The Campbell-Hausdorff Formula and Invariant Hyperfunctions

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Introduction

Let \( G \) be a Lie group and \( g \) its Lie algebra. We denote by \( V \) the underlying vector space of \( g \).

There is a canonical isomorphism between the ring \( Z(g) \) of the biinvariant differential operators on \( G \) and the ring \( I(g) \) of the constant coefficient operators on \( V \) which are invariant by the adjoint action of \( G \). When \( g \) is semi-simple, this is the "Harish-Chandra isomorphism"; for a general Lie algebra, this was established by Duflo [4].

We shall prove here, that when \( G \) is solvable the Duflo isomorphism extends to an isomorphism \( \Phi \) of the algebra of "local" invariant hyperfunctions under the group convolution and the algebra of invariant hyperfunctions on \( V \) under additive convolution (the exact result will be stated below). This gives a partial answer to a conjecture of Rais [12].

The existence of such an isomorphism \( \Phi \) is of importance for the harmonic analysis on \( G \), and for the study of the solvability of biinvariant operators on \( G \) (see [7]). It reflects and explains the "orbit method" ([8, 9]), i.e. the correspondence between orbits of \( G \) in \( V^* \), the dual vector space of \( V \), and unitary irreducible representations of \( G \): let \( T \) be an irreducible representation of \( G \), then the infinitesimal character of \( T \) is a character of the ring \( Z(g) \). Let \( \mathcal{O} \) be an orbit in \( V^* \), the map \( \rho_\mathcal{O}(f) = P(f) \, (f \in \mathcal{O}) \) is a character of the ring \( I(g) \) (\( I(g) \) being identified with the ring of invariant polynomials on \( V^* \)). The principle of the orbit method is to assign to a (good) orbit \( \mathcal{O} \) a representation \( T_\mathcal{O} \) of \( G \) (or \( g \)), whose infinitesimal character corresponds to \( \rho_\mathcal{O} \) via the isomorphism \( \Phi \). This is the technique used by M. Duflo to construct the ring isomorphism \( \Phi \).

Furthermore let \( t_\mathcal{O} \) be (when defined) the distribution on \( V \) which is the Fourier transform of the canonical measure on the orbit \( \mathcal{O} \), then \( t_\mathcal{O} \) is clearly an invariant positive eigendistribution of every operator \( P \) in \( I(g) \) of eigenvalue \( \rho_\mathcal{O}(P) \). Kirillov conjectured that the global character of the representation \( T_\mathcal{O} \) (when

* Supported in part by N.S.F. Contract number MCS76-02762
defined) should be intimately connected with the “orbit distribution” \( \Phi^{-1}(t_\mathfrak{g}) \), as proven in numerous cases. It is an essential result of Duflot [4] that these “orbit distributions” are indeed eigenfunctions for every bi-invariant operator \( P \) in \( \mathbb{Z}(\mathfrak{g}) \); as in Rais [11], this implies the local solvability of \( P \) [4].

We will here derived the existence of \( \Phi \) from a property of the Campbell-Hausdorff formula, that we conjecture and can prove in the solvable case. It is then a natural corollary of our conjecture, that bi-invariant operators are locally solvable and that “orbit distributions” are eigendistributions for \( \mathbb{Z}(\mathfrak{g}) \). Hence the correspondence between orbits and representations is already engraved in the structure of the multiplication law.

Let us describe with some details our technique and results: We denote by \( \mathfrak{g}_r \) the Lie algebra whose underlying vector space is \( V \) itself and in which the bracket \([*,*]_t \) is given by \([X,Y]_t=t[X,Y] \). Then \( \mathfrak{g}_r \) gives a deformation between \( \mathfrak{g} \) and the abelian Lie algebra, in which the fact is trivial.

In the course of the proof we encounter the following problem: Let \( \mathbf{L} \) be the free Lie algebra generated by two indeterminates \( x \) and \( y \) and \( \hat{\mathbf{L}} \) its completion. Since \( x+y-\log e^x e^y \) belongs to \([\mathbf{L},\mathbf{L}]\), by Campbell-Hausdorff formula, we can write it in \( x+y-\log e^x e^y = (1-e^{-ax})F+(e^{ad_x}-1)G \) for \( F \) and \( G \) in \( \hat{\mathbf{L}} \). \( F \) and \( G \) are not uniquely determined by this property.

**Conjecture.** For any Lie algebra \( \mathfrak{g} \) of finite dimension, we can find \( F \) and \( G \) such that they satisfy

a) \( x+y-\log e^x e^y = (1-e^{-ad_x})F+(e^{ad_x}-1)G \).

b) \( F \) and \( G \) give \( \mathfrak{g} \)-valued convergent power series on \((x,y)\in \mathfrak{g}\times\mathfrak{g} \).

c) \( \text{tr}((\text{ad}x)(\partial_x F);\mathfrak{g})+\text{tr}((\text{ad}y)(\partial_y G);\mathfrak{g}) = \frac{1}{2} \text{tr} \left( \frac{\text{ad}x}{e^{ad_x}-1} + \frac{\text{ad}y}{e^{ad_y}-1} - \frac{\text{ad}z}{e^{ad_z}-1}; \mathfrak{g} \right) \).

Here \( z = \log e^x e^y \) and \( \partial_x F \) (resp. \( \partial_y G \)) is the \textbf{End}(\mathfrak{g})-valued real analytic function defined by

\[
\mathfrak{g} \ni a \mapsto \frac{d}{dt} F(x+ta,y)|_{t=0}
\]

and \( \text{tr} \) denotes the trace of an endomorphism of \( \mathfrak{g} \).

When \( \mathfrak{g} \) is nilpotent, this conjecture is easily verified because \((\text{ad}x)(\partial_x F),\)
\(1-(\text{ad}x/(e^{ad_x}-1)) \) etc. are nilpotent endomorphisms of \( \mathfrak{g} \) so that their traces vanish. However, we get the following fact.

**Proposition 0.** If \( \mathfrak{g} \) is solvable, then Conjecture is true.

Let \( K \) be a non-empty closed cone in \( \mathfrak{g} \). Let \( \mathcal{S}(K) \) (resp. \( \hat{\mathcal{S}}(K) \)) be the vector space of the germs at the unit element \( e \in \mathbf{G} \) (resp. the orignon \( 0 \in \mathfrak{g} \)) of the functions (i.e. either distributions, or hyperfunctions or micro-functions) \( u(\mathfrak{g}) \) (resp. \( \bar{u}(\mathfrak{g}) \)) such that \( \text{supp} u \subset \text{exp} K \) (resp. \( \text{supp} \bar{u} \subset K \)) infinitesimally (see §2) and that
\( u(\mathfrak{g}h^{-1}) = |\det(\text{Ad}(g);\mathfrak{g})|^{-1} u(h) \) (resp. \( \bar{u}(\text{Ad}(g)x) = |\det(\text{Ad}(g);\mathfrak{g})|^{-1} \bar{u}(x) \)). We shall set \( j(x) = \det((1-e^{-ax})/\text{ad}x;\mathfrak{g}) \) for \( x \in \mathfrak{g} \) sufficiently near the origin. We
define the isomorphism \( \Phi : \mathcal{I}(K) \to \mathcal{J}(K) \) by \( (\Phi u)(x) = j(x)^{1/2} u(e^x) \) for \( u \in \mathcal{I}(K) \). If two closed cones \( K_1 \) and \( K_2 \) satisfy \( K_1 \cap (-K_2) = \{0\} \), then we can define the product \( \mathcal{I}(K_1) \times \mathcal{I}(K_2) \to \mathcal{I}(K_1 + K_2) \) (resp. \( \mathcal{J}(K_1) \times \mathcal{J}(K_2) \to \mathcal{J}(K_1 + K_2) \)) by the convolution \( * \), i.e.

\[
(u * v)(g) = \int_{\mathcal{G}} u(h) v(h^{-1} g) \, dh \quad \text{and} \quad (\tilde{u} * \tilde{v})(x) = \int_{\mathcal{G}} \tilde{u}(y) \tilde{v}(-y + x) \, dy.
\]

The exact statement which we shall prove is the following:

**Theorem.** If Conjecture is true for the group \( G \), then we have

\[
(\Phi u) * (\Phi v) = \Phi (u * v)
\]

for \( u \in \mathcal{I}(K_1) \) and \( v \in \mathcal{I}(K_2) \).

If we apply this theorem when \( v \) is supported at the origin, then we obtain the following corollary:

**Corollary 0.** Suppose that Conjecture is true for \( G \), then with any bi-invariant differential operator \( P \) on \( G \) we can associate a constant coefficient differential operator \( \tilde{P} \) on \( g \) so that \( \tilde{P} \Phi(u) = \Phi(Pu) \) holds for any \( u \in \mathcal{I}(g) \).

In paragraph 4, we will prove directly this particular case of our theorem. In fact, applying the same technique, we can prove a more precise result, giving a partial answer to a conjecture of Dixmier.

Let \( \gamma(P) = \beta(D(j^{1/2})P) \) the Duflo isomorphism from \( I(g) \) to \( Z(g) \), where \( \beta \) is the symmetrization map and \( D(j^{1/2}) \) the “differential” operator (of infinite order) defined by \( j^{1/2} \), let us look at the operator \( \gamma(P) \) as a bi-invariant differential operator on \( G \); we denote by \( (\exp)^\psi(\gamma(P)) \) the differential operator on \( g \) with analytic coefficients, which is the inverse image of \( \gamma(P) \) by the exponential mapping. Let \( D \) be the ring of the germs at 0 of differential operators with analytic coefficients. We consider the left ideal \( \mathcal{L} \) of \( D \) generated by the elements \( \langle[A, x], \partial_x \rangle + \text{tr(ad} A; g) \), \( A \in g \) (here \( \langle[A, x], \partial_x \rangle \) is the adjoint vector field given by \( \frac{d}{d\epsilon} \varphi(\exp \epsilon A - x)|_{\epsilon = 0} \). Every invariant distribution on \( g \) is annihilated by \( \mathcal{L} \).

So Corollary 0 is implied by:

**Corollary 1.** Suppose that Conjecture is true for \( G \), then

\[
(\exp)^\psi(\gamma(P)) - j(x)^{-\frac{1}{2}} P j(x)^{\frac{1}{2}} \in \mathcal{L}.
\]

Since Conjecture is solved in the solvable case the above theorem and its corollaries are true for a solvable group \( G \). Recall that the result stated in Corollary 0 holds for \( g \) semi-simple as proved by Harish-Chandra [6]. Howe [16] says that he proved Theorem for a nilpotent group \( G \) and a restricted class of functions \( u, v \).

Acknowledgement. We wish to thank Weita Chang, Dixmier, Duflo, Rais, and acknowledge that their questions and their work have stimulated our work.
§1

For The Theory of Microfunctions, we Refer to [1, 10, 15]. Let $G$ be a Lie group, $\mathfrak{g}$ its Lie algebra and $\exp: \mathfrak{g} \to G$ the exponential map. Let $M$ be a real analytic manifold on which $G$ acts real analytically. A hyperfunction $u(x)$ on $M$ is called a relative invariant with respect to a character $\chi$ of $G$ if $u(gx) = \chi(g)u(x)$ holds on $G \times M$. Here $u(gx)$ is the pull-back of $u$ by the map $r: G \times M \to M$ defined by $(g, x) \mapsto gx$, and $\chi(g)u(x)$ is the product of a real analytic function $\chi(g)$ on $G \times M$ and the pull-back of $u$ by the projection from $G \times M$ onto $M$. More generally, let $A$ be a subset of $M$, $G_A = \{g \in G; gA = A\}$. A hyperfunction $u(x)$ defined in a neighborhood $U$ of $A$ is called relative invariant locally at $A$ if there is a neighborhood $W$ of $G_A \times A$ such that $r(W) \subset U$ and that $u(gx) = \chi(g)u(x)$ on $W$.

For any $X \in \mathfrak{g}$, we denote by $D_X$ the vector field defined by $(D_X u)(x) = \frac{d}{dt} u(\exp(-tX)x)|_{t=0}$, and by $\delta \chi$ the derivative of $\chi$ (i.e. $\delta \chi(X) = \frac{d}{dt} \chi(\exp tX)|_{t=0}$).

Lemma 1.1. If $u$ is a relative invariant locally on $A$ hyperfunction then $(D_X + \delta \chi(X))u = 0$ in a neighborhood of $A$ for any $X \in \mathfrak{g}$.

Proof. We define the map $\phi: \mathbb{R} \times M \to G \times M$ by $(t, x) \mapsto (\exp(-tX), x)$. Then the pull-back of $u(gx)$ is the pull-back $(r \phi)^* u$ of $u$ by the map $r \circ \phi$, and the pull-back of $\chi(g)u(x)$ is $\chi(e^{-tX})u(x)$. Since $r \circ \phi$ has maximal rank, this is justified. Thus $(r \circ \phi)^* u = \chi(e^{-tX})u(x)$. If we differentiate the both-sides with respect to $t$, and restrict them at the variety $t = 0$ in $\mathbb{R} \times M$, we obtain $D_X u$ from the left hand side and $-\delta \chi(X)u$ from the right hand side. \(\text{Q.E.D.}\)

§2

Let $G$ be a Lie group, $\mathfrak{g}$ its Lie algebra and $\exp: \mathfrak{g} \to G$ the exponential map. We denote by $d\mathfrak{g}$ the left invariant Haar measure and by $d\mathcal{X}$ the Euclidean measure on $\mathfrak{g}$. After the normalization, $d\mathfrak{g}$ and $d\mathcal{X}$ are related under the exponential map by the formula: $d(e^\mathfrak{g}) = j(x)dx$ where $j(x) = \det((1 - e^{-ad\mathcal{X}})/ad\mathcal{X}; \mathfrak{g})$ in a neighborhood of $x = 0$, because the derivative of $\exp x$ at $x$ is given by $(1 - e^{-ad\mathcal{X}})/ad\mathcal{X}$ when we identify $TG$ with $G \times G$ by the left translation. We define the character $\chi_0(g)$ of $G$ by $|\det(\text{Ad}(g); \mathfrak{g})|$, we denote by $d\chi_0$ the corresponding character of $\mathfrak{g}$, i.e. $d\chi_0(x) = \text{tr}(\text{ad}x; \mathfrak{g})$.

Let $A$ and $B$ be subsets of a $C^1$-manifold $M$, $x$ a point in $M$. Take a local coordinate system $(x_1, \ldots, x_i)$ of $M$. The set of limits of the sequence $a_n(y_n - z_n)$ where $a_n \to 0$, $y_n \in A$, $z_n \in B$ and $y_n$, $z_n$ converge to $x$ when $n \to \infty$, is denoted by $C_x(A; B)$ regarded as a closed subset of the tangent space $T_xM$ of $M$ at $x$. $C_x(A; \{x\})$ is simply denoted by $C_x(A)$. If $f$ is a differential map from $M$ to a $C^1$ manifold $N$, then we have $(df)_x(C_x(A; B)) \subset C_{f(x)}(fA; fB)$. If $C_x(A; B) \subset \text{Ker } df(x) \subset \{0\}$, then there is a neighborhood $U$ of $x$ such that

$$(df)_x C_x(A; B) = C_{f(x)}(f(A \cap U); f(B \cap U)).$$
If $C_\epsilon(A,B) = \{0\}$, then $x$ is an isolated point of $A$ and $B$. $C_\epsilon(A,B) = \emptyset$ if and only if $A \cap B \neq x$.

Let $K$ be a closed cone in $g$. We shall denote by $\mathcal{J}(K)$ (resp. $\mathcal{J}^+(K)$) the space of the germs of function $u(g)$ (resp. $\tilde{u}(x)$) on $G$ (resp. on $g$) at $e \in G$ (resp. $0 \in g$) satisfying

\begin{equation}
C_\epsilon(supp u) \subset K \subset g = T_e G \quad (\text{resp. } C_\epsilon(supp \tilde{u}) \subset K \subset g = T_0 g)
\end{equation}

and

\begin{equation}
(2.2) \quad u \text{ is a relative invariant locally at } e \text{ with respect to the character } \chi_0(g)^{-1}.
\end{equation}

Let $K_1$ and $K_2$ be two closed cones in $g$ such that $K_1 \cap (-K_2) = \{0\}$. If $u \in \mathcal{J}(K_1)$ and $v \in \mathcal{J}(K_2)$, then $(supp u) \cap (supp v)^{-1}$ is contained in $\{e\}$ locally. Suppose that $u$ and $v$ are defined on a neighborhood $U_0$ of $e$. For any open neighborhood $W \subset U_0$ of $e$, we can find neighborhoods $W$ and $V$ of $e$ such that $W \subset U$, $W^{-1} \subset U$, $\{h \in W; h \in supp u, h^{-1} \in supp v\} \subset \{e\}$ and that the map $(g, h) \rightarrow g$ from $(g, h) \in V \times W; h^{-1} g \in supp v, h \in supp u$ to $V$ is a proper map. Hence we can define $(u \ast v)(g)$ by

$$\int_W u(h) v(h^{-1} g) \, dh \quad \text{on } g \in V.$$ 

This gives the bilinear homomorphism $\mathcal{J}(K_1) \times \mathcal{J}(K_2) \rightarrow \mathcal{J}(K_1 + K_2)$ because $C_\epsilon((supp u) \cdot (supp v)) \subset K_1 + K_2$. In the same way, we can define the convolution

$$\tilde{(u \ast v)}(x) = \int_S \tilde{u}(y) \tilde{v}(-y + x) \, dy$$

which gives the homomorphism $\mathcal{J}^+(K_1) \ast \mathcal{J}^+(K_2) \rightarrow \mathcal{J}^+(K_1 + K_2)$.

Note that if $u$ belongs to $\mathcal{J}(g)$, then we have $\chi_0(g) u(g) = u(g)$. In fact, if $u$ is restricted to the identity $u(g_1 g_2^{-1}) = \chi_0(g_1)^{-1} u(g_2)$ on the submanifold $\{g_1, g_2 \in G; g_2 = g_1^{-1}\}$, then we obtain the above identity. Hence we have $u(g) = \chi_0(g)^2 u(g)$ for any $g \in G$. We shall define the isomorphism $\Phi: \mathcal{J}(K) \rightarrow \mathcal{J}^+(K)$ by $\Phi(u)(x) = j(x)^{1/2} u(e^x)$. The above remark shows us $\Phi(u)(x) = \chi_0(e^x)^{1/2} j(x)^{1/2} u(e^x)$ for any $\lambda$.

For any $\tilde{u}(x)$ in $\tilde{\mathcal{J}}(g)$, we have $d \chi_0(x) \tilde{u}(x) = 0$. In fact, by Lemma 1.1, we have $\langle A, x \rangle, \partial_x \rangle \tilde{u}(x) = -d \chi_0(A) \tilde{u}(x)$ for any $A \in g$. Here, for any $g$-valued real analytic function $E(x)$ on $g$, $\langle E(x), \partial_x \rangle$ is the vector field defined by $\langle E(x), \partial_x \rangle u(x) = \frac{d}{dt} u(x + t E(x))|_{t = 0}$. Thus, we have the identity $\langle A, x \rangle, \partial_x \rangle \tilde{u}(x) = -d \chi_0(A) \tilde{u}(x)$ on $(x, A) \in g \times g$. If we restrict this on the submanifold $A = x$, we obtain $d \chi_0(x) \tilde{u}(x) = 0$. These observations also show the following:

Let us denote by $G_0$ the kernel of $\chi_0$ and $q_0$ its Lie algebra. Then, $G_0$ is a unimodular group. For any $u \in \mathcal{J}(g)$, we can find an absolute invariant $v$ on $G_0$ such that $u = v \delta(\chi_0)$. Similarly, for any $\tilde{u} \in \tilde{\mathcal{J}}(g)$, we can find an absolute invariant $\tilde{v}$ on $q_0$ such that $\tilde{u} = \tilde{v} \delta(\chi_0)$. Thus we can reduce the study of $\mathcal{J}(g)$ and $\tilde{\mathcal{J}}(g)$ into the case where the group is unimodular, although we will not employ this fact.
§ 3. We Shall Prove Theorem

Take two closed cones \( K_1 \) and \( K_2 \) of \( g \) such that \( K_1 \cap (-K_2) = \{0\} \) and two functions \( u \) in \( \mathcal{F}(K_1) \) and \( v \) in \( \mathcal{F}(K_2) \). Set \( w(g) = \int u(h) v(h^{-1} g) dh \), and \( \tilde{w} = \Phi u, \tilde{v} = \Phi v \).

In order to prove Theorem we shall compute \( \tilde{w} \).

\[
\tilde{w}(z) = j(z)^{\frac{1}{2}} \int_{g} u(e^{\tau}) v(h^{-1} e^{\tau}) dh
\]

\[
= j(z)^{\frac{1}{2}} \int_{g} u(e^{\tau}) v(e^{-\tau} e^{\tau}) j(x) dx
\]

\[
= j(z)^{\frac{1}{2}} \int_{g} dx \int_{g} dy u(e^{\tau}) v(e^{\tau}) j(x) \delta(y - \log e^{-\tau} e^{\tau}).
\]

**Lemma 3.1.** \( \delta(y - \log e^{-\tau} e^{\tau}) = j(y) j(z)^{-1} \delta(z - \log e^{\tau} e^{\tau}). \)

**Proof.** We have \( \delta(y - f(z)) = |Jf|^{-1} \delta(z - f^{-1}(y)) \) where \( Jf \) is the Jacobian of \( f \). Setting \( f(z) = \log e^{-x} e^{y} \), we shall apply this. We have, for \( a \in g \)

\[
f(z + ea) = \log e^{-x} e^{y + ea}
\]

which equals \( \log e^{-x} e^{y} \exp(e(1 - e^{-ad z})/ad z) \) modulo \( \varepsilon^2 \). As we can set \( y = \log e^{-x} e^{y} \), this is equal to

\[
\log e^{y} \exp(e(1 - e^{-ad z})/ad z) a = y + \varepsilon \frac{ad y}{1 - e^{-ad z}} \frac{1 - e^{-ad z}}{ad z} a \text{ modulo } \varepsilon^2.
\]

Thus we obtain \( Jf = \det \frac{ad y}{1 - e^{-ad z}} \frac{1 - e^{-ad z}}{ad z} \), which implies the desired result. Q.E.D.

By this lemma, we have

\[
(\tilde{u} * \tilde{v})(z) = \int \tilde{u}(x) \tilde{v}(y) \delta(z - x - y) dx dy.
\]

We want to prove that this integral equals

\[
(\tilde{u} \ast \tilde{v})(z) = \int \tilde{u}(x) \tilde{v}(y) \delta(z - x - y) dx dy.
\]

Given a vector space \( V \) and two functions \( \tilde{u} \) and \( \tilde{v} \) on \( V \), given a structure \( \mu \) of Lie algebra on \( V \), we want to prove for the Lie algebra \( g = (V, \mu) \) the equality:

\[
\int \left( \frac{j(x) j(y)}{j(z)} \right)^{\frac{1}{2}} \tilde{u}(x) \tilde{v}(y) \delta(z - \log e^{x} e^{y}) dx dy = \int \tilde{u}(x) \tilde{v}(y) \delta(z - x - y) dx dy.
\]

If we consider the Lie algebra \( g_t = (V, t \mu) \) i.e. \([x, y]_t = t[x, y] \), the first member of the equality becomes

\[
\varphi_t(z) = \int \left( \frac{j(tx) j(ty)}{j(tz)} \right)^{\frac{1}{2}} \tilde{u}(x) \tilde{v}(y) \delta \left( z - \frac{1}{t} \log e^{tx} e^{ty} \right) dx dy,
\]

(3.2)
and this must be equal to the second member which is the value of \( \varphi_t \) for \( t=0 \). Therefore it is enough to show that \( \varphi_t \) does not depend on \( t \), or equivalently \( \frac{\partial}{\partial t} \varphi_t = 0 \). Let us calculate this derivative.

**Lemma 3.2.** Let \( F(x, y) \) and \( G(x, y) \) be two \( \mathfrak{g} \)-valued real analytic functions on \( (x, y) \in \mathfrak{g} \times \mathfrak{g} \) defined in a neighborhood of the origin. Suppose that \( F(0, 0) = G(0, 0) = 0 \) and that

\[
\begin{align*}
 x + y - \log e^x e^y &= (1 - e^{-\text{ad}x}) F(x, y) + (e^{\text{ad}y} - 1) G(x, y).
\end{align*}
\]

Then, we have

\[
(3.3) \quad \frac{\partial}{\partial t} \frac{1}{t} \log e^x e^y = \left( \left\langle \left[ \frac{1}{t} F(t x, t y), \partial_x \right] \right\rangle + \left\langle \left[ y, \frac{1}{t} G(t x, t y), \partial_y \right] \right\rangle \right) \frac{1}{t} \log e^x e^y.
\]

Here \( \left\langle A(x), \partial_x \right\rangle \) is the derivation defined by

\[
(\left\langle A(x), \partial_x \right\rangle u)(x) = \frac{d}{d \varepsilon} u(x + \varepsilon A(x)) \bigg|_{\varepsilon = 0}.
\]

**Proof.** Set \( F_t = t^{-1} F(t x, t y) \) and \( G_t = t^{-1} G(t x, t y) \). Then, the right hand side of (3.3) is the value of

\[
t^{-1} \frac{d}{d \varepsilon} \log \exp(t x + \varepsilon [t x, F_t]) \exp(t y + \varepsilon [t y, G_t])
\]

at \( \varepsilon = 0 \). We shall calculate

\[
A = \exp(t x + \varepsilon [t x, F_t]) \exp(t y + \varepsilon [t y, G_t])
\]

modulo \( \varepsilon^2 \). We have

\[
\exp(t x + \varepsilon [t x, F_t]) = e^{tx} \exp \frac{1 - e^{-\text{ad}tx}}{\text{ad}(tx)} [t x, F_t]
\]

\[
= e^{tx} \exp(1 - e^{-\text{ad}tx}) F_t \quad \text{modulo } \varepsilon^2,
\]

and similarly \( \exp(t y + \varepsilon [t y, G_t]) = \exp(e^{\text{ad}ty} - 1) G_t \exp t y \) modulo \( \varepsilon^2 \). Thus, we have

\[
A = e^{tx} \exp(1 - e^{-\text{ad}tx}) F_t + (e^{\text{ad}ty} - 1) G_t e^{ty}
\]

\[
= e^{(t + \varepsilon)x} \exp \left( x + y - \frac{1}{t} \log e^{ty} e^{tx} \right) e^{ty}
\]

\[
= e^{(t + \varepsilon)x} e^{ty} \exp \left( y - \frac{1}{t} \log e^{tx} e^{ty} \right)
\]

\[
= e^{(t + \varepsilon)x} e^{(t + \varepsilon)y} \exp - \varepsilon \left( \frac{1}{t} \log e^{tx} e^{ty} \right) \quad \text{modulo } \varepsilon^2.
\]
We have therefore

\[
\log A = \log e^{(t + \varepsilon)x} e^{(t + \varepsilon)y} - \frac{\varepsilon}{t} \log e^{tx} e^{ty} \\
= \frac{t}{t + \varepsilon} \log e^{(t + \varepsilon)x} e^{(t + \varepsilon)y}.
\]

This implies Lemma 3.2. Q.E.D.

This lemma shows in particular

\[
(3.4) \quad \frac{\partial}{\partial t} \delta \left( z - \frac{1}{t} \log e^{tx} e^{ty} \right) = \left( \langle [x, F_i], \partial_x \rangle + \langle [y, G_i], \partial_y \rangle \right) \delta \left( z - \frac{1}{t} \log e^{tx} e^{ty} \right).
\]

Therefore, integrating by parts, we have the equality

\[
(3.5) \quad p_1 = \int \left( \frac{j(t) j(t y)}{j(t z)} \right)^{\frac{1}{3}} u(x) \bar{v}(y) \frac{\partial}{\partial t} \delta \left( z - \frac{1}{t} \log e^{tx} e^{ty} \right) \, dx \, dy
\]

\[
= - \int \left\{ \left( \langle [x, F_i], \partial_x \rangle + \langle [y, G_i], \partial_y \rangle + \text{div}_x [x, F_i] + \text{div}_{y} [y, G_i] \right) \left( \frac{j(t) j(t y)}{j(t z)} \right)^{\frac{1}{3}} u(x) \bar{v}(y) \right\} \delta \left( z - \frac{1}{t} \log e^{tx} e^{ty} \right) \, dx \, dy.
\]

Here \( \text{div}_x \) (resp. \( \text{div}_y \)) signifies the divergent with respect to the variable \( x \) (resp. \( y \)), i.e. the function \( \text{div}_x E(x) \) is the sum of the vector field \( \langle E(x), \partial_x \rangle \) and its formal adjoint.

If a function \( \varphi(x) \) satisfies \( \varphi(\text{Ad}(g)x) = \chi(g) \varphi(x) \) with a character \( \chi(g) \), then we have

\[
\left\langle [A, x], \partial_x \right\rangle \varphi = (\delta \chi)(A) \varphi(x) \quad \text{for} \quad A \in g.
\]

Here, \( \delta \chi \) is the derivative of \( \chi \). Hence, if \( \varphi \) is an absolute invariant, \( \varphi \) and \( \left\langle [A, x], \partial_x \right\rangle \) commute. Since \( (j(x) j(y) j(z))^\frac{1}{3} \) is an absolute invariant

\[
\left\langle [x, F_i], \partial_x \right\rangle + \left\langle [y, G_i], \partial_y \right\rangle + \text{div}_x [x, F_i] + \text{div}_y [y, G_i]
\]

commutes with this function. Since \( \tilde{u}(x) \) is a relative invariant with respect to the character \( |\det(\text{Ad}(g); g)^{-1}| \), we have

\[
\left\langle [A, x], \partial_x \right\rangle \tilde{u}(x) = - \text{tr}(\text{ad} A) \tilde{u}(x).
\]

Thus, we obtain

\[
(3.6) \quad p_1 = - \int (\text{tr}(\text{ad}(F_i + G_i), g) + \text{div}_x [x, F_i] + \text{div}_y [y, G_i]) (j(t x) j(t y) j(t z))^\frac{1}{3} \tilde{u}(x) \bar{v}(y) \delta \left( z - \frac{1}{t} \log e^{tx} e^{ty} \right) \, dx \, dy.
\]

Lemma 3.3. \( \frac{\partial}{\partial t} \log j(t x) = \text{tr} \left( \frac{\text{ad} x}{e^{\text{ad} x} - 1} - \frac{1}{t} \right) \).
Proof.

\[
\frac{\partial}{\partial t} \log \det \frac{1-e^{-\text{ad}t x}}{\text{ad}(t x)} = \text{tr} \frac{\text{ad} t x}{1-e^{-\text{ad}t x}} \frac{\partial}{\partial t} \frac{1-e^{-\text{ad}t x}}{\text{ad} t x} \\
= \text{tr} \left( \frac{\text{ad} x}{e^{\text{ad}x} - 1} \right).
\]

By this lemma we have

\[
\frac{\partial}{\partial t} \left( \frac{j(t x) j(t y)}{j(t z)} \right) = \frac{1}{2} \text{tr} \left( \frac{\text{ad} x}{e^{\text{ad}x} - 1} + \frac{\text{ad} y}{e^{\text{ad}y} - 1} - \frac{\text{ad} z}{e^{\text{ad}z} - 1} + 1 \right) \left( \frac{j(t x) j(t y)}{j(t z)} \right).
\]

We obtain finally

\[
\frac{\partial}{\partial t} \phi_i = - \left\{ \text{div}_x [x, F_i] + \text{div}_y [y, G_i] + \text{tr} \text{ad}(F_i + G_i) \right. \\
- \frac{1}{2} \text{tr} \left( \frac{\text{ad} x}{e^{\text{ad}x} - 1} + \frac{\text{ad} y}{e^{\text{ad}y} - 1} - \frac{\text{ad} z}{e^{\text{ad}z} - 1} + 1 \right) \\
- \left. \left( \frac{j(t x) j(t y)}{j(t z)} \right)^{1/2} \right\} \delta \left( z - \frac{1}{t} \log e^{x'} e^{y'} \right) \, dx \, dy.
\]

In order to see that \( \frac{\partial}{\partial t} \phi_i \) vanishes, it is enough to show

\[
(3.7) \quad \text{div}_x [x, F_i] + \text{div}_y [y, G_i] + \text{tr} \text{ad}(F_i + G_i) \\
- \frac{1}{2} \text{tr} \left( \frac{\text{ad} x}{e^{\text{ad}x} - 1} + \frac{\text{ad} y}{e^{\text{ad}y} - 1} - \frac{\text{ad} z}{e^{\text{ad}z} - 1} + 1 \right) = 0
\]

when \( z = \frac{1}{t} \log e^{x'} e^{y'} \). Since the left hand side of this formula is homogeneous of degree 1 when we assign degree -1 to \( t \) and degree 1 to \( x \) and \( y \), it is enough to show (3.7) when \( t = 1 \).

For a \( g \)-valued function \( A(x) \), let us denote by \( \partial_x A \) the endomorphism of \( g \) defined by \( g \ni a \mapsto \frac{d}{dt} A(x + t a) \big|_{t=0} \). Then \( \text{div}_x A(x) = \text{tr} \partial_x A(x) \).

Since \( \partial_x [x, A(x)] = (\text{ad} x) \partial_x A - \text{ad} A \), the formula (3.7) is equivalent to

\[
(3.8) \quad \text{tr}(\text{ad} x)(\partial_x F) + \text{tr}(\text{ad} y)(\partial_y G) = \frac{1}{2} \text{tr} \left( \frac{\text{ad} x}{e^{\text{ad}x} - 1} + \frac{\text{ad} y}{e^{\text{ad}y} - 1} - \frac{\text{ad} z}{e^{\text{ad}z} - 1} \right) + 1)
\]

with \( z = \log e^{x'} e^{y'} \). This completes the proof of Theorem.

§4. Bi-invariant Differential Operators

We consider the algebra \( I(g) \) of the \( G \)-invariant elements of \( S(g) \). We identify \( S(g) \) with the algebra of constant coefficient differential operators on \( g \), hence \( I(g) \) is identified with the ring of constant coefficient differential operators on \( g \).
invariant by the action of $G$. We consider the universal enveloping algebra $U(g)$ of $g$ and its center $Z(g)$. We identify $U(g)$ with the algebra of the left invariant differential operators, hence $Z(g)$ will be identified with the ring of biinvariant differential operators on $G$.

We denote by $\delta$ the Dirac function on $G$ supported at the unit $e$, then $u * \delta = \delta * u = u$. On the other hand, we have $P(u * v) = u * P \delta$ for $P \in U(g)$. This shows that $P u = u * P \delta$. We shall denote by the same letter $\delta$ the Dirac function on $g$ supported at the origin. Similarly if $P \in S(g)$, $P u = u * P \delta = P \delta * u$. We shall denote by $(\exp)^*(\exp)_*$ the pull-back of functions or differential operators on $G$ to those on $g$ (resp. the inverse of $(\exp)^*$), by the exponential map.

We shall denote by $\beta$ the linear mapping from $S(g)$ onto $U(g)$ obtained by symmetrization. We have $(\beta(P) \varphi)(x) = (P \bar{\varphi})(0)$ with $\bar{\varphi}(x) = \varphi(e^x)$, hence $(\beta(P) \varphi)(e^x) = (P \delta)(x)$.

For a real analytic function $f(x)$ on $g$ defined on a neighborhood of the origin, and $P \in S(g)$, we define

$$D(f) P \in S(g) \quad \text{by} \quad ((D(f) P) \delta)(x) = f(-x) P \delta(x),$$

or

$$((D(f) P) \varphi)(0) = P(x \mapsto f(x) \varphi(x))(0).$$

We shall denote by $\gamma$ the map from $I(g)$ onto $Z(g)$ defined by $P \mapsto \beta(D(j^\sharp) P)$. Duflo [4] has proved that for any Lie algebra $g$, $\gamma$ is an isomorphism of the rings $I(g)$ and $Z(g)$.

We have seen that for any $P \in I(g)$,

$$\chi_0(e^x) (P \delta)(x) = P \delta(x),$$

and hence $\chi_0(e^x)$ and $P$ commute. In fact,

$$\chi_0(e^{x-y}) (P \delta)(x-y) = (P \delta)(x-y)$$

and this implies

$$\chi_0(e^x) (P \delta)(x-y) = \chi_0(e^x) (P \delta)(x-y).$$

Let us denote by $g_0 = \{ A \in g; \text{tr ad } A = 0 \}$, this implies that $P \in S(g_0)$ (see also [3, 13]). In particular, we have $j(x)^\sharp (P \delta) x) = j(-x)^\sharp (P \delta, x)$, as $j(x) = (\det e^{-ad x}) j(-x)$. So we have $\Phi(\gamma(P) \delta) = P \delta$. If we take $v = \gamma(P) \delta$ then we can get from Theorem the following proposition.

**Proposition 4.1.** If Conjecture is true for $g$, then for every $\tilde{u} \in \tilde{I}(g)$ and $P \in I(g)$

$$(\exp)^* \gamma(P) \tilde{u} = (j(x)^{-\frac{1}{2}} P j(x)^{\sharp}) \tilde{u}.$$

(In particular $\gamma$ is an isomorphism of the ring $I(g)$ and $Z(g)$.)

However, we can get a more precise result applying the same method as in the preceding paragraphs. Let us denote by $D$ the ring of the germs of the differential operators at the origin.
Proposition 4.2. Suppose that Conjecture is true for \( \mathfrak{g} \), then for any \( P \in \mathfrak{I}(\mathfrak{g}) \)

\[ j(x)^{-\frac{1}{2}}(\exp^* \gamma(P)) j(x)^{-\frac{1}{2}} - P \]

is contained in the left ideal of \( \mathcal{D} \) generated by the \( \langle [A, x], \partial_x \rangle + \text{tr ad} A \)'s \( (A \in \mathfrak{g}) \).

(As we have \( \langle [A, x], \partial_x \rangle + \text{tr ad} A \) \( \tilde{u}(x) = 0 \) for every \( \tilde{u} \in \mathmathcal{F}(\mathfrak{g}) \), this implies Proposition 4.1.)

Proof. Remark that for \( P \in \mathcal{S}(\mathfrak{g}) \), \( \exp^* (\beta(P)) \) is the differential operator defined by

\[ ((\exp^* \beta(P)) u) (x) = P_y (u (\log e^x e^y)) \big|_{y = 0}, \]

where \( P_y \) means that \( P \) operates on the \( y \) variable. Hence

\[ Q = j(x)^{-\frac{1}{2}}(\exp^* \gamma(P)) j(x)^{-\frac{1}{2}} \]

is the operator:

\[ (Q u) (x) = P_y \left( \frac{j(x)^{\frac{1}{2}} j(y)^{\frac{1}{2}}}{j(\log e^x e^y)^{\frac{1}{2}}} u (\log e^x e^y) \right) \bigg|_{y = 0}. \]

As before we introduce the Lie algebra \( \mathfrak{g}_t \) and the corresponding operator \( Q_t \), then

\[ (Q_t u) (x) = P_y \left( \frac{j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}}}{j(\log e^{t x} e^{t y})^{\frac{1}{2}}} u \left( \frac{1}{t} \log e^{t x} e^{t y} \right) \right) \bigg|_{y = 0}. \]

Let us remark that if we define the left ideal \( \mathcal{L}_t \) of \( \mathcal{D} \) generated by the element \( \langle [x, A], \partial_x \rangle + \text{tr ad} A \; ; \; \mathfrak{g}_t \rangle \) then for \( t \neq 0 \) \( \mathcal{L}_t = \mathcal{L} \). Hence we have to prove that: \( Q_t - P \in \mathcal{L} \). As \( Q_0 = P \), it is sufficient to prove that \( \frac{\partial}{\partial t} Q_t \in \mathcal{L} \), where

\[ \left( \frac{\partial}{\partial t} Q_t \right) u (x) = \frac{\partial}{\partial t} (Q_t u) (x) \]

\[ = P_y \left( \frac{\partial}{\partial t} j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}}}{j(\log e^{t x} e^{t y})^{\frac{1}{2}}} u \left( \frac{1}{t} \log e^{t x} e^{t y} \right) \right) \bigg|_{y = 0}. \]

Let \( F \) and \( G \) be as in Lemma 3.2,

\[ F_t (x, y) = \frac{F(t x, t y)}{t}, \quad G_t (x, y) = \frac{G(t x, t y)}{t}, \]

and

\[ d(x, y, t) = \frac{1}{2} \text{tr} \left( \frac{\text{ad} x}{e^{t \text{ad} x} - 1} + \frac{\text{ad} y}{e^{t \text{ad} y} - 1} - \frac{\text{ad} z}{e^{t \text{ad} z} - 1} - \frac{1}{t} \right) \]

\[ - \text{tr} \left( (\text{ad} x) \partial_x F_t + (\text{ad} y) \partial_y G_t \right) \]

where \( z = \frac{\log e^x e^y}{t} \). Then we prove:
\begin{align*}
(4.1) \quad & \frac{\partial}{\partial t} \left( j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx} e^{ty})^{-\frac{1}{2}} u \left( \frac{1}{t} \log e^{tx} e^{ty} \right) \right) \\
& = d(x, y, t) j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx} e^{ty})^{-\frac{1}{2}} u \left( \frac{1}{t} \log e^{tx} e^{ty} \right) \\
& \quad + \sum_{i=1}^{n} \alpha_i(x, y, t) \left( \left\langle [e_i, z], \partial_z \right\rangle + \text{tr ad} e_i \cdot u \right) \left( \frac{1}{t} \log e^{tx} e^{ty} \right) \\
& \quad + \sum_{i=1}^{n} \left\langle [y, e_j], \partial_y \right\rangle \cdot \beta_i(x, y, t) u \left( \frac{1}{t} \log e^{tx} e^{ty} \right).
\end{align*}

Here, \( e_i \) (\( i = 1, 2, \ldots, n \)) is a basis of the Lie algebra \( \mathfrak{g} \), \( \left\langle [e_i, z], \partial_z \right\rangle \) denotes the adjoint field corresponding to \( e_i \), and \( \alpha_i(x, y, t) \) and \( \beta_i(x, y, t) \) are analytic functions defined near the origin.

To prove (4.1), we compute as in Lemma 3.2

\[
\frac{1}{t+\varepsilon} \log e^{(t+\varepsilon)x} e^{(t+\varepsilon)y} \quad \text{modulo } \varepsilon^2
\]

\[
= \frac{1}{t} \log e^{tx} e^{tx} e^{ty} e^{ty} e^{-\frac{\log e^{tx} e^{ty}}{t}}
\]

\[
= \frac{1}{t} \log e^{tx} e^{ty} e^t \left( e^{-\text{tr ad}_y(x+y-\frac{\log e^{tx} e^{ty}}{t})} \right)
\]

\[
= \frac{1}{t} \log e^{tx} e^{ty} e^t \left( e^{-\text{tr ad}_y((1-e^{-\text{tr ad}_x})F_t+(e^{-\text{tr ad}_y}-1)G_t)} \right).
\]

We write

\[
e^{-\text{tr ad}_y((1-e^{-\text{tr ad}_x})F_t+(e^{-\text{tr ad}_y}-1)G_t)}
\]

\[
= (1-e^{-\text{ad}(\text{tr ad}_x e^{ty})})F_t + (1-e^{-\text{tr ad}_y})(G_t-F_t).
\]

So we have

\[
\frac{d}{dt} \left( \frac{1}{t} \log e^{tx} e^{ty} \right) = \left\langle [z, F_t], \partial_z \right\rangle \left( \frac{1}{t} \log e^{tx} e^{ty} \right) + \left\langle [y, G_t-F_t], \partial_y \right\rangle \left( \frac{1}{t} \log e^{tx} e^{ty} \right)
\]

(if \( F_t(x, y) = \sum f_i(x, y, t) e_i \), and \( I(x) = x \))

\[
\left\langle [F_t, z], \partial_z \right\rangle \left( \frac{1}{t} \log e^{tx} e^{ty} \right) = \sum f_i(x, y) (\left\langle [e_i, z], \partial_z \right\rangle \cdot I) \left( \frac{1}{t} \log e^{tx} e^{ty} \right).
\]

We write

\[
\left. \frac{\partial}{\partial t} j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx} e^{ty})^{-\frac{1}{2}} u \left( \frac{1}{t} \log e^{tx} e^{ty} \right) \right|_{t=t_0}
\]

\[
= \left. \frac{\partial}{\partial t} j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j \left( \frac{\log e^{tx} e^{ty}}{t} \right)^{\frac{1}{2}} u \left( \frac{1}{t} \log e^{tx} e^{ty} \right) \right|_{t=t_0}
\]
\[ \frac{1}{2} \text{tr} \left( \frac{\text{ad} x}{e^{t \text{ad} x} - 1} + \frac{\text{ad} y}{e^{t \text{ad} y} - 1} - \frac{\text{ad} z}{e^{t \text{ad} z} - 1} - \frac{1}{t} \right) \\
\cdot j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}} j(t z)^{\frac{1}{2}} \left( \log e^{t x} e^{t y} \right)^{-\frac{1}{2}} u \left( \frac{1}{t} \log e^{t x} e^{t y} \right) \\
+ j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}} j(t z)^{\frac{1}{2}} \left( \frac{\partial}{\partial t} \left( t \log e^{t x} e^{t y} \right) \right)^{-\frac{1}{2}} u \left( \frac{1}{t} \log e^{t x} e^{t y} \right) \Bigg|_{t = t_0} \]

by Lemma 3.3.

Now if \((G_t - F_t)(x, y) = \sum \lambda_i(x, y, t) e_i\) we have

\[ \langle [y, G_t - F_t], \partial_y \rangle = \sum_{i=1}^n \langle [y, e_i], \partial_y \rangle \lambda_i(x, y, t) - \text{tr} \text{ad} y \partial_y (G_t - F_t) \]

As \(j\) is an absolute invariant, \(j\) commutes with the adjoint fields.

Hence from the preceding calculation, we obtain that the left hand side of (4.1) is equal to

\[ \left( \frac{1}{2} \text{tr} \left( \frac{\text{ad} x}{e^{t \text{ad} x} - 1} + \frac{\text{ad} y}{e^{t \text{ad} y} - 1} - \frac{\text{ad} z}{e^{t \text{ad} z} - 1} - \frac{1}{t} \right) \right) \text{tr} \text{ad} y \partial_y (G_t - F_t) \\
\cdot j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}} j(t z)^{\frac{1}{2}} \left( \log e^{t x} e^{t y} \right)^{-\frac{1}{2}} u \left( \frac{1}{t} \log e^{t x} e^{t y} \right) \\
+ j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}} j(t z)^{\frac{1}{2}} \left( \frac{\partial}{\partial t} \left( t \log e^{t x} e^{t y} \right) \right)^{-\frac{1}{2}} u \left( \frac{1}{t} \log e^{t x} e^{t y} \right) \\
+ \sum_{i=1}^n \langle [y, e_i], \partial_y \rangle \cdot \left( \beta_i(x, y, t) u \left( \frac{1}{t} \log(e^{t x} e^{t y}) \right) \right) \].

But, we have

\[ \frac{1}{2} \text{tr} \left( \frac{\text{ad} x}{e^{t \text{ad} x} - 1} + \frac{\text{ad} y}{e^{t \text{ad} y} - 1} - \frac{\text{ad} z}{e^{t \text{ad} z} - 1} - \frac{1}{t} \right) \text{tr} \text{ad} y \partial_y (G_t - F_t) \\
= d(x, y, t) + \text{tr(} \text{ad} y \partial_y F_t + \text{ad} x \partial_t F_t). \]

Let us remark here that if \(E\) is in \(\mathfrak{L}\), we have \(g \cdot E(x, y) = E(g x, g y)\) for every \(g \in G\). The operator \((\partial_x E) \text{ad} x + (\partial_y E) \text{ad} y\) is the linear operator

\[ c \mapsto \frac{d}{dc} E(x + c[x, c], y + c[y, c])|_{c=0}^{-} \]

\[ = \frac{d}{dc} E(\exp c \cdot x, \exp c \cdot y)|_{c=0}^{-} \]

\[ = \frac{d}{dc} \exp c \cdot E(x, y)|_{c=0}^{-} \]

\[ = - [E(x, y), c] \]

hence is the operator \(-\text{ad} E\).
We then obtain that the left side of (4.1) is equal to
\[
\begin{align*}
&d(x, y, t) j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}} j(\log e^{e^x} e^{e^y})^{-\frac{1}{2}} u \left( \frac{1}{t} \log e^{e^x} e^{e^y} \right) \\
&\quad + j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}} j(\log e^{e^x} e^{e^y})^{-\frac{1}{2}} \left( \langle [F_r, z], \partial_z \rangle + \text{tr \, ad} F_{\cdot} \cdot u \right) \left( \frac{1}{t} \log e^{e^x} e^{e^y} \right) \\
&\quad + \sum_{i=1}^n \langle [y, e_i], \partial_y \rangle \beta_i(x, y, t) u \left( \frac{1}{t} \log (e^{e^x} e^{e^y}) \right),
\end{align*}
\]
which is of the required form.

Now if our conjecture is true for \( \mathfrak{g} \), then we can find \( F \) and \( G \) such that \( d(x, y, t) = 0 \). Now we remark that if \( P \in \mathfrak{I}(g) \),
\[
P_y \langle [y, e_i], \partial_y \rangle = \langle [y, e_i], \partial_y \rangle P_y
\]
hence \( (P_y \langle [y, e_i], \partial_y \rangle \psi(y))|_{y=0} = 0 \). Let \( R_i(t) \) denote the differential operator
\[
(R_i(t) \varphi)(x) = P_y \left( \alpha_i(x, y, t) \varphi \left( \frac{1}{t} \log e^{e^x} e^{e^y} \right) \right)|_{y=0}.
\]
We obtain from (4.1)
\[
\frac{\partial}{\partial t} Q_e = \sum_{i=1}^n R_i(t) \langle [e_i, x], \partial_x \rangle + \text{tr \, ad} e_i,
\]
i.e. \( \frac{\partial}{\partial t} Q_e \in \mathcal{L} \). Q.E.D.

**Remark.** The same proof shows the corresponding fact for bi-invariant integral operators.

**Remark.** We will see in the next section that our conjecture is true for \( G \) solvable; we can easily deduce from Proposition 4.1, the fact that every bi-invariant operator on \( G \) is locally solvable, which was already obtained by Rouvière [14] and Duflo-Rais [5]. In fact \( P \) being invariant by the action of \( G \) we can find a fundamental solution for \( P \) which is invariant by \( G \). It follows that \( (\exp)^* \gamma(P) \) has a local fundamental solution. If \( G \) is exponential solvable, the maps \( F \) and \( G \) can be constructed in the whole space \( \mathfrak{g} \) hence the Propositions 4.1 and 4.2 hold on the whole space \( \mathfrak{g} \). So \( \exp^*(j(P)) \) has a fundamental solution on the space \( G \), (Weita Chang [2] has proven recently that every bi-invariant operator on an simply connected solvable group is globally solvable). We recall that M. Duflo has shown that every bi-invariant differential operator on a Lie group \( G \) is locally solvable [4].

§ 5. Proof of Proposition 0

First we shall translate our conjecture into another form. Let us write for an \( A \in \overline{\mathbf{L}} \)
\[
2(x + y - \log e^x e^y) = (x + y - \log e^x e^y) + A + (x + y - \log e^x e^y) - A.
\]
Hence we will consider $A \in \hat{L}$ such that $(x + y - \log e^x e^y) + A$ is divisible by $x$ (i.e. in $[x, \hat{L}]$) and $(x + y - \log e^x e^y) - A$ is divisible by $y$ (i.e. in $[y, \hat{L}]$). As $x + y - \log e^x e^y \equiv \frac{x}{2} [x, y]$ mod $[[L, L], L]$ and $[x, y]$ is divisible by $x$ and $y$, we may take $A$ in $[[L, L], L]$. We will write $x + y - \log e^x e^y + A = [x, P]$, $A - (x + y - \log e^x e^y) = [y, Q]$, choose $F = \frac{1}{2} \frac{x}{1 - e^{-ad x}} P$, $G = -\frac{1}{2} \frac{y}{e^{ad y} - 1} Q$ and translate our conjecture in terms of $A$.

We shall first give two preliminary lemmata.

**Lemma 5.1.**

i) $\partial_x \log e^x e^y = \frac{\text{ad } z}{e^{\text{ad } z} - 1} \frac{e^{\text{ad } x} - 1}{\text{ad } x}$

and

ii) $\partial_y \log e^x e^y = \frac{\text{ad } z}{1 - e^{-\text{ad } y}} \frac{1 - e^{-\text{ad } y}}{\text{ad } y}$.

*Here* $z = \log e^x e^y$.

**Proof.** We have, modulo $e^2$,

$$
\log e^{(x + \varepsilon a)} e^y = \log e^{\frac{\text{ad } x}{\text{ad } x - 1} a} e^y = \log e^{\frac{\text{ad } x}{\text{ad } x - 1} a} e^z = z + \varepsilon \frac{\text{ad } z}{\text{ad } x - 1} a.
$$

The formula ii) is shown in the same way. Q.E.D.

**Lemma 5.2.** Let $a \in g$, $f(\lambda)$ and $g(\lambda)$ two power series on $\lambda$. Then

$$
\text{tr} \left( f(ad x) \partial_x (g(ad x) a) \right) = \text{tr} \left( f(ad x) \frac{g(0) - g(ad x)}{ad x} \right). 
$$

**Proof.** By linearity, we may assume $g(\lambda) = \lambda^n$. If $n = 0$, the lemma is evident. Suppose $n \geq 1$. Then we have

$$
ad (x + \varepsilon c) a - (ad x)^n a = \varepsilon \sum_{k = 0}^{n-1} (ad x)^{n-1-k} (ad c) (ad x)^k a
$$

$$
= -\varepsilon \sum_{k = 0}^{n-1} (ad x)^{n-1-k} \text{ad} ((ad x)^k a) c \mod e^2.
$$

Thus we have

$$
\partial_x (g(ad x) a) = -\sum_{k = 0}^{n-1} (ad x)^{n-1-k} \text{ad} ((ad x)^k a).
$$

If $k > 0$, $\text{tr} f(ad x)(ad x)^{n-1-k} \text{ad} ((ad x)^k a)$ vanishes. In fact, if we set $b = (ad x)^{k-1} a$ and $\varphi(\lambda) = \lambda^{n-1-k} f(\lambda)$, then

$$
\text{tr} \varphi(ad x) \text{ad} ((ad x) b) = \text{tr} \varphi(ad x)(ad x ad b - ad b ad x) = 0.
$$

Therefore, we obtain

$$
\text{tr} f(ad x) \partial_x g(ad x) a = -\text{tr} f(ad x)(ad x)^{n-1} (ad a). \quad \text{Q.E.D.}
$$
Proposition 5.3. Conjecture is implied from the following: For any Lie algebra \( g \), we can find \( A \) in \([[[\hat{L}, \hat{L}], \hat{L}], \hat{L}]\) satisfying the conditions i), ii) and iii):

i) There is \( P \) in \( \hat{L} \) such that \( A + x + y - \log e^x e^y = [x, P] \) and that \( P \) gives a convergent power series on \( (x, y) \in g \times g \).

ii) There is \( Q \) in \( \hat{L} \) such that \( A - (x + y - \log e^x) = [y, Q] \) and that \( Q \) gives a convergent power series on \( (x, y) \in g \times g \).

iii) \( \frac{\text{tr}}{1 - e^{-\text{ad} x}} \frac{\text{ad} x}{\text{ad} y} A - \frac{\text{tr}}{e^{\text{ad} y} - 1} \frac{\text{ad} y}{\text{ad} z} A = \text{tr} \left( \frac{\text{ad} z}{e^{\text{ad} z} - 1} - 1 + \frac{1}{2} \frac{\text{ad} z}{\text{ad} y} \right) \),

where \( z = \log e^x e^y \).

Proof. We have \( x + y - \log e^x e^y = \frac{1}{2} [x, P] - \frac{1}{2} [y, Q] \). Let \( F = \frac{1}{2} \frac{\text{ad} x}{1 - e^{-\text{ad} x}} P \) and \( G = -\frac{1}{2} \frac{\text{ad} y}{e^{\text{ad} y} - 1} Q \). Then we have

\[
x + y - \log e^x e^y = (1 - e^{-\text{ad} x}) F + (e^{\text{ad} y} - 1) G.
\]

We have \([x, P] = 2(1 - e^{-\text{ad} x}) F\). Therefore, by Lemma 5.2, we have

\[
\frac{\text{tr}}{1 - e^{-\text{ad} x}} \frac{\text{ad} x}{\text{ad} x} [x, P] = 2 \frac{\text{tr}}{1 - e^{-\text{ad} x}} \frac{0 - (1 - e^{-\text{ad} x})}{\text{ad} x} \text{ad} F + 2 \frac{\text{tr}}{\text{ad} x} \frac{\text{ad} x}{\text{ad} x} \text{ad} F
\]

\[
= 2 \frac{\text{tr}}{\text{ad} x} \frac{\text{ad} x}{\text{ad} x} \text{ad} F - 2 \frac{\text{tr}}{\text{ad} F} F.
\]

Similarly, we have \( -\frac{\text{tr}}{e^{\text{ad} y} - 1} \frac{\text{ad} y}{\text{ad} y} [y, Q] = 2 \frac{\text{tr}}{\text{ad} y} \frac{\text{ad} y}{\text{ad} y} \text{ad} G - 2 \frac{\text{tr}}{\text{ad} G} G \). Set \( \tilde{z} = \log e^x e^y \), we have, by Lemma 5.1

\[
\frac{\text{ad} \tilde{z}}{1 - e^{-\text{ad} \tilde{z}}} + \frac{1 - e^{-\text{ad} \tilde{z}}}{\text{ad} x} \text{ad} A = \frac{\text{ad} \tilde{z}}{1 - e^{-\text{ad} \tilde{z}}} + \frac{1 - e^{-\text{ad} \tilde{z}}}{\text{ad} x} \text{ad} A.
\]

Hence, we obtain

\[
\frac{\text{tr}}{1 - e^{-\text{ad} x}} \frac{\text{ad} x}{\text{ad} x} [x, P] = \frac{\text{tr}}{1 - e^{-\text{ad} x}} \frac{\text{ad} x}{\text{ad} x} \text{ad} A + \text{tr} \left( \frac{\text{ad} \tilde{z}}{1 - e^{-\text{ad} \tilde{z}}} + \frac{1 - e^{-\text{ad} \tilde{z}}}{\text{ad} x} \text{ad} A \right)
\]

\[
= \frac{\text{tr}}{1 - e^{-\text{ad} x}} \frac{\text{ad} x}{\text{ad} x} \text{ad} A + \text{tr} \left( \frac{\text{ad} \tilde{z}}{1 - e^{-\text{ad} \tilde{z}}} + \frac{1 - e^{-\text{ad} \tilde{z}}}{\text{ad} x} \text{ad} A \right).
\]

In the same way, we have

\[
-\frac{\text{tr}}{e^{\text{ad} y} - 1} \frac{\text{ad} y}{\text{ad} y} [y, Q] = -\frac{\text{tr}}{e^{\text{ad} y} - 1} \frac{\text{ad} y}{\text{ad} y} \text{ad} A + \text{tr} \left( \frac{\text{ad} z}{e^{\text{ad} z} - 1} + \frac{1 - e^{-\text{ad} z}}{e^{\text{ad} y} - 1} \text{ad} z \right).
\]

Thus, we obtained

\[
\text{tr} (\text{ad} x)(\text{ad} F) + \text{tr} (\text{ad} y)(\text{ad} G)
\]

\[
= \text{tr} (\text{ad} F) + \text{tr} (\text{ad} G) + \frac{1}{2} \text{tr} \left( \frac{\text{ad} x}{1 - e^{-\text{ad} x}} \frac{\text{ad} x}{\text{ad} y} \text{ad} A - \frac{\text{ad} y}{e^{\text{ad} y} - 1} \text{ad} z \right)
\]

\[
+ \frac{1}{2} \text{tr} \left( \frac{\text{ad} x}{1 - e^{-\text{ad} x}} + \frac{\text{ad} y}{e^{\text{ad} y} - 1} - \frac{\text{ad} z}{e^{\text{ad} z} - 1} \text{ad} z \right).
\]
\[ \text{tr}(\text{ad } F) + \text{tr}(\text{ad } G) + \frac{1}{2} \text{tr} \left( \frac{\text{ad } x}{1 - e^{-\text{ad } x}} + \frac{\text{ad } y}{e^{\text{ad } y} - 1} - \frac{\text{ad } z}{1 - e^{-\text{ad } z}} - 1 + \frac{1}{2} \text{ad } z \right). \]

Since \( \frac{1}{1 - e^{-\lambda}} = \frac{1}{e^{\lambda} - 1} + \lambda \) and \( \text{tr} \text{ad } z = \text{tr}(\text{ad } x + \text{ad } y) \), this equals

\[ \text{tr}(\text{ad } F) + \text{tr}(\text{ad } G) + \frac{1}{4} \text{tr}(\text{ad } x - \text{ad } y) + \frac{1}{2} \text{tr} \left( \frac{\text{ad } x}{e^{\text{ad } x} - 1} + \frac{\text{ad } y}{e^{\text{ad } y} - 1} - \frac{\text{ad } z}{e^{\text{ad } z} - 1} - 1 \right). \]

Hence, it is enough to show that

(5.1) \( \text{tr}(\text{ad } F) + \text{tr}(\text{ad } G) = \frac{1}{4} \text{tr}(\text{ad } y - \text{ad } x). \)

However, adding a constant multiple of \( x \) (resp. \( y \)) to \( P \) (resp. \( Q \)), we may assume that \( P \) (resp. \( Q \)) is equal to \( \alpha x \) (resp. \( \beta x \)) modulo \( [\hat{L}, \hat{L}] \). However, if \( x \) and \( y \) are not \( \text{ad}^n e^x \equiv -\frac{1}{2} [x, y] \) modulo \( [[\hat{L}, \hat{L}], \hat{L}] \) and hence \( P \equiv \frac{1}{2} y \) (resp. \( Q \equiv \frac{1}{2} x \)). Thus, we have \( F \equiv \frac{1}{4} y \) (resp. \( G = -\frac{1}{4} x \)) modulo \( [\hat{L}, \hat{L}] \). Since \( \text{tr} \text{ad} [\hat{L}, \hat{L}] = 0 \), (5.1) is satisfied.

Q.E.D.

Let \( A \) satisfy i), ii), iii), of the Proposition 4.3. We may remark that \( A'(x, y) = \frac{1}{4}(A(x, y) - A(y, x) - A(-x, -y) - A(-y, -x)) \) satisfies also 1), 2), and 3). This follows from the following observations:

a) if \( m(x, y) = x + y - \log e^x e^y \), then \( m(x, y) = -m(-y, -x) \);

\[
m(x, y) - m(y, x) = \log e^x e^y - \log e^y e^x = (e^{\text{ad } x} - 1) \log e^x e^y = (1 - e^{\text{ad } y}) \log e^x e^y
\]

hence is divisible by \( x \) and \( y \).

b) if \( t(x, y) = \text{tr} \left( \frac{\text{ad } z}{e^{\text{ad } z} - 1} - 1 + \frac{\text{ad } z}{2} \right) \) then \( t(x, y) = t(y, x) = t(-x, -y) \).

c) for any \( E \in [\hat{L}, \hat{L}] \),

\[
\text{tr} \frac{\text{ad } x}{1 - e^{-\text{ad } x}} \partial_x E - \text{tr} \frac{\text{ad } y}{e^{\text{ad } y} - 1} \partial_y E = \text{tr} \frac{\text{ad } x}{e^{\text{ad } x} - 1} \partial_x E - \text{tr} \frac{\text{ad } y}{1 - e^{-\text{ad } y}} \partial_y E.
\]

In fact the difference is

\[
\text{tr}(\text{ad } x \partial_x E + \text{ad } y \partial_y E) = \text{tr}(\partial_x E \text{ad } x + \partial_y E \text{ad } y)
\]

\[= - \text{tr} \text{ad } E(x, y) \quad \text{(see 4.2)} \]

\[= 0 \quad \text{as } E \in [\hat{L}, \hat{L}]. \]

We will now construct \( A \) in \( [[[\hat{L}, \hat{L}], \hat{L}]] \) such that

\[ A(x, y) = -A(y, x) = -A(-x, -y) \]

and i) \( x + y - \log e^x e^y + A(x, y) = [x, P] \) and \( P \) gives a convergent power series on \( (x, y) \in \mathfrak{g} \times \mathfrak{g} \). (iii) follows then. If \( \mathfrak{g} \) is solvable we will be able to prove that \( A \) satisfies also the condition iii).
We consider now the condition i):

$$x + y - \log e^y e^x + A(x, y) = [x, P(x, y)].$$

Then for every $t$, we will have

$$tx + ty - \log e^y e^x + A(tx, ty) = t[x, P(tx, ty)].$$

Hence $\frac{\partial}{\partial t} (tx + ty - \log e^y e^x) + \frac{\partial}{\partial t} A(tx, ty) \in [x, \hat{L}]$ and $\frac{\partial}{\partial t} A(tx, ty)$ satisfies the same antisymmetry relation as $A$.

Let $\theta$ be the vector field $\langle x, \partial_x \rangle + \langle y, \partial_y \rangle$ (or the derivation of $\hat{L}$ defined by $\theta|L_n = n \text{id} L_n$ where $L_n$ is the space of elements of $L$ of degree $n$) then

$$t \frac{\partial}{\partial t} B(tx, ty) |_{t=1} = \theta B,$$

for $B \in \hat{L}$. We compute

$$\theta(x + y - \log e^y e^x) = x + y - \frac{\text{ad} \hat{z}}{e^\text{ad} \hat{z} - 1} \cdot y - \frac{\text{ad} \hat{z}}{1 - e^{-\text{ad} \hat{z}}} \cdot x$$

with $\hat{z} = \log e^y e^x$ and we will write $\theta(x + y - \log e^y e^x)$ as an antisymmetric element mod $[x, \hat{L}]$.

For any real analytic function $g(\lambda)$, we have $g(\text{ad} z) = e^{\text{ad} x} g(\text{ad} \hat{z}) e^{-\text{ad} x}$, in particular $g(\text{ad} \hat{z}) \cdot x \equiv g(\text{ad} z) \cdot x$ modulo $[x, \hat{L}]$ and

$$g(\text{ad} z) \cdot y \equiv g(\text{ad} \hat{z}) e^{-\text{ad} x} y \equiv g(\text{ad} \hat{z}) e^{-\text{ad} \hat{z}} \cdot y \mod [x, \hat{L}].$$

Hence we write modulo $[x, \hat{L}]$

$$\theta(x + y - \log e^y e^x) = \left(1 - \frac{\text{ad} \hat{z}}{1 - e^{-\text{ad} \hat{z}}} \right) \cdot x + y - \frac{\text{ad} \hat{z}}{1 - e^{-\text{ad} \hat{z}}} e^{-\text{ad} \hat{z}} \cdot y$$

$$\equiv f(\text{ad} \hat{z}) \cdot x + f(\text{ad} z) \cdot y,$$

where $f(\lambda) = \left(1 - \frac{\lambda}{1 - e^{-\lambda}} \right)$

$$\equiv f(\text{ad} \hat{z}) \cdot x - f(\text{ad} z) \cdot y + 2 f(\text{ad} z) \cdot y.$$

We write, as $f(0) = 0$,

$$f(\text{ad} z) \cdot y = \frac{f(\text{ad} z)}{e^{\text{ad} z} - 1} (e^{\text{ad} z} - 1) y$$

$$= \frac{f(\text{ad} z)}{e^{\text{ad} z} - 1} (e^{\text{ad} x} - 1) y,$$

therefore $f(\text{ad} z) \cdot y \equiv \left(\frac{f(\text{ad} \hat{z})}{e^{\text{ad} \hat{z}} - 1} - \frac{f(\text{ad} z)}{e^{\text{ad} z} - 1} \right) \cdot y$.

As $\left(\frac{f(\text{ad} \hat{z})}{e^{\text{ad} \hat{z}} - 1} - \frac{f(\text{ad} z)}{e^{\text{ad} z} - 1} \right) \cdot x \equiv 0$ we obtain that

$$\theta(x + y - \log e^y e^x) \equiv f(\text{ad} z) \cdot x - f(\text{ad} z) \cdot y + 2 \left(\frac{f(\text{ad} \hat{z})}{e^{\text{ad} \hat{z}} - 1} - \frac{f(\text{ad} z)}{e^{\text{ad} z} - 1} \right) \cdot (x + y).$$
Let us denote by \( \alpha(x, y) \) the second member of this equality. We have obviously \( \alpha(x, y) = -\alpha(y, x) \), hence if we define \( \beta(x, y) = \frac{1}{2}(\alpha(x, y) + \alpha(-y, -x)) \), \( \beta \) will satisfy the relation \( \beta(x, y) = -\beta(y, x) = -\beta(-x, -y) \) and \( \theta(x + y - \log e^xe^y) \equiv \beta(x, y) \mod [x, \hat{L}] \). We remark that the function \( h(\lambda) = \left(1 - \frac{\lambda}{1 - e^{-\lambda}}\right) \frac{1}{e^\lambda - 1} \) verifies \( h(\lambda) = -h(-\lambda) - 1 \) as \( \frac{1}{1 - e^{-\lambda}} = \frac{1}{e^\lambda - 1} + 1 \), hence

\[
\beta(x, y) = 2 \left( \frac{f(ad\hat{z})}{e^{ad\hat{z}} - 1} - \frac{f(adz)}{e^{adz} - 1} \right) \cdot (x + y) + \frac{1}{2} \left( f(ad\hat{z}) + f(-adz) \right) \cdot x
\]

\[-\frac{1}{2} \left( f(adz) + f(-ad\hat{z}) \right) \cdot y.
\]

We can therefore define \( A(x, y) \) by the differential equation:

\[
(5.2) \quad \theta A = 2 \left( 1 - \frac{adz}{1 - e^{-adz}} \right) \frac{1}{e^{adz} - 1} (x + y) - 2 \left( 1 - \frac{ad\hat{z}}{1 - e^{-ad\hat{z}}} \right) \frac{1}{e^{ad\hat{z}} - 1} \cdot (x + y)
\]

\[+ \frac{1}{2} \left( \frac{adz}{e^{adz} - 1} - 1 \right) \cdot x + \frac{1}{2} \left( \frac{ad\hat{z}}{1 - e^{-ad\hat{z}}} - 1 \right) \cdot x - \frac{1}{2} \left( \frac{ad\hat{z}}{e^{ad\hat{z}} - 1} - 1 \right) \cdot y
\]

\[-\frac{1}{2} \left( \frac{adz}{1 - e^{-adz}} - 1 \right) \cdot y,
\]

with the initial condition \( A(0, 0) = 0 \) (\( \hat{z} = \log e^xe^y \), \( z = \log e^xe^y \)). As the second member is a convergent power series at the origin, so is \( A(x, y) \).

The preceding calculation implies now 1) and 2) of the:

Lemma 5.4.

1) \( A(x, y) = -A(y, x) = -A(-x, -y) \),

2) \( x + y - \log e^xe^y + A \in [x, \hat{L}] \),

3) \( A \in [L, [\hat{L}, \hat{L}]]. \)

For 3) we remark that \( A \in [L, \hat{L}] \), and the properties \( A(x, y) = -A(-x, -y) \) implies that \( A \in [L, [\hat{L}, \hat{L}]] \). The lemma is proven. Q.E.D.

Let \( g \) be a power series of the two non commutative variables \( x \) and \( y \), i.e. \( g \) is in the completion of the tensor algebra \( \hat{T}(x, y) \) of the vector space \( \mathbb{C}x + \mathbb{C}y \). We denote by \( c(g) \) the image of \( g \) under the map \( \hat{T}(x, y) \to \hat{S}(x, y) = C[[[x, y]]], \) i.e. \( c(g) \) is a power series in the commutative variables \( x \) and \( y \).

Lemma 5.5. If \( g \) is solvable, \( \text{tr}(g(adx, ady)) \) depends only on \( c(g) \).

Proof. There is a basis of \( g^F \) where the operators \( adx, ady \) are lower triangular, then \( \text{ad} \cdot [x, y] = adx \cdot ady - ady \cdot adx \) have zeros on the diagonal, and the lemma follows.

Let us write \( A = p(adx, ady) \cdot [x, y] \), where \( p \) is a convergent power series in the non commutative variables \( x \) and \( y \).

Lemma 5.6. Let \( g \) be solvable, then

\[
\text{tr} \left( \frac{adx}{1 - e^{-adx}} \cdot \partial_x A - \frac{ady}{e^{ady} - 1} \cdot \partial_y A \right)
\]
\[
= - \text{tr} \left( e^{\text{ad}_z} - 1 \right) \left( \frac{\text{ad}_x}{e^{\text{ad}_x} - 1} \right) \left( \frac{\text{ad}_y}{e^{\text{ad}_y} - 1} \right) p(\text{ad}_x, \text{ad}_y).
\]

**Proof.** Let us consider the endomorphism

\[g \ni c \mapsto \frac{d}{d\varepsilon} p(\text{ad}_x + \varepsilon \text{ad}_c, \text{ad}_y) \cdot [x, y] \big|_{\varepsilon = 0};\]

this is a sum of terms of the form

\[p_1(\text{ad}_x, \text{ad}_y) \text{ad}_c p_2(\text{ad}_x, \text{ad}_y) \cdot [x, y] \]

\[= - p_1(\text{ad}_x, \text{ad}_y) \text{ad}(p_2(\text{ad}_x, \text{ad}_y) \cdot [x, y]) \cdot c.
\]

The trace of the endomorphism \(\frac{\text{ad}_x}{1 - e^{-\text{ad}_z}} p_1(\text{ad}_x, \text{ad}_y) \text{ad}(p_2(\text{ad}_x, \text{ad}_y) \cdot [x, y])\) vanishes by the preceding lemma. So the only term appearing in \(\text{tr} \frac{\text{ad}_x}{1 - e^{-\text{ad}_z}} \partial_x A\) will come from the trace of the endomorphism

\[c \mapsto \frac{d}{d\varepsilon} \frac{\text{ad}_x}{1 - e^{-\text{ad}_z}} p(\text{ad}_x, \text{ad}_y)[x + \varepsilon c, y] \big|_{\varepsilon = 0}.
\]

We obtain that the left side of the equality is:

\[- \text{tr} \left( \frac{\text{ad}_x}{1 - e^{-\text{ad}_z}} \text{ad}_y + \frac{\text{ad}_y}{e^{\text{ad}_y} - 1} \text{ad}_x \right) p(\text{ad}_x, \text{ad}_y) \]

\[= - \text{tr} \left( \frac{\text{ad}_x}{e^{\text{ad}_y} - 1} \right) \left( \frac{\text{ad}_y}{e^{\text{ad}_y} - 1} \right) (e^{\text{ad}_z} - 1) p(\text{ad}_x, \text{ad}_y).
\]

If we restrict our attention when \(g\) is solvable, we have to prove:

\[- \text{tr} \left( \frac{\text{ad}_x}{e^{\text{ad}_y} - 1} \right) \left( \frac{\text{ad}_y}{e^{\text{ad}_y} - 1} \right) (e^{\text{ad}_z} - 1) p(\text{ad}_x, \text{ad}_y) = \text{tr} \left( \frac{\text{ad}_z}{e^{\text{ad}_z} - 1} - 1 + \frac{1}{2} \text{ad}_z \right).
\]

Hence, considering the commutative ring \(\mathbb{C}[[x, y]]\) we need only to prove:

\[c(p)(x, y) = \left( 1 - \frac{x + y}{2} \right) \left( 1 - \frac{x + y}{e^{x+y} - 1} \right) \frac{1}{x e^x - 1} \frac{1}{e^y - 1}.
\]

We denote by \(q(x, y)\) the right hand side.

Let us consider the homomorphism \(h: [\hat{\mathfrak{L}}, \hat{\mathfrak{L}}] \to [\hat{\mathfrak{L}}, \hat{\mathfrak{L}}] / [[\hat{\mathfrak{L}}, \hat{\mathfrak{L}}], [\hat{\mathfrak{L}}, \hat{\mathfrak{L}}]]\) and let us write for \(m \in [\hat{\mathfrak{L}}, \hat{\mathfrak{L}}], m = \varphi(\text{ad}_x, \text{ad}_y) \cdot [x, y]\) then clearly \(h(m)\) depends only on \(c(\varphi)\). Therefore, for \(f(x, y) \in \mathbb{C}[[x, y]]\), we shall write \(f(\text{ad}_x, \text{ad}_y) [x, y]\) for the element \(\varphi(\text{ad}_x, \text{ad}_y) [x, y]\) modulo \([[\hat{\mathfrak{L}}, \hat{\mathfrak{L}}], [\hat{\mathfrak{L}}, \hat{\mathfrak{L}}]]\) with \(f = c(\varphi)\).

**Remark 5.7.** If \(f(x, y) \in \mathbb{C}[[x, y]]\) is such that \(f(\text{ad}_x, \text{ad}_y) \cdot [x, y] = 0\) modulo \([[\hat{\mathfrak{L}}, \hat{\mathfrak{L}}], [\hat{\mathfrak{L}}, \hat{\mathfrak{L}}]]\), then \(f(x, y) = 0\). In fact if \(\varphi(\text{ad}_x, \text{ad}_y) \cdot [x, y] \in [[\hat{\mathfrak{L}}, \hat{\mathfrak{L}}], [\hat{\mathfrak{L}}, \hat{\mathfrak{L}}]]\), with \(f = c(\varphi)\) then \(\text{tr} (\partial_x \varphi(\text{ad}_x, \text{ad}_y) \cdot [x, y]; g) = 0\) for any solvable Lie algebra \(g\).
On the other hand the same calculation as in Lemma 5.6 shows that

\[
\text{tr}(\partial_4(\varphi(\ad x, \ad y) \cdot [x, y]); g) = - \text{tr}(\varphi(\ad x, \ad y) \ad y; g).
\]

Considering the 2 dimension Lie algebra \(g\) with basis \(H, A\) and relation \([H, A] = A\), we have for \(x = x_1 H + x_2 A, y = y_1 H + y_2 A\),

\[
\text{tr}(\varphi(\ad x, \ad y) \ad y; g) = f(x_1, y_1) y_1,
\]

hence \(f(x_1, y_1) y_1 = 0\), and so is \(f\).

Proposition 0 will result from the following lemma.

**Lemma 5.8.** Let

\[
\alpha = \left(1 - \frac{\ad z}{e^{\ad z} - 1}\right) \frac{1}{\ad z} \cdot (x + y - z) + \frac{1}{2} z - \left(1 - \frac{\ad z}{e^{\ad z} - 1}\right) \frac{1}{\ad z} \cdot (x + y - z) - \frac{1}{2} z
\]

then

1) \(h(x) = q(\ad x, \ad y) \cdot [x, y]\),
2) \(h(x) = h(A)\).

**Proof.** 1) We have as \((x + y - z) \in [\hat{L}, \hat{L}]\),

\[
\alpha \equiv \left(1 - \frac{\ad z}{e^{\ad z} - 1}\right) \frac{1}{\ad z} \cdot (x + y - z) + \frac{1}{2} z
\]

\[- \left(1 - \frac{\ad z}{e^{\ad z} - 1}\right) \frac{1}{\ad z} \cdot (x + y - z) - \frac{1}{2} z \mod [[\hat{L}, \hat{L}], [\hat{L}, \hat{L}]]
\]

\[
\equiv \left(1 - \frac{\ad z}{e^{\ad z} - 1}\right) \frac{1}{\ad z} \cdot (z - \bar{z}) - \frac{1}{2} (z - \bar{z})
\]

and 1) will result from the following formula:

\[
(z - \bar{z}) \equiv \frac{\ad z}{e^{\ad z} - 1} \cdot \frac{e^{\ad x} - 1}{\ad x} \cdot \frac{e^{\ad y} - 1}{\ad y} \cdot [x, y] \mod [[\hat{L}, \hat{L}], [\hat{L}, \hat{L}]].
\]

**Proof of (5.3).** Let

\[
\varphi_1(x, y) = (e^x - e^{-y})^{-1} \left(\frac{e^x - 1}{x} - \frac{e^{-y} - 1}{y}\right)
\]

\[
\varphi_2(x, y) = (e^y - e^{-x})^{-1} \left(\frac{1 - e^{-x}}{x} - \frac{e^y - 1}{y}\right).
\]

then \(\varphi_1\) and \(\varphi_2\) are analytic functions at the origin. We have

a) \((x + y - z) \equiv \varphi_1(\ad x, \ad y) \cdot [x, y]\),

b) \((x + y - z) \equiv \varphi_2(\ad x, \ad y) \cdot [x, y] \mod [[\hat{L}, \hat{L}], [\hat{L}, \hat{L}]].

For a) we consider

\[
e^{\ad x} - e^{-\ad y})(x + y - z) = (e^{\ad x} - e^{-\ad y})(x + y)
\]
\( (e^{ad_x} - e^{-ad_y})(\tilde{z}) = e^{-ad_y}(e^{ad_{\tilde{z}}} - 1) \cdot \tilde{z} = 0 \) so
\[
(e^{ad_x} - e^{-ad_y})(x + y - \tilde{z}) = (e^{ad_x} - 1) + (1 - e^{-ad_y}) \cdot (x + y)
\]
\[
= \left( \frac{e^{ad_x} - 1}{ad_x - 1} - \frac{1 - e^{-ad_y}}{ad_y} \right) \cdot [x, y]
\]
and we obtain the equality a) by Remark 5.7. Now
\[
z - \tilde{z} \equiv (\varphi_1 - \varphi_2)(ad_x, ad_y) \cdot [x, y]
\]
but
\[
\varphi_1 = (e^z - 1)^{-1} \left( \frac{e^z - e^y}{x} - \frac{e^y - 1}{y} \right), \quad \varphi_2 = (e^z - 1)^{-1} \left( \frac{e^z - 1}{x} - \frac{e^z - e^x}{y} \right),
\]
with \( z = x + y \), and
\[
\varphi_1 - \varphi_2 = (e^z - 1)^{-1} \left( \frac{(e^z - 1)(e^y - 1)}{x} + \frac{(e^x - 1)(e^y - 1)}{y} \right)
\]
\[
= (e^z - 1)^{-1} \frac{z}{xy} (e^x - 1)(e^y - 1)
\]
and this proves Formula (5.3).
Let us prove 2) in Lemma 5.8. We let
\[
\zeta(x, y) = \left( 1 - \frac{ad \tilde{z}}{e^{ad \tilde{z}} - 1} \right) \frac{1}{ad \tilde{z}} (x + y - \tilde{z}) + \frac{1}{2} \tilde{z},
\]
then \( x + y - \tilde{z} \in [\mathfrak{L}, \mathfrak{L}] \), we have
\[
\zeta(tx, ty) = \left( 1 - \frac{t ad \tilde{z}}{e^{t ad \tilde{z}} - 1} \right) \frac{1}{t ad \tilde{z}} (tx + ty - \tilde{z}(tx, ty)) + \frac{1}{2} \tilde{z}(tx, ty)
\]
\[
= \left( 1 - \frac{t ad \tilde{z}}{e^{t ad \tilde{z}} - 1} \right) \frac{1}{ad \tilde{z}} \left( x + y - \frac{1}{t} \tilde{z}(tx, ty) \right) + \frac{1}{2} \tilde{z}(tx, ty)
\]
modulo \([[[[\mathfrak{L}, \mathfrak{L}], [\mathfrak{L}, \mathfrak{L}]], [\mathfrak{L}, \mathfrak{L}]], \mathfrak{L}]]\).
Here \( \tilde{z} \) still denotes \( \log e^y e^x \) and \( \tilde{z}(tx, ty) = \log e^{ty} e^{tx} \). We have
\[
\frac{\partial}{\partial t} \left( \left( 1 - \frac{t z}{e^{t z} - 1} \right) \frac{1}{z - t - 1} \right) = \frac{1}{e^z - 1} \frac{z}{1 - e^{-z} - 1}.
\]
So
\[
(\theta \zeta)(x, y) = \left( \frac{ad \tilde{z}}{1 - e^{-ad \tilde{y}}} - 1 \right) \frac{1}{ad \tilde{z} - 1} \cdot (x + y - \tilde{z}) + \left( 1 - \frac{ad \tilde{z}}{e^{ad \tilde{z}} - 1} \right) \frac{1}{ad \tilde{z}} \tilde{z} - \theta \tilde{z} + \frac{1}{2} \theta \tilde{z}
\]
\[
= \left( \frac{ad \tilde{z}}{1 - e^{-ad \tilde{z}}} - 1 \right) \frac{1}{ad \tilde{z} - 1} \cdot (x + y) - \left( 1 - \frac{ad \tilde{z}}{e^{ad \tilde{z}} - 1} \right) \frac{1}{ad \tilde{z}} \cdot \theta \tilde{z} + \frac{1}{2} \theta \tilde{z}
\]
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as

\[
\left( \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} - 1 \right) \cdot \frac{1}{e^{\text{ad} \bar{z}} - 1} \cdot \bar{z} = \frac{1}{2} \bar{z} = \left( \frac{1 - \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1}}{1 - e^{-\text{ad} \bar{z}}} \right) \cdot \frac{1}{\text{ad} \bar{z}} \cdot \bar{z}.
\]

Recalling that

\[
\theta \bar{z} = \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} \cdot y + \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} \cdot x = \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} \cdot (x + y) + \text{ad} \bar{z} \cdot x,
\]

we obtain:

\[
\theta \bar{z}(x, y) = \left( \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} - 1 \right) \frac{1}{e^{\text{ad} \bar{z}} - 1} \cdot (x + y) + \left( \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} - 1 \right) \frac{1}{e^{\text{ad} \bar{z}} - 1} \cdot (x + y)
\]

\[
+ \left( \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} - 1 \right) \frac{1}{e^{\text{ad} \bar{z}} - 1} \cdot \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} \cdot x + \frac{1}{2} \cdot \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} \cdot y + \frac{1}{2} \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} \cdot x
\]

\[
\equiv 2 \left( \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} - 1 \right) \frac{1}{e^{\text{ad} \bar{z}} - 1} \cdot (x + y) - \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} \cdot (x + y)
\]

\[
+ \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} \cdot x - x + \frac{1}{2} \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} \cdot y + \frac{1}{2} \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} \cdot x
\]

\[
\equiv 2 \left( \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} - 1 \right) \frac{1}{e^{\text{ad} \bar{z}} - 1} \cdot (x + y) + \frac{1}{2} \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} \cdot x - x + \frac{1}{2} \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} \cdot y.
\]

After antisymmetrization, we obtain

\[
\theta \alpha(x, y) \equiv 2 \left( 1 - \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} \right) \frac{1}{e^{\text{ad} \bar{z}} - 1} \cdot (x + y) - 2 \left( 1 - \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} \right) \frac{1}{e^{\text{ad} \bar{z}} - 1} \cdot (x + y)
\]

\[
+ \left( \frac{1}{2} \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} + \frac{1}{2} \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} - 1 \right) \cdot x - \left( \frac{1}{2} \frac{\text{ad} \bar{z}}{e^{\text{ad} \bar{z}} - 1} + \frac{1}{2} \frac{\text{ad} \bar{z}}{1 - e^{-\text{ad} \bar{z}}} - 1 \right) \cdot y
\]

\[
\equiv \theta A. \quad \text{c.q.f.d.}
\]

References

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Received December 15, 1977/April 10, 1978