On certain mean values and the value-distribution of logarithms of Dirichlet L-functions

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Abstract

We study the value-distribution of Dirichlet L-functions $L(s,\chi)$ in the half-plane $\sigma = \Re s > 1/2$. The main result is that a certain average of the logarithm of $L(s,\chi)$ with respect to χ, or of the Riemann zeta-function $\zeta(s)$ with respect to $\Im s$, can be expressed as an integral involving a density function, which depends only on σ and can be explicitly constructed. Several mean-value estimates on L-functions are essentially used in the proof in the case $1/2 < \sigma \leq 1$.

1 Introduction

Let $s = \sigma + i\tau$ be a complex variable, and $\zeta(s)$ the Riemann zeta-function. In the first half of the 20th century, Bohr (sometimes with Courant, Jessen or Landau) studied the distribution of values of $\log\zeta(s)$ and its derivative $(\zeta'/\zeta)(s)$ extensively. For example it was shown that, for any fixed $\sigma > 1$, the set of values $(\zeta'/\zeta)(\sigma + i\tau)$ $(\tau \in \mathbf{R})$ is everywhere dense in a certain region which is a circular area or an annulus on the complex plane $\mathbf{C}$. As for $\log\zeta(\sigma+i\tau)$, an analogous result holds for $\sigma > 1$, and if $1/2 < \sigma \leq 1$, the set of values of $\log\zeta(\sigma + i\tau)$ is, under a certain fixed choice of the branch of the logarithm, everywhere dense in $\mathbf{C}$ (see Chapter XI of Titchmarsh [20]). In [3], Bohr and Jessen proved the following limit theorem. Let R be an arbitrary rectangle in $\mathbf{C}$, with the edges parallel to the axes. For any $T > 0$, let $V_\sigma(T, R)$ be the Lebesgue measure of the set of all $\tau \in [-T, T]$ for which $\log\zeta(\sigma + i\tau) \in R$ holds. Then the theorem of Bohr and Jessen asserts the existence of the limit

$$W_\sigma(R) = \lim_{T\to\infty}(2T)^{-1}V_\sigma(T, R) \tag{1.1}$$

for any $\sigma > 1/2$. Moreover they proved that this limit can be written as

$$W_\sigma(R) = \int_R \mathcal{F}_\sigma(w)|dw|, \tag{1.2}$$

where $w = u + iv \in \mathbf{C}$, $|dw| = (2\pi)^{-1}dudv$ and $\mathcal{F}_\sigma$ is a continuous, everywhere non-negative function defined over $\mathbf{C}$. The proof of Bohr and Jessen depends on their own geometric study [2] on certain “infinite sums” of planer convex curves. Later, Jessen and Wintner [13], Borchsenius and Jessen [4] developed alternative approaches to the Bohr-Jessen theorem, based on the theory of Fourier transforms. A modern formulation of the Bohr-Jessen theorem, written in terms of weak convergence of probability measures, can be found in Laurinčikas' book [15]. Generalizations of the Bohr-Jessen theorem to more general zeta and L-functions were studied by the second-named author [16], [17], [18].

The behaviour of zeta or L-functions is, generally speaking, quite complicated, so it is natural to consider various types of averages to obtain some definite statements on the value-distribution of them. In the case of the Bohr-Jessen theorem, an average with respect to $\tau = \Im s$ is taken.

Recently, under the motivation of studying Euler-Kronecker constants of global fields (see [7], [8], [12]), averages with respect to characters have been studied by the first-named author [9]. Let K be a global field, χ a character on K, and $L(s,\chi)$ the associated L-function. The main aim of [9] is to prove the existence of the density function $M_\sigma(w)$ defined on $\mathbf{C}$ for which

$$\mathrm{Avg}_\chi \Phi\left(\frac{L'(s,\chi)}{L(s,\chi)}\right) = \int_{\mathbf{C}} M_\sigma(w)\Phi(w)|dw| \tag{1.3}$$

holds for a sufficiently wide class of test functions Φ, where $s = \sigma + i\tau$ is fixed, and Avg_χ means some average with respect to χ. In [9], the following three cases are considered:

(A) K is either the rational number field $\mathbf{Q}$, or an imaginary quadratic field, or a function field over a finite field $\mathbf{F}_q$, and χ are Dirichlet characters on K.

(B) K is a number field having at least two archimedean primes, and χ are normalized unramified Grössencharacters.

(C) $K = \mathbf{Q}$ and $\chi = \chi_{\tau'}$, where $\tau' \in \mathbf{R}$, is defined by $\chi_{\tau'}(p) = p^{-i\tau'}$ for each prime p.

Then in [9], among other things, formula (1.3) is established in the following situation:

(i) When $\sigma = \Re s > 1$, in each of case (A), (B), (C), formula (1.3) holds for any continuous Φ.

(ii) Formula (1.3) for the function field case in case (A) further holds for $\sigma > 3/4$ and Φ is any “character” ψ_z with $z \in \mathbf{C}$ defined by

$$\psi_z(w) = \exp(i\Re(\overline{z}w)); \tag{1.4}$$

or $\sigma > 1/2$ and Φ is any polynomial in z, $\overline{z}$ (and furthermore, for $\sigma > 3/4$ if $\Phi \in L^1 \cap L^\infty$ and the Fourier transform of Φ has compact support, or for $\sigma > 5/6$ if Φ is a standard function in the sense of Weil [21]).

The rigorous meaning of Avg_χ will be given later, but the meaning in the case (C) is to be mentioned here. It is given by

$$\mathrm{Avg}_\chi \phi(\chi_{\tau'}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \phi(\chi_{\tau'})d\tau' \tag{1.5}$$

for any integrable function $\phi(\chi_{\tau'})$ of τ'. In case (C), the associated L-function is

$$\prod_p \left(1-\chi_{\tau'}(p)p^{-s}\right)^{-1} = \prod_p \left(1-p^{-s-i\tau'}\right)^{-1}, \tag{1.6}$$

which is nothing but the Riemann zeta-function $\zeta(s+i\tau')$. Therefore, in this case, the left-hand side of (1.3) is equal to

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Phi\left(\frac{\zeta'}{\zeta}(s+i\tau')\right) d\tau'. \tag{1.7}$$

In particular, if we could choose $\Phi = \mathbf{1}_R$, the characteristic function of the rectangle R, then the integral in (1.7) is the measure of the set of all $\tau' \in [-T,T]$ for which $(\zeta'/\zeta)(s+i\tau') \in R$ holds. Consequently (1.3) in this case would give an analogue of (1.2) for ζ'/ζ.

However, actually, formula (1.3) in case (C) has been shown only for $\sigma > 1$ in [9]. One of the reasons is that, since the Riemann hypothesis (RH, for brevity) has not been proved for the Riemann zeta-function, we cannot exclude the possibility of the existence of zeros in the strip $1/2 < \sigma < 1$, which causes a trouble. In the function field case we know that the analogue of RH is true, so we can go into the critical strip. But there exists another difficulty; still in the function field case, what we have shown in [9] is a partial answer ((ii) above). This is because some relevant estimates proved in [9] is not sufficiently strong.

On the other hand, in the case of $\log\zeta(s)$, Bohr and Jessen proved (1.1) and (1.2) for any $\sigma > 1/2$, without assuming RH. A technical reason of their success is that they used mean value estimates of certain related Dirichlet series quite ingeniously.

Therefore, if we aim to obtain an analogue of (1.3) for the $\log L$ case, we might go further. We search for some analogue of $M_\sigma(w)$ in the $\log L$ case, which we denote by $\mathcal{M}_\sigma(w)$, for which

$$\mathrm{Avg}_\chi \Phi(\log L(s,\chi)) = \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi(w)|dw| \tag{1.8}$$

holds.

In the present paper we will mainly study the case when $K = \mathbf{Q}$, but in the former half of the paper we will work in a more general situation.

In Section 2 we will state our main theorem. The density function $\mathcal{M}_\sigma(w)$ will be constructed and studied in Section 3. After discussing the case $\sigma > 1$ briefly in Section 4, we will proceed to the study of the case

$1/2 < \sigma \leq 1$. In Section 5 we will prepare some auxiliary estimations of relevant Fourier coefficients. The proof of the main thoerem for $1/2 < \sigma \leq 1$ will be described in Sections 6 to 9.

If we assume GRH (the generalized Riemann hypothesis for L-functions), or restrict ourselves to the function field case, then we can even treat the mean values of $\psi(\log L(s,\chi))$ for any *quasi-characters* ψ of $\mathbf{C}$, and this leads us to some stronger conclusions (cf. [11]).

In the following sections, ε denotes an arbitrarily small positive number, not necessarily the same at each occurrence. The Vinogradov symbol $f \ll g$ means $f = O(g)$. The symbol $|A|$ means the cardinality of the set A.

2 Statement of the main result

In Section 1 we mentioned that cases (A), (B), and (C) are studied in [9]. In the present paper our main concern is the case $K = \mathbf{Q}$, therefore we pick up only the following two cases:

(C) $K = \mathbf{Q}$ and $\chi = \chi_{\tau'}$. The meaning of Avg_χ is (1.5), and the associated L-function is $\zeta(s + i\tau')$ as was shown in (1.6).

(A,$\mathbf{Q}$) $K = \mathbf{Q}$ and χ are Dirichlet characters with prime conductors. The associated L-function is the Dirichlet L-function $L(s,\chi)$.

It is necessary to fix the branch of $\log L$. When $\sigma > 1$, the L-function has the Euler product expression

$$L(s,\chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}, \tag{2.1}$$

and so in this half-plane we define

$$\log L(s,\chi) = -\sum_p \mathrm{Log}(1 - \chi(p)p^{-s}), \tag{2.2}$$

where Log means the principal branch.

In the strip $D = \{s \; ; \; 1/2 < \sigma \leq 1\}$, there is the possibility of the existence of zeros of $L(s,\chi)$, since we do not assume GRH. We remove all segments $B_j(\chi) = \{s = \sigma + i\tau_j \; ; \; 1/2 < \sigma \leq \sigma_j\}$ from D, where $\sigma_j + i\tau_j$ are possible zeros (and a possible pole) of $L(s,\chi)$ in D, and put

$$G_\chi = D \setminus \bigcup_j B_j(\chi).$$

At any point $s_0 = \sigma_0 + i\tau_0 \in G_\chi$, we define the value of $\log L(s_0,\chi)$ by the analytic continuation along the horizontal path $\{s = \sigma + i\tau_0 \; ; \; \sigma \geq \sigma_0\}$. In the case when χ is the trivial character $\mathbf{1}$, that is the case of $\zeta(s)$, we write $G = G_{\mathbf{1}}$. When we consider case (C), we fix this G, while in case (A,$\mathbf{Q}$), G_χ varies when χ varies.

In the case (A,$\mathbf{Q}$), the meaning of Avg_χ is as follows. For any prime f, let $X(f)$ be the set of all primitive Dirichlet characters whose conductor is f, and $X'(f)$ be a subset of $X(f)$ for which

$$\lim_{f\to\infty}\frac{|X'(f)|}{|X(f)|}=1 \tag{2.3}$$

holds. Consider any complex-valued function $\phi(\chi)$ of χ which is defined for each $\chi\in X'(f)$ for each prime f. Let

$$\mathrm{Avg}_{X'(f)}\phi(\chi)=\frac{\sum_{\chi\in X'(f)}\phi(\chi)}{|X(f)|} \tag{2.4}$$

and

$$\mathrm{Avg}_{f\le m}\phi(\chi)=\frac{\sum_{f\le m}\mathrm{Avg}_{X'(f)}\phi(\chi)}{\sum_{f\le m}1}, \tag{2.5}$$

where m is a positive integer, and f runs over all prime numbers not larger than m. Then, the meaning of Avg_χ in this case is

$$\mathrm{Avg}_\chi\phi(\chi)=\lim_{m\to\infty}\left(\mathrm{Avg}_{f\le m}\phi(\chi)\right). \tag{2.6}$$

Note that, if ϕ is bounded, this average will not change if we choose $X'(f)$ smaller keeping condition (2.3). Note also the following. As long as ϕ is bounded, the limit value (2.6) remains the same if the denominator of the right-hand side of (2.4) is replaced by $|X'(f)|$ (which looks more natural but is less convenient).

At the end of this section we will prove the following

Proposition 1 *Fix any s with $1/2<\Re s\le 1$. Then*

$$X'(f)=X'(f,s):=\{\chi\in X(f)\ ;\ s\in G_\chi\}$$

satisfies (2.3).

In view of this proposition, hereafter we fix $X'(f)$ as follows. When $\Re s>1$, simply put $X'(f)=X(f)$. When $1/2<\Re s\le 1$, choose $X'(f)=X'(f,s)$ as that defined by this proposition, and define $L(s,\chi)$ for each $\chi\in X'(f)$ as above by the analytic continuation inside G_χ.

The main aim of the present paper is to prove the following theorem.

Theorem 1 *Let $s=\sigma+i\tau\in\mathbf{C}$ be fixed, with $\sigma=\Re s>1/2$. There exists a density function $\mathcal{M}_\sigma(w)$, which is a continuous non-negative function defined on $\mathbf{C}$, for which*

$$\mathrm{Avg}_\chi\Phi(\log L(s,\chi))=\int_{\mathbf{C}}\mathcal{M}_\sigma(w)\Phi(w)|dw| \tag{2.7}$$

holds in both the cases (C) *and* (A,$\mathbf{Q}$)*. The test function* Φ *is one of the following* (*or any finite linear combination of them*):

(i) Φ *is any continuous bounded function ,*

(ii) Φ *is the characteristic function of either a compact subset of* $\mathbf{C}$ *or the complement of such a subset.* (*Consequently we find that* $\mathcal{M}_\sigma(w)$ *is equal to* $\mathcal{F}_\sigma(w)$ *in* (1.2).)

In the above theorem, and also in what follows, when we state a formula for Avg_χ, it will always include the claim that the limit exists.

Note that, when $\sigma > 1$, Φ can be any continuous function; see Theorem 2 in Section 4.

In Case (A), the condition of Φ can be relaxed considerably if we assume GRH ([11]). The main point of the present paper is that we can prove our theorem *unconditionally.*

In Case (C), the meaning of Avg_χ is given by (1.5), hence (2.7) is

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Phi(\log\zeta(s+i\tau'))d\tau' = \int_{\mathbf{C}}\mathcal{M}_\sigma(w)\Phi(w)|dw|. \tag{2.8}$$

On the left-hand side, $\log\zeta(s+i\tau')$ is not defined when $s+i\tau'$ is a zero or the pole of $\zeta(s)$, but the integral is well-defined. Therefore in case (C) it is not necessary to exclude such situation.

In Case (A,$\mathbf{Q}$), the meaning of Avg_χ is (2.6). Since $|X(f)| = f-2$ for any prime f and $\sum_{f\le m}1 = \pi(m)$, the number of primes not larger than m, assertion (2.7) in this case is

$$\lim_{m\to\infty}\frac{1}{\pi(m)}\sum_{\substack{2<f\le m\\ f:\text{prime}}}\frac{1}{f-2}\sum_{\chi\in X'(f)}\Phi(\log L(s,\chi)) = \int_{\mathbf{C}}\mathcal{M}_\sigma(w)\Phi(w)|dw|. \tag{2.9}$$

We conclude this section with the proof of Proposition 1. It is an immediate corollary of the following

Proposition 2 *For any fixed* $T\ge 2$ *and* $1/2<\sigma_0\le 1$*, let* $X''(f)$ *be the set of all* $\chi\in X_f$ *such that* $L(s,\chi)$ *has no zeros* s *with* $\Re s\ge\sigma_0$ *and* $|\Im s|\le T$*. Then* $\lim_{f\to\infty}|X''(f)|/|X(f)| = 1$.

Proof. Let $N(\sigma_0,T,\chi)$ denote the number of zeros of $L(s,\chi)$ with $\Re s\ge\sigma_0$ and $|\Im s|\le T$. Then Theorem 12.1 of Montgomery [19] asserts

$$\sum_{\chi\in X(f)}N(\sigma_0,T,\chi)\ll (fT)^{A(\sigma_0)}(\log fT)^{14}$$

with some $A(\sigma_0)<1$. (The choice given there is $A(\sigma_0)=3(1-\sigma_0)/(2-\sigma_0)$ (resp. $2(1-\sigma_0)/\sigma_0$) for $1/2<\sigma_0\le 4/5$ (resp. $4/5\le\sigma_0\le 1$).) Therefore

$$\begin{aligned}\frac{|X(f)\setminus X''(f)|}{|X(f)|} &\le \frac{1}{|X(f)|}\sum_{\chi\in X(f)}N(\sigma_0,T,\chi)\\ &\ll f^{A(\sigma_0)-1}T^{A(\sigma_0)}(\log fT)^{14},\end{aligned} \tag{2.10}$$

which tends to 0 as f tends to ∞. This proves the proposition.

3 The construction of the density function and its Fourier dual

In the following three sections we assume that K is a global field, and χ is a Dirichlet character on K. Though in the latter half of the present paper we only need the case when $K = \mathbf{Q}$, we work with a more general situation because of our later purposes. The associated L-function is defined by

$$L(s,\chi) = \prod_{\wp} \left(1 - \chi(\wp)N(\wp)^{-s}\right)^{-1},$$

where $\wp$ runs over non-archimedean primes of K and $N(\wp)$ is the norm of $\wp$. Let $\sigma > 0$, and let P be a finite set of non-archimedean primes. Define

$$L_P(s,\chi) = \prod_{\wp\in P} \left(1 - \chi(\wp)N(\wp)^{-s}\right)^{-1} \tag{3.1}$$

and

$$\log L_P(s,\chi) = -\sum_{\wp\in P} \mathrm{Log}\left(1 - \chi(\wp)N(\wp)^{-s}\right). \tag{3.2}$$

Let

$$T = \{t \in \mathbf{C}\ ;\ |t| = 1\}, \qquad T_P = \prod_{\wp\in P} T,$$

and define $g_{\sigma,P} : T_P \to \mathbf{C}$ by

$$g_{\sigma,P}(\mathbf{t}_P) = \sum_{\wp\in P} g_{\sigma,\wp}(t_\wp) \tag{3.3}$$

with $\mathbf{t}_P = (t_\wp)_{\wp\in P} \in T_P$ and

$$g_{\sigma,\wp}(t_\wp) = -\mathrm{Log}\left(1 - t_\wp N(\wp)^{-\sigma}\right). \tag{3.4}$$

Then, if P is coprime with the modulus of χ, we can write

$$\log L_P(s,\chi) = g_{\sigma,P}\left(\chi_P N(P)^{-i\tau}\right), \tag{3.5}$$

where $\chi_P = (\chi(\wp))_{\wp\in P} \in T_P$ and $N(P)^{-i\tau} = (N(\wp)^{-i\tau})_{\wp\in P} \in T_P$.

We first prove the existence of the density function $\mathcal{M}_{\sigma,P}$ which is characterized by the following proposition.

Proposition 3 *For any $\sigma > 0$, there exists a function (or Schwartz distribution if $|P| = 1$) $\mathcal{M}_{\sigma,P} : \mathbf{C} \to \mathbf{R}$, which satisfies*

$$\int_{\mathbf{C}} \mathcal{M}_{\sigma,P}(w)\Phi(w)|dw| = \int_{T_P} \Phi(g_{\sigma,P}(\mathbf{t}_P))d^*\mathbf{t}_P \tag{3.6}$$

for any continuous function Φ *on* $\mathbf{C}$, *where* $d^*\mathbf{t}_P$ *is the normalized Haar measure on* T_P. *The function* $\mathcal{M}_{\sigma,P}$ *is compactly supported, non-negative,* $\mathcal{M}_{\sigma,P}(\overline{w}) = \mathcal{M}_{\sigma,P}(w)$, *and*

$$\int_{\mathbf{C}} \mathcal{M}_{\sigma,P}(w)|dw| = 1. \tag{3.7}$$

This is the analogue of Theorem 1 of [9] in the L'/L case. A different point is that, in the L'/L case the corresponding $g_{\sigma,\wp}$ function has the property of sending the unit circle to another circle, but in the present case the image of the $g_{\sigma,\wp}$ function is a certain convex curve, not a circle.

We first consider the case when P consists of only one element, $P = \{\wp\}$. In this case $T_\wp = T$, $t_\wp = e^{i\theta} \in T_\wp$, and $d^*t_\wp = (2\pi)^{-1}d\theta$. Let $z = re^{i\theta} \in \mathbf{C}$ ($0 \le r < 1$, $0 \le \theta < 2\pi$), and $w = w(z) = -\mathrm{Log}(1 - re^{i\theta})$. Fix a number $\rho_{\sigma,\wp}$ satisfying $N(\wp)^{-\sigma} < \rho_{\sigma,\wp} < 1$, and denote by $A(\sigma,\wp)$ the open region surrounded by the curve $w = -\mathrm{Log}(1 - \rho_{\sigma,\wp}e^{i\theta})$. Then $w = w(z)$ gives a one-to-one correspondence from the open disc $\{z\ ;\ |z| < \rho_{\sigma,\wp}\}$ to $A(\sigma,\wp)$. Since the Jacobian of this mapping is $r/|1 - re^{i\theta}|^2$, we have

$$\begin{aligned}\int_{T_\wp} \Phi(g_{\sigma,\wp}(t_\wp))d^*t_\wp &= \frac{1}{2\pi}\int_0^{2\pi} \Phi\left(-\mathrm{Log}(1 - N(\wp)^{-\sigma}e^{i\theta})\right) d\theta \\ &= \frac{1}{2\pi}\int\int_{A(\sigma,\wp)} \Phi(w)\delta(r - N(\wp)^{-\sigma})\frac{|1 - re^{i\theta}|^2}{r}dudv, \end{aligned} \tag{3.8}$$

where $\delta(\cdot)$ stands for the Dirac delta distribution and $w = u+iv$. Therefore, if we define

$$\mathcal{M}_{\sigma,\wp}(w) = \frac{|1 - re^{i\theta}|^2}{r}\delta(r - N(\wp)^{-\sigma}) \tag{3.9}$$

for $w \in A(\sigma,\wp)$ and $\mathcal{M}_{\sigma,\wp}(w) = 0$ otherwise, then the right-hand side of (3.8) is equal to

$$\int_{\mathbf{C}} \mathcal{M}_{\sigma,\wp}(w)\Phi(w)|dw|,$$

hence (3.6) for $P = \{\wp\}$ follows.

For general P, we can construct the $\mathcal{M}_{\sigma,P}$ satisfying (3.6) by the convolution product, that is, if $P = P' \cup \{\wp\}$, defined by

$$\mathcal{M}_{\sigma,P}(w) = \int_{\mathbf{C}} \mathcal{M}_{\sigma,P'}(w')\mathcal{M}_{\sigma,\wp}(w - w')|dw'|. \tag{3.10}$$

The other statements of Proposition 3 are clear from the construction.

Remark 1. Formula (3.6) in Proposition 3 is valid also if Φ is the characteristic function of either a compact subset of $\mathbf{C}$ or the complement of such a subset. This can be shown by approximating the characteristic function by suitable continuous functions. (cf. Section 4.3 of [10].)

Remark 2. Let U be a compact subset of $\mathbf{C}$. By Remark 1 we can choose $\Phi = \mathbf{1}_U$, the characteristic function of U. Then (3.6) implies

$$\int_U \mathcal{M}_{\sigma,P}(w)|dw| = \mathrm{Vol}(g_{\sigma,P}^{-1}(U)),$$

where the volume on the right-hand side is measured by $d^*\mathbf{t}_P$. Therefore the support of $\mathcal{M}_{\sigma,P}$ is the image of the mapping $g_{\sigma,P}$.

Next we consider the Fourier transform of $\mathcal{M}_{\sigma,P}$. Let $\psi_z(w)$ be as in (1.4), and define

$$\widetilde{\mathcal{M}}_{\sigma,\wp}(z) = \int_{\mathbf{C}} \mathcal{M}_{\sigma,\wp}(w)\psi_z(w)|dw|. \tag{3.11}$$

By Proposition 3 we see that

$$\begin{aligned}\widetilde{\mathcal{M}}_{\sigma,\wp}(z) &= \int_T \psi_z(g_{\sigma,\wp}(t_\wp))d^*t_\wp \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp\left(i\Re\left\{\overline{z}(-\mathrm{Log}(1-e^{i\theta}N(\wp)^{-\sigma}))\right\}\right)d\theta.\end{aligned} \tag{3.12}$$

Applying Theorem 13 of Jessen-Wintner [13], we obtain

$$\widetilde{\mathcal{M}}_{\sigma,\wp}(z) = O((1+|z|)^{-1/2}) \tag{3.13}$$

if $N(\wp)$ is sufficiently large, say, $N(\wp) > N^*$. Also it is clear from (3.12) that

$$|\widetilde{\mathcal{M}}_{\sigma,\wp}(z)| \le 1 \tag{3.14}$$

for any $\wp$. Therefore, if we define

$$\widetilde{\mathcal{M}}_{\sigma,P}(z) = \prod_{\wp\in P} \widetilde{\mathcal{M}}_{\sigma,\wp}(z) \tag{3.15}$$

for general P, we have

$$\widetilde{\mathcal{M}}_{\sigma,P}(z) = O((1+|z|)^{-|P^*|/2}), \tag{3.16}$$

where $P^* = \{\wp \in P\ ;\ N(\wp) > N^*\}$, and

$$|\widetilde{\mathcal{M}}_{\sigma,P}(z)| \le 1 \tag{3.17}$$

for any P.

Let P_0 be a finite set of non-archimedean primes with $|P_0^*| > 4$. Then from (3.16) and (3.17) we have

(a) $\widetilde{\mathcal{M}}_{\sigma,P_0} \in L^t$ for any $t \in [1,+\infty]$,

(b) $|\widetilde{\mathcal{M}}_{\sigma,P}(z)| \leq |\widetilde{\mathcal{M}}_{\sigma,P_0}(z)|$ for any $P \supset P_0$.

These (a), (b) correspond to (a), (b) in Section 3.11 of [9].

Let $y > 0$, and consider the case $P = P_y = \{\wp\ ;\ N(\wp) \leq y\}$. Our next aim is to prove the fact corresponding to Section 3.11 (c) (or Theorem 4 in Section 3.6) of [9], that is, $\widetilde{\mathcal{M}}_{\sigma,P}(z)$ converges to a certain function $\widetilde{\mathcal{M}}_\sigma(z)$ uniformly in any compact set when $y \to \infty$.

In the L'/L case, the corresponding statement was proved in [9] by using an explicit infinite series expression of the Fourier transform of the density function involving Bessel functions. In the present case we apply a different method, similar to the argument developed in Section 3 of [18].

Let $\zeta = N(\wp)^{-\sigma}e^{i\theta}$. Then $w = w(\zeta) = -\mathrm{Log}(1-\zeta)$ is holomorphic in ζ for $|\zeta| < 1$. Hence $\Re w$, $\Im w$ are harmonic in ζ, and so is

$$\Re(\overline{z}w) = \Re z \Re w + \Im z \Im w.$$

By the mean value theorem for harmonic functions we have

$$\frac{1}{2\pi}\int_0^{2\pi} \Re(\overline{z}w)d\theta = 0. \tag{3.18}$$

From (3.12) and (3.18) we can write

$$\widetilde{\mathcal{M}}_{\sigma,\wp}(z) - 1 = \frac{1}{2\pi}\int_0^{2\pi} \{\exp(i\Re(\overline{z}w)) - 1 - i\Re(\overline{z}w)\}\, d\theta. \tag{3.19}$$

Since $|e^{ix} - 1 - ix| \ll x^2$ for any real x (by the Taylor expansion for small $|x|$, and by the fact $|e^{ix}| = 1$ for large $|x|$), we have

$$\begin{aligned} |\widetilde{\mathcal{M}}_{\sigma,\wp}(z) - 1| &\ll \int_0^{2\pi} |\Re(\overline{z}w)|^2 d\theta \\ &\leq |z|^2 \int_0^{2\pi} |w|^2 d\theta \ll |z|^2 N(\wp)^{-2\sigma}. \end{aligned} \tag{3.20}$$

Let $P = P_y$, $P' = P_{y'}$, where $y' > y$. Denote all the elements of the set $P' \setminus P$ by $\wp_1, \ldots, \wp_n$, and put $P(j) = P \cup \{\wp_1, \ldots, \wp_j\}$. Then

$$\begin{aligned} |\widetilde{\mathcal{M}}_{\sigma,P'}(z) - \widetilde{\mathcal{M}}_{\sigma,P}(z)| &\leq \sum_{j=1}^n |\widetilde{\mathcal{M}}_{\sigma,P(j)}(z) - \widetilde{\mathcal{M}}_{\sigma,P(j-1)}(z)| \\ = \sum_{j=1}^n |\widetilde{\mathcal{M}}_{\sigma,P(j-1)}(z)| \cdot |\widetilde{\mathcal{M}}_{\sigma,\wp_j}(z) - 1| &\ll |z|^2 \sum_{j=1}^n N(\wp_j)^{-2\sigma} \end{aligned} \tag{3.21}$$

by (3.17) and (3.20).

Now we assume $\sigma > 1/2$. Then the sum on the right-hand side of the above tends to 0 when $y \to \infty$. Therefore we can conclude:

(c) When $y \to \infty$, $\widetilde{\mathcal{M}}_{\sigma,P}(z)$ $(P = P_y)$ is convergent to a certain function $\widetilde{\mathcal{M}}_\sigma(z)$ uniformly in $\{z\ ;\ |z| \leq a\}$ for any $a > 0$.

From (a), (b) and (c), similarly to Section 3.11 of [9], we can now obtain

Proposition 4 *When $P = P_y$ and $y \to \infty$, $\widetilde{\mathcal{M}}_{\sigma,P}(z)$ converges to $\widetilde{\mathcal{M}}_\sigma(z)$ uniformly in $\sigma \geq 1/2 + \varepsilon$ (for any $\varepsilon > 0$) and $z \in \mathbf{C}$. The limit function $\widetilde{\mathcal{M}}_\sigma(z)$ is hence continuous in σ and z. Moreover, for each $\sigma > 1/2$, the function $\widetilde{\mathcal{M}}_\sigma(z)$ in z belongs to L^t $(1 \leq t \leq +\infty)$, and the above convergence is also L^t-convergence.*

Furthermore, from (3.17) we have

$$|\widetilde{\mathcal{M}}_\sigma(z)| \leq 1, \tag{3.22}$$

while from (3.16) we have

$$\widetilde{\mathcal{M}}_\sigma(z) = O((1 + |z|)^{-n}) \tag{3.23}$$

for any $n \geq 1$.

From the definition (3.15) we see that $\widetilde{\mathcal{M}}_{\sigma,P}$ is the Fourier transform of $\mathcal{M}_{\sigma,P}$, and hence

$$\mathcal{M}_{\sigma,P}(w) = \int_{\mathbf{C}} \widetilde{\mathcal{M}}_{\sigma,P}(z)\psi_{-w}(z)|dz|. \tag{3.24}$$

We now prove that, as $y \to \infty$, the function $\mathcal{M}_{\sigma,P}$ converges to

$$\mathcal{M}_\sigma(w) = \int_{\mathbf{C}} \widetilde{\mathcal{M}}_\sigma(z)\psi_{-w}(z)|dz|. \tag{3.25}$$

The integral on the right-hand side converges absolutely because of (3.23). From (3.24) and (3.25) we have

$$|\mathcal{M}_{\sigma,P}(w) - \mathcal{M}_\sigma(w)| \leq \int_{\mathbf{C}} |\widetilde{\mathcal{M}}_{\sigma,P}(z) - \widetilde{\mathcal{M}}_\sigma(z)||dz|. \tag{3.26}$$

Let $\varepsilon > 0$, and fix a P_0 with $|P_0^*| > 4$. In view of (3.16), (b) and (3.23), we can find a sufficiently large $R = R(\varepsilon, \sigma, P_0) > 0$ for which

$$\int_{|z| \geq R} |\widetilde{\mathcal{M}}_{\sigma,P}(z) - \widetilde{\mathcal{M}}_\sigma(z)||dz| < \varepsilon \tag{3.27}$$

holds for any $P \supset P_0$. Furthermore Proposition 4 implies that there exists a sufficiently large $y = y(\varepsilon, \sigma, R) > 0$ for which

$$|\widetilde{\mathcal{M}}_{\sigma,P}(z) - \widetilde{\mathcal{M}}_\sigma(z)| < \frac{\varepsilon}{2\pi R^2} \tag{3.28}$$

holds for any z, where $P = P_y$. From (3.26), (3.27) and (3.28) we have

$$|\mathcal{M}_{\sigma,P}(w) - \mathcal{M}_\sigma(w)| < \frac{\varepsilon}{2\pi R^2} \int_{|z|<R} |dz| + \varepsilon = 2\varepsilon \tag{3.29}$$

for $P = P_y$. This implies the first assertion of the following proposition.

Proposition 5 *Let $\sigma > 1/2$. When $P = P_y$ and $y \to \infty$, $\mathcal{M}_{\sigma,P}(w)$ converges to $\mathcal{M}_\sigma(w)$ uniformly in w. The limit function $\mathcal{M}_\sigma(w)$ is continuous in w, non-negative, tends to 0 when $|w| \to \infty$, $\mathcal{M}_\sigma(\overline{w}) = \mathcal{M}_\sigma(w)$, and*

$$\int_{\mathbf{C}} \mathcal{M}_\sigma(w)|dw| = 1. \tag{3.30}$$

The functions $\mathcal{M}_\sigma$ and $\widetilde{\mathcal{M}}_\sigma$ are Fourier duals of each other.

The remaining part of the proposition: Non-negativity and $\mathcal{M}_\sigma(\overline{w}) = \mathcal{M}_\sigma(w)$ easily follow from Proposition 3. Since $\mathcal{M}_{\sigma,P}$ is compactly supported for any finite P (Proposition 3), from (3.29) we see that $\mathcal{M}_\sigma(w) \to 0$ as $|w| \to \infty$. From (3.7) and the uniformity of convergence we have

$$\int_{\mathbf{C}} \mathcal{M}_\sigma(w)|dw| \leq 1. \tag{3.31}$$

Hence $\mathcal{M}_\sigma \in L^1$, so its Fourier transform is continuous, to be identical with $\widetilde{\mathcal{M}}_\sigma$ pointwisely. Therefore

$$\widetilde{\mathcal{M}}_\sigma(z) = \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\psi_z(w)|dw|. \tag{3.32}$$

In particular,

$$\int_{\mathbf{C}} \mathcal{M}_\sigma(w)|dw| = \widetilde{\mathcal{M}}_\sigma(0). \tag{3.33}$$

But $\widetilde{\mathcal{M}}_{\sigma,\wp}(0) = 1$ by (3.12), so $\widetilde{\mathcal{M}}_{\sigma,P}(0) = 1$ for any P, and hence also $\widetilde{\mathcal{M}}_\sigma(0) = 1$. This completes the proof of Proposition 5.

Remark 3. The existence of the density function was already proved by Theorem 19 of [13], at least in Case (C). Here we prefer, however, the above more analytic way of construction. Some more properties of $\widetilde{\mathcal{M}}_\sigma(z)$ (and its two-variable version) are studied in Section 4 of [11].

4 The value-distribution in the case $\Re s > 1$

In this section we consider the case when $\sigma = \Re s > 1$. Here we discuss all the cases (A), (B), (C) stated in Section 1. The meaning of Avg_χ in case (A), when K is an imaginary quadratic field or a function field, is similar to (2.6). In the function field case, we fix one prime divisor $\wp_\infty$ which is treated as being archimedean. The character χ runs over all Dirichlet characters on K, whose conductor is a prime divisor, and satisfying $\chi(\wp_\infty) = 1$. The definition of characters in case (C) was given in Section 1. For the details in case (B), see Section 4 of [9]. For any χ, by $\mathbf{f}_\chi$ we mean the conductor of χ. We first quote the following lemma.

Lemma 1 (Lemma 4.3.1 of [9]) *Let χ runs over any one of the above indicated families of characters on K, but in case* (A)*, exclude finitely many χ such that $\mathbf{f}_\chi \in P$. Then we have*

$$\mathrm{Avg}_\chi(\Psi(\chi_P)) = \int_{T_P} \Psi(\mathbf{t}_P) d^*\mathbf{t}_P \tag{4.1}$$

for any continuous function $\Psi : T_P \to \mathbf{C}$.

Based on this lemma, we can prove the following theorem.

Theorem 2 *For any $s \in \mathbf{C}$ with $\sigma = \Re s > 1$, in each of case* (A), (B), (C),

$$\mathrm{Avg}_\chi \Phi(\log L(s,\chi)) = \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi(w)|dw| \tag{4.2}$$

holds for any continuous function Φ on $\mathbf{C}$.

This corresponds to Theorem 6 of [9]. Since the proof goes just analogously, we sketch briefly.

Choosing $\Psi = \Phi \circ g_{\sigma,P}$ in Lemma 1 and combining with Proposition 3, we obtain

$$\begin{aligned} \mathrm{Avg}_\chi(\Phi(\log L_P(s,\chi)) &= \int_{T_P} \Phi(g_{\sigma,P}(\mathbf{t}_P)) d^*\mathbf{t}_P \\ &= \int_{\mathbf{C}} \mathcal{M}_{\sigma,P}(w)\Phi(w)|dw|. \end{aligned} \tag{4.3}$$

In Lemma 1 we excluded finitely many χ, but it does not affect the value of Avg_χ.

Since $\sigma > 1$, the image of $g_{\sigma,P}$ remains bounded when $|P| \to \infty$. This implies, by Remark 2, that the support of $\mathcal{M}_\sigma$ is also bounded. Therefore, to prove Theorem 2, we may assume that Φ is compactly supported, hence is uniformly continuous. Moreover, $\log L_P(s,\chi)$ tends to $\log L(s,\chi)$ when $|P| \to \infty$ uniformly in any compact subset of the half-plane $\sigma > 1$. Therefore letting $|P| \to \infty$ on the both sides of (4.3), we obtain (4.2), because on the right-hand side $\mathcal{M}_{\sigma,P}(w)$ tends to $\mathcal{M}_\sigma(w)$ by Proposition 5. This completes the proof of Theorem 2.

Remark 4. Our definition of Avg_χ in the case (A,$\mathbf{Q}$) is, in the present paper, given by (2.6). However it is possible to consider a simpler form of average, that is

$$\lim_{N(\mathbf{f})\to\infty} \frac{1}{|X(\mathbf{f})|} \sum_{\chi \in X(\mathbf{f})} \phi(\chi) \tag{4.4}$$

(where $N(\mathbf{f})$ is the norm of $\mathbf{f}$ and $X(\mathbf{f})$ is the set of all characters of conductor $\mathbf{f}$). It is possible to prove the analogue of Theorem 2 for the average of form (4.4). See Theorem 4 of [11].

5 Some estimation of Fourier coefficients

Let $z_1, z_2 \in \mathbf{C}$, and

$$\psi_{z_1,z_2}(w) = \exp\left(\frac{i}{2}(z_1\overline{w} + z_2 w)\right). \tag{5.1}$$

Note that $\psi_z(w) = \psi_{z,\overline{z}}(w)$. The purpose of this section is to study the coefficients of the Fourier expansion of $\psi_{z_1,z_2}(g_{\sigma,\wp}(t_\wp))$. This is an analogue of Section 5 of [9], where the same problem is discussed for

$$g_{\sigma,\wp}(t_\wp) = \frac{t_\wp \log N(\wp)}{t_\wp - N(\wp)^\sigma} \qquad (L'/L \text{ case}), \tag{5.2}$$

which is used for the study of $(L'/L)(s,\chi)$. We will prove estimates analogous to Corollary 5.2.13 and Corollary 5.2.18 of [9]. In [9], those corollaries are proved only in the case $z_2 = \overline{z_1}$. Therefore in this section we treat the $\log L$ case and the L'/L case in a parallel manner, in order to prove the results for general z_1 and z_2 in both cases.

In this section we use the abbreviation $q = N(\wp)^\sigma$ ($\sigma > 0$), $\lambda = \log N(\wp)$, $t = t_\wp$, $g = g_{\sigma,\wp}$. Then

$$g(t) = \begin{cases} -\log(1 - t/q) & (\log L \text{ case}), \\ \lambda t/(t-q) & (L'/L \text{ case}). \end{cases}$$

Denote by $g(t) = \sum_{n=1}^\infty a_n (t/q)^n$ the power series expansion of $g(t)$ in the region $|t| < q$. Then $a_n = 1/n$ ($\log L$ case), or $= -\lambda$ (L'/L case). Hence the power series expansion of $g(t)^k$ ($k \geq 1$) is given by

$$g(t)^k = \sum_{n=1}^\infty a_n^{(k)} (t/q)^n, \tag{5.3}$$

where

$$a_n^{(k)} = \sum_{\substack{n = n_1 + \cdots + n_k \\ n_\nu \geq 1}} a_{n_1} \cdots a_{n_k}$$

is equal to

$$\sum_{\substack{n = n_1 + \cdots + n_k \\ n_\nu \geq 1}} \frac{1}{n_1 \cdots n_k} \ (\log L \text{ case}); \quad (-\lambda)^k \sum_{\substack{n = n_1 + \cdots + n_k \\ n_\nu \geq 1}} 1 \ (L'/L \text{ case}). \tag{5.4}$$

In particular,

$$a_n^{(k)} = 0 \quad \text{if} \quad k > n. \tag{5.5}$$

Note that

$$|a_n^{(k)}| \leq \sum_{\substack{n = n_1 + \cdots + n_k \\ n_\nu \geq 1}} 1 = \binom{n-1}{k-1} \tag{5.6}$$

in the $\log L$ case, while

$$|a_n^{(k)}| \leq \lambda^k \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \tag{5.7}$$

for the L'/L case.

For $z \in \mathbf{C}$ and $|t| < q$, we have

$$\exp\left(\frac{i}{2} z g(t)\right) = 1 + \sum_{k=1}^{\infty} \frac{(iz/2)^k}{k!} g(t)^k. \tag{5.8}$$

Substituting (5.3), (5.4) and (5.5) into the right-hand side, we have

$$\exp\left(\frac{i}{2} z g(t)\right) = \sum_{n=0}^{\infty} \lambda_n(z)(t/q)^n, \tag{5.9}$$

with

$$\lambda_n(z) = \begin{cases} G_n^*(iz/2) & (\log L \text{ case}), \\ G_n(-\lambda iz/2) & (L'/L \text{ case}) \end{cases} \tag{5.10}$$

for $n \geq 0$, where

$$G_n(x) = \sum_{k=1}^{n} \frac{1}{k!} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} x^k \quad (n \geq 1); \quad G_0(x) = 1, \tag{5.11}$$

and

$$G_n^*(x) = \sum_{k=1}^{n} \frac{1}{k!} \Bigg(\sum_{\substack{n=n_1+\cdots+n_k \\ n_\nu \geq 1}} \frac{1}{n_1 \cdots n_k} \Bigg) x^k \ \ (n \geq 1); \quad G_0^*(x) = 1. \tag{5.12}$$

Because of (5.6), we have

$$0 \leq G_n^*(x) \leq G_n(x) \qquad (x \geq 0). \tag{5.13}$$

When $|t| = 1$, from (5.9) we have

$$\psi_{z_1,z_2}(g(t)) = \sum_{n \in \mathbf{Z}} A(n; z_1, z_2) t^n, \tag{5.14}$$

where

$$A(n; z_1, z_2) = A_{\sigma,\wp}(n; z_1, z_2) = \sum_{\substack{l,m \geq 0 \\ l-m=n}} \frac{\lambda_l(z_2)\lambda_m(z_1)}{q^{l+m}}. \tag{5.15}$$

These are the Fourier coefficients of $\psi_{z_1,z_2}(g(t))$. Therefore

$$A(n; z_1, z_2) = \int_T \psi_{z_1,z_2}(g(t)) t^{-n} d^*t. \tag{5.16}$$

Let $Z = \max\{|z_1|, |z_2|\}$, and define

$$x_0 = \begin{cases} Z/2 & (\log L \text{ case}), \\ \lambda Z/2 & (L'/L \text{ case}). \end{cases} \tag{5.17}$$

We now prove the following estimate.

Proposition 6

$$|A(n; z_1, z_2)| \le \frac{1}{q^{|n|}} G_{|n|}(x_0) \exp\left(\frac{2x_0}{q-1}\right). \tag{5.18}$$

Proof. First of all, since

$$A(-n; z_1, z_2) = A(n; z_2, z_1), \tag{5.19}$$

we may assume that $n \ge 0$. From (5.10), (5.13) and the facts $|G_n(z)| \le G_n(|z|)$, $|G_n^*(z)| \le G_n^*(|z|)$, we have $|\lambda_n(z_j)| \le G_n(x_0)$ $(j = 1, 2)$. Hence from (5.15) we have

$$|A(n; z_1, z_2)| \le \sum_{\substack{l,m\ge 0 \\ l-m=n}} \frac{G_l(x_0) G_m(x_0)}{q^{l+m}} = \frac{1}{q^n} \sum_{m \ge 0} \frac{G_m(x_0) G_{m+n}(x_0)}{q^{2m}}. \tag{5.20}$$

Let

$$L_m(x) = \sum_{k=0}^{m} \frac{1}{k!} \begin{pmatrix} m \\ k \end{pmatrix} x^k. \tag{5.21}$$

Then

$$\begin{aligned} L_m(x) &= 1 + \sum_{k=1}^{m-1} \frac{1}{k!} \left\{ \begin{pmatrix} m-1 \\ k \end{pmatrix} + \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} \right\} x^k + \frac{1}{m!} x^m \\ &= \sum_{k=0}^{m-1} \frac{1}{k!} \begin{pmatrix} m-1 \\ k \end{pmatrix} x^k + \sum_{k=1}^{m} \frac{1}{k!} \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} x^k \\ &= L_{m-1}(x) + G_m(x), \end{aligned}$$

hence

$$L_m(x) = \sum_{\mu=0}^{m} G_\mu(x). \tag{5.22}$$

Lemma 2 *For non-negative integers m, n and $x \ge 0$, we have*

$$G_{m+n}(x) \le G_n(x) L_m(x). \tag{5.23}$$

The lemma is obvious when $n = 0$, so we assume $n \geq 1$. The coefficient of x^k in $G_{m+n}(x)$ is

$$\frac{1}{k!}\begin{pmatrix} m+n-1 \\ k-1 \end{pmatrix} = \frac{1}{k!}\sum_{\substack{0\leq\mu\leq m, 1\leq\nu\leq n \\ \mu+\nu=k}} \begin{pmatrix} m \\ \mu \end{pmatrix}\begin{pmatrix} n-1 \\ \nu-1 \end{pmatrix},$$

which is

$$\leq \sum_{\substack{0\leq\mu\leq m, 1\leq\nu\leq n \\ \mu+\nu=k}} \frac{1}{\mu!\nu!}\begin{pmatrix} m \\ \mu \end{pmatrix}\begin{pmatrix} n-1 \\ \nu-1 \end{pmatrix}.$$

But the latter is the coefficient of x^k in the expansion of $G_n(x)L_m(x)$, hence the assertion of Lemma 2 follows.

Applying this lemma to (5.20), we obtain

$$|A(n; z_1, z_2)| \leq \frac{G_n(x_0)}{q^n}\sum_{m\geq 0}\frac{G_m(x_0)}{q^m}\frac{L_m(x_0)}{q^m}. \tag{5.24}$$

Here we quote (3.8.16) of [9]:

$$\sum_{m=0}^{\infty} G_m(x)t^m = \exp\left(\frac{xt}{1-t}\right) \qquad (|t| < 1). \tag{5.25}$$

Using this with $t = 1/q$, we have

$$\sum_{m=0}^{\infty}\frac{G_m(x_0)}{q^m} = \exp\left(\frac{x_0}{q-1}\right), \tag{5.26}$$

and also, combining with (5.22), we have

$$\begin{aligned} \frac{L_m(x_0)}{q^m} &= \frac{1}{q^m}\sum_{\mu=0}^{m} G_\mu(x_0) \\ &\leq G_0(x_0) + \frac{G_1(x_0)}{q} + \cdots + \frac{G_m(x_0)}{q^m} \\ &\leq \sum_{\mu=0}^{\infty}\frac{G_\mu(x_0)}{q^\mu} = \exp\left(\frac{x_0}{q-1}\right). \end{aligned} \tag{5.27}$$

Substituting (5.27) into the right-hand side of (5.24), and then using (5.26), we obtain the assertion of Proposition 6.

The following mean-value estimate of the Fourier coefficients is also useful.

Proposition 7 *If $q = N(\wp)^\sigma > \sqrt{2}$, we have*

$$\sum_{n\in\mathbf{Z}} |A(n; z_1, z_2)|(|n| + 1) \leq \exp\left(\frac{Cx_0}{q-1}\right) \tag{5.28}$$

with an absolute constant $C > 0$.

Proof. Multiplying both sides of (5.25) by t and differentiating, we have

$$1+\sum_{m\geq 1}(m+1)G_m(x)t^m=\left(1+\frac{tx}{(1-t)^2}\right)\exp\left(\frac{xt}{1-t}\right). \qquad (5.29)$$

Using (5.19) and Proposition 6, we have

$$\begin{aligned}
&\sum_{n\in\mathbf{Z}}|A(n;z_1,z_2)|(|n|+1)\\
&= |A(0;z_1,z_2)|+\sum_{n=1}^{\infty}(n+1)\left(|A(n;z_1,z_2)|+|A(n;z_2,z_1)|\right)\\
&\leq \left(1+2\sum_{n=1}^{\infty}(n+1)G_n(x_0)q^{-n}\right)\exp\left(\frac{2x_0}{q-1}\right). \qquad (5.30)
\end{aligned}$$

Here we apply (5.29) with $t=q^{-1}$ to find that the right-hand side of (5.30) is equal to

$$\left\{2\left(1+\frac{qx_0}{(q-1)^2}\right)\exp\left(\frac{x_0}{q-1}\right)-1\right\}\exp\left(\frac{2x_0}{q-1}\right). \qquad (5.31)$$

If $q>\sqrt{2}$, then

$$\frac{q}{q-1}=1+\frac{1}{q-1}<1+\frac{1}{\sqrt{2}-1}=2+\sqrt{2}<4,$$

so

$$1+\frac{qx_0}{(q-1)^2}<\exp\left(\frac{qx_0}{(q-1)^2}\right)=\exp\left(\frac{q}{q-1}\cdot\frac{x_0}{q-1}\right)<\exp\left(\frac{4x_0}{q-1}\right).$$

Using this inequality and the fact $2e^a-1\leq e^{2a}$ (valid for any $a\in\mathbf{R}$), we see that (5.31) is

$$\begin{aligned}
&\leq \left\{2\exp\left(\frac{5x_0}{q-1}\right)-1\right\}\exp\left(\frac{2x_0}{q-1}\right)\\
&\leq \exp\left(\frac{10x_0}{q-1}\right)\exp\left(\frac{2x_0}{q-1}\right)=\exp\left(\frac{12x_0}{q-1}\right),
\end{aligned}$$

which implies Proposition 7 with $C=12$.

Remark 5. In the $\log L$ case, we used (5.13) to reduce the argument to discussion on $G_n(x)$. If we use $G_n^*(x)$ itself, we can show

$$|A(n;z_1,z_2)|\leq\frac{1}{q^{|n|}}G_{|n|}^*(x_0)\exp\left(-2x_0\log(1-q^{-1})\right)\quad(\log L\text{ case}) \qquad (5.32)$$

instead of Proposition 6, and can improve the value of the constant C in Proposition 7.

Let $\mathbf{n}_P = (n_\wp)_{\wp \in P}$, and define

$$A_{\sigma,P}(\mathbf{n}_P; z_1, z_2) = \prod_{\wp \in P} A_{\sigma,\wp}(n_\wp; z_1, z_2). \tag{5.33}$$

Then, by (3.3) and (5.14),

$$\begin{aligned} \psi_{z_1,z_2}(g_{\sigma,P}(\mathbf{t}_P)) &= \prod_{\wp \in P} \left(\sum_{n_\wp \in \mathbf{Z}} A_{\sigma,\wp}(n_\wp; z_1, z_2) t_\wp^{n_\wp} \right) \\ &= \sum_{\mathbf{n}_P \in \mathbf{Z}_P} A_{\sigma,P}(\mathbf{n}_P; z_1, z_2) \mathbf{t}_P^{\mathbf{n}_P}, \end{aligned} \tag{5.34}$$

where

$$\mathbf{t}_P^{\mathbf{n}_P} = \prod_{\wp \in P} t_\wp^{n_\wp}, \qquad \mathbf{Z}_P = \prod_{\wp \in P} \mathbf{Z}.$$

Therefore $A_{\sigma,P}(\mathbf{n}_P; z_1, z_2)$ are the Fourier coefficients of $\psi_{z_1,z_2}(g_{\sigma,P}(\mathbf{t}_P))$. From (5.16) we have

$$A_{\sigma,P}(\mathbf{n}_P; z_1, z_2) = \int_{T_P} \psi_{z_1,z_2}(g_{\sigma,P}(\mathbf{t}_P)) \mathbf{t}_P^{-\mathbf{n}_P} d^*\mathbf{t}_P. \tag{5.35}$$

On the other hand, from (3.12) and (3.15) we have

$$\widetilde{\mathcal{M}}_{\sigma,P}(z) = \int_{T_P} \psi_z(g_{\sigma,P}(\mathbf{t}_P)) d^*\mathbf{t}_P. \tag{5.36}$$

Comparing this with (5.35), we find that

$$\widetilde{\mathcal{M}}_{\sigma,P}(z) = A_{\sigma,P}(\mathbf{0}; z, \overline{z}), \tag{5.37}$$

where $\mathbf{0} = (0)_{\wp \in P}$.

6 Case (C) for $\Phi = \psi_z$

Now we start the proof of our main theorem in the strip $1/2 < \sigma \leq 1$. We first consider the case when $\Phi = \psi_z$. Then the right-hand side of (2.7) is $\widetilde{\mathcal{M}}_\sigma(z)$ by (3.32). Therefore our aim is to prove

$$\mathrm{Avg}_\chi \psi_z(\log L(s, \chi)) = \widetilde{\mathcal{M}}_\sigma(z). \tag{6.1}$$

In this section we will prove (6.1) in case (C).

In Case (C), the left-hand side of (6.1) is

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \psi_z(\log \zeta(s + i\tau')) d\tau' \tag{6.2}$$

(see (2.8)). Write $s = \sigma + i\tau$. Then (6.2) is equal to

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T+\tau}^{T+\tau} \psi_z(\log\zeta(\sigma + i\tau'))d\tau'.$$

Since $|\psi_z(\log\zeta(\sigma+i\tau'))| = 1$, the contribution of the intervals $[-T, -T+\tau]$, $[T, T+\tau]$ can be ignored; in other words, it is sufficient to prove

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \psi_z(\log\zeta(\sigma + i\tau'))d\tau' = \widetilde{\mathcal{M}}_\sigma(z). \tag{6.3}$$

Let $P = P_y$ be the set of prime numbers not greater than y. Define

$$\zeta_P(s) = \prod_{p\in P}(1 - p^{-s})^{-1} \tag{6.4}$$

and

$$\log\zeta_P(s) = -\sum_{p\in P} \operatorname{Log}(1 - p^{-s}). \tag{6.5}$$

The starting point of our proof is the inequality

$$\begin{aligned}
&\left|\frac{1}{2T}\int_{-T}^{T} \psi_z(\log\zeta(\sigma + i\tau'))d\tau' - \widetilde{\mathcal{M}}_\sigma(z)\right| \\
&\leq \left|\frac{1}{2T}\int_{-T}^{T} \psi_z(\log\zeta(\sigma + i\tau'))d\tau' - \frac{1}{2T}\int_{-T}^{T} \psi_z(\log\zeta_P(\sigma + i\tau'))d\tau'\right| \\
&\quad + \left|\frac{1}{2T}\int_{-T}^{T} \psi_z(\log\zeta_P(\sigma + i\tau'))d\tau' - \widetilde{\mathcal{M}}_{\sigma,P}(z)\right| \\
&\quad + \left|\widetilde{\mathcal{M}}_{\sigma,P}(z) - \widetilde{\mathcal{M}}_\sigma(z)\right| \\
&= X_P(z) + Y_P(z) + Z_P(z),
\end{aligned} \tag{6.6}$$

say. To prove (6.3), it suffices to show that, under a suitable choice of $y = y(T)$, $X_P(z)$, $Y_P(z)$ and $Z_P(z)$ tend to 0 when $T \to \infty$.

First we consider $X_P(z)$. Fix a number σ_0 satisfying $1/2 < \sigma_0 < 1$. Let ε_1 be a fixed small positive number satisfying $0 < 3\varepsilon_1 < \sigma_0 - 1/2$, and put $\alpha_1 = \sigma_0 - 2\varepsilon_1$.

Proposition 8 *The estimate*

$$X_P(z) \ll |z|\left\{y^{1/2-\alpha_1+\varepsilon} + T^{1/2-\alpha_1+\varepsilon}\exp\left(C_1\left(\frac{y}{\log y}\right)^{1/2}\right)\right\} + T^{-1} \tag{6.7}$$

holds uniformly in $\sigma_0 \leq \sigma \leq 1$, where C_1 is an absolute positive constant, and the constant implied by the Vinogradov symbol depends only on σ_0 and ε.

Proof. First, by using the fact $|\psi_z| = 1$ and the inequality

$$|\psi_z(w) - \psi_z(w')| \leq |z| \cdot |w - w'| \tag{6.8}$$

((6.5.19) of [9]), we have

$$\begin{aligned}
&X_P(z) \\
&\leq \frac{1}{2T}\int_{-2}^{2} 2d\tau' + \frac{1}{2T}\int_{I(T)} |\psi_z(\log\zeta(\sigma+i\tau')) - \psi_z(\log\zeta_P(\sigma+i\tau'))|d\tau' \\
&\leq \frac{4}{T} + \frac{|z|}{2T}\int_{I(T)} |\log\zeta(\sigma+i\tau') - \log\zeta_P(\sigma+i\tau')|d\tau',
\end{aligned} \tag{6.9}$$

where $I(T) = [-T, -2] \cup [2, T]$. Let δ_1 be a sufficiently small fixed positive constant, and define

$$\begin{aligned}
I_P^1(T) &= \{\tau' \in I(T) \; ; \; |\log\zeta(\sigma+i\tau') - \log\zeta_P(\sigma+i\tau')| \geq \delta_1\}, \\
I_P^2(T) &= \{\tau' \in I(T) \; ; \; |\log\zeta(\sigma+i\tau') - \log\zeta_P(\sigma+i\tau')| < \delta_1\}.
\end{aligned}$$

Then from (6.9) we have

$$X_P(z) \leq \frac{4}{T} + \frac{|z|}{2T}(X_1 + X_2), \tag{6.10}$$

where

$$X_j = \int_{I_P^j(T)} |\log\zeta(\sigma+i\tau') - \log\zeta_P(\sigma+i\tau')|d\tau' \qquad (j = 1, 2).$$

Consider X_2. Let

$$f_P(\sigma+i\tau') = \frac{\zeta(\sigma+i\tau')}{\zeta_P(\sigma+i\tau')} - 1.$$

When $\tau' \in I_P^2(T)$, $|\Im\log\zeta(\sigma+i\tau') - \Im\log\zeta_P(\sigma+i\tau')|$ is small. On the other hand, if $|f_P(\sigma+i\tau')| < \delta_1$, then the argument of $\zeta(\sigma+i\tau')/\zeta_P(\sigma+i\tau')$ is small. Therefore in this case

$$\log\zeta(\sigma+i\tau') - \log\zeta_P(\sigma+i\tau') = \mathrm{Log}(1 + f_P(\sigma+i\tau')),$$

hence

$$|\log\zeta(\sigma+i\tau') - \log\zeta_P(\sigma+i\tau')| \ll |f_P(\sigma+i\tau')|. \tag{6.11}$$

Since this inequality clearly holds in the case $|f_P(\sigma+i\tau')| \geq \delta_1$ also, we now obtain

$$\begin{aligned}
X_2 &\ll \int_{I_P^2(T)} |f_P(\sigma+i\tau')|d\tau' \\
&\leq \left(\int_{I(T)} 1d\tau'\right)^{1/2}\left(\int_{I(T)} |f_P(\sigma+i\tau')|^2 d\tau'\right)^{1/2}.
\end{aligned} \tag{6.12}$$

Now we quote (the second half of) Lemma 5 of [18], which asserts that

$$\frac{1}{2T}\int_{J(T)}|f_P(\sigma+i\tau')|^2 d\tau' \ll y^{1-2\alpha_1+\varepsilon}+T^{1-2\alpha_1+\varepsilon}\exp\left(C_1\left(\frac{y}{\log y}\right)^{1/2}\right) \tag{6.13}$$

holds uniformly in $\alpha_1 \leq \sigma \leq 2$ with an absolute constant $C_1 > 0$, where $J(T) = [-T,-1]\cup[1,T]$. (Note that the notation N in [18] should be read as $\pi(y) \sim y/\log y$ in our present notation, and $(y/\log y)^{1-2\alpha_1+\varepsilon}$ can be estimated as $\ll y^{1-2\alpha_1+\varepsilon}$.) Applying (6.13) to the right-hand side of (6.12), we obtain

$$\frac{1}{T}X_2 \ll y^{1/2-\alpha_1+\varepsilon}+T^{1/2-\alpha_1+\varepsilon}\exp\left(\frac{C_1}{2}\left(\frac{y}{\log y}\right)^{1/2}\right). \tag{6.14}$$

Next consider the integral X_1. For any non-negative integer l, define

$$H_P^l(T) = \{\tau' \in I(T)\ ;\ 2^l\delta_1 \leq |\log\zeta(\sigma+i\tau')-\log\zeta_P(\sigma+i\tau')| < 2^{l+1}\delta_1\}$$

and denote by $h_P^l(T)$ the (Lebesgue) measure of $H_P^l(T)$. Then

$$\begin{aligned} X_1 &= \sum_{l=0}^{\infty}\int_{H_P^l(T)}|\log\zeta(\sigma+i\tau')-\log\zeta_P(\sigma+i\tau')|d\tau' \\ &\leq \delta_1\sum_{l=0}^{\infty}2^{l+1}h_P^l(T). \end{aligned} \tag{6.15}$$

For any $\eta \geq \delta_1$, let

$$K_P(T,\eta) = \{\tau' \in I(T)\ ;\ |\log\zeta(\sigma+i\tau')-\log\zeta_P(\sigma+i\tau')| \geq \eta\}$$

and denote by $k_P(T,\eta)$ the (Lebesgue) measure of $K_P(T,\eta)$. Then

$$T^{-1}k_P(T,\eta) \leq \frac{32}{\pi\varepsilon_1^2}\eta^{-2}\int_{\alpha_1}^{\beta_1}\left(\frac{1}{2T}\int_{J(T)}|f_P(\sigma+i\tau')|^2 d\tau'\right)d\sigma, \tag{6.16}$$

where $\beta_1 = 2(1+C_2\eta^{-1})$ with a certain absolute constant C_2. This is (4.9) of [18]. On the right-hand side of (4.9) of [18] there is a term $3/T$, but it is not necessary to add that term to the right-hand side of (6.16), because that term in [18] comes from the contribution of $\tau' \in [-2,2]$. The first half of Lemma 5 of [18] asserts

$$\frac{1}{2T}\int_{J(T)}|f_P(\sigma+i\tau')|^2 d\tau' \ll \sigma^{-1}y^{1-2\sigma+\varepsilon}+\sigma^{-1}T^{-1}y^{2-2\sigma+\varepsilon}, \tag{6.17}$$

which is valid uniformly in $2 \le \sigma \le \beta_1$. Applying (6.13) and (6.17) to the right-hand side of (6.16) for $\alpha_1 \le \sigma \le 2$ and $2 \le \sigma \le \beta_1$ respectively, we have

$$\begin{aligned} &T^{-1} k_P(T,\eta) \\ &\quad \ll \eta^{-2} \Big\{ y^{1-2\alpha_1+\varepsilon} + T^{1-2\alpha_1+\varepsilon} \exp\left(C_1 \left(\frac{y}{\log y} \right)^{1/2} \right) \\ &\qquad + y^{-3+\varepsilon} \log \beta_1 + T^{-1} y^{-2+\varepsilon} \log \beta_1 \Big\}. \end{aligned} \tag{6.18}$$

Since $\eta \ge \delta_1$, we have $\beta_1 \le 2(1 + C_2 \delta_1^{-1})$, and hence $\log \beta_1$ can be absorbed in the implied constant because δ_1 is fixed. Since $h_P^l(T) \le k_P(T, 2^l \delta_1)$, from (6.15) and (6.18) we have

$$\begin{aligned} \frac{1}{T} X_1 &\ll \sum_{l=0}^{\infty} 2^{-l} \Big\{ y^{1-2\alpha_1+\varepsilon} + T^{1-2\alpha_1+\varepsilon} \exp\left(C_1 \left(\frac{y}{\log y} \right)^{1/2} \right) \\ &\qquad + y^{-3+\varepsilon} \log \beta_1 + T^{-1} y^{-2+\varepsilon} \log \beta_1 \Big\} \\ &\ll y^{1-2\alpha_1+\varepsilon} + T^{1-2\alpha_1+\varepsilon} \exp\left(C_1 \left(\frac{y}{\log y} \right)^{1/2} \right). \end{aligned} \tag{6.19}$$

Combining (6.10), (6.14) and (6.19), we obtain Proposition 8.

Now we proceed to the study of $Y_P(z)$. By using (5.34) we have

$$\begin{aligned} \psi_z(\log \zeta_P(\sigma + i\tau')) &= \psi_z(g_{\sigma,P}(\chi_P)) \\ &= \sum_{\mathbf{n}_P \in \mathbf{Z}_P} A_{\sigma,P}(\mathbf{n}_P; z, \overline{z}) \chi_P^{\mathbf{n}_P}, \end{aligned}$$

where $\chi_P = (\chi(p))_{p \in P}$ and $\chi(p) = \chi_{\tau'}(p) = p^{-i\tau'}$, and so

$$\begin{aligned} &\frac{1}{2T} \int_{-T}^{T} \psi_z(\log \zeta_P(\sigma + i\tau')) d\tau' \\ &\quad = \sum_{\mathbf{n}_P \in \mathbf{Z}_P} A_{\sigma,P}(\mathbf{n}_P; z, \overline{z}) \frac{1}{2T} \int_{-T}^{T} \prod_{p \in P} e^{-i\tau' n_p \log p} d\tau'. \end{aligned} \tag{6.20}$$

Write $r = \pi(y)$ and $P = \{p_1, \ldots, p_r\}$. Since $n_{p_1} \log p_1 + \cdots + n_{p_r} \log p_r = 0$ if and only if $n_{p_1} = \cdots = n_{p_r} = 0$, the integral on the right-hand side of (6.20) is

$$= \left. \frac{e^{-i\tau'(n_{p_1} \log p_1 + \cdots + n_{p_r} \log p_r)}}{-i(n_{p_1} \log p_1 + \cdots + n_{p_r} \log p_r)} \right|_{\tau'=-T}^{T}$$

for any $\mathbf{n}_P \ne \mathbf{0}$. Therefore

$$\frac{1}{2T} \int_{-T}^{T} \psi_z(\log \zeta_P(\sigma + i\tau')) d\tau'$$

$$= \quad A_{\sigma,P}(\mathbf{0}; z, \overline{z}) + O\left(\frac{1}{T}\sum_{\substack{\mathbf{n}_P\in\mathbf{Z}_P\\ \mathbf{n}_P\neq\mathbf{0}}}\frac{|A_{\sigma,P}(\mathbf{n}_P; z, \overline{z})|}{|n_{p_1}\log p_1+\cdots+n_{p_r}\log p_r|}\right). \tag{6.21}$$

Since the first term on the right-hand side is equal to $\widetilde{\mathcal{M}}_{\sigma,P}(z)$ by (5.37), we obtain

$$Y_P(z) \ll \frac{1}{T}\sum_{\substack{\mathbf{n}_P\in\mathbf{Z}_P\\ \mathbf{n}_P\neq\mathbf{0}}}\frac{|A_{\sigma,P}(\mathbf{n}_P; z, \overline{z})|}{|n_{p_1}\log p_1+\cdots+n_{p_r}\log p_r|}. \tag{6.22}$$

By estimating the right-hand side of the above, we prove

Proposition 9 *The estimate*

$$Y_P(z) \ll \frac{1}{T}\exp\left(C_3\left(|z|\frac{y^{3/2-\sigma}}{\log y}+\frac{y}{\log y}\right)\right)$$

holds uniformly in $\sigma_0 \le \sigma \le 1$, where C_3 is an absolute positive constant.

Proof. We denote the positive members of $\{n_{p_1},\ldots,n_{p_r}\}$ by $k_1,\ldots,k_u$, and the negative members by $-l_1,\ldots,-l_v$. Then $u+v\le r$. Further we define $p(i)$ and $q(j)$ by $k_i = n_{p(i)}$ and $-l_j = n_{q(j)}$ $(1\le i\le u, 1\le j\le v)$. Then

$$\begin{aligned}&|n_{p_1}\log p_1+\cdots+n_{p_r}\log p_r|\\ &= \left|\sum_{i=1}^{u}k_i\log p(i)-\sum_{j=1}^{v}l_j\log q(j)\right|\\ &= \left|\log\left(1+\left(\frac{p(1)^{k_1}\cdots p(u)^{k_u}}{q(1)^{l_1}\cdots q(v)^{l_v}}-1\right)\right)\right|.\end{aligned} \tag{6.23}$$

Let δ_2 be a sufficiently small fixed positive constant, and denote by $\mathbf{Z}_P^{(1)}$ the set of all $\mathbf{n}_P\in\mathbf{Z}_P\setminus\{\mathbf{0}\}$ for which

$$\left|\frac{p(1)^{k_1}\cdots p(u)^{k_u}}{q(1)^{l_1}\cdots q(v)^{l_v}}-1\right|\ge\delta_2$$

holds. Put $\mathbf{Z}_P^{(2)}=\mathbf{Z}_P\setminus\{\mathbf{Z}_P^{(1)}\cup\mathbf{0}\}$, and divide (6.22) as

$$Y_P(z)\ll\frac{1}{T}\left(\sum_{\mathbf{n}_P\in\mathbf{Z}_P^{(1)}}+\sum_{\mathbf{n}_P\in\mathbf{Z}_P^{(2)}}\right)=\frac{1}{T}(Y_1+Y_2), \tag{6.24}$$

say.

When $\mathbf{n}_P \in \mathbf{Z}_P^{(1)}$, we have

$$|n_{p_1} \log p_1 + \cdots + n_{p_r} \log p_r| \geq \min\{\log(1+\delta_2), -\log(1-\delta_2)\} \gg 1.$$

Therefore

$$\begin{aligned} Y_1 &\ll \sum_{\mathbf{n}_P \in \mathbf{Z}_P^{(1)}} |A_{\sigma,P}(\mathbf{n}_P; z, \overline{z})| \\ &\leq \sum_{\mathbf{n}_P \in \mathbf{Z}_P} |A_{\sigma,P}(\mathbf{n}_P; z, \overline{z})| \\ &= \prod_{p\in P} \left(\sum_{n_p \in \mathbf{Z}} |A_{\sigma,p}(n_p; z, \overline{z})| \right) \end{aligned} \tag{6.25}$$

by (5.33). Applying Proposition 7 with $x_0 = |z|/2$, we obtain

$$Y_1 \ll \prod_{p\in P} \exp\left(\frac{C|z|}{2(p^\sigma - 1)}\right) = \exp\left(\frac{C|z|}{2} \sum_{p\leq y} \frac{1}{p^\sigma - 1}\right). \tag{6.26}$$

By using the prime number theorem and partial summation we can easily see that the sum in the right-most side of (6.26) is $\ll \eta(y)$, where

$$\eta(y) = \eta(\sigma, y) = \begin{cases} y^{1-\sigma}(\log y)^{-1} & \text{if} \quad 0 < \sigma < 1, \\ \log\log y & \text{if} \quad \sigma = 1. \end{cases} \tag{6.27}$$

Therefore

$$\frac{1}{T} Y_1 \ll \frac{1}{T} \exp\left(C_4 |z| \eta(y)\right) \tag{6.28}$$

with an absolute constant $C_4 > 0$.

Next consider Y_2. When $\mathbf{n}_P \in \mathbf{Z}_P^{(2)}$, we have

$$\begin{aligned} &|n_{p_1} \log p_1 + \cdots + n_{p_r} \log p_r| \\ &\quad \gg \left| \frac{p(1)^{k_1} \cdots p(u)^{k_u}}{q(1)^{l_1} \cdots q(v)^{l_v}} - 1 \right| \\ &\quad = \left| \frac{p(1)^{k_1} \cdots p(u)^{k_u} - q(1)^{l_1} \cdots q(v)^{l_v}}{q(1)^{l_1} \cdots q(v)^{l_v}} \right| \\ &\quad \geq \frac{1}{q(1)^{l_1} \cdots q(v)^{l_v}}, \end{aligned}$$

where the last inequality follows because $\mathbf{n}_P \neq \mathbf{0}$. Therefore

$$Y_2 \ll \sum_{\mathbf{n}_P \in \mathbf{Z}_P^{(2)}} q(1)^{l_1} \cdots q(v)^{l_v} |A_{\sigma,P}(\mathbf{n}_P; z, \overline{z})|. \tag{6.29}$$

Since

$$1 - \delta_2 < \frac{p(1)^{k_1} \cdots p(u)^{k_u}}{q(1)^{l_1} \cdots q(v)^{l_v}} < 1 + \delta_2 \tag{6.30}$$

holds for $\mathbf{n}_P \in \mathbf{Z}_P^{(2)}$, we have

$$\begin{aligned} & q(1)^{l_1} \cdots q(v)^{l_v} \\ &= (q(1)^{l_1} \cdots q(v)^{l_v})^{1/2} (q(1)^{l_1} \cdots q(v)^{l_v})^{1/2} \\ &\ll (p(1)^{k_1} \cdots p(u)^{k_u})^{1/2} (q(1)^{l_1} \cdots q(v)^{l_v})^{1/2} \\ &= \left(p_1^{|n_{p_1}|} \cdots p_r^{|n_{p_r}|} \right)^{1/2}. \end{aligned}$$

Therefore from (6.29) and (5.33) we have

$$Y_2 \ll \sum_{\mathbf{n}_P \in \mathbf{Z}_P^{(2)}} \prod_{p \in P} p^{|n_p|/2} |A_{\sigma,p}(n_p; z, \overline{z})|. \tag{6.31}$$

Applying Proposition 6, we have

$$\begin{aligned} Y_2 &\ll \sum_{\mathbf{n}_P \in \mathbf{Z}_P} \prod_{p \in P} p^{(1/2-\sigma)|n_p|} G_{|n_p|}(|z|/2) \exp\left(\frac{|z|}{p^\sigma - 1} \right) \\ &= \prod_{p \in P} \left(\sum_{n_p \in \mathbf{Z}} p^{(1/2-\sigma)|n_p|} G_{|n_p|}(|z|/2) \right) \exp\left(\frac{|z|}{p^\sigma - 1} \right). \end{aligned} \tag{6.32}$$

Evaluating the right-hand side by (5.25), we obtain

$$\begin{aligned} \frac{1}{T} Y_2 &\ll \frac{1}{T} \prod_{p \in P} 2 \exp\left(\frac{|z|}{2(p^{\sigma-1/2} - 1)} + \frac{|z|}{p^\sigma - 1} \right) \\ &\ll \frac{1}{T} 2^r \exp\left(C_5 |z| \eta(\sigma - 1/2, y) \right) \end{aligned} \tag{6.33}$$

with an absolute constant $C_5 > 0$. Since $r = \pi(y) \ll y/\log y$, the assertion of Proposition 9 follows from (6.24), (6.28) and (6.33).

Now, from Propositions 8, 9 and (6.6), we obtain

$$\begin{aligned} & \left| \frac{1}{2T} \int_{-T}^{T} \psi_z(\log \zeta(\sigma + i\tau)) d\tau - \widetilde{\mathcal{M}}_\sigma(z) \right| \\ &\ll |z| \left\{ y^{1/2-\alpha_1+\varepsilon} + T^{1/2-\alpha_1+\varepsilon} \exp\left(C_1 \left(\frac{y}{\log y} \right)^{1/2} \right) \right\} \\ &\quad + \frac{1}{T} \exp\left(C_3 \left(|z| \frac{y^{3/2-\sigma}}{\log y} + \frac{y}{\log y} \right) \right) + Z_P(z). \end{aligned} \tag{6.34}$$

Proposition 4 implies that $Z_P(z) \to 0$ as $y \to \infty$, uniformly in z. We now choose

$$y = y(T) = (\log T)^{\omega_1} \qquad (0 < \omega_1 < 1). \tag{6.35}$$

Then, when T tends to ∞, $y = y(T)$ also tends to ∞, hence $Z_P(z) \to 0$. On the other hand, since $\omega_1 < 1$, we have $\omega_1(3/2 - \sigma) < 1$ for any $\sigma > 1/2$. Thus the two exponential factors on the right-hand side of (6.34) are $O(T^\varepsilon)$ for any $\varepsilon > 0$. Therefore, if ε is sufficiently small, then all the terms on the right-hand side of (6.34) tend to 0 as $T \to \infty$. This completes the proof of (6.3).

Remark 6. For any fixed $R > 0$, (6.34) implies that the convergence in (6.3), that is the case $\Phi = \psi_z$ of (2.8), is uniform in $|z| \leq R$.

7 Case (A,Q) for $\Phi = \psi_z$

Now we proceed to the study of Case (A,**Q**). Let $1/2 < \sigma_0 < 1$, $0 < 3\varepsilon_1 < \sigma_0 - 1/2$, $\alpha_1 = \sigma_0 - 2\varepsilon_1$ as in Section 6. Further put $\alpha_0 = \sigma_0 - \varepsilon_1$ and $\alpha_2 = 1/2 + \varepsilon_1$. Then $1/2 < \alpha_2 < \alpha_1 < \alpha_0 < \sigma_0 < 1$. All of these constants are regarded to be fixed. In this section we fix a point $s = \sigma + i\tau$ in the strip $\sigma_0 \leq \Re s \leq 1$, and will prove (2.9) for this s and $\Phi = \psi_z$. We begin with the analogue of (6.6), that is

$$\begin{aligned}
&\left|\frac{1}{\pi(m)} \sum_{f \leq m} \frac{1}{f-2} \sum_{\chi \in X'(f)} \psi_z(\log L(s,\chi)) - \widetilde{\mathcal{M}}_\sigma(z)\right| \\
\leq\ & \left|\frac{1}{\pi(m)} \sum_{f \leq m} \frac{1}{f-2} \sum_{\chi \in X'(f)} \psi_z(\log L(s,\chi))\right. \\
&\qquad \left. -\frac{1}{\pi(m)} \sum_{f \leq m} \frac{1}{f-2} \sum_{\chi \in X'(f)} \psi_z(\log L_P(s,\chi))\right| \\
&\qquad +\left|\frac{1}{\pi(m)} \sum_{f \leq m} \frac{1}{f-2} \sum_{\chi \in X'(f)} \psi_z(\log L_P(s,\chi)) - \widetilde{\mathcal{M}}_{\sigma,P}(z)\right| \\
&\qquad +\left|\widetilde{\mathcal{M}}_{\sigma,P}(z) - \widetilde{\mathcal{M}}_\sigma(z)\right| \\
=\ & X_P(z) + Y_P(z) + Z_P(z), \qquad (7.1)
\end{aligned}$$

say, where $P = P_y = \{p_1, \ldots, p_r\}$. Note that, in this section, f always denotes a prime (> 2).

First we estimate $X_P(z)$. For this purpose we use the method in Section 4 of [18], whose idea actually goes back to Bohr [1].

Let c be a positive constant, and define the domain

$$H(\tau) = \{s' = \sigma' + i\tau' \ ; \ \sigma' > \alpha_0, \ \tau - c < \tau' < \tau + c\},$$

and the function

$$R_P(s',\chi) = \log L(s',\chi) - \log L_P(s',\chi)$$

on $G_\chi(\alpha_1) = G_\chi \cap \{\sigma > \alpha_1\}$. Let δ be a fixed small positive constant. By $\varphi_P^\delta(\tau, \chi)$ we mean the function whose value is $= 0$ if $H(\tau) \subset G_\chi(\alpha_1)$ and $|R_P(s', \chi)| < \delta$ for any $s' \in H(\tau)$, and $= 1$ otherwise. When $\sigma' > 1$, we have

$$R_P(s', \chi) \ll \sum_{p>y} p^{-\sigma'} \ll \frac{1}{\sigma' - 1},$$

hence we can find a $\beta_0 = \beta_0(\delta) > 1$, independent of χ, for which

$$|R_P(s', \chi)| < \delta \tag{7.2}$$

holds for any s' with $\sigma' \geq \beta_0$. Put $\beta_1 = \beta_1(\delta) = 2\beta_0$, and

$$\begin{aligned} Q_1(\tau) &= \{s' = \sigma' + i\tau' \ ; \ \alpha_1 \leq \sigma' \leq \beta_1, \ \tau - 2c \leq \tau' \leq \tau + 2c\}, \\ Q_0(\tau) &= \{s' = \sigma' + i\tau' \ ; \ \alpha_0 < \sigma' < \beta_0, \ \tau - c < \tau' < \tau + c\}, \end{aligned}$$

so that $Q_0(\tau) = H(\tau) \cap \{\sigma' < \beta_0\}$, and $\overline{Q_0(\tau)} \subset Q_1(\tau)$. Define the function

$$f_P(s', \chi) = \frac{L(s', \chi)}{L_P(s', \chi)} - 1$$

on $Q_1(\tau)$.

Lemma 3 *If $|f_P(s', \chi)| < \delta/2$ for any $s' \in Q_0(\tau)$, then $\varphi_P^\delta(\tau, \chi) = 0$.*

This is just a simple generalization of Hilfssatz 5 of [1], but we give a proof here for the convenience of readers.

By (7.2), it suffices to show that $Q_0(\tau) \subset G_\chi(\alpha_1)$ and that $|R_P(s', \chi)| < \delta$ for any $s' \in Q_0(\tau)$.

Let $s' \in Q_0(\tau)$. Since δ is small, the assumption $|f_P(s', \chi)| < \delta/2$ implies $L(s', \chi) \neq 0$, so $Q_0(\tau) \subset G_\chi(\alpha_1)$. By using the assumption again, we see that the argument of $L(s', \chi)/L_P(s', \chi)$ remains between $-\pi/2$ and $\pi/2$ when $s' \in Q_0(\tau)$. Therefore

$$R_P(s', \chi) = \mathrm{Log}\frac{L(s', \chi)}{L_P(s', \chi)} = \mathrm{Log}(1 + f_P(s', \chi)), \tag{7.3}$$

which gives $|R_P(s', \chi)| \leq 2|f_P(s', \chi)| < \delta$. Hence the lemma.

The following simple function-theoretic lemma is also necessary.

Lemma 4 (Hilfssatz 4 of Bohr [1]) *Let Γ, Γ' be two closed curves on the complex plane, and D, D' the open regions surrounded by Γ, Γ', respectively. Assume $\Gamma \cup D \subset D'$. If $f(s')$ is holomorphic on D' and*

$$\int\int_{D'} |f(s')|^2 d\sigma' d\tau' < \pi \left(\frac{d(\Gamma, \Gamma')}{2}\right)^2 a^2,$$

where $d(\Gamma, \Gamma') = \inf\{|z - z'| \ ; \ z \in \Gamma, z' \in \Gamma'\}$, then $|f(s')| < a$ for any $s' \in \Gamma \cup D$.

Put

$$F_P(\tau,\chi)=\int\int_{Q_1(\tau)}|f_P(s',\chi)|^2 d\sigma' d\tau'.$$

The distance between the boundary of $Q_1(\tau)$ and that of $Q_0(\tau)$ is $\min\{\varepsilon_1, c\}$, which we denote by ε_2. Since $f_p(s',\chi)$ is holomorphic on $Q_1(\tau)$, Lemma 4 implies that, if

$$F_P(\tau,\chi)<\pi\left(\frac{\varepsilon_2}{2}\right)^2\left(\frac{\delta}{2}\right)^2 \tag{7.4}$$

holds, then $|f_P(s',\chi)|<\delta/2$ for $s'\in Q_0(\tau)$, and so, by Lemma 3, $\varphi_P^{\delta}(\tau,\chi)=0$.

For any prime f, define

$$X_1(f)=X_1(f,s):=\left\{\chi\in X'(f)\ ;\ F_P(\tau,\chi)\geq\pi\left(\frac{\varepsilon_2}{2}\right)^2\left(\frac{\delta}{2}\right)^2\right\},$$
$$X_2(f)=X_2(f,s):=\left\{\chi\in X'(f)\ ;\ F_P(\tau,\chi)<\pi\left(\frac{\varepsilon_2}{2}\right)^2\left(\frac{\delta}{2}\right)^2\right\}.$$

Divide

$$\begin{aligned}&\sum_{\chi\in X'(f)}\left(\psi_z(\log L(s,\chi))-\psi_z(\log L_P(s,\chi))\right)\\ &=\sum_{\chi\in X_1(f)}+\sum_{\chi\in X_2(f)}=S_1(f)+S_2(f),\end{aligned} \tag{7.5}$$

say.

Consider $S_2(f)$. When $\chi\in X_2(f)$, we find $|f_P(s',\chi)|<\delta/2$ for $s'\in Q_0(\tau)$ as we have seen before, especially $Q_0(\tau)\subset G_\chi(\alpha_1)$. Applying (6.8) we obtain

$$|S_2(f)|\leq|z|\sum_{\chi\in X_2(f)}|\log L(s,\chi)-\log L_P(s,\chi)|. \tag{7.6}$$

Combining this with (7.3), we obtain

$$\begin{aligned}|S_2(f)| &\ll |z|\sum_{\chi\in X_2(f)}|f_P(s,\chi)|\\ &\ll |z|f^{1/2}\left(\sum_{\chi\in X_2(f)}|f_P(s,\chi)|^2\right)^{1/2}.\end{aligned} \tag{7.7}$$

On the mean square of $|f_P(s,\chi)|$, we can show the following lemma.

Lemma 5 *For any prime f, we have*

$$\begin{aligned}\sum_{\chi\in X(f)} |f_P(s',\chi)|^2 &\ll fy^{1-2\sigma'} \\ &+ \quad f^{(1-\sigma')/(1-\alpha_2)}\exp\left(B_0\frac{y^{1-\alpha_2}}{\log y}\right)\left(1+\frac{|\tau'|+1}{f^{2\alpha_2}}\right) \qquad (7.8)\end{aligned}$$

(*with a certain absolute positive constant B_0*) *for any s' satisfying $\alpha_2 \le \Re s' \le \beta_1$, uniformly in this region.*

We postpone the proof of this lemma to the next section. Here we assume the assertion of Lemma 5. If $\sigma' \ge \alpha_1$, then the right-hand side of (7.8) is

$$\begin{aligned} &\le \quad fy^{1-2\alpha_1} + f^{(1-\alpha_1)/(1-\alpha_2)}\exp\left(B_0\frac{y^{1-\alpha_2}}{\log y}\right)\left(1+\frac{|\tau'|+1}{f^{2\alpha_2}}\right) \\ &= \quad A(\tau', f, y), \qquad (7.9)\end{aligned}$$

say. Then from (7.7), (7.8) and (7.9) we obtain

$$|S_2(f)| \ll |z| f^{1/2} A(\tau, f, y)^{1/2}. \qquad (7.10)$$

Next consider $S_1(f)$. We see that

$$\begin{aligned}\pi\left(\frac{\varepsilon_2}{2}\right)^2\left(\frac{\delta}{2}\right)^2|X_1(f)| \quad &\le \quad \sum_{\chi\in X_1(f)} F_P(\tau,\chi) \\ &= \quad \int\int_{Q_1(\tau)}\sum_{\chi\in X_1(f)}|f_P(s',\chi)|^2 d\sigma' d\tau'. \qquad (7.11)\end{aligned}$$

Applying Lemma 5, we see that the right-hand side of (7.11) is

$$\ll A(\tau, f, y)\int\int_{Q_1(\tau)} d\sigma' d\tau',$$

and the last integral is $O(1)$. Since ε_2, δ are also fixed, we find $|X_1(f)| \ll A(\tau, f, y)$. Therefore, noting $|\psi_z| = 1$, we obtain

$$|S_1(f)| \le 2|X_1(f)| \ll A(\tau, f, y). \qquad (7.12)$$

From (7.10) and (7.12), we can conclude

Proposition 10

$$X_P(z) \ll \frac{1}{\pi(m)}\sum_{f\le m}\frac{1}{f}\left(|z| f^{1/2} A(\tau, f, y)^{1/2} + A(\tau, f, y)\right).$$

The method of estimating $Y_P(z)$ is analogous to that in Section 6 of [9], in which the function field case has been treated. Assume $y \leq m$, which will be confirmed later (see (7.27)). Then

$$\begin{aligned}
&\frac{1}{\pi(m)} \sum_{f \leq m} \frac{1}{f-2} \sum_{\chi \in X'(f)} \psi_z(\log L_P(s,\chi)) \\
&= \frac{1}{\pi(m)} \sum_{f \leq y} \frac{1}{f-2} \sum_{\chi \in X'(f)} \psi_z(\log L_P(s,\chi)) \\
&\quad + \frac{1}{\pi(m)} \sum_{y < f \leq m} \frac{1}{f-2} \sum_{\chi \mathrm{mod} f} \psi_z(\log L_P(s,\chi)) \\
&\quad - \frac{1}{\pi(m)} \sum_{y < f \leq m} \frac{1}{f-2} \sum_{\substack{\chi \mathrm{mod} f \\ \chi \notin X'(f)}} \psi_z(\log L_P(s,\chi)) \\
&= J_0^{(m)} + J_1^{(m)} + J_2^{(m)}, \qquad (7.13)
\end{aligned}$$

say. When $f > y$ then $(f, P) = 1$, so from (3.5) and (5.34) we have

$$\begin{aligned}
\psi_z(\log L_P(s,\chi)) &= \psi_z(g_{\sigma,P}(\chi_P P^{-i\tau})) \\
&= \sum_{\mathbf{n}_P \in \mathbf{Z}_P} A_{\sigma,P}(\mathbf{n}_P; z, \overline{z}) \chi_P^{\mathbf{n}_P} P^{-i\tau \mathbf{n}_P}, \qquad (7.14)
\end{aligned}$$

where

$$\chi_P^{\mathbf{n}_P} = \prod_{p \in P} \chi(p)^{n_p}, \qquad P^{-i\tau \mathbf{n}_P} = \prod_{p \in P} p^{-i\tau n_p}.$$

Hence

$$\sum_{\chi \mathrm{mod} f} \psi_z(\log L_P(s,\chi)) = \sum_{\mathbf{n}_P \in \mathbf{Z}_P} A_{\sigma,P}(\mathbf{n}_P; z, \overline{z}) P^{-i\tau \mathbf{n}_P} \sum_{\chi \mathrm{mod} f} \chi_P^{\mathbf{n}_P}. \qquad (7.15)$$

Define

$$P_1^{\mathbf{n}_P} = \prod_{\substack{p \in P \\ n_p > 0}} p^{n_p}, \qquad P_2^{\mathbf{n}_P} = \prod_{\substack{p \in P \\ n_p < 0}} p^{-n_p}.$$

Then the inner sum on the right-hand side of (7.15) is $f - 1$ if $P_1^{\mathbf{n}_P} \equiv P_2^{\mathbf{n}_P}$ (mod f), and 0 otherwise. Therefore

$$\begin{aligned}
J_1^{(m)} &= \frac{1}{\pi(m)} \sum_{y < f \leq m} \sum_{\substack{\mathbf{n}_P \in \mathbf{Z}_P \\ P_1^{\mathbf{n}_P} \equiv P_2^{\mathbf{n}_P} \ (\mathrm{mod} f)}} A_{\sigma,P}(\mathbf{n}_P; z, \overline{z}) P^{-i\tau \mathbf{n}_P} \\
&\quad + \frac{1}{\pi(m)} \sum_{y < f \leq m} \frac{1}{f-2} \sum_{\substack{\mathbf{n}_P \in \mathbf{Z}_P \\ P_1^{\mathbf{n}_P} \equiv P_2^{\mathbf{n}_P} \ (\mathrm{mod} f)}} A_{\sigma,P}(\mathbf{n}_P; z, \overline{z}) P^{-i\tau \mathbf{n}_P} \\
&= J_{11}^{(m)} + J_{12}^{(m)}, \qquad (7.16)
\end{aligned}$$

say. We can write

$$J_{11}^{(m)} = \sum_{\mathbf{n}_P \in \mathbf{Z}_P} \mathcal{E}^{(m)}(\mathbf{n}_P) A_{\sigma,P}(\mathbf{n}_P; z, \overline{z}) P^{-i\tau \mathbf{n}_P}, \tag{7.17}$$

where

$$\mathcal{E}^{(m)}(\mathbf{n}_P) = \frac{1}{\pi(m)} |\{f; \text{prime}\ ;\ y < f \leq m,\ f|(P_1^{\mathbf{n}_P} - P_2^{\mathbf{n}_P})\}|.$$

In particular it is clear that $\mathcal{E}^{(m)}(\mathbf{0}) = 1 - \pi(m)^{-1}\pi(y)$. Therefore the term corresponding to $\mathbf{n}_P = \mathbf{0}$ on the right-hand side of (7.17) is

$$\left(1 - \frac{\pi(y)}{\pi(m)}\right) \widetilde{\mathcal{M}}_{\sigma,P}(z)$$

by (5.37). Noting (3.17), we see that this is equal to

$$\widetilde{\mathcal{M}}_{\sigma,P}(z) + O\left(\frac{y}{\pi(m)\log y}\right). \tag{7.18}$$

On the other hand, if $\mathbf{n}_P \neq \mathbf{0}$, we can show

$$\mathcal{E}^{(m)}(\mathbf{n}_P) \ll \frac{1}{\pi(m)} \left(\prod_{p \in P} (|n_p| + 1)\right) \log y. \tag{7.19}$$

In fact, writing the number of distinct prime divisors of a positive integer n as $\omega(n)$, it is well known that

$$\omega(n) \ll \frac{\log n}{\log\log n},$$

hence

$$\omega(|P_1^{\mathbf{n}_P} - P_2^{\mathbf{n}_P}|) \ll \frac{\log(|P_1^{\mathbf{n}_P} - P_2^{\mathbf{n}_P}|)}{\log\log(|P_1^{\mathbf{n}_P} - P_2^{\mathbf{n}_P}|)}.$$

Combining this with $|(P_1^{\mathbf{n}_P} - P_2^{\mathbf{n}_P})| \leq P^{|\mathbf{n}_P|}$, we obtain

$$\omega(|P_1^{\mathbf{n}_P} - P_2^{\mathbf{n}_P}|) \ll \frac{\log P^{|\mathbf{n}_P|}}{\log\log P^{|\mathbf{n}_P|}} \ll \log P^{|\mathbf{n}_P|} \leq \sum_{p \in P} |n_p| \log y, \tag{7.20}$$

from which (7.19) immediately follows.

Therefore

$$\begin{aligned} &\sum_{\mathbf{n}_P \in \mathbf{Z}_P \setminus \{\mathbf{0}\}} \mathcal{E}^{(m)}(\mathbf{n}_P) A_{\sigma,P}(\mathbf{n}_P; z, \overline{z}) P^{-i\tau \mathbf{n}_P} \\ &\ll \frac{1}{\pi(m)} (\log y) \prod_{p \in P} \sum_{n_p \in \mathbf{Z}} (|n_p| + 1) |A_{\sigma,p}(n_p; z, \overline{z})|, \end{aligned}$$

which is further estimated as

$$\begin{aligned}&\leq \frac{1}{\pi(m)}(\log y)\exp\left(\frac{C|z|}{2}\sum_{p\in P}\frac{1}{p^{\sigma}-1}\right)\\&\ll \frac{1}{\pi(m)}(\log y)\exp\left(C_4|z|\eta(y)\right)\end{aligned}$$

by using Proposition 7 and the prime number theorem, as in (6.26)−(6.28). Substituting (7.18) and the above estimate into (7.17), we obtain

$$J_{11}^{(m)}=\widetilde{\mathcal{M}}_{\sigma,P}(z)+O\left(\frac{y}{\pi(m)\log y}\right)+O\left(\frac{\log y}{\pi(m)}\exp\left(C_4|z|\eta(y)\right)\right). \tag{7.21}$$

The term $J_{12}^{(m)}$ can be expressed similarly to (7.17), only replacing $\mathcal{E}^{(m)}(\mathbf{n}_P)$ by

$$\widetilde{\mathcal{E}}^{(m)}(\mathbf{n}_P)=\frac{1}{\pi(m)}\sum_{\substack{y<f\leq m\\ P_1^{\mathbf{n}_P}\equiv P_2^{\mathbf{n}_P}\ (\mathrm{mod} f)}}\frac{1}{f-2}.$$

Since trivially we have

$$\widetilde{\mathcal{E}}^{(m)}(\mathbf{n}_P)\leq\frac{1}{\pi(m)}\sum_{f\leq m}\frac{1}{f-2}\ll\frac{\log\log m}{\pi(m)}, \tag{7.22}$$

we obtain

$$\begin{aligned}J_{12}^{(m)}&\ll\frac{\log\log m}{\pi(m)}\prod_{p\in P}|A_{\sigma,p}(n_p;z,\overline{z})|\\&\ll\frac{\log\log m}{\pi(m)}\exp\left(C_4|z|\eta(y)\right)\end{aligned} \tag{7.23}$$

again by using Proposition 7 and the prime number theorem.

Next, using $|\psi_z|=1$, we have

$$J_0^{(m)}\ll\frac{1}{\pi(m)}\sum_{f\leq y}1\ll\frac{y}{\pi(m)\log y}. \tag{7.24}$$

As for $J_2^{(m)}$, by using $|\psi_z|=1$ and (2.10) (with $T=2|\tau|$), we have

$$\begin{aligned}J_2^{(m)}&\ll\frac{1}{\pi(m)}\sum_{y<f\leq m}\frac{|X(f)\setminus X'(f)|+1}{|X(f)|}\\&\ll\frac{1}{\pi(m)}\sum_{y<f\leq m}f^{A(\sigma_0)-1}|\tau|^{A(\sigma_0)}(\log 2f|\tau|)^{14}.\end{aligned}$$

Replacing $(\log 2f|\tau|)^{14}$ by $(\log 2m|\tau|)^{14}$ and using partial summation, we obtain

$$J_2^{(m)}\ll\frac{(m|\tau|)^{A(\sigma_0)}(\log 2m|\tau|)^{14}}{\pi(m)\log m}. \tag{7.25}$$

From (7.13), (7.16), (7.21), (7.23), (7.24) and (7.25), we obtain

Proposition 11

$$
\begin{aligned}
& Y_P(z) \ll \frac{y}{\pi(m)\log y} + \frac{\log y}{\pi(m)} \exp\left(C_4|z|\eta(y)\right) \\
+ \quad & \frac{\log\log m}{\pi(m)} \exp\left(C_4|z|\eta(y)\right) + \frac{(m|\tau|)^{A(\sigma_0)}(\log 2m|\tau|)^{14}}{\pi(m)\log m}. \qquad (7.26)
\end{aligned}
$$

Now we choose

$$
y = y(m) = (\log m)^{\omega_2} \qquad (0 < \omega_2 < 2). \qquad (7.27)
$$

Then $\omega_2(1-\sigma) < 1$ for any $\sigma > 1/2$, so the two exponential factors on the right-hand side of (7.26) are $O(m^{\varepsilon})$ for any $\varepsilon > 0$. Therefore from (7.26) we have

$$
Y_P(z) \ll m^{-1+A(\sigma_0)+\varepsilon}, \qquad (7.28)
$$

hence $Y_P(z) \to 0$ as $m \to \infty$. Note that the implied constant in (7.28) is uniform in $|z| \le R$ for any fixed $R > 0$.

The exponential factor in definition (7.9) of $A(\tau', f, y)$ is also $O(m^{\varepsilon})$ under the above choice of y, hence

$$
\begin{aligned}
& \frac{1}{\pi(m)} \sum_{f \le m} \frac{1}{f} A(\tau, f, y) \\
\ll \quad & \frac{1}{\pi(m)} \sum_{f \le m} \left\{ (\log m)^{\omega_2(1-2\alpha_1)} + f^{(1-\alpha_1)/(1-\alpha_2)-1} m^{\varepsilon} \right\} \\
\ll \quad & (\log m)^{\omega_2(1-2\alpha_1)} + m^{-(\alpha_1-\alpha_2)/(1-\alpha_2)+\varepsilon}, \qquad (7.29)
\end{aligned}
$$

with the implied constant depending on τ. Similarly,

$$
\begin{aligned}
& \frac{1}{\pi(m)} \sum_{f \le m} f^{-1/2} A(\tau, f, y)^{1/2} \\
& \quad \ll \quad (\log m)^{\omega_2(1/2-\alpha_1)} + m^{-(\alpha_1-\alpha_2)/2(1-\alpha_2)+\varepsilon}. \qquad (7.30)
\end{aligned}
$$

Combining (7.29), (7.30) with Proposition 10, we find that $X_P(z) \to 0$ as $m \to \infty$. We also know that $Z_P(z) \to 0$ as $y \to \infty$, uniformly in z, by Proposition 4. Therefore from (7.1) we now obtain (2.9) for $\Phi = \psi_z$.

Remark 7. The above argument shows that the convergence in the case $\Phi = \psi_z$ of (2.9) is uniform in $|z| \le R$ for any $R > 0$.

8 A mean value estimate

In this section we supply a proof of Lemma 5. Except for the final part of this section, f can be any positive integer, not necessarily a prime. Recall $P =$

$P_y = \{p_1, \dots, p_r\}$. Let $\chi \in X(f)$, and write the Dirichlet series expansion of $f_P(s', \chi)$ in the region $\Re s' > 1$ as

$$f_P(s', \chi) = \frac{L(s', \chi)}{L_P(s', \chi)} - 1 = \sum_{n=1}^{\infty} b_n(\chi) n^{-s'}. \tag{8.1}$$

Then we find that $b_n(\chi) = \chi(n)$ if $n > 1$ and $(n, p_1 \cdots p_r) = 1$, and $b_n(\chi) = 0$ otherwise. Take an s' satisfying $\sigma' = \Re s' \geq \alpha_1$, and let $\xi \geq 1$, $c_0 > \max\{0, 1 - \sigma'\}$. Define

$$h_n(\chi) = b_n(\chi) \exp\left(-(n/\xi)^{\sigma' - 1/2}\right).$$

Then

$$\begin{aligned} &\sum_{n=1}^{\infty} h_n(\chi) n^{-s'} \\ &= \frac{1}{2\pi i(\sigma' - 1/2)} \int_{(c_0)} \Gamma\left(\frac{w}{\sigma' - 1/2}\right) f_P(s' + w, \chi) \xi^w dw, \end{aligned} \tag{8.2}$$

where the path of integration is the vertical line $\Re w = c_0$. This follows easily from (8.1).

Shift the path of integration to $\Re w = \alpha_2 - \sigma'$. The residue of the integrand at $w = 0$ is $(\sigma' - 1/2) f_P(s', \chi)$. Therefore, putting $w = \alpha_2 - \sigma' + iv$ we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} h_n(\chi) n^{-s'} = f_P(s', \chi) \\ &+ O\left(\frac{1}{\sigma' - 1/2} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{\alpha_2 - \sigma' + iv}{\sigma' - 1/2}\right) f_P(\alpha_2 + i(\tau' + v), \chi) \right| \xi^{\alpha_2 - \sigma'} dv\right) \end{aligned} \tag{8.3}$$

The O-term on the right-hand side is, by Stirling's formula, estimated as

$$\ll \xi^{\alpha_2 - \sigma'} \int_{-\infty}^{\infty} e^{-B_1|v|} |f_P(\alpha_2 + i(\tau' + v), \chi)| dv,$$

where B_1 is a positive constant depending on α_1, α_2. This is further estimated as

$$\begin{aligned} &\leq \xi^{\alpha_2 - \sigma'} \left(\int_{-\infty}^{\infty} e^{-B_1|v|} dv\right)^{1/2} \left(\int_{-\infty}^{\infty} e^{-B_1|v|} |f_P(\alpha_2 + i(\tau' + v), \chi)|^2 dv\right)^{1/2} \\ &\ll \xi^{\alpha_2 - \sigma'} \left(\int_{-\infty}^{\infty} e^{-B_1|v|} |f_P(\alpha_2 + i(\tau' + v), \chi)|^2 dv\right)^{1/2}. \end{aligned} \tag{8.4}$$

From (8.3) and (8.4), we have

$$\sum_{\chi \in X(f)} |f_P(s', \chi)|^2$$

$$\ll \xi^{2(\alpha_2-\sigma')}\int_{-\infty}^{\infty}e^{-B_1|v|}\sum_{\chi\in X(f)}|f_P(\alpha_2+i(\tau'+v),\chi)|^2dv$$

$$+\sum_{\chi\in X(f)}\left|\sum_{n=1}^{\infty}h_n(\chi)n^{-s'}\right|^2+|D_0(s')|^2. \tag{8.5}$$

We estimate the first term on the right-hand side of (8.5). First we note that

$$\sum_{\chi\in X(f)}|f_P(\alpha_2+i(\tau'+v),\chi)|^2$$

$$\ll \exp\left(B_2\frac{y^{1-\alpha_2}}{\log y}\right)\sum_{\chi \bmod f}|L(\alpha_2+i(\tau'+v),\chi)|^2+f \tag{8.6}$$

with an absolute constant $B_2>0$. In fact, since

$$L_P(\alpha_2+i\tau',\chi)^{-1}\le \exp\left(B_2'\sum_{p\le y}p^{-\alpha_2}\right)\le \exp\left(B_2\frac{y^{1-\alpha_2}}{\log y}\right),$$

estimate (8.6) easily follows from the definition of f_P.

Let $\varphi(f)$ be Euler's function. We prove the following

Lemma 6 *For $\Re s'=\alpha_2$, we have*

$$\sum_{\chi \bmod f}|L(s',\chi)|^2\ll\frac{\varphi(f)}{f^{2\sigma'}}(f+|\tau'|)+\varphi(f).$$

Proof. This is an analogue of a result of Gallagher [6], in which the same type of result was given for $\Re s'=1/2$. Let $\zeta(s',\alpha)=\sum_{n=0}^{\infty}(n+\alpha)^{-s'}$ be the Hurwitz zeta-function, and $\zeta_1(s',\alpha)=\zeta(s',\alpha)-\alpha^{-s'}$. We begin with the expression

$$L(s',\chi)=f^{-s'}\sum_{a=1}^{f}\chi(a)\zeta(s',a/f).$$

By using the orthogonality of characters we have

$$\sum_{\chi \bmod f}|L(s',\chi)|^2=\frac{\varphi(f)}{f^{2\sigma'}}\sum_{\substack{1\le a\le f\\(a,f)=1}}|\zeta(s',a/f)|^2$$

$$\ll \frac{\varphi(f)}{f^{2\sigma'}}\sum_{\substack{1\le a\le f\\(a,f)=1}}|\zeta_1(s',a/f)|^2+\varphi(f). \tag{8.7}$$

A key inequality of Gallagher [6] (cf. his proof [5] of the large sieve inequality) is

$$\sum_{a=1}^{f}|\zeta_1(s',a/f)|^2\le f\int_0^1|\zeta_1(s',\alpha)|^2d\alpha+2\int_0^1\left|\zeta_1(s',\alpha)\frac{\partial}{\partial\alpha}\zeta_1(s',\alpha)\right|d\alpha \tag{8.8}$$

([6], formula (9)). Since $(\partial/\partial\alpha)\zeta_1(s',\alpha) = -s'\zeta_1(s'+1,\alpha)$, this factor is $O(|\tau'|+1)$ for $\Re s' = \alpha_2$. (This is uniform in α, because we consider not ζ but ζ_1.) Therefore, by using Schwarz' inequality, we have

$$\sum_{a=1}^{f} |\zeta_1(s',a/f)|^2 \ll f\int_0^1 |\zeta_1(s',\alpha)|^2 d\alpha + (|\tau'|+1)\left(\int_0^1 |\zeta_1(s',\alpha)|^2 d\alpha\right)^{1/2} \tag{8.9}$$

Concerning the integral appearing on the right-hand side, we know

$$\int_0^1 |\zeta_1(s',\alpha)|^2 d\alpha = O(1) \tag{8.10}$$

for $\Re s' = \alpha_2$ (Koksma and Lekkerkerker [14]). Applying (8.10) to (8.9), we obtain

$$\sum_{a=1}^{f} |\zeta_1(s',a/f)|^2 \ll f + (|\tau'|+1), \tag{8.11}$$

and combining this with (8.7), we obtain the assertion of Lemma 6.

Using (8.6) and Lemma 6, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-B_1|v|} \sum_{\chi\in X(f)} |f_P(\alpha_2+i(\tau'+v),\chi)|^2 dv \\
&\ll \int_{-\infty}^{\infty} e^{-B_1|v|} \\
&\qquad \times\left\{\exp\left(B_2\frac{y^{1-\alpha_2}}{\log y}\right)\left(\frac{\varphi(f)}{f^{2\alpha_2}}(f+|\tau'+v|)+\varphi(f)\right)+f\right\}dv \\
&= \left\{\exp\left(B_2\frac{y^{1-\alpha_2}}{\log y}\right)\varphi(f)(f^{1-2\alpha_2}+1)+f\right\}\int_{-\infty}^{\infty} e^{-B_1|v|}dv \\
&\qquad +\exp\left(B_2\frac{y^{1-\alpha_2}}{\log y}\right)\frac{\varphi(f)}{f^{2\alpha_2}}\int_{-\infty}^{\infty} e^{-B_1|v|}|\tau'+v|dv \\
&\ll \exp\left(B_2\frac{y^{1-\alpha_2}}{\log y}\right)\varphi(f)\left(1+\frac{|\tau'|+1}{f^{2\alpha_2}}\right)+f.
\end{aligned} \tag{8.12}$$

Next consider the second term on the right-hand side of (8.5). Since $h_n(\chi)=0$ if $n\le y$, by using the orthogonality of characters we have

$$\begin{aligned}
&\sum_{\chi\in X(f)}\left|\sum_{n=1}^{\infty} h_n(\chi)n^{-s'}\right|^2 \le \sum_{\chi \bmod f}\left|\sum_{n=1}^{\infty} h_n(\chi)n^{-s'}\right|^2 \\
&\le \varphi(f)\sum_{\substack{m,n>y \\ m\equiv n (\bmod f)}} \exp\left(-\left(\frac{m}{\xi}\right)^{\sigma'-1/2}-\left(\frac{n}{\xi}\right)^{\sigma'-1/2}\right)(mn)^{-\sigma'} \\
&= \varphi(f)\left\{\sum_{m=n}+\sum_{m>n}+\sum_{m<n}\right\} = \varphi(f)(\Sigma_1+\Sigma_2+\Sigma_3),
\end{aligned} \tag{8.13}$$

say. Clearly

$$\Sigma_1 \le \sum_{n>y} n^{-2\sigma'} \ll y^{1-2\sigma'}. \tag{8.14}$$

Next, $\Sigma_3 = \Sigma_2$, and

$$\begin{aligned}\Sigma_2 = & \sum_{n>y} \exp\left(-\left(\frac{n}{\xi}\right)^{\sigma'-1/2}\right) n^{-\sigma'} \\ \times & \sum_{l\ge 1} \exp\left(-\left(\frac{n+lf}{\xi}\right)^{\sigma'-1/2}\right)(n+lf)^{-\sigma'}.\end{aligned} \tag{8.15}$$

The inner sum can be estimated as

$$\le \int_0^\infty \exp\left(-\left(\frac{n+vf}{\xi}\right)^{\sigma'-1/2}\right)(n+vf)^{-\sigma'}dv \ll f^{-1}\xi^{1-\sigma'},$$

hence

$$\begin{aligned}\Sigma_2 \ll & f^{-1}\xi^{1-\sigma'} \sum_{n>y} \exp\left(-\left(\frac{n}{\xi}\right)^{\sigma'-1/2}\right) n^{-\sigma'} \\ \le & f^{-1}\xi^{1-\sigma'} \int_0^\infty \exp\left(-\left(\frac{v}{\xi}\right)^{\sigma'-1/2}\right) v^{-\sigma'}dv \ll f^{-1}\xi^{2(1-\sigma')}.\end{aligned} \tag{8.16}$$

From (8.13), (8.14) and (8.16) we obtain

$$\sum_{\chi\in X(f)} \left|\sum_{n=1}^\infty h_n(\chi)n^{-s'}\right|^2 \ll \varphi(f)(y^{1-2\sigma'} + f^{-1}\xi^{2(1-\sigma')}). \tag{8.17}$$

Now let f be a prime, hence $\varphi(f) = f-1$. Substituting (8.12) and (8.17) into the right-hand side of (8.5), we obtain

$$\begin{aligned}\sum_{\chi\in X(f)} |f_P(s',\chi)|^2 \ll & f\xi^{2(\alpha_2-\sigma')} \exp\left(B_2\frac{y^{1-\alpha_2}}{\log y}\right)\left(1+\frac{|\tau'|+1}{f^{2\alpha_2}}\right) \\ + & fy^{1-2\sigma'} + \xi^{2(1-\sigma')}.\end{aligned} \tag{8.18}$$

Choosing the value of the parameter ξ as $\xi = f^{1/2(1-\alpha_2)}$, we obtain the assertion of Lemma 5. The proof of (2.9) for $\Phi = \psi_z$ is thus complete.

9 Completion of the proof

So far we have proved Theorem 1 in the special case $\Phi = \psi_z$. Now we prove the theorem for general test function Φ of type (i) or (ii).

By $f^\wedge$ (resp. $f^\vee$) we denote the Fourier (resp. inverse Fourier) transform of f. Let Λ be the set of all functions $f: \mathbf{C} \to \mathbf{C}$ such that $f, f^\wedge \in L^1 \cap L^\infty$ and $(f^\wedge)^\vee = f$. We first consider the case when $\Phi \in \Lambda$. The argument in this case is similar to that in Section 6.7 of [9].

Assume $\Phi \in \Lambda$. Then $(\Phi^\wedge)^\vee = \Phi$, that is

$$\Phi(w) = \int_{\mathbf{C}} \Phi^\wedge(z)\psi_{-z}(w)|dz|. \tag{9.1}$$

On the other hand, from Propositions 4 and 5 we see that $\mathcal{M}_\sigma \in \Lambda$. Therefore

$$\begin{aligned}\int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi(w)|dw| &= \int_{\mathbf{C}} \overline{\mathcal{M}_\sigma{}^\wedge(z)}\Phi^\wedge(z)|dz| \\ &= \int_{\mathbf{C}} \widetilde{\mathcal{M}}_\sigma(-z)\Phi^\wedge(z)|dz| \end{aligned}\tag{9.2}$$

(the second equality clearly follows from $\widetilde{\mathcal{M}}_\sigma = \mathcal{M}_\sigma{}^\wedge$ and (3.32)). From (9.1) and (9.2) we obtain

$$\begin{aligned}&\left|\mathrm{Avg}_{f\le m}\Phi(\log L(s,\chi)) - \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi(w)|dw|\right| \\ &\quad\le \int_{\mathbf{C}} |\Phi^\wedge(z)| \left|\mathrm{Avg}_{f\le m}\psi_{-z}(\log L(s,\chi)) - \widetilde{\mathcal{M}}_\sigma(-z)\right| |dz| \\ &\quad= \int_{\mathbf{C}} |\Phi^\wedge(-z)| \left|\mathrm{Avg}_{f\le m}\psi_{z}(\log L(s,\chi)) - \widetilde{\mathcal{M}}_\sigma(z)\right| |dz| \end{aligned}\tag{9.3}$$

in Case (A,$\mathbf{Q}$). We divide the integral on the right-hand side into two parts:

$$\int_{|z|\le R} + \int_{|z|>R} \tag{9.4}$$

(where $R > 0$). From (3.22) we have

$$\left|\mathrm{Avg}_{f\le m}\psi_{z}(\log L(s,\chi)) - \widetilde{\mathcal{M}}_\sigma(z)\right| \le 2.$$

Using this inequality and the fact $\Phi^\wedge \in L^1$, we see that, for any $\varepsilon > 0$, we can find a sufficiently large $R = R(\varepsilon)$ for which the second integral of (9.4) is smaller than $\varepsilon/2$. Then we use the case $\Phi = \psi_z$ of Theorem 1. We have already shown that the convergence is uniform in $|z| \le R$ (Remarks 6 and 7). Therefore, under the choice of a sufficiently large m, the first integral of (9.4) can also be smaller than $\varepsilon/2$. This completes the proof of Theorem 1 for $\Phi \in \Lambda$ in Case (A,$\mathbf{Q}$). In Case (C), we replace $\mathrm{Avg}_{f\le m}$ by

$$\mathrm{Avg}_{|\tau|\le T}\phi(\chi_\tau) = \frac{1}{2T}\int_{-T}^{T} \phi(\chi_\tau)d\tau$$

and argue as above to obtain the same conclusion.

It is known that the Schwartz space $\mathcal{S}$, whch consists of all C^∞-functions f such that $|w|^k D(f)$ tends to 0 as $|w| \to \infty$ for any $k \geq 0$ and any partial derivative of any order, is a subset of Λ. In particular, now we have verified Theorem 1 for any compactly supported C^∞-function.

Any compactly supported continuous function (or even any continuous function which tends to 0 as $|w| \to \infty$) can be approximated uniformly by compactly supported C^∞-functions, and furthermore, the characteristic function of any compact subset of $\mathbf{C}$ can be approximated by compactly supported continuous functions. Therefore Case (ii) of Theorem 1 follows. These arguments are rather standard, and are presented in detail in Section 4.3 of [10], so we omit the details here.

Finally, let Φ be any bounded continuous function. For any $R > 0$, there exists a compactly supported continuous function Φ_R, such that $\Phi_R(w) = \Phi(w)$ for $|w| \leq R$ and $|\Phi_R(w)| \leq |\Phi(w)|$ everywhere. We have already shown that Theorem 1 holds for compactly supported continuous functions, hence

$$\lim_{m\to\infty} \left(\mathrm{Avg}_{f\leq m}\Phi_R(\log L(s,\chi))\right) = \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi_R(w)|dw| \tag{9.5}$$

holds in Case (A,$\mathbf{Q}$). The right-hand side can be divided as

$$\int_{|w|\geq R} \mathcal{M}_\sigma(w)(\Phi_R(w) - \Phi(w))|dw| + \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi(w)|dw|,$$

and, when $R \to \infty$, the first term of the above tends to 0 because of (3.30). Therefore

$$\lim_{R\to\infty} \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi_R(w)|dw| = \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi(w)|dw|. \tag{9.6}$$

The sequence $\{\mathrm{Avg}_{f\leq m}\Phi(\log L(s,\chi))\}_{m=1}^\infty$ is bounded, hence we can find an accumulation point α. What we have to show is that this α is unique, and is equal to the right-hand side of (9.6). Let $\{\mathrm{Avg}_{f\leq m_1}\Phi(\log L(s,\chi))\}_{m_1=1}^\infty$ be a subsequence whose limit is α. The sequence

$$\left\{\mathrm{Avg}_{f\leq m_1}(\Phi(\log L(s,\chi)) - \Phi_R(\log L(s,\chi)))\right\}_{m_1=1}^\infty$$

is then convergent. Denoting its limit by $\beta(R)$, we have

$$\beta(R) = \alpha - \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi_R(w)|dw|. \tag{9.7}$$

Let $\mathrm{ch}_R(w)$ be the characteristic function of the set $\{w\ ;\ |w| \geq R\}$. Case (ii) of Theorem 1 implies

$$\begin{aligned}\lim_{m\to\infty} \left(\mathrm{Avg}_{f\leq m}\mathrm{ch}_R(\log L(s,\chi))\right) &= \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\mathrm{ch}_R(w)|dw| \\ &= \int_{|w|\geq R} \mathcal{M}_\sigma(w)|dw|, \end{aligned} \tag{9.8}$$

which tends to 0 as $R \to \infty$ by (3.30). Since $|\Phi - \Phi_R| \ll \mathrm{ch}_R$, we find that $\beta(R) \to 0$ as $R \to \infty$. Therefore, taking the limit $R \to \infty$ on the both sides of (9.7), we obtain

$$\alpha = \lim_{R\to\infty} \int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi_R(w)|dw|. \tag{9.9}$$

The desired result in Case (A,$\mathbf{Q}$) follows from (9.6) and (9.9). In Case (C), as before, we replace $\mathrm{Avg}_{f\le m}$ by $\mathrm{Avg}_{|\tau|\le T}$ and argue similarly. The proof of Theorem 1 is thus complete.

We note that in Case (C), the assertion (ii) of our Theorem 1 includes, as a special case, the classical result (1.1), (1.2) of Bohr and Jessen.

On the other hand, if we first assume the result of Bohr and Jessen, it is possible to deduce Case (C) of our Theorem 1 from their result. In fact, if Φ is a compactly supported C^∞-function, then Φ can be approximated uniformly by some finite linear combination of characteristic functions of rectangles, hence the result follows from the result of Bohr and Jessen. Then the general case of Theorem 1 follows as above.

References

[1] H. Bohr, Zur Theorie der Riemann'schen Zetafunktion im kritischen Streifen, Acta Math. **40** (1915), 67-100.

[2] H. Bohr and B. Jessen, Om Sandsynlighedsfordelinger ved Addition af convekse Kurver, Dan. Vid. Selsk. Skr. Nat. Math. Afd. (8)**12** (1929), 1-82.

[3] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion, I, Acta Math. **54** (1930), 1-35; II, ibid. **58** (1932), 1-55.

[4] V. Borchsenius and B. Jessen, Mean motions and values of the Riemann zeta function, Acta Math. **80** (1948), 97-166.

[5] P. X. Gallagher, The large sieve, Mathematika **14** (1967), 14-20.

[6] P. X. Gallagher, Local mean value and density estimates for Dirichlet L-functions, Indag. Math. **37** (1975), 259-264.

[7] Y. Ihara, On the Euler-Kronecker constants of global fields and primes with small norms, in "Algebraic Geometry and Number Theory", Progr. Math. **253**, Birkhäuser, Boston, 2006, pp.407-451.

[8] Y. Ihara, The Euler-Kronecker invariants in various families of global fields, in "Arithmetic, Geometry, and Coding Theory (AGCT 2005)", F. Rodier and S. Vladut (eds.), Séminaires et Congrès **21**, Soc. Math. France, to appear.

[9] Y. Ihara, On "M-functions" closely related to the distribution of L'/L-values, Publ. RIMS, Kyoto Univ. **44** (2008), 893-954.

[10] Y. Ihara and K. Matsumoto, On L-functions over function fields: Power-means of error-terms and distribution of L'/L-values, in "Algebraic Number Theory and Related Topics 2008", RIMS Kokyuroku Bessatsu, to appear.

[11] Y. Ihara and K. Matsumoto, On $\log L$ and L'/L for L-functions and the associated "M-functions": Connections in optimal cases, preprint.

[12] Y. Ihara, V. Kumar Murty and M. Shimura, On the logarithmic derivatives of Dirichlet L-functions at $s = 1$, Acta Arith., to appear.

[13] B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function, Trans. Amer. Math. Soc. **38** (1935), 48-88.

[14] J. F. Koksma and C. G. Lekkerkerker, A mean-value theorem for $\zeta(s, w)$, Indag. Math. **14** (1952), 446-452.

[15] A. Laurinčikas, Limit Theorems for the Riemann Zeta-Function, Kluwer Acad. Publ., Dordrecht, 1996.

[16] K. Matsumoto, A probabilistic study on the value-distribution of Dirichlet series attached to certain cusp forms, Nagoya Math. J. **116** (1989), 123-138.

[17] K. Matsumoto, Value-distribution of zeta-functions, in "Analytic Number Theory", Proc. Japanese-French Sympos. held in Tokyo, 1988, K. Nagasaka and E. Fouvry (eds.), Lecture Notes in Math. **1434**, Springer-Verlag, 1990, pp.178-187.

[18] K. Matsumoto, Asymptotic probability measures of zeta-functions of algebraic number fields, J. Number Theory **40** (1992), 187-210.

[19] H. L. Montgomery, Topics in Multiplicative Number Theory, Lecture Notes in Math. **227**, Springer-Verlag, 1971.

[20] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, 2nd ed. (revised by D. R. Heath-Brown), Oxford, 1986.

[21] A. Weil, Basic Number Theory, Springer-Verlag, 1967.

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