The Invariant Holonomic System on a Semisimple Lie Group

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0. Introduction

0.1.

Let G be a connected reductive algebraic group defined over $\mathbb{C}$ and $\mathfrak{G}$ its Lie algebra. The center $\mathfrak{Z}(\mathfrak{G})$ of the universal enveloping algebra $U(\mathfrak{G})$ is identified with the ring of bi-invariant differential operators on G. Let θ_G be the sheaf of vector fields on G. The adjoint action of G on G induces the Lie algebra homomorphism

$$\mathrm{Ad}: \mathfrak{G} \to \Gamma(G, \theta_G). \tag{0.1}$$

We shall denote by $\mathscr{D}_G$ the ring of differential operators on G. For any character $\chi: \mathfrak{Z}(\mathfrak{G}) \to \mathbb{C}$, let $\mathscr{M}_\chi$ be the $\mathscr{D}_G$-module $\mathscr{D}_G/(\mathscr{D}_G \mathrm{Ad}(\mathfrak{G}) + \sum_{P \in \mathfrak{Z}(\mathfrak{G})} \mathscr{D}_G(P - \chi(P)))$.

If $G_{\mathbf{R}}$ is a real form of G, any invariant eigendistribution satisfies the system of differential equations $\mathcal{M}_\chi$. The property of $\mathcal{M}_\chi$ is deeply investigated by Harish-Chandra [H-C]). In this article, we shall give the proof of the following theorem on $\mathcal{M}_\chi$.

Theorem 1. *(i) $\mathcal{M}_\chi$ is a regular holonomic $\mathcal{D}_G$-module. (ii) $\mathcal{M}_\chi$ is the minimal extension (see Section 2.2) of $\mathcal{M}_\chi|G_{\mathrm{reg}}$. Here G_{reg} is the open set of semisimple regular elements of G.*

The version of this theorem in the Lie algebra case is already proven in [H-K], and we shall use this result to prove Theorem 1. When χ is a trivial infinitesimal character, this theorem is shown in [H-K′] by a completely different method.

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0.2.

Let T^*G be the cotangent bundle of G and we identify T^*G with $G\times\mathfrak{G}^*$. Let $N(\mathfrak{G}^*)$ be the set of nilpotent elements of $\mathfrak{G}^*$. Let V be the common zeroes of the principal symbol of $\mathrm{Ad}(\mathfrak{G})$. Hence we have

$$V\cong\{(g,\xi)\in G\times\mathfrak{G}^*;\ g\cdot\xi=\xi\}. \tag{0.2}$$

Here $g\cdot\xi$ is the coadjoint action on $\mathfrak{G}^*$. We shall also prove the following theorem.

Theorem 2. (*i*) (Richardson [R]) *V is irreducible with dimension* $\dim G+\operatorname{rank} G$.

(*ii*) $\dim(V\cap(G\times N(\mathfrak{G}^*)))=\dim G$.

(*iii*) *The characteristic cycle of $\mathcal{M}_\chi$ coincides with $V\cdot(G\times N(\mathfrak{G}^*))$.*

The statement (*i*) is proven by Richardson ([R]).

0.3.

We define a $\mathcal{D}_G$-module $\mathcal{N}_G$ by $\mathcal{N}_G=\mathcal{D}_G/\mathcal{D}_G\,\mathrm{Ad}(\mathfrak{G})$. Then $\mathfrak{Z}(\mathfrak{G})$ acts on $\mathcal{N}_G$ by the right multiplication.

For a $\mathfrak{Z}(\mathfrak{G})$-module M, we set

$$\mathcal{N}_G(M)=\mathcal{N}_G\underset{\mathfrak{Z}(\mathfrak{G})}{\otimes}M. \tag{0.3}$$

Hence if M is the one-dimensional $\mathfrak{Z}(\mathfrak{G})$-module corresponding to a character χ of $\mathfrak{Z}(\mathfrak{G})$, $\mathcal{N}_G(M)$ is nothing but $\mathcal{M}_\chi$. We shall also prove the following theorem.

Theorem 3. *(i) $\mathcal{N}_G$ is flat over $\mathfrak{Z}(\mathfrak{G})$.*

(ii) $\mathcal{E}xt^j_{\mathcal{D}_G}(\mathcal{N}_G, \mathcal{D}_G) = 0$ for $j \neq \dim G - \operatorname{rank} G$.

(iii) Assume further G is semisimple. Then for any $\mathfrak{Z}(\mathfrak{G})$-module M, we have $\mathcal{H}^0_{N(G)}(\mathcal{N}_G(M)) = 0$, where $N(G)$ is the set of unipotent elements of G.

1. Proof of Theorems 1 and 3

1.1.

We shall prove these theorems by induction on dim G. We can reduce to the case where G has a trivial center. The following lemma is proven by Harish-Chandra.

Lemma 1.1 ([H-C]). *Let S be a nonempty closed subset of G invariant by the adjoint action. If S contains no semisimple element other than 1, then S is contained in $N(G)$.*

1.2.

Hereafter we assume that G has a trivial center. Let us take a semisimple element $a \neq 1$, and let G' be the centralizer of a and $\mathfrak{G}'$ its Lie algebra. Then by the hypothesis of induction, Theorems 1 and 3 are true for G'.

We can easily prove

Lemma 1.2. *$\pi^{-1}(a) \cap T^*_{G'}G \cap V \subset T^*_G G$. Here π is the projection $T^*G \to G$ and $T^*_{G'}G$ is the conormal bundle.*

Corollary 1.3. *$\mathcal{N}_G|_{G'}$ is generated by $u_G|_{G'}$ as a $\mathcal{D}_{G'}$-module on a neighborhood of a. Here u_G is the canonical generator of $\mathcal{N}_G$.*

1.3.

Lemma 1.4. *If Theorem 3 is true for G, then $\mathcal{H}^0_S(\mathcal{N}_G) = 0$ for any closed nowhere dense subset S of G.*

In fact, we have ch $\mathscr{H}^0_S(\mathscr{N}_G) \subset \pi^{-1}(S) \cap V$, and hence codim ch $\mathscr{H}^0_S(\mathscr{N}_G) >$ dim $G -$ rank G. Since $\mathscr{E}xt^j(\mathscr{N}_G, \mathscr{D}_G) = 0$ for $j \neq \dim G - \operatorname{rank} G$, $\mathscr{H}^0_S(\mathscr{N}_G) = 0$ (Theorem 2.12 [K′]).

We can also prove easily by induction on dim M.

Lemma 1.5. *If Theorem 1 is true for G, then for any finite-dimensional $\mathfrak{Z}(\mathfrak{G})$-module M, $\mathscr{N}_G(M)$ is regular holonomic and it is the minimal extension of $\mathscr{N}_G(M)|_{G_{\mathrm{reg}}}$.*

1.4.

Set $\nu(g) = \det(\mathrm{Ad}(g) - 1; \mathfrak{G}/\mathfrak{G}')$ for $g \in G'$. Then $u_{G'} \mapsto \nu^{1/2}(u_G|_{G'})$ defines a $\mathscr{D}_{G'}$-linear homomorphism on a neighborhood of a:

$$\mathscr{N}_{G'} \to \mathscr{N}_G|_{G'}. \tag{1.1}$$

The hypothesis of induction along with Lemma 1.4 implies

$$\mathscr{H}^0_{G' \setminus G_{\mathrm{reg}}}(\mathscr{N}_{G'}) = 0. \tag{1.2}$$

Since (1.1) is surjective by Corollary 1.3 and bijective on $G' \cap G_{\mathrm{reg}}$, (1.2) implies that (1.1) is an isomorphism on a neighborhood of a.

Let us embed $\mathfrak{Z}(\mathfrak{G})$ into $\mathfrak{Z}(\mathfrak{G}')$. Then $\mathfrak{Z}(\mathfrak{G}')$ is a free $\mathfrak{Z}(\mathfrak{G})$-module of finite rank. By Harish-Chandra [H-C], (1.1) is $\mathfrak{Z}(\mathfrak{G})$-linear on $G' \cap G_{\mathrm{reg}}$. Hence (1.2) implies the following lemma.

Lemma 1.6. *$\mathscr{N}_{G'}$ and $\mathscr{N}_G|_{G'}$ are isomorphic as $(\mathscr{D}_G, \mathfrak{Z}(\mathfrak{G}))$-bimodules on a neighborhood of a.*

1.5.

Now, we shall show Theorem 3 and

$$\mathscr{H}^0_{G \setminus G_{\mathrm{reg}}}(\mathscr{M}_\chi) = 0 \tag{1.3}$$

on a neighborhood of a.

Lemma 1.7. *If $\mathscr{L}$ is a coherent $\mathscr{D}_G$-module such that* ch $\mathscr{L} \subset V$ *and $\mathscr{L}|_{G'} = 0$, then $\mathscr{L} = 0$ on a neighborhood of a.*

This follows immediately from Lemma 1.4 and Theorem 2.6.17 in [K].

For a $\mathfrak{Z}(\mathfrak{G})$-module M, set $M' = M \underset{\mathfrak{Z}(\mathfrak{G})}{\otimes} \mathfrak{Z}(\mathfrak{G}')$. Then we have

$$\mathcal{T}or_j^{\mathfrak{Z}(\mathfrak{G}')}(\mathcal{N}_{G'}, M') \simeq \mathcal{T}or_j^{\mathfrak{Z}(\mathfrak{G})}(\mathcal{N}_G, M)|_{G'}.$$

Hence we have $\mathcal{T}or_j(\mathcal{N}_G, M) = 0$ for $j \neq 0$. The other statements follow in a similar way.

1.6.

By using Lemma 1.1, Theorem 3 and (1.3) is true outside $N(G)$.

2. Proof of Theorem 1 (Continued)

2.1.

In order to describe $\mathcal{M}_\chi|_{G_{\mathrm{reg}}}$, let us choose a Cartan subgroup T of G. Let t be its Lie algebra, Δ the root system for $(\mathfrak{G}, t)$ and W the Weyl group. For $\alpha \in \Delta$, let ξ_α be the corresponding character of T. Set $T_{\mathrm{reg}} = T \cap G_{\mathrm{reg}}$. Let $\varphi : \mathfrak{Z}(\mathfrak{G}) \to U(t)^W$ be the canonical isomorphism and we identify $U(t)$ with the ring of invariant differential operators on T. Then by Harish-Chandra ([H-C]), $\mathcal{M}_\chi|_{T_{\mathrm{reg}}}$ is equal to the system of differential equations

$$D^{-1/2}(\varphi(P) - \chi(P))D^{1/2}u = 0 \qquad \text{for } P \in \mathfrak{Z}(\mathfrak{G}), \tag{2.1}$$

where

$$D = \prod_{\alpha \in \Delta} (\xi_\alpha^{1/2} - \xi_\alpha^{-1/2}).$$

2.2.

Since (2.1) is regular holonomic, $\mathcal{M}_\chi|_{G_{\mathrm{reg}}}$ is regular holonomic. Let ${}^{\pi}(\mathcal{M}_\chi|_{G_{\mathrm{reg}}})$ be its minimal extension, i.e., a regular holonomic $\mathcal{D}_G$-module such that ${}^{\pi}(\mathcal{M}_\chi|_{G_{\mathrm{reg}}})|_{G_{\mathrm{reg}}} \cong \mathcal{M}_\chi|_{G_{\mathrm{reg}}}$ and such that it has neither non-zero submodule nor quotient whose support is contained in G_{reg}. By Harish-Chandra ([H-C]), we have

$$\mathcal{M}_\chi \text{ has no non-zero quotient whose support is contained in } G \backslash G_{\mathrm{reg}}. \tag{2.2}$$

Hence we have a canonical homomorphism

$$\mathcal{M}_\chi \to {}^{\pi}(\mathcal{M}_\chi|_{G_{\mathrm{reg}}}). \tag{2.3}$$

This is evidently surjective.

2.3.

By the result of Section 1, $\operatorname{supp}(\mathscr{H}^0_{G\setminus G_{\mathrm{reg}}}(\mathscr{M}_\chi))$ is contained in the set $N(G)$ of unipotent elements, and hence (2.3) is an isomorphism outside $N(G)$. Thus it is enough to show that (2.3) is an isomorphism on a neighborhood of 1.

2.4.

Let us take a small neighborhood U of 0 in $\mathfrak{G}$ such that $\exp:\mathfrak{G}\to G$ is an isomorphism from U onto $V=\exp(U)$. Let $\tilde{\mathscr{M}}_\chi$ be the $\mathscr{D}_{\mathfrak{G}}$-module $\mathscr{D}_{\mathfrak{G}}/(\mathscr{D}_{\mathfrak{G}}\operatorname{ad}\mathfrak{G}+\sum_{P\in S(\mathfrak{G})^G}\mathscr{D}_{\mathfrak{G}}(P-\chi(P)))$. Here ad is the homomorphism $\mathfrak{G}\to\Gamma(\mathfrak{G};\,\theta_g)$ given by the adjoint action. We identify $\mathfrak{Z}(\mathfrak{G})$ with $S(\mathfrak{G})^G$ and $S(\mathfrak{G})^G$ with the ring of G-invariant constant-coefficient differential operators on $\mathfrak{G}$. By [H-K], $\tilde{\mathscr{M}}_\chi$ is the minimal extension of $\tilde{\mathscr{M}}_\chi|_{\mathfrak{G}_{\mathrm{reg}}}$. Here $\mathfrak{G}_{\mathrm{reg}}$ is the set of regular semisimple elements of $\mathfrak{G}$. Moreover by [H-C], we have

$$(\exp)_*\mathrm{Hom}(\tilde{\mathscr{M}}_\chi\mathscr{O}_{\mathfrak{G}_{\mathrm{an}}})\cong\mathrm{Hom}_{\mathscr{D}_G}(\mathscr{M}_\chi,\,\mathscr{O}_{G_{\mathrm{an}}}).$$

on $V\cap G_{\mathrm{reg}}$. Here G_{an} and $\mathfrak{G}_{\mathrm{an}}$ are the underlying complex manifolds. This implies

$$({}^{\pi}\mathscr{M}_\chi|_{G_{\mathrm{reg}}})|_V\cong\exp_*(\tilde{\mathscr{M}}_\chi|_U). \tag{2.4}$$

2.5.

Now, we shall use the same argument as in [H-K].

Let $\mathscr{L}$ be the kernel of (2.3) and S the support of $\mathscr{L}$. Then one can easily show (by a similar argument as in [H-K], Section 6.6) that codim $S\geq 2$. Assume $S\ni e$. We take a generic point g of $S\cap V$. Since $\mathscr{E}xt^1_{\mathscr{D}_{\mathfrak{G}}}(\tilde{\mathscr{M}}_\chi,\,\mathscr{B}_{(\exp)^{-1}S|\mathfrak{G}})_{\log(g)}=0$ by Lemma 6.7.1 [H-K],

$$0\to\mathscr{L}\to\mathscr{M}_\chi\to{}^{\pi}(\mathscr{M}_{\chi|\mathfrak{G}_{\mathrm{reg}}})\to 0 \tag{2.5}$$

splits on a neighborhood of g. Hence $\mathscr{M}_\chi$ has a non-zero quotient supported in $G\setminus G_{\mathrm{reg}}$. This contradicts Harish-Chandra's result (2.2).

3. Proof of Theorem 3 (Continued)

3.1.

We proved already Theorem 3 outside the set $N(G)$ of unipotent elements. In the sequel, we use the following simple result.

Lemma 3.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_G$-module such that* $\operatorname{ch}\mathcal{M}\subset \pi^{-1}(N(g))\cap V$. *Then $\mathcal{M}$ is holonomic.*

In fact $\pi^{-1}N(G)\cap V$ is Lagrangean because $N(G)$ has finitely many G-orbits.

3.2.

We shall first prove Theorem 3(*iii*) by induction on dim M. If $\dim M=0$, this is true by Lemma 1.5. Hence we shall assume $\dim M>0$. Let M' be the largest submodule of M with $\dim M'=0$, and let $M''=M/M'$. Since $0\to\mathcal{N}_G(M')\to\mathcal{N}_G(M)\to\mathcal{N}_G(M'')\to 0$ is exact outside $N(G)$ and $\mathcal{H}^0_{N(G)}(\mathcal{N}_G(M'))=0$, this is exact on the whole of G. Hence in order to prove $\mathcal{H}^0_{N(G)}(\mathcal{N}_G(M))=0$, it is enough to show $\mathcal{H}^0_{N(G)}(\mathcal{N}_G(M''))=0$. Replacing M with M'', we shall assume $M'=0$ from the beginning. Then there exists $P\in\mathfrak{Z}(\mathfrak{G})$ such that $P-c$ acts injectively on M for any $c\in\mathbb{C}$. Then for any non-zero polynomial $b(P)$, $\dim(M/b(P)M)<\dim M$. Set $\mathcal{L}=\mathcal{H}^0_{N(G)}(\mathcal{N}_G(M))$. Then by Lemma 3.1, $\mathcal{L}$ is holonomic and hence End($\mathcal{L}$) is finite-dimensional. Therefore there exists a non-zero polynomial $b(P)$ such that $b(P)\mathcal{L}=0$. Since $b(P)$ acts injectively on $\mathcal{N}_G(M)$ outside $N(G)$, $b(P)$ acts injectively in $\mathcal{N}_G(M)/\mathcal{L}$. Moreover the kernel of $b(P)$ in $\mathcal{N}_G(M)$ is $\mathcal{L}$. Hence $\mathcal{L}\to\mathcal{N}_G(M/b(P)M)$ is injective. Since $\mathcal{H}^0_{N(G)}(\mathcal{N}_G(M/b(P)M))=0$ by the hypothesis of induction, we have $\mathcal{L}=0$.

3.3.

In order to prove Theorem 3(*ii*), we shall prove the following generalized statement.

Lemma 3.2. *If G is semisimple,* $\mathcal{E}xt^j(\mathcal{N}_G(M),\mathcal{D}_G)=0$ *for* $j\neq$ $\dim G-\dim M$ *for any finitely generated Cohen-Macaulay $\mathfrak{Z}(\mathcal{G})$-module M.*

Proof. We may assume that the center of G is trivial. Since $\operatorname{codim}\operatorname{ch}(\mathcal{N}_G(M))\geq\dim G-\dim M$, we have $\mathcal{E}xt^j(\mathcal{N}_G(M),\mathcal{D}_G)=0$ for $j<\dim G-\dim M$. Since $\dim\operatorname{proj}M=\operatorname{rank}G-\dim M$, M has a free resolution of length $m=\operatorname{rank}G-\dim M$: $0\leftarrow M\leftarrow L_0\leftarrow L_1\leftarrow\cdots\leftarrow L_M\leftarrow 0$. Hence we have a resolution outside $N(G)$: $0\leftarrow\mathcal{N}_G(M)\leftarrow\mathcal{N}_G(L_0)\leftarrow\cdots\leftarrow\mathcal{N}_G(L_M)\leftarrow 0$. Since $\mathcal{E}xt^j(\mathcal{N}_G,\mathcal{D}_G)=0$ outside $N(G)$ for $j>\dim G-\operatorname{rank}G$, we have

$$\text{supp}\ \mathscr{E}xt^j(\mathscr{N}_G(M), \mathscr{D}_G) \subset N(G)$$
$$\text{for}\ j > \dim G - \text{rank}\ G + m = \dim G - \dim M.$$

Hence $\mathscr{E}xt^j(\mathscr{N}_G(M), \mathscr{D}_G)$ is holonomic by Lemma 3.1. Now we shall proceed with the proof by induction on dim M. If $\dim M = 0$, this is trivial. If $\dim M > 0$, take $P \in \mathfrak{Z}(\mathfrak{G})$ such that $\dim(\text{supp}\ M \cap \{P = c\}) = \dim M - 1$ for any $c \in \mathbf{C}$. Then for any non-zero polynomial $b(P)$, $\mathscr{E}xt^j(\mathscr{N}_G(M/b(P)M), \mathscr{D}_G) = 0$ for $j > \dim G - \dim M + 1$ by the hypothesis of induction. Hence $b(P)$ acts surjectively on $\mathscr{L} = \mathscr{E}xt^j(\mathscr{N}_G(M), \mathscr{D}_G)$ for $j > \dim G - \dim M$. Since $\mathscr{L}$ is holonomic, there exists a non-zero $b(P)$ such that $b(P)\mathscr{L} = 0$. This implies $\mathscr{L} = 0$.

3.4.

We shall prove Theorem 3(i). In order to see this, it is enough to show that for an injective morphism $M' \to M$ of finitely generated $Z(\mathfrak{G})$-modules, $\mathscr{N}_G(M') \to \mathscr{N}_G(M)$ is injective. Since this is injective outside $N(G)$, this follows from Theorem 3(iii).

4. The Proof of Theorem 2

4.1.

Theorem 2(i) is a result of Richardson [R]. Let us prove Theorem 2(ii). Let q be the projection from V to $\mathfrak{G}^*$. Let S be a G-orbit of $N(\mathfrak{G}^*)$. Let $\xi \in S$. Then $\pi(V \cap q^{-1}(\xi)) = G_\xi$. Hence $\dim G_\xi + \dim S = \dim G$. This shows that $V \cap q^{-1}(S)$ is a non-singular manifold of dim G. Since $N(\mathfrak{G}^*)$ has finitely many G-orbits, $V \cap q^{-1}(N(\mathfrak{G}^*))$ has pure dimension dim G.

4.2.

Let us prove Theorem 2(iii). We may assume that G has a trivial center. By Proposition 4.8.3 and Theorem 6.1 in [H-K], the characteristic cycle of $\exp_*(\tilde{\mathscr{M}}_\chi)$ is $V \cdot (G \times N(\mathfrak{G}^*))$ on a neighborhood of 1 (with the notation in Section 2.4). Hence, by the result in Section 2, Theorem 2(iii) is true on a neighborhood of 1.

4.3.

If Theorem 2 is true, then for a finite-dimensional $\mathfrak{Z}(\mathfrak{G})$-module M, the characteristic cycle of $\mathcal{N}_G(M)$ is $(\dim M) V\cdot (G\times N(\mathfrak{G}^*))$.

4.4.

By Lemma 1.1, it is enough to show $\underline{\mathrm{ch}}\ \mathcal{M}_\chi = V\cdot (G\times N)$ on a neighborhood of a semisimple element a. Here $\underline{\mathrm{ch}}$ denotes the characteristic cycle. Let G' be the centralizer of a. Let ρ be the projection $G'\times_G T^*G\to T^*G'$ and $\tilde{\omega}$ the embedding $G'\times_G T^*G\to T^*G$. Then since $\mathcal{M}'$ is non-characteristic, we have $\underline{\mathrm{ch}}(\mathcal{M}_\chi|_{G'}) = \rho_*\tilde{\omega}^{-1}(\underline{\mathrm{ch}}\ \mathcal{M}_\chi)$ (see Chapter II, Section 6 [K]). Hence it is enough to show

$$\underline{\mathrm{ch}}(\mathcal{M}_\chi|_{G'}) = \rho_*\tilde{\omega}^{-1}(V\cdot G\times N).$$

Let V' and N' be the sets defined as V and N replacing G with G'. Since $\mathcal{M}_\chi|_{G'}$ is isomorphic to $\mathcal{M}_{G'}(M)$ with $M = \mathbf{C} \otimes_{\mathfrak{Z}(\mathfrak{G}')} \mathfrak{Z}(\mathfrak{G})$ by Lemma 1.4, we have

$$\begin{aligned}\underline{\mathrm{ch}}(\mathcal{M}_\chi|_{G'}) &= \underline{\mathrm{ch}}(\mathcal{M}_{G'}(M))\\ &= (\dim_{\mathbf{C}} M)\cdot V'\cdot (G'\times N'). \end{aligned}\tag{4.1}$$

Note that $\dim_{\mathbf{C}} M = \#(W/W')$. Here W and W' are the Weyl group of G and G' respectively. Hence it is enough to show

Lemma 4.1. *On a neighborhood of a,*

$$\rho_*\tilde{\omega}^{-1}(V\cdot G\times N) = \#(W/W')(V'\cdot G'\times N').$$

Proof. Set $\mathfrak{u} = \{\xi\in\mathfrak{G}^*;\ a\xi = \xi\}$. Then, $\mathfrak{u}\to\mathfrak{G}^*$ is an isomorphism. We shall show that

$$(G'\times\mathfrak{G}^*)\cap V\subset G'\times\mathfrak{u} \text{ on a neighborhood of } a. \tag{4.2}$$

In fact, it is enough to show that $g\colon \mathfrak{G}^*/\mathfrak{u}\to\mathfrak{G}^*/\mathfrak{u}$ has no eigenvalue 1 if $g\in G'$ is sufficiently near a. This is evident. Hence $(G'\times\mathfrak{G}^*)\cap V$ is isomorphic to V'. Take homogeneous functions $f_1,\ldots,f_r$ ($r = \operatorname{rank} G$) on $\mathfrak{G}^*$ such that $\mathbf{C}[f_1,\ldots,f_r] = S(\mathfrak{G})^G$. Then

$$\tilde{\omega}^{-1}(V\cdot N) = ((G'\times\mathfrak{G}^*)\cap V)\cap(f_1=\cdots=f_r=0).$$

Hence $\rho_* \tilde{\omega}^{-1}(V \cdot N) = V' \cap (f_1 = \cdots = f_r = 0)$. Since $S(\mathfrak{G}')^{G'}$ is a free module over $S(\mathfrak{G})^G$ of rank $\#(W/W')$, $\mathfrak{G}'^* \cap (f_1 = \cdots = f_r = 0) = \#(W/W')N'$. Q.E.D.

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