{2} < \sigma < 1, \\ T \log T & \cdots \sigma = 1, \\ T & \cdots \sigma > 1, \end{cases} \tag{3.4.2}$$

*where $\approx$ means that the ratio is bounded by two positive real constants depending only on $(K$ and $)\,\sigma$ when $T$ is sufficiently large.*

A stronger statement should be within easy reach, but this will be left to future publications.

Another remark is on the connection between the sum of inverse of the square of the imaginary part of regular zeros of $\tilde{M}_\sigma(z)$ and the variance

$$\mu_\sigma = \int M_\sigma(z)|z|^2|dz| \tag{3.4.3}$$

$$= \sum_{\wp} \frac{(\log N(\wp))^2}{N(\wp)^{2\sigma} - 1} = \sum_{\wp} (1 - N(\wp)^{-2\sigma}) r_{\sigma,\wp}^2$$

(cf. [5]). By (3.2.15), we expect that if $K$, $\sigma$ are such that $N(\wp)^\sigma > 1.54007$ for all $\wp$, then all regular zeros $\rho$ of $\tilde{M}_\sigma(z)$ are simple and

$$\mu_\sigma = 2 \sum_{\rho:\, regular} \mathrm{Im}(\rho)^{-2}. \tag{3.4.4}$$

**3.5 – Examples** When $K = \mathbf{Q}$ and $\sigma = 1$ resp. $\sigma = 1/2 + 1/20$, the tables of the *regular* zeros $yi$ ($y > 0$, small) of $\tilde{M}_\sigma(z)$ are as shown below, on the left (resp. right). The prime $p$ on the list indicates from which local factor each zero comes from. As for

| $y$ | p | | y | p |
|---|---|---|---|---|
| 6.35 | 3 | | 3.45 | 5 |
| 6.49 | 2 | | 3.48 | 7 |
| 7.38 | 5 | | 3.67 | 11 |
| 8.59 | 7 | | 3.73 | 3 |
| 11.00 | 11 | | 3.77 | 13 |
| 12.01 | 2 | | 3.98 | 17 |
| 12.17 | 13 | | 4.08 | 19 |
| 13.53 | 3 | | 4.25 | 23 |

the irregular zeros, for $\sigma = 1$ there is none, and for $\sigma = 11/20$, we found the following 3 quadruples of irregular zeros. Very probably, these exhaust all the irregular zeros in this case.

$$\begin{aligned} &\pm 1.738 \pm 13.267 \cdot i && (from\ p = 2) \\ &\pm 2.906 \pm 6.715 \cdot i && (from\ p = 2) \\ &\pm 1.28 \pm 6.415 \cdot i && (from\ p = 3) \end{aligned} \tag{3.5.1}$$

## 4 The Plancherel Volume

### 4.1 – General Remarks

Let $\mathbf{R}^d = \{x = (x_1, ..., x_d);\ x_i \in \mathbf{R}\,(1 \le i \le d)\}$ be the $d$-dimensional Euclidean space, and let $|dx| = (dx_1...dx_d)/(2\pi)^{d/2}$ be the self-dual Haar measure with respect to the self-dual pairing $e^{i<x,x'>}$ of $\mathbf{R}^d$, where $< x, x' >= \sum_{i=1}^d x_i x_i'$. Write, as usual, $|x| =< x, x >^{1/2}$. For a density measure $M(x)|dx|$ on $\mathbf{R}^d$ with center $O$ for which the standard formulas in Fourier analysis hold, namely;

(4.1.1) $$M(x) \ge 0, \qquad \int M(x)|dx| = 1;$$

(4.1.2) $$\int M(x)x_i|dx| = 0 \qquad (1 \le i \le d);$$

(4.1.3) $$\tilde{M}(y) = \int M(x)e^{i<x,y>}|dx|, \qquad M(x) = \int \tilde{M}(y)e^{-i<x,y>}|dy|;$$

(4.1.4) $$\nu := \nu_M = \int M(x)^2|dx| = \int |\tilde{M}(y)|^2|dy| \qquad \text{(The Plancherel Formula)},$$

we compare the two invariants

(4.1.5) $$\mu := \mu_M = \int M(x)|x|^2|dx| \qquad \text{(The variance)}$$

and the above $\nu_M$ which is also equal to

(4.1.6) $$\nu_M = M(x) * M(-x)\ |_{x=0}\ .$$

Here, $*$ denotes the convolution product with respect to $|dx|$. Thus, $\nu_M$ may be regarded as the density at the origin of the differences of two points in the measure space $(\mathbf{R}^d, M(x)|dx|)$.

In general, the two invariants, the average $\mu$ of the square of the distance from the center and the density $\nu$ at the origin of $x - x'$ $(x, x' \in \mathbf{R}^d)$, both with respect to the given density measure $M(x)|dx|$, are unrelated invariants. But the product

(4.1.7) $$\mu\nu^{\frac{2}{d}}$$

seems to be an interesting basic invariant. Note that this is invariant by the scalar transform

(4.1.8) $$M(x) \longmapsto c^d M(cx)$$

for any $c > 0$, In fact, $\mu$ (resp. $\nu$) is multiplied by $c^{-2}$ (resp. $c^d$). Let us pay attention to the following three special cases and the theorem to come thereafter.

**Example 1** If $M(x)|dx|$ is Gaussian, i.e., $M(x) = ce^{-a|x|^2}$ $(a,\, c > 0)$, then

$$\mu\nu^{\frac{2}{d}} = \frac{d}{2}. \tag{4.1.9}$$

In particular, the two dimensional Gaussian distribution satisfies $\mu\nu = 1$.

Indeed, we have $c = (2a)^{d/2}$ by (4.1.1), and $\mu = d/(2a)$, $\nu = a^{d/2}$.

**Example 2** If $M(x) = c$ $(|x| \leq R)$ and $= 0$ $(|x| > R)$, where $c,\, R > 0$, then

$$\mu\nu^{\frac{2}{d}} = \frac{2d}{d+2}\Gamma(\frac{d}{2}+1)^{\frac{2}{d}}. \tag{4.1.10}$$

In particular, when $d = 2$, we again have $\mu\nu = 1$.

Indeed, $c = 2^{d/2}\Gamma(\frac{d}{2}+1)R^{-d}$, $\mu = \frac{d}{d+2}R^2$, $\nu = 2^{d/2}\Gamma(\frac{d}{2}+1)R^{-d}$.

Thus, when $d = 2$, $\mu\nu = 1$ holds in these two special cases.

**Example 3** Define the function $f_d^*(r)$ of $r \geq 0$ by

$$f_d^*(r) = \begin{cases} \frac{d(d+2)}{2}\gamma_d \cdot (1-r^2) \cdots 0 \leq r \leq 1, \\ 0 \qquad\qquad\qquad\qquad\quad \cdots r \geq 1, \end{cases} \tag{4.1.11}$$

where

$$\gamma_d = (2\pi)^{\frac{d}{2}}/\mathrm{Vol}(\mathrm{S}_{d-1}) = 2^{\frac{d}{2}-1}\Gamma(d/2), \tag{4.1.12}$$

$\mathrm{Vol}(\mathrm{S}_{d-1})$ being the Euclidean volume of the $(d-1)$-dimensional unit sphere. And for any fixed $c > 0$, consider the function $M(x) = c^d \cdot f_d^*(c|x|)$ on $\mathbf{R}^d$. Then $M(x)$ also satisfies (4.1.1)(4.1.2) and we have

$$\mu\nu^{\frac{2}{d}} = \frac{2d}{d+4}\left(\frac{4\Gamma(\frac{d+4}{2})}{d+4}\right)^{\frac{2}{d}}. \tag{4.1.13}$$

Indeed, $\mu = c^{-2}\mu_d^*$ and $\nu = c^d\nu_d^*$, where

$$\mu_d^* = \frac{d}{d+4}, \qquad \nu_d^* = \frac{2d(d+2)}{d+4}\gamma_d. \tag{4.1.14}$$

**Theorem** 2 *For each $d \geq 1$ and each function $M(x)$ on $\mathbf{R}^d$ satisfying* (4.1.1)(4.1.2) *we have, for $\mu = \mu_M$ and $\nu = \nu_M$,*

$$\mu\nu^{\frac{2}{d}} \geq \frac{2d}{d+4}\left(\frac{4\Gamma(\frac{d+4}{2})}{d+4}\right)^{\frac{2}{d}}. \tag{4.1.15}$$

*Moreover, the equality holds if and only if $M(x)$ is the function given in* Example 3.

This inequality was not known to the author at the time of the Symposium. It was noticed and proved in December (shortly before the submittence of this article). This being fundamental and simple, the author is not yet sure whether it has not been known. The minimum-giving Example 3 was found by using small deformations, which leads to a simple differential equation of order 1. And once found, the proof is simple. At any rate, I give a sketch of the proof below.

**Sketch of Proof** Let $M(x)$ be as at the beginning of this subsection, with their invariants $\mu$, $\nu$. We shall prove

$$\mu\nu^{\frac{2}{d}} \geq (\mu_d^*)(\nu_d^*)^{\frac{2}{d}} \tag{4.1.16}$$

We may assume that $M(x)$ is rotation invariant, because averaging over $|x| = r$ does not change $\mu$, while $\nu$ either decreases or remains the same. Therefore, $M(x) = f(|x|)$ with some non-negative real valued function $f(r)$ of $r \geq 0$, and

$$\frac{1}{\gamma_d}\int_0^\infty f(r)r^{d-1}dr = 1, \quad \frac{1}{\gamma_d}\int_0^\infty f(r)r^{d+1}dr = \mu, \quad \frac{1}{\gamma_d}\int_0^\infty f(r)^2r^{d-1}dr = \nu. \tag{4.1.17}$$

By a suitable scalar transform (4.1.8) we may assume that $\mu$ is any given positive real number, and so we assume $\mu = \mu_d^*$. We then have

$$\frac{1}{\gamma_d}\int_0^1 f(r)(1-r^2)r^{d-1}dr \geq 1 - \mu_d^* > 0 \tag{4.1.18}$$

from (4.1.17), because the corresponding integral over $(1, \infty)$ is obviously non-positive. Now the Schwarz inequality gives

$$\left(\int_0^1 f_d^*(r)^2r^{d-1}dr\right)\left(\int_0^1 f(r)^2r^{d-1}dr\right) \geq \left(\int_0^1 f_d^*(r)f(r)r^{d-1}dr\right)^2. \tag{4.1.19}$$

Since the first integral on the left hand side is nothing but $\gamma_d\nu_d^*$, by inserting the explicit formula (4.1.11) for $f_d^*(r)$ and by using (4.1.14),(4.1.18) we directly obtain exactly the desired inequality $\nu \geq \nu_d^*$. The last statement of Theorem 2 is clear from the above proof.

**Corollary 4.1.20** *Let* $d = 2$. *Then* $\mu\nu \geq \frac{8}{9}$.

On the other hand, there is no upper bound for $\mu\nu^{2/d}$; indeed, if the support of $M(x)$ is concentrated to the sphere with center $O$ and radius $r$, then $\mu$ is close to $r^2$ while $\nu$ can be as large as possible.

**4.2 – The case of $M_\sigma(z)$ on C** In this case, the center is the origin, and the $\mu$-invariant is given by (3.4.3). The $\nu$-invariant, on the other hand, is not easy to compute; we use the second formula in (4.1.4) using the Fourier dual. Numerical computations suggest that the product $\mu\nu$ is often quite close to 1. For example, when $K = \mathbf{Q}$ (resp. $\mathbf{Q}(\sqrt{-1})$) and $\sigma = 1$, then $1 - \mu\nu = 0.017...$ (resp. 0.018...). So far, the only simple theorem that we were able to prove for this product, besides that $\mu\nu \geq 8/9$ (Cor 4.1.20), is:

**Theorem 3** *Let $K = \mathbf{F}_q(t)$, the rational function field of one variable over a finite field $\mathbf{F}_q$, and $\wp_\infty$ be the infinite prime w.r.t. $t$. As in [5], exclude from $M_\sigma(z)$ and $\tilde{M}_\sigma(z)$ the local $\wp_\infty$ factor. Call $\mu_{\sigma,q}$ (resp. $\nu_{\sigma,q}$) the corresponding $\mu$ ( resp. $\nu$ )-invariant. Then for any fixed $\sigma > 1/2$,*

$$\lim_{q \mapsto \infty} (\mu_{\sigma,q}\nu_{\sigma,q}) = 1. \tag{4.2.1}$$

In the following subsection, we shall give some more explanations on why $\mu\nu$ is close to 1 and how the above theorem may be derived.

**4.3 – Comparison with the Gaussian distribution**

The reason for $\mu\nu$ being close to 1 consists of the following. Roughly speaking,

(i) when $r$ is not so large, the function

$$\tilde{M}^{(2)}(r) := \frac{1}{2\pi} \int_0^{2\pi} |\tilde{M}_\sigma(re^{i\theta})|^2 d\theta \tag{4.3.1}$$

of $r$ is "quite close" to $e^{-\frac{\mu}{2}r^2}$, and on the other hand,

(ii) when $r$ is large, $\tilde{M}^{(2)}(r)$ is "very small".
Note that

$$\nu = \int_0^\infty \tilde{M}^{(2)}(r) r dr, \tag{4.3.2}$$

$$\mu^{-1} = \int_0^\infty e^{-\frac{\mu}{2}r^2} r dr. \tag{4.3.3}$$

As for (i), for example, when $K = \mathbf{Q}$, $\sigma = 1$, the two graphs for $0 \leq r \leq 4$ are so close that they are visually identical when laid one upon the other. As for (ii), the expected order of magnitude of $-\log \tilde{M}^{(2)}(r)$ is not $\frac{\mu}{2}r^2$ but smaller, probably $O(r^{(\sigma+\epsilon)^{-1}})$. But the contribution of this part to $\nu$ being small, this difference does not count much.

Precise quantitative versions of (i)(ii) can be derived from the following inequalities for the local factors $\tilde{M}_{\sigma,\wp}(z)$.

(i)' If $N(\wp)^\sigma > c_0$ and $|z| \leq r_{\sigma,\wp}^{-1}$, then

$$1 - \frac{1}{2}\left(1 + \frac{c_1}{N(\wp)^{2\sigma}}\right) r_{\sigma,\wp}^2 |z|^2 \leq |\tilde{M}_{\sigma,\wp}(z)|^2 \leq \exp\{-\frac{1}{2}\left(1 - \frac{1}{N(\wp)^{2\sigma}}\right) r_{\sigma,\wp}^2 |z|^2\} \tag{4.3.4}$$

where $c_0$, $c_1$ are some absolute positive constants.

(ii)'

$$|\tilde{M}_{\sigma,\wp}(z)|^2 \leq \frac{2}{\pi} \exp(c_2 N(\wp)^{-2\sigma})(r_{\sigma,\wp}|z|)^{-1} \tag{4.3.5}$$

holds for any $z \in \mathbf{C}$, where $c_2$ is some absolute positive constant.

These can be effectively used when the contribution of the set of *primes with the smallest norm* (in $K$) is growingly large. This is the case when $K$ is the rational function field over $\mathbf{F}_q$, with $q \mapsto \infty$, and this is how we can prove Theorem 3 above.

According to the suggestion of one of our colleagues, we have given below a list of (much) more papers than those directly cited in the text. Most of them are more related to the preceeding work [5] of the author on which the present work is based.

## References


[1] H.Bohr, B.Jessen, Über die Wertverteilung der Riemannschen Zetafunktion, Erste Mitteilung, *Acta Math.* **54** (1930),1-35; Zweite Mitteilung, *ibid.* **58** (1932),1-55.

[2] T.Hattori, K.Matsumoto, A limit theorem for Bohr-Jessen's probability measures of the Riemann zeta-function, *J. reine angew. Math.* **507**(1999), 219-232.

[3] Y.Ihara, On the Euler-Kronecker constants of global fields and primes with small norms, in: *Algebraic Geometry and Number Theory, In Honor of Vladimir Drinfeld's 50th Birthday* (V.Ginzburg, ed.), Progress in Math. **253** (2006), 407-451. Birkhäuser.

[4] Y.Ihara, The Euler-Kronecker invariants in various families of global fields, to appear in *Proc. of AGCT 2005* (Arithmetic Geometry and Coding Theory 10) (F.Rodier et al. ed.), Séminaires et congrès, Soc.math.de France.

[5] Y.Ihara, On "M-functions" closely related to the distribution of L'/L-values, to appear in *Publ. RIMS, Kyoto University*; (a short resumé with the same title can be found in: RIMS Kōkyūroku, Bessatsu B4, *Proc. Symp. on Algebraic Number theory and Related Topics*; Dec. 2007 (K.Hashimoto et al., ed.)).

[6] Y.Ihara, V.K.Murty and M.Shimura, On the logarithmic derivatives of Dirichlet L-functions at $s = 1$, Preprint, 2007.

[7] B.Jessen,A.Wintner, Distribution functions and the Riemann zeta function, *Trans.Amer.Math.Soc.* **38** (1935), 48-88

[8] A.Laurinčitas, Limit Theorems for the Riemann Zeta-function, Kluwer, 1996.

[9] E.Lukacs, Characteristic functions (Second Edition), Griffin, London (1970).

[10] E.Lukacs, R.G.Laha, Appications of characteristic functions, Charles Griffin and Co., London (1964).

[11] K.Matsumoto, Discrepancy estimates for the value-distribution of the Riemann zeta-function I *Acta Arith.* **48**(1987),167-190; III *ibid* **50**(1988),315-337.

[12] K.Matsumoto, Value distribution of zeta-functions, in: *Analytic Number Theory, Proc. of the Japanese-French Symposium held in Tokyo, Oct 1988* (K.Nagasaka et al. ed.), Lecture Notes in Math **1434** (1990), Springer Verlag.

[13] K.Matsumoto, Asymptotic probability measures of zeta-functions of algebraic number fields, *J.Number Theory* **40**(1992), 187-210.

[14] E.C.Titchmarsh, The theory of the Riemann zeta-function, Oxford (1951).

[15] G.N.Watson, A treatise on the Theory of Bessel functions, Cambridge (1922).

[16] A.Weil, Basic Number Theory, GMW144 Springer-Verlag, Berlin Heidelberg New York (1967).