On the value-distribution of $\log L$ and L'/L

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1 The general formulation

The purpose of this article is to survey recent results of the authors on the value-distribution of $\log L$ and L'/L for certain L-functions. Details [9, 10, 11] will be published elsewhere. We begin with the general formulation of the problem.

Let K be a global field, X a certain family of infinitely many "characters" χ defined on K, and $L(s,\chi)$ the "L-function" associated with χ, where $s=\sigma+i\tau$ is a complex variable. Our aim is to construct some "density function" $M_\sigma(w)$ (resp. $\mathcal{M}_\sigma(w)$) which describes the value-distribution of L'/L (resp. $\log L$) in the sense that

$$\mathrm{Avg}_\chi \Phi\left(\frac{L'}{L}(s,\chi)\right)=\int_{\mathbf{C}} M_\sigma(w)\Phi(w)|dw| \tag{1.1}$$

or

$$\mathrm{Avg}_\chi \Phi\left(\log L(s,\chi)\right)=\int_{\mathbf{C}} \mathcal{M}_\sigma(w)\Phi(w)|dw| \tag{1.2}$$

holds, where s is fixed, Avg_χ is an average with respect to $\chi\in X$ in some suitable sense, Φ is a "test function" (some function with good properties defined on $\mathbf{C}$), and $|dw|=(2\pi)^{-1}dudv$ $(w=u+iv)$.

In the case of the Riemann zeta-function $\zeta(s)$, the value-distribution theory was cultivated by H. Bohr and his colleagues in the first half of the 20th century. Bohr treated the average with respect to the imaginary part of the variable; in other words, he considered the case when characters are defined by $\chi_{\tau'}(p)=p^{-i\tau'}$ for each prime p. In this case the associated L-function is

$$\prod_p (1-\chi_{\tau'}(p)p^{-s})^{-1}=\prod_p(1-p^{-s-i\tau'})^{-1}=\zeta(s+i\tau'), \tag{1.3}$$

and the average with respect to $\chi_{\tau'}$ implies the mean value with respect to τ'. In the $\log\zeta$ case, Bohr and Jessen [1] proved the following theorem. Let $\sigma>1/2$, $T>0$, R an arbitrary rectangle in $\mathbf{C}$ with the edges parallel to the axes, and $V_\sigma(T,R)$ the Lebesgue measure of the set of all $\tau' \in [-T,T]$ for which $\log\zeta(\sigma+i\tau') \in R$ holds (under a certain fixed choice of the branch). Then they proved that there exists a continuous, everywhere non-negative function $\mathcal{F}_\sigma(w)$ for which

$$\lim_{T\to\infty}\frac{1}{2T}V_\sigma(T,R)=\int_R \mathcal{F}_\sigma(w)|dw| \tag{1.4}$$

holds for any R. Therefore, if we define the average by

$$\mathrm{Avg}_\chi\phi(\chi_{\tau'})=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\phi(\chi_{\tau'})d\tau' \tag{1.5}$$

(where $\phi(\chi_{\tau'})$ is any integrable function of τ'), we can see that (1.4) of Bohr and Jessen is a special case of (1.2), with $\Phi=\mathbf{1}_R$, the characteristic function of R, and $\mathcal{M}_\sigma=\mathcal{F}_\sigma$.

On the other hand, it is also important to consider some different type of averages. In particular, various averages of Dirichlet L-functions with respect to Dirichlet characters have been studied by many mathematicians. Some probabilistic limit theorems in this direction were obtained by P. D. T. A. Elliott and E. Stankus in 1970s. (See Chapter 8 of Laurinčikas [14].)

Recently, motivated by a study on Euler-Kronecker constants of global fields ([5], [6], [12]), the first author [8] searched for the density function (the "M-function") $M_\sigma(w)$ which satisfies (1.1) in various situations. In the next section we review this work briefly.

In what follows, the symbol $|A|$ means the cardinality of the set A. The meaning of Vinogradov's symbol $f \ll g$ is the same as that of Landau's symbol $f=O(g)$.

2 The situations we are working in

In [8], the following three situations are considered:

(A) $K=\mathbf{Q}$ (the rational number field), or an imaginary quadratic field, or a function field over any finite field. In the last case, we fix one prime divisor $\wp_\infty$ of degree 1 which plays the same role as the unique archimedean prime in the first two cases. The characters are Dirichlet characters χ on K, normalized by the condition $\chi(\wp_\infty)=1$ in the last case.

(B) K is a number field with at least two archimedean primes, and the characters are some family of unramified Grössencharacters.

(C) $K=\mathbf{Q}$, and the characters are $\chi_{\tau'}$ defined in Section 1.

We explain the meaning of "average" in Case (A). Let $\mathbf{f}$ be a prime divisor of K, and $N(\mathbf{f})$ its norm. By $X(\mathbf{f})$ we mean the set of all Dirichlet characters whose conductor is $\mathbf{f}$. Let

$$\mathrm{Avg}_{X(\mathbf{f})}\phi(\chi)=\frac{1}{|X(\mathbf{f})|}\sum_{\chi\in X(\mathbf{f})}\phi(\chi) \tag{2.1}$$

(where ϕ is any complex-valued function) and

$$\mathrm{Avg}^{(1)}_{N(\mathbf{f})\leq m}\phi(\chi) = \frac{\sum_{N(\mathbf{f})\leq m}\mathrm{Avg}_{X(\mathbf{f})}\phi(\chi)}{\sum_{N(\mathbf{f})\leq m}1}, \tag{2.2}$$

where m is a positive integer, and $\mathbf{f}$ runs over all prime divisors whose norm does not exceed m. Then the meaning of Avg_χ in Case (A), studied in [8], is

$$\mathrm{Avg}^{(1)}_{\chi}\phi(\chi) = \lim_{m\to\infty}\left(\mathrm{Avg}^{(1)}_{N(\mathbf{f})\leq m}\phi(\chi)\right). \tag{2.3}$$

(We attach the suffix (1) here, because later we will also consider different types of averages.) The meaning of the average in Case (C) is (1.5). As for the meaning of the average in Case (B), we refer to [8].

The associated L-function is defined as the usual Euler product, but in the function field case, without the $\wp_\infty$-component; that is,

$$L(s,\chi) = \prod_{\wp\neq\wp_\infty}(1-\chi(\wp)N(\wp)^{-s})^{-1},$$

where $\wp$ runs over prime divisors and $N(\wp)$ is the norm of $\wp$.

In the domain of absolute convergence of L-functions, the following theorem is proved in [8]:

Theorem 1 *We can construct explicitly the "M-function" $M_\sigma(w)$ on* $\mathbf{C}$ *for any* $\sigma > 1/2$*, such that when* $\sigma = \Re s > 1$, (1.1) *holds for all the cases* (A), (B) *and* (C)*, for any continuous function* Φ*.*

Needless to say, more interesting is to consider the situation in the critical strip, which is surely much more difficult. One obvious obstacle is the generalized Riemann hypothesis (GRH), because in the L'/L case the L-function is in the denominator. Therefore, in [8], the function field case is mainly studied, because in this case the GRH has been solved.

In [8], formula (1.1) is proved in Case (A) for function fields K, in each of the following two cases.

(a) $\sigma > 3/4$, and Φ is any *character* $\mathbf{C} \to T = \{t \in \mathbf{C}\ ; |t| = 1\}$, i.e., $\Phi = \psi_z$ with some $z \in \mathbf{C}$ defined by

$$\psi_z(w) = \exp(i\Re(\overline{z}w)); \tag{2.4}$$

(b) $\sigma > 1/2$, and Φ is any *polynomial* in two variables z, $\bar{z}$;

see Theorem 7(ii)(iii) of [8]. The point is that instead of the characteristic function of rectangles as in previous investigations, we consider characters and polynomials. The result for characters will have some applications to more general case of Φ, as we shall see.

However, we already mentioned in Section 1 that the result (1.4) of Bohr and Jessen is proved for any $\sigma > 1/2$. Therefore we may expect that the results such as (a) is also valid for any $\sigma > 1/2$.

3 Unconditional results for the $\log L$ case, $K = \mathbf{Q}$

The reason why Bohr-Jessen succeeded in proving their result for any $\sigma > 1/2$ is that they used some mean value results on relevant Dirichlet series. Since then, mean value theorems have been frequently used successfully in the value-distribution theory (cf. Laurinčikas [14], Steuding [17]). Therefore it is natural to expect that a suitable usage of mean value results will improve the above (a) and (b), or will give some progress even in the number field case.

One possible way of research is to develop the $\log L$ analogue of the theory in [8] and combine it with the Bohr-Jessen theory. The original argument of Bohr and Jessen is rather geometric, based on various properties of planer convex curves, but later, more analytic (or Fourier-theoretic) approaches to the Bohr-Jessen type theorems have been developed; see Jessen-Wintner [13], Borchsenius-Jessen [2], the second author [15] etc. Therefore a strategy is to apply the methods in those papers to the $\log L$ analogue of the theory of [8] which is also rather Fourier-theoretic. This has been carried out in [10], and the following theorems are proved.

First, in the domain of absolute convergence, as an analogue of Theorem 1, we obtain:

Theorem 2 *We can construct explicitly the "M-function" $\mathcal{M}_\sigma(w)$ on $\mathbf{C}$ for any $\sigma > 1/2$, such that when $\sigma = \Re s > 1$,* (1.2) *holds for all the cases* (A), (B) *and* (C), *for any continuous function Φ.*

In the strip $1/2 < \sigma \le 1$, in [10] we prove the result only in the case $K = \mathbf{Q}$, so only the case (A) (with $K = \mathbf{Q}$) or (C).

First of all we have to fix the branch of the logarithm. Let $D = \{s\ ;\ 1/2 < \sigma \le 1\}$, and remove from D all segments

$$B_j(\chi) = \{s = \sigma + i\tau_j\ ;\ 1/2 < \sigma \le \sigma_j\},$$

where $\sigma_j + i\tau_j$ are possible zeros (and a possible pole) of $L(s, \chi)$ in D, and denote the remaining set by G_χ. At any point $s_0 = \sigma_0 + i\tau_0 \in G_\chi$, we define the value of $\log L(s_0, \chi)$ by the analytic continuation along the horizontal path $\{s = \sigma + i\tau_0\ ;\ \sigma \ge \sigma_0\}$.

In Case (A) (with $K = \mathbf{Q}$), if we use definition (2.3) of the average, then we should exclude all points $s \in B_j(\chi)$ for all j and all χ from our consideration, which is quite unsatisfactory. To avoid this trouble, we modify the definition of the average as follows. Let $X(f)$ be the set of all primitive Dirichlet characters whose conductor is f, and $X'(f)$ be a subset of $X(f)$ for which

$$\lim_{f \to \infty} \frac{|X'(f)|}{|X(f)|} = 1 \tag{3.1}$$

holds. Let

$$\mathrm{Avg}_{X'(f)}\phi(\chi) = \frac{1}{|X(f)|}\sum_{\chi\in X'(f)}\phi(\chi),$$

and define $\mathrm{Avg}^{(2)}_{f\le m}\phi(\chi)$ by replacing $\mathrm{Avg}_{X(f)}$ in (2.2) by $\mathrm{Avg}_{X'(f)}$, and define $\mathrm{Avg}^{(2)}_{\chi}\phi(\chi)$ by replacing $\mathrm{Avg}^{(1)}_{f\le m}$ in (2.3) by $\mathrm{Avg}^{(2)}_{f\le m}$. It is important that, if ϕ is bounded, then the value of $\mathrm{Avg}^{(2)}_{\chi}\phi(\chi)$ will not change when we choose smaller $X'(f)$, keeping condition (3.1).

By using a zero density theorem on L-functions (Montgomery [16]), we can check that, for any s with $1/2 < \Re s \le 1$, the set

$$X'(f) = X'(f,s) := \{\chi \in X(f)\ ;\ s \in G_\chi\} \tag{3.2}$$

satisfies (3.1). Under this choice of $X'(f)$ and the above definition of the average, we can show the following theorem.

Theorem 3 *Let $1/2 < \sigma = \Re s \le 1$. In Case* (A) (*with $K = \mathbf{Q}$*) *or Case* (C), (1.2) *holds for Φ which is one of the following* (*or any finite linear combination of them*):

(i) *Φ is any bounded continuous function.*

(ii) *Φ is the characteristic function of either a compact subset of $\mathbf{C}$ or the complement of such a subset.*

The special case $\Phi = \psi_z$ (see (2.4)) of (i) is actually the basic case in our proof.

In Case (A) (with $K = \mathbf{Q}$), (1.2) can be read as

$$\lim_{m\to\infty}\frac{1}{\pi(m)}\sum_{\substack{2<f\le m\\ f:\text{prime}}}\frac{1}{f-2}\sum_{\chi\in X'(f)}\Phi(\log L(s,\chi)) = \int_{\mathbf{C}}\mathcal{M}_\sigma(w)\Phi(w)|dw|, \tag{3.3}$$

where $\pi(m)$ is the number of primes not larger than m.

In Case (C), the meaning of the average is (1.5), and (1.2) can be read as

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Phi(\log\zeta(s+i\tau'))d\tau' = \int_{\mathbf{C}}\mathcal{M}_\sigma(w)\Phi(w)|dw|. \tag{3.4}$$

If $s + i\tau' \notin G_{\mathbf{1}}$ (where $\mathbf{1}$ is the trivial character), then $\log\zeta(s+i\tau')$ is not defined, but still the integral is well-defined for the above type of Φ, so it is not necessary to exclude such situation.

4 The construction of "M-functions"

In this and the next section we assume $K = \mathbf{Q}$, and sketch how to prove Theorems 2 and 3. In this section we explain how to construct the density function (the "M-function"). The argument is basically an analogue of that developed in [8], where the existence of the "M-function" in the L'/L case was established.

Let P be a finite set of primes, and define

$$L_P(s,\chi) = \prod_{p\in P}(1-\chi(p)p^{-s})^{-1}.$$

The first aim is to construct the density function $\mathcal{M}_{\sigma,P}(w)$ for which

$$\mathrm{Avg}_\chi \Phi(\log L_P(s,\chi)) = \int_{\mathbf{C}} \mathcal{M}_{\sigma,P}(w)\Phi(w)|dw| \tag{4.1}$$

holds for any continuous function Φ.

Let $T = \{t \in \mathbf{C}\ ;\ |t| = 1\}$, and $T_P = \prod_{p\in P} T$. Define the function $g_{\sigma,P} : T_P \to \mathbf{C}$ by

$$g_{\sigma,P}(\mathbf{t}_P) = -\sum_{p\in P}\log(1-t_p p^{-\sigma}), \qquad \mathbf{t}_P = (t_p)_{p\in P} \in T_P. \tag{4.2}$$

Then we see that

$$\log L_P(s,\chi) = g_{\sigma,P}(\chi_P P^{-i\tau}), \tag{4.3}$$

where $\chi_P = (\chi(p))_{p\in P} \in T_P$ and $P^{-i\tau} = (p^{-i\tau})_{p\in P} \in T_P$. Here we quote the following lemma:

$$\mathrm{Avg}_\chi \Psi(\chi_P) = \int_{T_P} \Psi(\mathbf{t}_P)d^*\mathbf{t}_P \tag{4.4}$$

holds for any continuous function $\Psi : T_P \to \mathbf{C}$, where $d^*\mathbf{t}_P$ is the normalized Haar measure on T_P (Lemma 4.3.1 of [8]; in Case (A), exclude those χ whose conductor is $\in P$). This lemma is a reflection of the uniform distribution of χ_P on T_P, and can be proved by using the orthogonality relation of characters in Case (A), and the Kronecker-Weyl theorem in Case (C). Applying this lemma with $\Psi = \Phi \circ g_{\sigma,P}$, and combining with (4.3), we obtain

$$\mathrm{Avg}_\chi \Phi(\log L_P(s,\chi)) = \int_{T_P} \Phi(g_{\sigma,P}(\mathbf{t}_P))d^*\mathbf{t}_P. \tag{4.5}$$

Therefore, to prove (4.1), it is enough to find $\mathcal{M}_{\sigma,P}(w)$ which satisfies

$$\int_{T_P} \Phi(g_{\sigma,P}(\mathbf{t}_P))d^*\mathbf{t}_P = \int_{\mathbf{C}} \mathcal{M}_{\sigma,P}(w)\Phi(w)|dw|. \tag{4.6}$$

When P consists of only one prime $P=\{p\}$, by direct calculations we find that

$$\mathcal{M}_{\sigma,\{p\}}(w) = \frac{|1-r_p e^{i\theta_p}|^2}{r_p}\delta(r_p - p^{-\sigma}) \tag{4.7}$$

suffices, where r_p, θ_p are determined by $w = -\log(1 - r_p e^{i\theta_p})$ and $\delta(\cdot)$ denotes the Dirac delta distribution. (When there is no solution r_p, θ_p then $\mathcal{M}_{\sigma,\{p\}}(w) = 0$.) For general (finite) P, we define $\mathcal{M}_{\sigma,P}(w)$ by the convolution product

$$\mathcal{M}_{\sigma,P}(w) = \int_{\mathbf{C}} \mathcal{M}_{\sigma,P'}(w')\mathcal{M}_{\sigma,\{p\}}(w - w')|dw'| \tag{4.8}$$

if $P = P' \cup \{p\}$. Then, if $|P| \geq 2$, this is a function in the usual sense, compactly supported, non-negative, and

$$\int_{\mathbf{C}} \mathcal{M}_{\sigma,P}(w)|dw| = 1. \tag{4.9}$$

It is not difficult to see that this $\mathcal{M}_{\sigma,P}(w)$ satisfies (4.6), and hence (4.1), for any continuous Φ.

Now we choose $P = P_y := \{p : \text{prime} \ ; \ p \leq y\}$. The next step is the proof of the existence of the limit

$$\mathcal{M}_{\sigma}(w) = \lim_{y\to\infty} \mathcal{M}_{\sigma,P}(w). \tag{4.10}$$

For this purpose we consider the Fourier transform

$$\widetilde{\mathcal{M}}_{\sigma,P}(z) = \prod_{p\in P} \int_{\mathbf{C}} \mathcal{M}_{\sigma,\{p\}}(w)\psi_z(w)|dw|. \tag{4.11}$$

Since each integral on the right-hand side is $O((1 + |z|)^{-1/2})$ (Theorem 13 of Jessen and Wintner [13]), we have $\widetilde{\mathcal{M}}_{\sigma,P}(z) = O((1 + |z|)^{-|P|/2})$. Using this fact, we can show that there exists the limit

$$\widetilde{\mathcal{M}}_{\sigma}(z) = \lim_{y\to\infty} \widetilde{\mathcal{M}}_{\sigma,P}(z), \tag{4.12}$$

and the above convergence is uniform in any compact subset of $\mathbf{C}$ (by applying the method in Section 3 of [15]). From (4.12) we can deduce the existence of the limit (4.10) for any $\sigma > 1/2$. It is continuous in both σ and z, non-negative, and

$$\int_{\mathbf{C}} \mathcal{M}_{\sigma}(w)|dw| = 1. \tag{4.13}$$

It is compactly supported when $\sigma > 1$, while tends to 0 as $|w| \to \infty$ when $1/2 < \sigma \leq 1$. The functions $\mathcal{M}_{\sigma}$ and $\widetilde{\mathcal{M}}_{\sigma}$ are Fourier duals of each other.

It is worthwhile noting that $\widetilde{\mathcal{M}}_{\sigma}$ has the Dirichlet series expansion

$$\widetilde{\mathcal{M}}_{\sigma}(z) = \sum_{n=1}^{\infty} \lambda_z(n)\lambda_{\overline{z}}(n)n^{-2\sigma}, \tag{4.14}$$

where $\lambda_z(n)$ is defined by

$$L(s,\chi)^{iz/2} = \sum_{n=1}^{\infty} \lambda_z(n)\chi(n)n^{-s}. \tag{4.15}$$

Since $\lambda_z(n) \ll n^{\epsilon}$ for any $\epsilon > 0$, the series (4.14) is convergent absolutely for $\sigma > 1/2$ (see [11]).

5 Mean value theorems play a role

Now our aim is to show that the "M-function" $\mathcal{M}_\sigma$, constructed in the preceding section, indeed satisfies (1.2) for any $\sigma > 1/2$ and for a reasonably large family of test functions Φ. We briefly outline the proof for Case (C). A fundamental idea in [10] (as well as in [8]) is to consider the case $\Phi = \psi_z$ first. In this case the right-hand side of (3.4) is $\widetilde{\mathcal{M}}_\sigma(z)$, and so, what we have to show is that

$$\left| \frac{1}{2T} \int_{-T}^{T} \psi_z(\log \zeta(\sigma + i\tau'))d\tau' - \widetilde{\mathcal{M}}_\sigma(z) \right| \tag{5.1}$$

tends to 0 as $T \to \infty$. (We can easily see that $\zeta(s + i\tau')$ can be replaced by $\zeta(\sigma + i\tau')$.) The quantity (5.1) can be estimated as

$$\begin{aligned}
\leq\ & \left| \frac{1}{2T} \int_{-T}^{T} \psi_z(\log \zeta(\sigma + i\tau'))d\tau' - \frac{1}{2T} \int_{-T}^{T} \psi_z(\log \zeta_P(\sigma + i\tau'))d\tau' \right| \\
& + \left| \frac{1}{2T} \int_{-T}^{T} \psi_z(\log \zeta_P(\sigma + i\tau'))d\tau' - \widetilde{\mathcal{M}}_{\sigma,P}(z) \right| \\
& + |\widetilde{\mathcal{M}}_{\sigma,P}(z) - \widetilde{\mathcal{M}}_\sigma(z)| \\
=\ & X_P(z) + Y_P(z) + Z_P(z),
\end{aligned} \tag{5.2}$$

say.

If we choose $y = y(T)$ such that $y \to \infty$ as $T \to \infty$, then (4.12) implies that $Z_P(z) \to 0$ as $T \to \infty$.

To estimate $X_P(z)$, we use upper bound estimates of the mean square of

$$f_P(\sigma + i\tau') = \frac{\zeta(\sigma + i\tau')}{\zeta_P(\sigma + i\tau')} - 1,$$

which were proved in [15]. In the general theory of Dirichlet series, an asymptotic formula for the mean square of Dirichlet series due to Carlson [3] is known. The proof of the mean square estimates given in [15] is inspired by a proof of Carlson's theorem described in Titchmarsh [18].

Consider $Y_P(z)$. From (4.3) (in the case $L = \zeta$, $\chi = \chi_{\tau'}$) we have

$$\psi_z(\log \zeta_P(\sigma + i\tau')) = \psi_z(g_{\sigma,P}(\chi_P)).$$

We use the Fourier expansion

$$\psi_z(g_{\sigma,P}(\mathbf{t}_P)) = \sum_{\mathbf{n}_P \in \mathbf{Z}_P} A_{\sigma,P}(\mathbf{n}_P, z)\mathbf{t}_P^{\mathbf{n}_P}, \tag{5.3}$$

where

$$\mathbf{Z}_P = \prod_{p \in P} \mathbf{Z}, \quad \mathbf{n}_P = (n_p)_{p \in P} \in \mathbf{Z}_P, \quad \mathbf{t}_P^{\mathbf{n}_P} = \prod_{p \in P} t_p^{n_p} \in T_P,$$

and $A_{\sigma,P}(\mathbf{n}_P, z)$ are the Fourier coefficients. Among them, the constant term is $A_{\sigma,P}(\mathbf{0}, z) = \widetilde{\mathcal{M}}_{\sigma,P}(z)$ (where $\mathbf{0} = (0)_{p\in P}$), and hence

$$\begin{aligned} Y_P(z) &= \left| \sum_{\substack{\mathbf{n}_P\in\mathbf{Z}_P \\ \mathbf{n}_P\neq\mathbf{0}}} A_{\sigma,P}(\mathbf{n}_P, z)\frac{1}{2T}\int_{-T}^{T}\prod_{p\in P} e^{-i\tau' n_p\log p}d\tau' \right| \\ &\ll \frac{1}{T}\sum_{\substack{\mathbf{n}_P\in\mathbf{Z}_P \\ \mathbf{n}_P\neq\mathbf{0}}} \left| \frac{A_{\sigma,P}(\mathbf{n}_P, z)}{\sum_{p\in P} n_p\log p} \right|. \end{aligned} \tag{5.4}$$

We can show a certain upper bound of each Fourier coefficient $A_{\sigma,P}(\mathbf{n}_P, z)$, and also a mean value estimate of them (by the method similar to Section 5 of [8]). Applying those estimates to the right-hand side of (5.4), we obtain an upper bound of $Y_P(z)$.

Lastly, choosing $y = (\log T)^\omega$, where ω is a small positive constant, we find that both $X_P(z)$ and $Y_P(z)$ tend to 0 as $T \to \infty$. Now we are done in the case $\Phi = \psi_z$. Moreover we can observe that, in this case, the convergence to the limit $\widetilde{\mathcal{M}}_\sigma(z)$ is uniform in $|z| \le R$ for any $R > 0$. Noting this uniformity, we can deduce the general case of (i) and (ii) of Theorem 3 from this case by passage to the Fourier dual and by suitable approximations.

In Case (A) with $K = \mathbf{Q}$, the basic structure of the argument is similar. The quantity corresponding to $X_P(z)$ is estimated by using the idea of Section 4 of [15], combined with a mean square estimate of

$$f_P(s,\chi) = \frac{L(s,\chi)}{L_P(s,\chi)} - 1$$

with respect to χ. The latter estimate is reduced to a mean square estimate of Dirichlet L-functions, which can be shown as an analogue of Gallagher's result [4]. The quantity corresponding to $Y_P(z)$ is treated by the method in Section 6 of [8].

6 The function field case

In the previous sections we have seen how to apply mean value estimates to obtain sharp results in the $\log L$ case, $K = \mathbf{Q}$. The same idea can surely be applied to other cases. In this section we consider the case when K is a function field, and explain how to improve and generalize the results (a) (b) mentioned in Section 2. We hope this is not just a case study but will give some insight into and prospect for the number field cases.

In a recent article [9] we have shown that in the function field case, formula (1.1) (with the meaning (2.3) of the average) holds under the condition:

(c) $\sigma > 1/2$, and Φ is any continuous function with at most polynomial growth.

Of course condition (c) contains both (a) and (b). The main point of the proof is an upper bound estimate of power mean values

$$\mathrm{Avg}_{X(\mathbf{f})}|g(s,\chi,y)|^{2k} \tag{6.1}$$

($k > 0$ fixed) of

$$g(s,\chi,y) = \frac{L'}{L}(s,\chi) - \frac{L'_P}{L_P}(s,\chi), \tag{6.2}$$

where $P = P_y := \{\wp : \text{prime divisor on } K, \neq \wp_\infty, N(\wp) \leq y\}$. To evaluate (6.1), we write

$$\begin{aligned} g(s,\chi,y) &= \frac{1}{2\pi i}\left\{\int_{\Re w=c} - \int_{\Re w=-\epsilon}\right\}\Gamma(w)g(s+w,\chi,y)X^w dw \\ &= Int_+ - Int_-, \end{aligned} \tag{6.3}$$

say, where $c > \max(0, 1-\sigma)$, ϵ is a small positive number, $X \geq 1$. We estimate each of $|Int_-|$ using the Weil Riemann Hypothesis [20], and the power mean values of $|Int_+|$ by using orthogonality relation for characters. A suitable choice of X depending on $N(\mathbf{f})$ will give a simple and rather surprising result to the effect that the mean value (6.1) tends to 0 as $y \to \infty$, *uniformly* in $\mathbf{f}$. Combining this with (b) mentioned above in Section 2, we can prove (1.1) for any Φ satisfying condition (c) directly (first for $\Phi \in C^1$ with compact support, and then for general Φ by approximation); that is, it is not necessary to treat the case $\Phi = \psi_z$ before treating the general case. This is a big difference from the proof of Theorem 3.

The above result is another example of successful application of mean value theorems. However, even condition (c) is not the weakest condition we have ever obtained. To state a further stronger result, we first define another variant of average by

$$\mathrm{Avg}^{(3)}_\chi \phi(\chi) = \lim_{\substack{\mathbf{f}:\text{prime} \\ N(\mathbf{f})\to\infty}} \left(\mathrm{Avg}_{X(\mathbf{f})}\phi(\chi)\right). \tag{6.4}$$

Then the following theorem is proved in [11]:

Theorem 4 *Let K be a function field, and by Avg_χ we mean the average defined by* (6.4)*. Then both* (1.1) *and* (1.2) *hold for any $\sigma > 1/2$ and any Φ satisfying* (ii) (*in Theorem* 3) *or*

(iii) Φ *is any continuous function with at most exponential growth, i.e., $\Phi(w) \ll e^{a|w|}$ holds for some $a > 0$.*

The proof of Theorem 4 is again different from both in [10] and in [9]. In [11] we also treat the number field case, which will be discussed in the next section.

7 Under the GRH

According to the well-known principle on the analogy between function fields and number fields, we may expect that the analogue of Theorem 4 holds for number fields, at least under the assumption of the GRH. In fact, in [11] we prove the following theorem.

Theorem 5 *Let K be the rational number field or an imaginary quadratic field, and consider Case* (A) *with the meaning* (6.4) *of the average. We assume the GRH for $L(s,\chi)$. Then both* (1.1) *and* (1.2) *hold for any $\sigma > 1/2$ and any Φ satisfying* (ii) *or* (iii).

In [11], Theorem 4 and Theorem 5 are proved simultaneously. Therefore hereafter we assume that K is either $\mathbf{Q}$, an imaginary quadratic field, or a function field, and assume the GRH when K is a number field. We have seen in the previous sections that mean value theorems play a key role in the proof of our theorems. In particular, in Section 5 we mentioned that a part of our proof was inspired by Carlson's theorem. In [11], we prove a certain variant of Carlson's theorem, which is quite essential in our argument.

To explain our variant of Carlson's theorem, first recall the expression (4.14) for $\widetilde{\mathcal{M}}_\sigma(z)$. Here we introduce the following more general Dirichlet series:

$$\widetilde{\mathcal{M}}_s(z_1,z_2) = \sum_D \lambda_{z_1}(D)\lambda_{z_2}(D)N(D)^{-2s}, \tag{7.1}$$

where s, z_1 and z_2 are complex variables, D runs over all integral divisors of K, $N(D)$ is the norm of D, and $\lambda_z(D)$ is defined by

$$L(s,\chi)^{iz/2} = \sum_D \lambda_z(D)\chi(D)N(D)^{-s}. \tag{7.2}$$

It is also possible to define the same type of function $\widetilde{M}_s(z_1,z_2)$ in the L'/L case, which was already done in [8]. Basic properties and the behaviour of $\widetilde{M}_s(z_1,z_2)$ (resp. $\widetilde{\mathcal{M}}_s(z_1,z_2)$) have been discussed in [8], [7] (resp. [11]).

We also consider the *quasi-characters* $\psi_{z_1,z_2} : \mathbf{C} \to \mathbf{C}^\times$ parametrized by $z_1, z_2 \in \mathbf{C}$ defined by

$$\psi_{z_1,z_2}(w) = \exp\left(\frac{i}{2}(z_1\overline{w} + z_2 w)\right),$$

which will be the "basic case" for Φ in the proof of Theorems 4 and 5. Our Carlson's theorem can be written as follows.

Theorem 6 *Let K be as in Theorem* 4 *or Theorem* 5*, and assume the GRH if K is a number field. Then*

$$\mathrm{Avg}^{(3)}_\chi \psi_{z_1,z_2}\left(\frac{L'}{L}(s,\chi)\right) = \widetilde{M}_\sigma(z_1,z_2) \tag{7.3}$$

and

$$\mathrm{Avg}_\chi^{(3)} \psi_{z_1,z_2}(\log L(s,\chi)) = \widetilde{\mathcal{M}}_\sigma(z_1,z_2) \tag{7.4}$$

hold uniformly in $|z_1|, |z_2| \le R$ and for $s = \sigma + i\tau$ with $\sigma \ge 1/2 + \epsilon$ and, in the number field case, $|\tau| \le T$.

To prove this theorem, it is necessary to obtain a sufficiently sharp upper bound of $(L'/L)(s,\chi)$ for $\sigma > 1/2$. Such an upper bound can be shown by using the polynomial expression of the L-function in the function field case, and by using Weil's explicit formula ([19]) in the number field case.

The necessary estimate in the $\log L$ case can be deduced by integrating the result in the L'/L case.

When $z_2 = \overline{z_1}$, we see that $\psi_{z_1,z_2} = \psi_{z_1}$. Therefore Theorem 6 gives (1.1) and (1.2) in the case $\Phi = \psi_{z_1}$. From this result, we obtain the assertions of Theorems 4 and 5 again by suitable approximations. The proof that Φ can be of exponential growth follows by comparing the following two facts: first,

$$M_\sigma(w),\ \mathcal{M}_\sigma(w) \ll \exp(-\lambda|w|^2) \tag{7.5}$$

for any $\lambda > 0$, and secondly,

$$\mathrm{Avg}_{X(\mathbf{f})} \exp\left(a\left|\frac{L'}{L}(s,\chi)\right|\right),\ \mathrm{Avg}_{X(\mathbf{f})} \exp(a|\log L(s,\chi)|) \ll 1 \tag{7.6}$$

for any $a > 0$. The first is a simple generalization of a result of Jessen and Wintner [13], while the second can be shown from Theorem 6 (here it is important that Theorem 6 is proved not only for $z_2 = \overline{z_1}$, but for general z_1 and z_2).

The conditions for Φ in our unconditional Theorem 3 is obviously much restrictive than the conditions in Theorem 5. It seems very difficult to prove the claim of Theorem 5 unconditionally at present, but we believe that the methods developed in [9, 10, 11] will give a base for further progress in the future.

References

[1] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion, I, Acta Math. **54** (1930), 1-35; II, ibid. **58** (1932), 1-55.

[2] V. Borchsenius and B. Jessen, Mean motions and values of the Riemann zeta function, Acta Math. **80** (1948), 97-166.

[3] F. Carlson, Contributions à la théorie des séries de Dirichlet, Note I, Ark. Mat. Astr. Fysik **16**, no.18 (1922), 19pp.

[4] P. X. Gallagher, Local mean value and density estimates for Dirichlet L-functions, Indag. Math. **37** (1975), 259-264.

[5] Y. Ihara, On the Euler-Kronecker constants of global fields and primes with small norms, in "Algebraic Geometry and Number Theory", Progr. Math. **253**, Birkhäuser, Boston, 2006, pp.407-451,

[6] Y. Ihara, The Euler-Kronecker invariants in various families of global fields, in "Arithmetic, Geometry, and Coding Theory (AGCT 2005)", F. Rodier and S. Vladut (eds.), Séminaires et Congrès **21**, Soc. Math. France, to appear.

[7] Y. Ihara, Some density functions and their Fourier transforms arising from number theory, in "Number Theory and Probability Theory", H. Sugita (ed.), RIMS Kôkyûroku, Kyoto Univ., 2008, pp.28-42.

[8] Y. Ihara, On "M-functions" closely related to the distribution of L'/L-values, Publ. RIMS Kyoto Univ. **44** (2008), 893-954.

[9] Y. Ihara and K. Matsumoto, On L-functions over function fields: Power-means of error-terms and distribution of L'/L-values, RIMS preprint series **1637** (2008), submitted.

[10] Y. Ihara and K. Matsumoto, On certain mean values and the value-distribution of logarithms of Dirichlet L-functions, preprint.

[11] Y. Ihara and K. Matsumoto, On $\log L$ and L'/L for L-functions and the associated "M-functions": Connections in optimal cases, preprint.

[12] Y. Ihara, V. Kumar Murty and M. Shimura, On the logarithmic derivatives of Dirichlet L-functions at $s = 1$, Acta Arith., to appear.

[13] B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function, Trans. Amer. Math. Soc. **38** (1935), 48-88.

[14] A. Laurinčikas, Limit Theorems for the Riemann Zeta-Function, Kluwer Acad. Publ., Dordrecht, 1996.

[15] K. Matsumoto, Asymptotic probability measures of zeta-functions of algebraic number fields, J. Number Theory **40** (1992), 187-210.

[16] H. L. Montgomery, Topics in Multiplicative Number Theory, Lecture Notes in Math. **227**, Springer-Verlag, 1971.

[17] J. Steuding, Value-Distribution of L-Functions, Lecture Notes in Math. **1877**, Springer-Verlag, 2007.

[18] E. C. Titchmarsh, The Theory of Functions, 2nd ed., Oxford Univ. Press, Oxford, 1939.

[19] A. Weil, Sur les "formules explicites" de la théorie des nombres premiers, Comm. Sém. Math. Lund, Suppl. (1952), 252-265; Collected Works, vol.2, pp.48-62.

[20] A. Weil, Courbes Algébriques et Variétés Abéliennes, Hermann, Paris, 1971.

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