The Universal Verma Module and the $b$-Function

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§ 0. Introduction

In this paper, we study the universal Verma module and apply this to the determination of the $b$-functions of the invariants on the flag manifold.

Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{n}$ the nilpotent radical of $\mathfrak{b}$ and $\mathfrak{h}$ a Cartan subalgebra in $\mathfrak{b}$. Let $V$ be a finite-dimensional irreducible representation of $\mathfrak{g}$ and let $u$ be a lowest weight vector of $V$. Then there exists $f \in U(\mathfrak{h})$ and a commutative diagram

$$
\begin{array}{ccc}
U(\mathfrak{g}) \otimes \mathbb{C}_{U(\mathfrak{n})} & \rightarrow & U(\mathfrak{g}) \otimes V_{U(\mathfrak{n})} \\
\downarrow f & & \downarrow g \\
U(\mathfrak{g}) \otimes \mathbb{C}_{U(\mathfrak{n})} & \rightarrow & U(\mathfrak{g}) \otimes \mathbb{C}_{U(\mathfrak{n})}
\end{array}
$$

(0.1)

where $g$ is given by the $n$-linear morphism from $V$ to $\mathbb{C}$ sending $u$ to $1$. Note that $\text{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}) \cong U(\mathfrak{h})$.

The first problem is to determine the minimal $f$ with such a property. In order to state the answer to this problem, we shall introduce further notations. Let $\mathcal{A}$ be the root system for $(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \mathcal{A}$, let $h_\alpha$ be the coroot of $\alpha$. Let $\mathcal{A}^+$ be the set of positive roots given by $\mathfrak{b}$ and $\rho$ the half-sum of positive roots. Let $-\mu$ be the lowest weight of $V$.

**Theorem.** There exists a commutative diagram (0.1), with

$$f = \prod_{\alpha \in \mathcal{A}^+} (h_\alpha + h_\alpha(\rho) + 1, h_\alpha(\mu))$$

where $(x, n) = x(x+1) \cdots (x+n-1)$. Conversely for any commutative diagram (0.1), $f$ is a multiple of $\prod_{\alpha \in \mathcal{A}^+} (h_\alpha + h_\alpha(\rho) + 1, h_\alpha(\mu))$.

By using this theorem, we can calculate the $b$-functions on the flag manifold. Let $G$ be a simply connected algebraic group with Lie algebra $\mathfrak{g}$, and let $B$ and $N$ be the subgroup of $G$ with Lie algebras $\mathfrak{b}$ and $\mathfrak{n}$, respectively, and let $B_-$ be the opposite Borel subgroup.

Received January 17, 1984.
Then the semi-group of $B_+ \times B$-semi-invariants $f$ on $G$, i.e. regular functions $f$ on $G$ which satisfies $f(b'gb) = \chi'(b')\chi(b)f(g)$ for $b' \in B_+$, $g \in G$, $b \in B$ with characters $\chi'$ and $\chi$ of $B_+$ and $B$, is parametrized by the set $P_+$ of dominant integral weights. More precisely, for $\lambda \in P_+$, let $V_\lambda$ be a finite-dimensional irreducible representation of $G$ with highest weight $\lambda$, $v_\lambda$ a highest weight vector of $V_\lambda$, and $\bar{v}_\lambda$ a lowest weight vector of the dual $V_\lambda^*$ of $V_\lambda$. We normalize them such that $\langle v_\lambda, \bar{v}_\lambda \rangle = 1$. Then, the regular function $f^\lambda$ given by

$$f^\lambda(g) = \langle gv_\lambda, \bar{v}_\lambda \rangle$$

is a semi-invariant, and any semi-invariant is a constant multiple of some $f^\lambda$. We have

$$f^{\lambda + \nu}(g) = f^{\lambda}(g)f^{\nu}(g).$$

**Theorem.** For any dominant integral weight $\mu$, we can find a differential operator $P_\mu$ on $G$ such that

$$P_\mu f^{\lambda + \nu} = b_\mu(\lambda)f^\lambda$$

for any $\lambda$.

Here

$$b_\mu(\lambda) = \prod_{\alpha \in \Delta_+} (h_\alpha(\lambda + \rho), h_\alpha(\mu)).$$

**Notations**

- $\mathbb{Z}_+$: the set of non-negative integers.
- $\mathbb{Z}_{++}$: the set of positive integers.
- $\mathfrak{g}$: a semi-simple Lie algebra over $\mathbb{C}$.
- $\mathfrak{b}$: a Borel subalgebra of $\mathfrak{g}$.
- $\mathfrak{n}$: $[\mathfrak{b}, \mathfrak{b}]$.
- $\mathfrak{h}$: a Cartan subalgebra of $\mathfrak{b}$.
- $\mathfrak{b}_-$: the opposite Borel subalgebra of $\mathfrak{b}$ such that $\mathfrak{b}_- \cap \mathfrak{b} = \mathfrak{h}$.
- $\mathfrak{n}_-$: $[\mathfrak{b}_-, \mathfrak{b}_-]$.
- $\Delta$: the root system of $(\mathfrak{g}, \mathfrak{h})$.
- $\Delta^+$: the set of positive roots given by $\mathfrak{b}$.
- $h_\alpha$: the coroot of $\alpha \in \Delta$.
- $s_\alpha$: the reflection $\lambda \mapsto \lambda - h_\alpha(\lambda)\alpha$.
- $W$: the Weyl group of $(\Delta, h^\vee)$.
- $Q_+(\Delta) = \sum_{\alpha \in \Delta^+} \mathbb{Z}_+ \alpha$.
- $Q(\Delta) = \sum_{\alpha \in \Delta} \mathbb{Z} \alpha$.
- $P_+ = \{ \lambda \in h^*; h_\alpha(\lambda) \in \mathbb{Z}_+ \text{ for any } \alpha \in \Delta^+ \}$.
- $\rho = (\sum_{\alpha \in \Delta^+} \alpha)/2$.
Universal Verma Module

$S(Δ^+)$ : the set of simple roots of $Δ^+$.
$U(∗)$ : the universal enveloping algebra
$U(\mathfrak{g})$ : $U(\mathfrak{g}) = C$, $U_j(\mathfrak{g}) = U_{j-1}(\mathfrak{g}) \mathfrak{g} + U_{j-1}(\mathfrak{g})$
$R$ : $S(\mathfrak{h}) = U(\mathfrak{h})$
$c$ : the canonical homomorphism $\mathfrak{h} \rightarrow R$
$U_\mathfrak{c}(∗)$ : $R \otimes_c U(∗)$
$R_{c + μ}$ : for $μ \in \mathfrak{h}^*$, the $U_\mathfrak{c}(\mathfrak{b})$-module $U_\mathfrak{c}(\mathfrak{b})/(U_\mathfrak{c}(\mathfrak{b})n + \sum_{h \in \mathfrak{h}} U_\mathfrak{c}(\mathfrak{b})(h - c(h) - μ(h)))$
$1_{c + μ}$ : the canonical generator of $R_{c + μ}$
$C_2$ : for $λ \in \mathfrak{h}^*$, the $U(\mathfrak{b})$-module $U(\mathfrak{b})/(U(\mathfrak{b})n + \sum_{h \in \mathfrak{h}} U(\mathfrak{b})(h - \lambda(h)))$
$Z(\mathfrak{g})$ : the center of $U(\mathfrak{g})$
$λ_λ$ : the central character $Z(\mathfrak{g}) \rightarrow C$ of $U(\mathfrak{g}) \otimes_{U(\mathfrak{c})} C_{1 - μ}$; $λ_λ = λ_{wλ}$ for $w \in W$
$V_λ$ : for $λ \in P_+$, a finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight $λ$
$v_λ$ : a highest weight vector of $V_λ$
$v_{-λ}$ : a lowest weight vector of $V^*_λ$
$(x, m) : x(x+1) \cdots (x+m - 1)$
$G, B, N, B_-, N_-, T$: the group with $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}$, $\mathfrak{n}_-$, and $\mathfrak{h}$ as their Lie algebras.

S1. The universal Verma module

For a ring $R$ and a Lie algebra $\mathfrak{a}$ over $C$, we write $U_\mathfrak{a}(\mathfrak{a})$ for $R \otimes_c U(\mathfrak{a}) = U(R \otimes_c \mathfrak{a})$. Hereafter we take $S(\mathfrak{h}) = U(\mathfrak{h})$ for $R$, where $\mathfrak{h}$ is a Cartan subalgebra of a semi-simple Lie algebra $\mathfrak{g}$. Let $c$ be the canonical injection from $\mathfrak{h}$ into $R$. We define $R_c$ by $R_c = U_R(\mathfrak{b})/U_R(\mathfrak{b})n + \sum_{h \in \mathfrak{h}} U_R(\mathfrak{b})(h - c(h))$. Then $R_c$ is isomorphic to $R$ as $R$-module. We write $1_c$ for the canonical generator of $R_c$.

Definition 1.1. We call $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R_c$ the universal Verma module.

As a $\mathfrak{g}$-module, $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R_c$ is isomorphic to $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} C$. For $λ \in \mathfrak{h}^*$, let $C_λ$ be the $U(\mathfrak{b})$-module given by $U(\mathfrak{b})/(U(\mathfrak{b})n + \sum_{h \in \mathfrak{h}} U(\mathfrak{b})(h - \lambda(h)))$. We regard $C_λ$ also as an $R$-module by $R \rightarrow U(\mathfrak{b})$. Then $C_λ \otimes_R (U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R_c)$ is nothing but the Verma module with highest weight $λ$. Note that the universal Verma module is, as an $R$-module, isomorphic to $R \otimes_c U(\mathfrak{n}_{-})$, and in particular it is a free $R$-module.

For $μ \in \mathfrak{h}^*$, we write $R_{c + μ}$ for the $U_R(\mathfrak{b})$-module $C_μ \otimes_c R_c$.

The following lemma is almost obvious.

Lemma 1.2. $\text{End}_{U_R(\mathfrak{g})}(U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R_c) = R$.

Now, we choose a non-degenerate $W$-invariant symmetric bilinear
form \((\lambda, \mu)\) on \(\mathfrak{h}^*\).

**Lemma 1.3.** For \(\mu \in \mathfrak{h}^*\), let \(f_\mu\) be the function on \(\mathfrak{h}^*\) given by

\[
f_\mu(\lambda) = (\lambda + \mu + \rho, \lambda + \mu + \rho) - (\lambda + \rho, \lambda + \rho) = 2(\mu, \lambda + \rho) + (\mu, \mu).
\]

and regard this as an element of \(R\).

Then we have

\[
f_\mu \text{ Ext}^j_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_c, U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_{c + \rho}) = 0 \quad \text{for any } j.
\]

**Proof.** The Laplacian \(\Delta \in \mathfrak{h}(\mathfrak{g})\) acts on \(U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_c\) by the multiplication of \((\lambda + \rho, \lambda + \rho)\) and on \(U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_{c + \rho}\) by \((\lambda + \mu + \rho, \lambda + \mu + \rho)\). Hence \((\lambda + \mu + \rho, \lambda + \mu + \rho) - (\lambda + \rho, \lambda + \rho)\) annihilates \(\text{Ext}^j\).

Q.E.D.

Now, let \(F\) be a finite-dimensional \(\mathfrak{h}\)-module generated by a weight vector \(u\) of a weight \(\lambda_0 \in \mathfrak{h}^*\). Hence \(\mathfrak{h}\) acts semisimply on \(F\). We shall choose a decreasing finite filtration \(\{F^i\}\) of \(F\) by \(\mathfrak{h}\)-modules such that

\[
F^0 = F
\]

(1.1) \(F^i/F^{i+1}\) has a unique weight \(\lambda_j\).

(1.3) \(\lambda_j \neq \lambda_{j'}\) for \(j \neq j'\).

Therefore, we have \(F^1 \cong \mathfrak{h} F\) and \(F^0/F^1 \cong C_{\lambda_0}\). Hence there exists an isomorphism

\[
\varphi_1 : U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_{c + \lambda_0} \tilde{\longrightarrow} U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_c \otimes F^0/F^1).
\]

Now, we shall construct a commutative diagram

\[
\begin{array}{ccc}
U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_{c + \lambda_0} & \xrightarrow{\varphi_j} & U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_c \otimes F^0/F^1) \\
\downarrow f_j & & \downarrow \\
U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_{c + \lambda_0} & \xrightarrow{\varphi_1} & U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_c \otimes F^0/F^1)
\end{array}
\]

(1.4)\(j\):

with \(f_j \in R\), by the induction on \(j\).

Assuming that (1.4)\(j\) has been already constructed \((j \geq 1)\), we shall construct (1.4)\(j+1\). We have an exact sequence

\[
0 \longrightarrow U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_c \otimes F^j/F^{j+1}) \longrightarrow U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_c \otimes F^0/F^{j+1}) \longrightarrow \]
This gives an exact sequence

\[ \text{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes \mathcal{R}_{e+\lambda_0}, U_R(\mathfrak{g}) \otimes (R_c \otimes F^0/F^{j+1})) \]

\[ \rightarrow \text{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes \mathcal{R}_{e+\lambda_0}, U_R(\mathfrak{g}) \otimes (R_c \otimes F^0/F^{j+1})) \]

\[ \rightarrow \text{Ext}^1_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes \mathcal{R}_{e+\lambda_0}, U_R(\mathfrak{g}) \otimes (R_c \otimes F^j/F^{j+1})). \]

On the other hand, \( F^j/F^{j+1} \) is a direct sum of copies of \( R_{e+\lambda_j} \). Therefore, by Lemma 1.3, we have

\[ g_j \text{ Ext}^1_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes \mathcal{R}_{e+\lambda_0}, U_R(\mathfrak{g}) \otimes (R_c \otimes F^j/F^{j+1})) = 0 \]

where \( g_j \in R \) is given by \( g_j(\lambda) = (\lambda + \lambda_J + \rho, \lambda + \lambda_j + \rho) - (\lambda + \lambda_0 + \rho, \lambda + \lambda_0 + \rho) \). Hence \( g_j \delta(\varphi_j)=0 \), which shows that \( g_j \varphi_j \) lifts to \( \psi : U_R(\mathfrak{g}) \otimes U_R(\mathfrak{g}) \mathcal{R}_{e+\lambda_0} \rightarrow U_R(\mathfrak{g}) \otimes U_R(\mathfrak{g}) (R_c \otimes F^0/F^{j+1}) \).

If \( \psi \) is divisible by \( g_j \), then \( \varphi_j \) itself lifts and we obtain (1.4)_{j+1} with \( f_{j+1} = f_j \).

Assume that \( \psi \) is not divisible by \( g_j \). For \( \lambda \in \mathfrak{g}^* \), let us denote by \( \psi(\lambda) \) the specialization of \( \psi \), i.e. \( \mathcal{C}_\lambda \otimes_R \psi \). Then, for a generic point \( \lambda \) of \( g_j^{-1}(0) \), \( \psi(\lambda) \neq 0 \). Hence we obtain a diagram

\[
\begin{array}{ccc}
U(\mathfrak{g}) \otimes (\mathcal{C}_\lambda \otimes F^j/F^{j+1}) & \rightarrow & U(\mathfrak{g}) \otimes (\mathcal{C}_\lambda \otimes F^j/F^{j+1}) \\
\uparrow & & \uparrow \\
U(\mathfrak{g}) \otimes \mathcal{C}_{\lambda+\lambda_0} & \rightarrow & U(\mathfrak{g}) \otimes (\mathcal{C}_\lambda \otimes F^0/F^j) \\
\downarrow & & \downarrow \\
g_j(\lambda) \varphi_j(\lambda) & \rightarrow & \psi(\lambda)
\end{array}
\]

(1.5)

Since \( g_j(\lambda)=0 \), we obtain a nonzero homomorphism \( h : U(\mathfrak{g}) \otimes_U(\mathfrak{g}) C_{\lambda+\lambda_0} \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}) (C_\lambda \otimes F^j/F^{j+1}) \). Since \( U(\mathfrak{g}) \otimes_U(\mathfrak{g}) (C_\lambda \otimes F^j/F^{j+1}) \) is a direct sum of copies of \( U(\mathfrak{g}) \otimes_U(\mathfrak{g}) C_{\lambda+\lambda_0} \), the central character of \( U(\mathfrak{g}) \otimes_U(\mathfrak{g}) C_{\lambda+\lambda_0} \) and that of \( U(\mathfrak{g}) \otimes_U(\mathfrak{g}) C_{\lambda+\lambda_j} \) must coincide. Hence there exists \( w \in W \) such that \( w(\lambda+\lambda_0+\rho) = \lambda+\lambda_j+\rho \). This shows that \( w(\lambda+\lambda_0+\rho) = \lambda+\lambda_j+\rho \) holds for any \( \lambda \in g_j^{-1}(0) \). Since \( \lambda_j \neq \lambda_0, w \neq 1 \). Since \( w \) fixes the hyperplane \( (\lambda, \lambda_j - \lambda_0) = 0 \), \( w \) must be the reflection \( s_\alpha \) for some \( \alpha \in \Delta^+ \). Hence we obtain

\[ 0 = \lambda + \lambda_j + \rho - s_\alpha(\lambda + \lambda_0 + \rho) = \lambda_j - \lambda_0 + h_\alpha(\lambda + \lambda_0 + \rho) \alpha. \]
This implies that $\lambda_j = \lambda_0 + k\alpha$ for some $k \in \mathbb{C}$. Since $\lambda_j - \lambda_0 \in Q_+(\Delta) \setminus \{0\}$, $k$ is a strictly positive integer. Moreover, $h_a(\lambda + \lambda_0 + \rho) + k = 0$ holds on $g_j^{-1}(0)$. Hence $g_j$ is a constant multiple of $h_a(\lambda + \lambda_0 + \rho) + k$.

Summing up, we obtain

**Lemma 1.4.** (i) If $\lambda_j$ is not of the form $\lambda_0 + k\alpha$ with $\alpha \in \Delta$, $k \in \mathbb{Z}_{++}$, then $\varphi_j$ lifts to $\varphi_{j+1} : U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{h})} \mathbb{R}e_{+\lambda_0} \rightarrow U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{h})} (\mathbb{R}e \otimes F^0/F^{j+1})$.

(ii) If $\lambda_j = \lambda_0 + k\alpha$ for some $\alpha \in \Delta$ and $k \in \mathbb{Z}_{++}$, then $(c(h_a) + h_a(\lambda_0 + \rho) + k)\varphi_j$ lifts to $\varphi_{j+1}$.

Repeating this procedure we obtain

**Theorem 1.5.** There exists a commutative diagram

$$
\begin{array}{ccc}
U_R(\mathfrak{g}) & \otimes_{U_R(\mathfrak{h})} \mathbb{R}e_{+\lambda_0} & \rightarrow
U_R(\mathfrak{g}) \otimes (\mathbb{R}e \otimes F) \\
\downarrow f & & \downarrow \\
U_R(\mathfrak{g}) & \otimes_{U_R(\mathfrak{h})} \mathbb{R}e_{+\lambda_0} & \rightarrow
U_R(\mathfrak{g}) \otimes (\mathbb{R}e \otimes F^0/F^1).
\end{array}
$$

(1.6)

Here $f = \prod_{(\alpha, k) \in \Sigma(F)} (h_a + h_a(\lambda_0 + \rho) + k)$ and $\Sigma(F)$ is the set of pairs $(\alpha, k)$ of positive root $\alpha$ and a positive integer $k$ such that $\lambda_0 + k\alpha$ is a weight of $F$.

**Example 1.6.** We set $F_k = U(\mathfrak{g})/(U(\mathfrak{h})\mathfrak{h} + U(\mathfrak{n})\mathfrak{n})$. Let $K$ be the quotient field of $R$. Then for any $k$, there exists a unique

$$
\varphi_k : U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{h})} \mathbb{R}e \rightarrow U_R(\mathfrak{g}) \otimes (\mathbb{R}e \otimes F_k)
$$

such that the following diagram commutes

$$
\begin{array}{ccc}
U_R(\mathfrak{g}) & \otimes_{U_R(\mathfrak{h})} \mathbb{R}e & \rightarrow
U_R(\mathfrak{g}) \otimes (\mathbb{R}e \otimes F_k) \\
1 & & \downarrow \\
U_R(\mathfrak{g}) & \otimes_{U_R(\mathfrak{h})} (\mathbb{R}e \otimes F_0).
\end{array}
$$

Hence, taking the projective limit, we obtain

$$
\hat{\varphi} : U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{h})} \lim_k U_R(\mathfrak{g}) \otimes (\mathbb{R}e \otimes F_k).
$$

When $\mathfrak{g} = \mathfrak{sl}_2$, we shall calculate $\hat{\varphi}$. Let us take the generator $X_+, X_-$, $h$ such that $[h, X_\pm] = \pm 2X_\pm$, $[X_+, X_-] = h$. Set $\lambda = c(h)$. We can write $P = \hat{\varphi}(1)$ in the following form

$$
P = \sum_{j=0}^\infty a_j X_+^j \otimes X_-^j (1 \otimes 1)
$$
with $a_0 = 1$. Then

\[ X_+ P = \sum a_j X_+ X_j \otimes X_j (1_e \otimes 1) \]
\[ = \sum a_j X_j \otimes X_j (1_e \otimes 1) + \sum j a_j X_j^{-1} (h - j + 1) \otimes X_j (1_e \otimes 1) \]
\[ = \sum a_j X_j \otimes X_j (1_e \otimes 1) + \sum j (\lambda + j + 1) a_j X_j^{-1} \otimes X_j (1_e \otimes 1). \]

Here we have used the relation $[X_+, X_j] = jX_j^{-1} (h - j + 1)$.

Hence we obtain the recursion formula

\[ a_j = -\frac{1}{j(\lambda + j + 1)} a_{j-1} \quad \text{for } j \geq 1. \]

Solving this, we obtain

\[ P = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (\lambda + 2, j)} X_j \otimes X_j (1_e \otimes 1). \]  

(1.7)

Let $V_\mu^*$ be a finite-dimensional irreducible representation of $\mathfrak{g}$ with a lowest weight $-\mu$ and $v_{-\mu}$ a lowest weight vector. As well-known, $-\mu + k\alpha$ is a weight of $V_\mu^*$ if and only if $0 \leq k \leq h_\alpha(\mu)$. Hence Theorem 1.5 implies the following Theorem.

**Theorem 1.7.** There exists a homomorphism

\[ \varphi_0 : U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_e \longrightarrow U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_{e+\mu} \otimes V_\mu^*) \]

such that $g \circ \varphi_0 = \prod_{\alpha \in \Lambda^+} (h_\alpha + h_\alpha(\rho) + 1, h_\alpha(\mu))$, where $g : U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_{e+\mu} \otimes V_\mu^*) \to U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_e$ is given by $g(1 \otimes 1_{e+\mu} \otimes v_{-\mu}) = 1 \otimes 1_e$.

Now, we shall show the converse.

**Proposition 1.8.** For any homomorphism

\[ \varphi : U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_e \longrightarrow U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_{e+\mu} \otimes V_\mu^*), \]

set $f = g \circ \varphi \in R$. Then $f$ is a multiple of $\prod_{\alpha \in \Lambda^+} (h_\alpha + h_\alpha(\rho) + 1, h_\alpha(\mu))$.

*Proof.* Note that $h_\alpha + h_\alpha(\rho) + k = c(h_\alpha + h_\alpha(\rho) + k')$ with $\alpha, \alpha' \in \Lambda^+, k, k' \in \mathbb{C}$ implies, $\alpha = \alpha'$, $k = k'$. Hence we can construct another $\varphi$ such that $g \circ \varphi$ is the greatest common divisor of $f$ and $\prod (h_\alpha + h_\alpha(\rho) + 1, h_\alpha(\mu))$. Therefore, we may assume from the beginning that $f$ is a divisor of $\prod (h_\alpha + \rho(h_\alpha) + 1, h_\alpha(\mu))$.

Set $M = U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_{e+\mu} \otimes V_\mu^*) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} V_\mu^*$ and let $M_j$ be the image of $U_j(\mathfrak{g}) \otimes V_\mu^*$ in $M$. Then we can easily show
\[ \text{gr } M = \bigoplus M_j / M_{j-1} = (S(\mathfrak{g}) / S(\mathfrak{g})_n) \otimes_{\mathfrak{g}^*} V^*_\mu \]

as an \( n \)-module.

Now, \( \nu = \varphi(1) \) is a non-zero element of \( M \) which is \( n \)-invariant. Let \( j \) be the smallest integer such that \( \nu \in M_j \) and let \( \bar{\nu} \) be the image of \( \nu \) in \( M_j / M_{j-1} \). Then \( \bar{\nu} \) is also \( n \)-invariant. By the Killing form we identify \( \mathfrak{g} \) and \( \mathfrak{g}^* \). Then \( S(\mathfrak{g}) / S(\mathfrak{g})_n \) is isomorphic to \( C[\mathfrak{b}] \), the polynomial ring of \( \mathfrak{b} \). Hence we can regard \( \bar{\nu} \) as a \( V^*_\mu \)-valued function on \( \mathfrak{b} \), and we denote it \( \mathcal{V} \).

By the assumption, \( \nu \) has the form

\[ \nu = f \otimes v_{-\mu} \mod U(\mathfrak{b}_-) \otimes_n V^*_\mu. \]

Hence \( j \geq \deg f \) and we have either

\[ \begin{align*}
  (1.8) & \quad j > \deg f \quad \text{and} \quad \mathcal{V} | \bar{\nu} = 0 \\
  (1.9) & \quad j = \deg f \quad \text{and} \quad \mathcal{V}(h) = \bar{f}(h) v_{-\mu} \quad \text{for} \quad h \in \mathfrak{h}.
\end{align*} \]

Here \( \bar{f} \) is the homogeneous part of \( f \). Since \( N[\mathfrak{h}] \) is an open dense subset of \( \mathfrak{b} \), \( \mathcal{V} | \bar{\nu} = 0 \) implies \( \mathcal{V} = 0 \). Hence the first case (1.8) does not occur and we have (1.9).

Let \( S(\Delta^+) \) be the set of simple roots. For \( \alpha \in \Delta \), let \( x_{\alpha} \) be a root vector with root \( \alpha \). We normalize as \( [x_{\alpha}, x_{-\alpha}] = h_\alpha \). We set

\[ \begin{align*}
  x_+ &= \sum_{\alpha \in \Delta^+} x_{\alpha} \\
  x_- &= \sum_{\alpha \in \Delta^+} x_{-\alpha}.
\end{align*} \]

We take the element \( h_0 \in \mathfrak{h} \) such that \( h_0(\alpha) = 2 \) for \( \alpha \in \Delta^+ \). Then \( h_0 = \sum_{\alpha \in \Delta^+} h_\alpha \). Now, we can show easily \( [h_0, x_\pm] = \mp 2x_\pm \), \( [x_+, x_-] = h_0 \) and hence \( \langle h_0, x_+, x_- \rangle \) forms a Lie algebra isomorphic to \( sl_2 \). We have

\[ e^{tx_+} h_0 = h_0 - 2tx_+. \]

Therefore, we obtain

\[ \begin{align*}
  \mathcal{V}(ah_0 - 2x_+) &= \mathcal{V}(a e^{a^{-1}x_+} h_0) = e^{a^{-1}x_+} \mathcal{V}(ah_0) \\
  &= \bar{f}(ah_0) e^{a^{-1}x_+} v_{-\mu} \\
  &= \sum_{k \geq 0} \frac{(a^{-1})^k}{k!} \bar{f}(ah_0) x_+^k v_{-\mu}.
\end{align*} \]

The representation theory of \( sl_2 \) implies that \( x_+^k v_{-\mu} \neq 0 \) for \( 0 \leq k \leq h_0(\mu) \) and \( x_+^k v_{-\mu} = 0 \) for \( k > h_0(\mu) \). Since \( \mathcal{V}(ah_0 - 2x_+) \) is a polynomial in \( a \), \( \bar{f}(ah_0) a^{-h_0(\mu)} \) is also a polynomial in \( a \). Moreover \( \bar{f}(h_0) \neq 0 \) because \( \bar{f} \) is a
Universal Verma Module

factor of $\prod h_a(x)$. This shows that

$$\deg f = \deg \frac{f}{\prod h_a(x)} = \sum_{a \in \Delta^+} h_a(x).$$

Hence $f$ is $\prod (h_a + h_a(x) + 1, h_a(x))$ up to constant multiple. Q.E.D.

For a $\mathfrak{g}$-module $V$ and a $\mathfrak{h}$-module $F$, we have a canonical isomorphism

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} (F \otimes V) \rightarrow V \otimes_F (U(\mathfrak{g}) \otimes U(\mathfrak{g})) \tag{1.10}$$

by $1 \otimes (f \otimes v) \rightarrow v \otimes (1 \otimes f)$ for $v \in V$, $f \in F$.

Similarly, we have

$$U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_{c+p} \otimes V_{\mu}^*) \rightarrow V_{\mu}^* \otimes_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes R_{c+p}) \tag{1.11}$$

Therefore, we have

$$\text{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes R_c, U_R(\mathfrak{g}) \otimes (R_{c+p} \otimes V_{\mu}^*)) = \text{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes R_c, V_{\mu}^* \otimes (U_R(\mathfrak{g}) \otimes R_{c+p}))$$

$$= \text{Hom}_{U_R(\mathfrak{g})} (V_{\mu} \otimes (U_R(\mathfrak{g}) \otimes R_c), U_R(\mathfrak{g}) \otimes R_{c+p})$$

$$= \text{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes (R_c \otimes V_{\mu}), U_R(\mathfrak{g}) \otimes R_{c+p}).$$

We choose a lowest weight vector $v_{-\mu}$ of $V_{\mu}^*$ and a highest weight vector $v_\mu$ of $V_{\mu}$, normalized by $\langle v_\mu, v_{-\mu} \rangle = 1$. We define $g: U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_{c+p} \otimes V_{\mu}^*) \rightarrow U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_c$ and $h: U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_{c+p} \rightarrow U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} (R_c \otimes V_{\mu})$ by $g(1 \otimes 1_{c+p} \otimes v_{-\mu}) = 1 \otimes 1_c$ and $h(1 \otimes 1_{c+p}) = 1 \otimes 1_c \otimes v_\mu$.

**Theorem 1.9.** Assume that

$$\varphi \in \text{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes R_c, U_R(\mathfrak{g}) \otimes (R_{c+p} \otimes V_{\mu}^*))$$

and

$$\psi \in \text{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \otimes (R_c \otimes V_{\mu}), U_R(\mathfrak{g}) \otimes R_{c+p})$$

correspond by the isomorphism (1.12). Set $f = g \circ \varphi \in R$ and $f' = \psi \circ h \in R$. Then, we have

$$f' = \prod_{a \in \Delta^+} \frac{h_a + h_a(x) + 1, h_a(x)}{h_a + h_a(x) + \mu} f \tag{1.13}$$
Proof. For $\lambda \in \mathfrak{h}^*$, we shall denote by $\varphi(\lambda)$, $\psi(\lambda)$, $h(\lambda)$ and $g(\lambda)$ their specializations at $\lambda$. Identifying $V^*_\mu \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{2+\mu})$ with $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (C_{2+\mu} \otimes V^*_\mu)$, etc., we have commutative diagrams

$$
U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_2 \xrightarrow{\varphi(\lambda)} V^*_\mu \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_{2+\mu}) \xrightarrow{\psi(\lambda)} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_2
$$

and

$$
U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_{2+\mu} \xrightarrow{h(\lambda)} V^*_\mu \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_2) \xrightarrow{\psi(\lambda)} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_{2+\mu}.
$$

Letting $\lambda$ be a dominant integral weight and employing the homomorphism $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_2 \rightarrow V_2$, etc. we obtain

$$
V_2 \xrightarrow{\bar{\varphi}} V^*_\mu \otimes V_{2+\mu} \xrightarrow{\bar{g}} V_2
$$

(1.14)

and

$$
V_{2+\mu} \xrightarrow{h} V_2 \xrightarrow{\bar{h}} V^*_\mu \otimes V_{2+\mu}
$$

(1.15)

Here $\bar{g}$ and $\bar{h}$ are characterized by $g(u_{-\mu} \otimes v_{2+\mu}) = u_2$ and $h(v_{2+\mu}) = u_{\mu} \otimes v_2$. Moreover, $\bar{\varphi}$ and $\bar{\psi}$ are related by

$$(c \otimes \text{id}_{V_{2+\mu}})(w \otimes \bar{\varphi}(v)) = \bar{\psi}(w \otimes v) \quad \text{for } v \in V_2 \text{ and } w \in V_\mu,$$

where $c$ is the contraction $V_\mu \otimes V^*_\mu \rightarrow \mathbb{C}$.

Now, $V_\mu \otimes V_2$ contains $V_{2+\mu}$ with multiplicity 1. Let us denote by $p$ the projector form $V_\mu \otimes V_2$ onto $h(V_{2+\mu})$, and regard this as an endomorphism of $V_\mu \otimes V_2$. Then by (1.15), we have

$$\bar{h} \circ \bar{\psi} = f'(\lambda)p.$$

On the other hand, we have a commutative diagram
where $\iota: C \to V^*_\mu \otimes V_\mu$ is the canonical injection. Therefore we have

$$f(\lambda) \text{id}_{V_\lambda} = f'(\lambda) (c \otimes V_\lambda) \circ (V^*_\mu \otimes p) \circ (\iota \otimes V_\lambda).$$

Taking the trace, we have

$$f(\lambda) \dim V_\lambda = f'(\lambda) \text{tr}_{Y_\lambda} (c \otimes V_\lambda) \circ (V^*_\mu \otimes p) \circ (\iota \otimes V_\lambda).$$

In order to calculate the right-hand side, we shall take bases $\{w_j\}$ of $V_\lambda$, $\{u_k\}$ of $V_\mu$ and their dual bases $\{w^*_j\}$ and $\{u_k^*\}$. Then

$$\begin{align*}
(c \otimes V_\lambda) \circ (V^*_\mu \otimes p) \circ (\iota \otimes V_\lambda)(w_j) &= \sum_k (c \otimes V_\lambda) \circ (V^*_\mu \otimes p)(u_k^* \otimes u_k \otimes w_j) \\
&= \sum_k (c \otimes V_\lambda)(u_k^* \otimes p(u_k \otimes w_j)).
\end{align*}$$

Hence we obtain

$$\text{tr}_{Y_\lambda} (c \otimes V_\lambda) \circ (V^*_\mu \otimes p) \circ (\iota \otimes V_\lambda)$$

$$= \sum_{j,k} \langle w^*_j, (c \otimes V_\lambda)(u_k^* \otimes p(u_k \otimes w_j)) \rangle$$

$$= \sum_{j,k} \langle u_k^* \otimes w^*_j, p(u_k \otimes w_j) \rangle$$

$$= \text{tr}_{V^*_\mu \otimes V_\lambda} p = \dim V_{\lambda + \rho}.$$

By (1.16), we obtain

$$f(\lambda) \dim V_\lambda = f'(\lambda) \dim V_{\lambda + \rho}.$$

Then the assertion follows from Weyl’s dimension formula

$$\dim V_\lambda = \prod_{\alpha \in \Delta^+} \frac{h_n(\lambda + \rho)}{h_n(\rho)}.$$

Q.E.D.

**Corollary 1.10.** For a dominant integral weight $\mu$, there exists a commutative diagram

$$\begin{aligned}
U_R(g) \otimes R_{e + \mu}^+ &\xrightarrow{f} U_R(g) \otimes (R_e \otimes V_\mu) \\
\psi \downarrow &\quad \Downarrow \psi \\
U_R(g) \otimes (R_e \otimes V_\mu) &\xrightarrow{f} U_R(g) \otimes R_{e + \mu}
\end{aligned}$$
where \( f = \prod_{a \in J^+} (h_a + h_a(\rho), h_a(\mu)) \) and \( h(1 \otimes 1_{c+\rho}) = 1 \otimes 1_c \otimes v_\mu \).

**Remark 1.11.** This corollary is also obtained either by a similar argument as the proof of Theorem 1.5 or directly from Theorem 1.7 by the following argument. First note that for any \( U_R(b) \)-module \( F \), we have

\[
R \operatorname{Hom}_{U_R(b)} (U_R(g) \otimes F, U_R(g)) = U_R(g) \otimes R \operatorname{Hom}_{U_R(b)} (F, U_R(b)).
\]

On the other hand, for a finite dimensional \( b \)-module \( V \)

\[
R \operatorname{Hom}_{U_R(b)} (R_c \otimes V, U_R(b)) = R_{-c-2\rho} \otimes V^* [-\dim b]
\]

where \( R_{-c-2\rho} \) is the \( U_R(b) \)-module \( R \) with weight \(-c-2\rho\). Hence the commutative diagram

\[
\begin{array}{ccc}
U_R(g) \otimes R_c & \longrightarrow & U_R(g) \otimes (R_c + \rho \otimes V^*_\rho) \\
\downarrow f' & & \downarrow \\
U_R(g) \otimes R_c & \longrightarrow & U_R(g) \otimes V^*_\rho
\end{array}
\]

with \( f' = \prod_a (h_a + h_a(\rho) + 1, h_a(\mu)) \) gives

\[
U_R(g) \otimes R_{-c-2\rho} \leftarrow U_R(g) \otimes (R_{-c-\rho-2\rho} \otimes V^*_\rho) \leftarrow U_R(g) \otimes R_{-c-2\rho}.
\]

Now, the isomorphism \( h \mapsto -h - h(2\rho + \mu) \) gives Corollary 1.10.

**§ 2. The \( b \)-functions of \( B_- \times B \)-semi-invariants**

For a dominant integral weight \( \lambda \), let \( V_\lambda \) be an irreducible representation of \( g \) with highest weight \( \lambda \). Let \( v_\lambda \) be a highest weight vector of \( V_\lambda \) and \( v_{-\lambda} \) the lowest weight vector of \( V_\lambda^* \), normalized by \( \langle v_\lambda, v_{-\lambda} \rangle = 1 \).

Let \( f^\lambda \) be the regular function on \( G \) defined by

\[
f^\lambda(g) = \langle g v_\lambda, v_{-\lambda} \rangle.
\]

Then \( f^\lambda \) is \( B_- \times B \)-semi-invariant such that

\[
f^\lambda(b'gb) = x_{-\lambda}^-(b')x_{\lambda}^+(b)f^\lambda(g) \quad \text{for } g \in G, b' \in B_- \text{ and } b \in B,
\]

\[
(2.1) \quad f^\lambda(g) = \langle g v_\lambda, v_{-\lambda} \rangle.
\]

Then \( f^\lambda \) is \( B_- \times B \)-semi-invariant such that

\[
f^\lambda(b'gb) = x_{-\lambda}^-(b')x_{\lambda}^+(b)f^\lambda(g) \quad \text{for } g \in G, b' \in B_- \text{ and } b \in B,
\]

\[
(2.2) \quad f^\lambda(b'gb) = x_{-\lambda}^-(b')x_{\lambda}^+(b)f^\lambda(g)
\]

for \( g \in G, b' \in B_- \) and \( b \in B \),
where $\chi_i^±$ is the character of $B$ and $B_-$ such that

$$\chi_i^±(e^h) = e^{\pm(h)} \quad \text{for } h \in \mathfrak{h}.$$  

Moreover we have

$$(2.3) \quad f^i(e) = 1.$$  

Note that any $B_- \times B$-semi-invariant with character $\chi_i^± \otimes \chi_i^±$ is a constant multiple of $f^i$ and any $B_- \times B$-semi-invariant has a character $\chi_i^± \otimes \chi_i^λ$ for some $λ \in P^+$. This follows from the well-known formula

$$\mathcal{G}(G) = \bigoplus_{λ \in P^+} V_λ^* \otimes V_λ.$$  

In particular, we have

$$(2.4) \quad f^{i+2}(g) = f^i(g)f^i(g).$$  

**Theorem 2.1.** For any dominant integral weight $μ$, there exists a differential operator $P_μ$ such that

$$(2.5) \quad P_μ f^{i+μ} = b_μ(λ)f^λ \quad \text{for any } λ.$$  

Here $b_μ(λ) = \prod_{α \in \Delta^+} (h_α(λ + μ), h_α(μ)).$

**Proof.** Let us denote by $\mathcal{D}$ the sheaf of differential operators on $G$. Then the right-action of $G$ on itself gives a homomorphism $R: U(\mathfrak{g}) \to \mathcal{D}(G)$. In particular, $R(U(\mathfrak{g}))$ is the set of left invariant differential operators on $G$.

By Corollary 1.10, there exists an $n$-invariant element $P$ of $V_μ^* \otimes (U_λ(\mathfrak{g}) \otimes U_λ(\mathfrak{g}) R_{i+μ})$ with weight $c$, whose coefficient of $v_λ$ is $\prod_{α \in \Delta^+} (c(h_α) + h_α(μ), h_α(μ))$. Hence $P$ is written in the following form

$$P = \sum_{j=0}^{N} v_j \otimes P_j \otimes 1_{i+μ}$$

where

$$(2.6) \quad v_0 = v_{-μ}, \quad P_0 = \prod_{α \in \Delta^+} (h_α + h_α(μ - μ), h_α(μ))$$

and

$$(2.7) \quad v_j \in \mathfrak{n} V_μ^*, \quad P_j \in U(\mathfrak{b}_-) \mathfrak{n}_- \quad \text{for } j \geq 1.$$  

We shall define the differential operator $P_μ$ on $G$ by

$$(2.8) \quad (P_μu)(g) = \sum_j \langle v_j, gu_j \rangle (R(P_j)u)(g).$$
Lemma 2.2. For any $y \in \mathfrak{n}$, we have
\[ [R(y), P_{\mu}] \in \mathcal{D}(G)R(\mathfrak{n}). \]

Proof. We have $[R(y), \langle v_{\mu}, gv_j \rangle] = \langle v_{\mu}, gvy_j \rangle$. Hence we have
\[
([R(y), P_{\mu}]u)(g) = \sum_j \langle g^{-1}v_{\mu}, yv_j \rangle (R(P_j)u)(g) \\
+ \sum_j \langle g^{-1}v_{\mu}, v_j \rangle (R([y, P_j])u)(g).
\]

Since $\sum_j v_j \otimes P_j \otimes 1_{e+\rho}$ is $\mathfrak{n}$-invariant, we have
\[
\sum_j yv_j \otimes P_j \otimes 1_{e+\rho} + \sum_j v_j \otimes [y, P_j] \otimes 1_{e+\rho} = 0
\]
in
\[
V_{\mu}^* \otimes U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{g})} R_{e+\rho} = V_{\mu}^* \otimes (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}).
\]

Therefore we can write, as the identity in $V_{\mu}^* \otimes U(\mathfrak{g})$,
\[
\sum_j yv_j \otimes P_j + \sum v_j \otimes [y, P_j] = \sum w_k \otimes S_k
\]
with $w_k \in V_{\mu}^*$ and $S_k \in U(\mathfrak{g})\mathfrak{n}$. This shows
\[
([R(y), P_{\mu}]u)(g) = \sum_k \langle g^{-1}v_{\mu}, w_k \rangle (R(S_k)u)(g).
\]

Since $R(S_k) \in \mathcal{D}(G)R(\mathfrak{n})$, we have the desired result. Q.E.D.

By this lemma, we have for $y \in \mathfrak{n}$
\[
R(y)P_{\mu} f^{2+\rho} = [R(y), P_{\mu}] f^{2+\rho} + P_{\mu} R(y) f^{2+\rho} = 0
\]
because $f^{2+\rho}$ is right invariant by $N$. Therefore $P_{\mu} f^{2+\rho}$ is also right $N$-invariant. Since $B_- N$ is an open dense subset of $G$, it is sufficient to show (2.5) on $B_-$. Now for $g \in B_-$, we have
\[
(P_{\mu} f^{2+\rho})(g) = \sum_j \langle v_{\mu}, gv_j \rangle (R(P_j) f^{2+\rho})(g).
\]

Note that all $P_j$ belongs to $U(b_-)$ and $P_j \in U(b_-)\mathfrak{n}_-$ for $j \neq 0$. Since $f^{2+\rho}(n_- h) = f^{2+\rho}(hn_-) = h^{2+\rho}$ for $h \in T$ and $n_- \in N_-$, $f^{2+\rho}|_{\mathfrak{n}_-}$ is right $N_-$-invariant. This shows $R(P_j) f^{2+\rho}|_{B_-} = 0$ for $j \neq 0$. It is easy to see for $g \in B_-$
\[
R(P_0) f^{2+\rho}(g) = \prod_{\alpha} (h_\alpha(\lambda + \mu) + h_\alpha(\rho - \mu), h_\alpha(\mu)) f^{2+\rho} \\
= b_\rho(\lambda) f^{2+\rho}
\]
and \( \langle v_\mu, g v_0 \rangle = 1/r \).

This completes the proof of Theorem 2.1.

**Remark 2.3.** We can show \( b_\mu(\lambda) \) in Theorem 2.1 is the best possible one. This follows from the similar argument as Proposition 1.8, or we can use the result in [3]. In fact if \( w_0 \) is the longest element of \( W \), then \( T_{H - w_0 b}^b G \) is a good Lagrangian variety in the sense in [3], which is equivalent to saying that \( n \) is a prehomogeneous vector space over \( \mathbb{B} \). Hence we can show the degree of the local \( b \)-function is \( \sum_{a \in \mathcal{A}} h_a(\mu) \).

**Bibliography**


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