Inventiones math. 47, 249–272 (1978)

Inventiones mathematicae

The Campbell-Hausdorff Formula and Invariant Hyperfunctions

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Introduction

Let $\mathbf{G}$ be a Lie group and $\mathfrak{g}$ its Lie algebra. We denote by V the underlying vector space of $\mathfrak{g}$.

There is a canonical isomorphism between the ring $\mathbf{Z}(\mathfrak{g})$ of the biinvariant differential operators on $\mathbf{G}$ and the ring $\mathbf{I}(\mathfrak{g})$ of the constant coefficient operators on V which are invariant by the adjoint action of $\mathbf{G}$. When $\mathfrak{g}$ is semi-simple, this is the "Harish-Chandra isomorphism"; for a general Lie algebra, this was established by Duflo [4].

We shall prove here, that when $\mathbf{G}$ is solvable the Duflo isomorphism extends to an isomorphism Φ of the algebra of "local" invariant hyperfunctions under the group convolution and the algebra of invariant hyperfunctions on V under additive convolution (the exact result will be stated below). This gives a partial answer to a conjecture of Rais [12].

The existence of such an isomorphism Φ is of importance for the harmonic analysis on $\mathbf{G}$, and for the study of the solvability of biinvariant operators on $\mathbf{G}$ (see [7]). It reflects and explains the "orbit method" ([8, 9]), i.e. the correspondence between orbits of $\mathbf{G}$ in V^*, the dual vector space of V, and unitary irreducible representations of $\mathbf{G}$: let T be an irreducible representation of $\mathbf{G}$, then the infinitesimal character of T is a character of the ring $\mathbf{Z}(\mathfrak{g})$. Let $\mathcal{O}$ be an orbit in V^*, the map $\rho_{\mathcal{O}}(P)=P(f)$ $(f\in\mathcal{O})$ is a character of the ring $\mathbf{I}(\mathfrak{g})$ ($\mathbf{I}(\mathfrak{g})$ being identified with the ring of invariant polynomials on V^*). The principle of the orbit method is to assign to a (good) orbit $\mathcal{O}$ a representation $T_{\mathcal{O}}$ of $\mathbf{G}$ (or $\mathfrak{g}$), whose infinitesimal character corresponds to $\rho_{\mathcal{O}}$ via the isomorphism Φ. This is the technique used by M. Duflo to construct the ring isomorphism Φ.

Furthermore let $t_{\mathcal{O}}$ be (when defined) the distribution on V which is the Fourier transform of the canonical measure on the orbit $\mathcal{O}$, then $t_{\mathcal{O}}$ is clearly an invariant positive eigendistribution of every operator P in $\mathbf{I}(\mathfrak{g})$ of eigenvalue $\rho_{\mathcal{O}}(P)$. Kirillov conjectured that the global character of the representation $T_{\mathcal{O}}$ (when

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* Supported in part by N.S.F. Contract number MCS 76-02762

0020-9910/78/0047/0249/$4.80

defined) should be intimately connected with the "orbit distribution" $\Phi^{-1}(t_{\mathcal{O}})$, as proven in numerous cases. It is an essential result of Duflo [4] that these "orbit distributions" are indeed eigenfunctions for every biinvariant operator P in $\mathbf{Z}(\mathfrak{g})$; as in Rais [11], this implies the local solvability of P [4].

We will here derived the existence of Φ from a property of the Campbell-Hausdorff formula, that we conjecture and can prove in the solvable case. It is then a natural corollary of our conjecture, that biinvariant operators are locally solvable and that "orbit distributions" are eigendistributions for $\mathbf{Z}(\mathfrak{g})$. Hence the correspondence between orbits and representations is already engraved in the structure of the multiplication law.

Let us describe with some details our technique and results: We denote by $\mathfrak{g}_t$ the Lie algebra whose underlying vector space is V itself and in which the bracket $[*,*]_t$ is given by $[X,Y]_t=t[X,Y]$. Then $\mathfrak{g}_t$ gives a deformation between $\mathfrak{g}$ and the abelian Lie algebra, in which the fact is trivial.

In the course of the proof we encounter the following problem: Let $\mathbf{L}$ be the free Lie algebra generated by two indeterminates x and y and $\hat{\mathbf{L}}$ its completion. Since $x+y-\log e^y e^x$ belongs to $[\hat{\mathbf{L}},\hat{\mathbf{L}}]$, by Campbell-Hausdorff formula, we can write it in $x+y-\log e^y e^x=(1-e^{-\operatorname{ad} x})F+(e^{\operatorname{ad} y}-1)G$ for F and G in $\hat{\mathbf{L}}$. F and G are not uniquely determined by this property.

Conjecture. *For any Lie algebra $\mathfrak{g}$ of finite dimension, we can find F and G such that they satisfy*

a) $x+y-\log e^y e^x=(1-e^{-\operatorname{ad} x})F+(e^{\operatorname{ad} y}-1)G.$

b) *F and G give $\mathfrak{g}$-valued convergent power series on $(x,y)\in\mathfrak{g}\times\mathfrak{g}$.*

c) $\operatorname{tr}((\operatorname{ad} x)(\partial_x F);\mathfrak{g})+\operatorname{tr}((\operatorname{ad} y)(\partial_y G);\mathfrak{g})$

$$=\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}+\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}-\frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1}-1;\mathfrak{g}\right).$$

Here $z=\log e^x e^y$ and $\partial_x F$ (resp. $\partial_y G$) *is the* $\mathbf{End}(\mathfrak{g})$*-valued real analytic function defined by*

$$\mathfrak{g}\ni a\mapsto\frac{d}{dt}F(x+ta,y)|_{t=0}\quad\left(\text{resp. }\mathfrak{g}\ni a\mapsto\frac{d}{dt}G(x,y+ta)|_{t=0}\right),$$

and tr *denotes the trace of an endomorphism of* $\mathfrak{g}$.

When $\mathfrak{g}$ is nilpotent, this conjecture is easily verified because $(\operatorname{ad} x)(\partial_x F)$, $1-(\operatorname{ad} x/(e^{\operatorname{ad} x}-1))$ etc. are nilpotent endomorphisms of $\mathfrak{g}$ so that their traces vanish. However, we get the following fact.

Proposition 0. *If $\mathfrak{g}$ is solvable, then* Conjecture *is true.*

Let K be a non-empty closed cone in $\mathfrak{g}$. Let $\mathscr{I}(K)$ (resp. $\tilde{\mathscr{I}}(K)$) be the vector space of the germs at the unit element $e\in\mathbf{G}$ (resp. the origon $0\in\mathfrak{g}$) of the functions (i.e. either distributions, or hyperfunctions or micro-functions) $u(g)$ (resp. $\tilde{u}(x)$) such that $\operatorname{supp} u\subset\exp K$ (resp. $\operatorname{supp}\tilde{u}\subset K$) infinitesimally (see § 2) and that $u(ghg^{-1})=|\det(\operatorname{Ad}(g);\mathfrak{g})|^{-1}u(h)$ (resp. $\tilde{u}(\operatorname{Ad}(g)x)=|\det(\operatorname{Ad}(g);\mathfrak{g})|^{-1}\tilde{u}(x)$). We shall set $j(x)=\det((1-e^{-\operatorname{ad} x})/\operatorname{ad} x;\mathfrak{g})$ for $x\in\mathfrak{g}$ sufficiently near the origin. We define the isomorphism Φ: $\mathscr{I}(K) \to \tilde{\mathscr{I}}(K)$ by $(\Phi u)(x) = j(x)^{1/2} u(e^x)$ for $u \in \mathscr{I}(K)$. If two closed cones K_1 and K_2 satisfy $K_1 \cap (-K_2) = \{0\}$, then we can define the product $\mathscr{I}(K_1) \times \mathscr{I}(K_2) \to \mathscr{I}(K_1 + K_2)$ (resp. $\tilde{\mathscr{I}}(K_1) \times \tilde{\mathscr{I}}(K_2) \to \tilde{\mathscr{I}}(K_1 + K_2)$) by the convolution $*$, i.e.

$$(u * v)(g) = \int_{\mathbf{G}} u(h)\, v(h^{-1} g)\, dh \quad \text{and} \quad (\tilde{u} * \tilde{v})(x) = \int_{\mathfrak{g}} \tilde{u}(y)\, \tilde{v}(-y + x)\, dy.$$

The exact statement which we shall prove is the following:

Theorem. *If* Conjecture *is true for the group* $\mathbf{G}$, *then we have*

$$(\Phi u) * (\Phi v) = \Phi(u * v)$$

for $u \in \mathscr{I}(K_1)$ *and* $v \in \mathscr{I}(K_2)$.

If we apply this theorem when v is supported at the origin, then we obtain the following corollary:

Corollary 0. *Suppose that* Conjecture *is true for* $\mathbf{G}$, *then with any biinvariant differential operator* P *on* $\mathbf{G}$ *we can associate a constant coefficient differential operator* $\tilde{P}$ *on* $\mathfrak{g}$ *so that* $\tilde{P}\Phi(u) = \Phi(Pu)$ *holds for any* $u \in \mathscr{I}(\mathfrak{g})$.

In paragraph 4, we will prove directly this particular case of our theorem. In fact, applying the same technique, we can prove a more precise result, giving a partial answer to a conjecture of Dixmier.

Let $\gamma(P) = \beta(D(j^{1/2})P)$ the Duflo isomorphism from $\mathbf{I}(\mathfrak{g})$ to $\mathbf{Z}(\mathfrak{g})$, where β is the symmetrization map and $D(j^{1/2})$ the "differential" operator (of infinite order) defined by $j^{1/2}$, let us look at the operator $\gamma(P)$ as a biinvariant differential operator on $\mathbf{G}$; we denote by $(\exp)^*(\gamma(P))$ the differential operator on $\mathfrak{g}$ with analytic coefficients, which is the inverse image of $\gamma(P)$ by the exponential mapping. Let D be the ring of the germs at 0 of differential operators with analytic coefficients. We consider the left ideal $\mathscr{L}$ of D generated by the elements $\langle [A, x], \partial_x \rangle + \operatorname{tr}(\operatorname{ad} A; \mathfrak{g})$, $A \in \mathfrak{g}$ (here $\langle [A, x], \partial_x \rangle$ is the adjoint vector field given by $\frac{d}{d\varepsilon} \varphi(\exp \varepsilon A \cdot x)|_{\varepsilon = 0}$). Every invariant distribution on $\mathfrak{g}$ is annihilated by $\mathscr{L}$. So Corollary 0 is implied by:

Corollary 1. *Suppose that* Conjecture *is true for* $\mathbf{G}$, *then*

$$(\exp)^*(\gamma(P)) - j(x)^{-\frac{1}{2}} P j(x)^{\frac{1}{2}} \in \mathscr{L}.$$

Since Conjecture is solved in the solvable case the above theorem and its corollaries are true for a solvable group $\mathbf{G}$. Recall that the result stated in Corollary 0 holds for $\mathfrak{g}$ semi-simple as proved by Harish-Chandra [6]. Howe [16] says that he proved Theorem for a nilpotent group $\mathbf{G}$ and a restricted class of functions u, v.

Acknowledgement. We wish to thank Weita Chang, Dixmier, Duflo, Rais, and acknowledge that their questions and their work have stimulated our work.

§1

For The Theory of Microfunctions, we Refer to [1, 10, 15]. Let $\mathbf{G}$ be a Lie group, $\mathfrak{g}$ its Lie algebra and exp: $\mathfrak{g} \to G$ the exponential map. Let $\mathbf{M}$ be a real analytic manifold on which $\mathbf{G}$ acts real analytically. A hyperfunction $u(x)$ on $\mathbf{M}$ is called a *relative invariant* with respect to a character χ of G if $u(gx) = \chi(g)u(x)$ holds on $G \times \mathbf{M}$. Here $u(gx)$ is the pull-back of u by the map r: $\mathbf{G} \times \mathbf{M} \to \mathbf{M}$ defined by $(g, x) \mapsto gx$, and $\chi(g)u(x)$ is the product of a real analytic function $\chi(g)$ on $\mathbf{G} \times \mathbf{M}$ and the pull-back of u by the projection from $\mathbf{G} \times \mathbf{M}$ onto $\mathbf{M}$. More generally, let A be a subset of M, $\mathbf{G}_A = \{g \in \mathbf{G};\ gA = A\}$. A hyperfunction $u(x)$ defined in a neighborhood U of A is called *relative invariant* locally at A if there is a neighborhood W of $\mathbf{G}_A \times A$ such that $\mathrm{r}(W) \subset U$ and that $u(gx) = \chi(g)u(x)$ on W.

For any $X \in \mathfrak{g}$, we denote by D_X the vector field defined by $(D_X u)(x) = \frac{d}{dt} u(\exp(-tX)x)|_{t=0}$, and by $\delta\chi$ the derivative of χ $\left(\text{i.e. } \delta\chi(X) = \frac{d}{dt}\chi(\exp tX)|_{t=0}\right)$.

Lemma 1.1. *If u is a relative invariant locally on A hyperfunction then $(D_X + \delta\chi(X))u = 0$ in a neighborhood of A for any $X \in \mathfrak{g}$.*

Proof. We define the map φ: $\mathbb{R} \times \mathbf{M} \to \mathbf{G} \times \mathbf{M}$ by $(t, x) \mapsto (\exp(-tX), x)$. Then the pull-back of $u(gx)$ is the pull-back $(r\varphi)^* u$ of u by the map $r \circ \varphi$, and the pull-back of $\chi(g)u(x)$ is $\chi(e^{-tX})u(x)$. Since $r \circ \varphi$ has maximal rank, this is justified. Thus $(r \circ \varphi)^* u = \chi(e^{-tX})u(x)$. If we differentiate the both-sides with respect to t, and restrict them at the variety $t = 0$ in $\mathbb{R} \times \mathbf{M}$, we obtain $D_X u$ from the left hand side and $-\delta\chi(X)u$ from the right hand side. Q.E.D.

§2

Let $\mathbf{G}$ be a Lie group, $\mathfrak{g}$ its Lie algebra and exp: $\mathfrak{g} \to \mathbf{G}$ the exponential map. We denote by $\mathbf{dg}$ the left invariant Haar measure and by $\mathbf{dx}$ the Euclidean measure on $\mathfrak{g}$. After the normalization, $\mathbf{dg}$ and $\mathbf{dx}$ are related under the exponential map by the formula: $d(e^x) = j(x)\,dx$ where $j(x) = \det((1 - e^{-\operatorname{ad} x})/\operatorname{ad} x; \mathfrak{g})$ in a neighborhood of $x = 0$, because the derivative of $\exp x$ at x is given by $(1 - e^{-\operatorname{ad} x})/\operatorname{ad} x$ when we identify $\mathbf{TG}$ with $\mathfrak{g} \times \mathbf{G}$ by the left translation. We define the character $\chi_0(g)$ of G by $|\det(\operatorname{Ad}(g); \mathfrak{g})|$, we denote by $d\chi_0$ the corresponding character of $\mathfrak{g}$, i.e. $d\chi_0(x) = \operatorname{tr}(\operatorname{ad} x; \mathfrak{g})$.

Let A and B be subsets of a C^1-manifold $\mathbf{M}$, x a point in $\mathbf{M}$. Take a local coordinate system $(x_1, \ldots, x_l)$ of $\mathbf{M}$. The set of limits of the sequence $a_n(y_n - z_n)$ where $a_n > 0$, $y_n \in A$, $z_n \in B$ and y_n, z_n converge to x when $n \to \infty$, is denoted by $\mathbf{C}_x(A; B)$ regarded as a closed subset of the tangent space $\mathbf{T}_x\mathbf{M}$ of $\mathbf{M}$ at x. $\mathbf{C}_x(A; \{x\})$ is simply denoted by $\mathbf{C}_x(A)$. If f is a differential map from $\mathbf{M}$ to a C^1 manifold N, then we have $(df)_x(\mathbf{C}_x(A; B)) \subset \mathbf{C}_{f(x)}(fA; fB)$. If $\mathbf{C}_x(A; B) \cap \operatorname{Ker} df(x) \subset \{0\}$, then there is a neighborhood U of x such that

$$(df)_x\, \mathbf{C}_x(A; B) = \mathbf{C}_{f(x)}(f(A \cap U); f(B \cap U)).$$

If $\mathbf{C}_x(A;B)=\{0\}$, then x is an isolated point of $\bar{A}$ and $\bar{B}$. $\mathbf{C}_x(A;B)=\emptyset$ if and only if $\bar{A}\cap\bar{B}\not\ni x$.

Let K be a closed cone of $\mathfrak{g}$. We shall denote by $\mathscr{I}(K)$ (resp. $\tilde{\mathscr{I}}(K)$) the space of the germs of function $u(g)$ (resp. $\tilde{u}(x)$) on $\mathbf{G}$ (resp. on $\mathfrak{g}$) at $e\in\mathbf{G}$ (resp. $0\in\mathfrak{g}$) satisfying

(2.1) $\mathbf{C}_e(\operatorname{supp} u)\subset K\subset\mathfrak{g}=\mathbf{T}_e\mathbf{G}$ (resp. $C_0(\operatorname{supp}\tilde{u})\subset K\subset\mathfrak{g}=\mathbf{T}_0\,\mathfrak{g}$)

and

(2.2) u is a relative invariant locally at e with respect to the character $\chi_0(g)^{-1}$.

Let K_1 and K_2 be two closed cones in $\mathfrak{g}$ such that $K_1\cap(-K_2)=\{0\}$. If $u\in\mathscr{I}(K_1)$ and $v\in\mathscr{I}(K_2)$, then $(\operatorname{supp} u)\cap(\operatorname{supp} v)^{-1}$ is contained in $\{e\}$ locally. Suppose that u and v are defined on a neighborhood U_0 of e. For any open neighborhood $U\subset U_0$ of e, we can find neighborhoods W and V of e such that $W\subset U$, $W^{-1}\subset U$, $\{h\in W; h\in\operatorname{supp} u, h^{-1}\in\operatorname{supp} v\}\subset\{e\}$ and that the map $(g,h)\mapsto g$ from $\{(g,h)\in V\times W; h^{-1}g\in\operatorname{supp} v, h\in\operatorname{supp} u\}$ to V is a proper map. Hence we can define $(u*v)(g)$ by

$$\int_W u(h)\,v(h^{-1}g)\,dh \quad \text{on } g\in V.$$

This gives the bilinear homomorphism $\mathscr{I}(K_1)\times\mathscr{I}(K_2)\to\mathscr{I}(K_1+K_2)$ because $\mathbf{C}_e((\operatorname{supp} u)\cdot(\operatorname{supp} v))\subset K_1+K_2$. In the same way, we can define the convolution

$$(\tilde{u}*\tilde{v})(x)=\int_{\mathfrak{g}}\tilde{u}(y)\,\tilde{v}(-y+x)\,dy$$

which gives the homomorphism $\tilde{\mathscr{I}}(K_1)\times\tilde{\mathscr{I}}(K_2)\to\tilde{\mathscr{I}}(K_1+K_2)$.

Note that if u belongs to $\mathscr{I}(\mathfrak{g})$, then we have $\chi_0(g)u(g)=u(g)$. In fact, if we restrict the identity $u(g_1 g g_1^{-1})=\chi_0(g_1)^{-1}u(g)$ on the submanifold $\{(g_1,g)\in\mathbf{G}\times\mathbf{G};\ g_1=g^{-1}\}$, then we obtain the above identity. Hence we have $u(g)=\chi_0(g)^{\lambda}u(g)$ for any $\lambda\in\mathbb{C}$. We shall define the isomorphism Φ: $\mathscr{I}(K)\to\tilde{\mathscr{I}}(K)$ by $(\Phi u)(x)=j(x)^{\frac{1}{2}}u(e^x)$. The above remark shows us $(\Phi u)(x)=\chi_0(e^x)^{\lambda}j(x)^{\frac{1}{2}}u(e^x)$ for any λ.

For any $\tilde{u}(x)$ in $\tilde{\mathscr{I}}(\mathfrak{g})$, we have $d\chi_0(x)\,\tilde{u}(x)=0$. In fact, by Lemma 1.1, we have $\langle[A,x],\partial_x\rangle\,\tilde{u}(x)=-d\chi_0(A)\,\tilde{u}(x)$ for any $A\in\mathfrak{g}$. Here, for any $\mathfrak{g}$-valued real analytic function $E(x)$ on $\mathfrak{g}$, $\langle E(x),\partial_x\rangle$ is the vector field defined by $\langle E(x),\partial_x\rangle\,u(x)=\frac{d}{dt}u(x+tE(x))|_{t=0}$. Thus, we have the identity $\langle[A,x],\partial_x\rangle\,\tilde{u}(x)=-d\chi_0(A)\,\tilde{u}(x)$ on $(x,A)\in\mathfrak{g}\times\mathfrak{g}$. If we restrict this on the submanifold $A=x$, we obtain $d\chi_0(x)\,\tilde{u}(x)=0$. These observations also show the following:

Let us denote by $\mathbf{G}_0$ the kernel of χ_0 and $\mathfrak{g}_0$ its Lie algebra. Then, $\mathbf{G}_0$ is a unimodular group. For any $u\in\mathscr{I}(\mathfrak{g})$, we can find an absolute invariant v on $\mathbf{G}_0$ such that $u=v\,\delta(\chi_0)$. Similarly, for any $\tilde{u}\in\tilde{\mathscr{I}}(\mathfrak{g})$, we can find an absolute invariant $\tilde{v}$ on $\mathfrak{g}_0$ such that $\tilde{u}=\tilde{v}\,\delta(d\chi_0)$. Thus we can reduce the study of $\mathscr{I}(\mathfrak{g})$ and $\tilde{\mathscr{I}}(\mathfrak{g})$ into the case where the group is unimodular, although we will not employ this fact.

§3. We Shall Prove Theorem

Take two closed cones K_1 and K_2 of $\mathfrak{g}$ such that $K_1 \cap (-K_2) = \{0\}$ and two functions u in $\mathscr{I}(K_1)$ and v in $\mathscr{I}(K_2)$. Set $w(g) = \int_{\mathbf{G}} u(h)\, v(h^{-1} g)\, dh$, and $\tilde{u} = \Phi u$, $\tilde{v} = \Phi v$, $\tilde{w} = \Phi w$.

In order to prove Theorem we shall compute $\tilde{w}$.

$$\begin{aligned}\tilde{w}(z) &= j(z)^{\frac{1}{2}} \int_{\mathbf{G}} u(h)\, v(h^{-1} e^z)\, dh \\ &= j(z)^{\frac{1}{2}} \int_{\mathfrak{g}} u(e^x)\, v(e^{-x} e^z)\, j(x)\, dx \\ &= j(z)^{\frac{1}{2}} \int_{\mathfrak{g}} dx \int_{\mathfrak{g}} dy\, u(e^x)\, v(e^y)\, j(x)\, \delta(y - \log e^{-x} e^z).\end{aligned}$$

Lemma 3.1. $\delta(y - \log e^{-x} e^z) = j(y)\, j(z)^{-1}\, \delta(z - \log e^x e^y)$.

Proof. We have $\delta(y - f(z)) = |Jf|^{-1}\, \delta(z - f^{-1}(y))$ where Jf is the Jacobian of f. Setting $f(z) = \log e^{-x} e^z$, we shall apply this. We have, for $a \in \mathfrak{g}$

$$f(z + \varepsilon a) = \log e^{-x} e^{z + \varepsilon a}$$

which equals $\log e^{-x} e^z \exp(\varepsilon(1 - e^{-\operatorname{ad} z})/\operatorname{ad} z)\, a$ modulo ε^2. As we can set $y = \log e^{-x} e^z$, this is equal to

$$\log e^y \exp(\varepsilon(1 - e^{-\operatorname{ad} z})/\operatorname{ad} z)\, a = y + \varepsilon \frac{\operatorname{ad} y}{1 - e^{-\operatorname{ad} y}} \frac{1 - e^{-\operatorname{ad} z}}{\operatorname{ad} z} a \quad \text{modulo } \varepsilon^2.$$

Thus we obtain $Jf = \det \dfrac{\operatorname{ad} y}{1 - e^{-\operatorname{ad} y}} \dfrac{1 - e^{-\operatorname{ad} z}}{\operatorname{ad} z}$, which implies the desired result. Q.E.D.

By this lemma, we have

$$\begin{aligned}(3.1)\quad \tilde{w}(z) &= \iint u(e^x)\, v(e^y)\, j(x)\, j(y)\, j(z)^{-\frac{1}{2}}\, \delta(z - \log e^x e^y)\, dx\, dy \\ &= \iint \left(\frac{j(x)\, j(y)}{j(z)}\right)^{\frac{1}{2}} \tilde{u}(x)\, \tilde{v}(y)\, \delta(z - \log e^x e^y)\, dx\, dy.\end{aligned}$$

We want to prove that this integral equals

$$(\tilde{u} * \tilde{v})(z) = \int \tilde{u}(x)\, \tilde{v}(y)\, \delta(z - x - y)\, dx\, dy.$$

Given a vector space V and two functions $\tilde{u}$ and $\tilde{v}$ on V, given a structure μ of Lie algebra on V, we want to prove for the Lie algebra $\mathfrak{g} = (V, \mu)$ the equality:

$$\int \left(\frac{j(x)\, j(y)}{j(z)}\right)^{\frac{1}{2}} \tilde{u}(x)\, \tilde{v}(y)\, \delta(z - \log e^x e^y)\, dx\, dy = \int \tilde{u}(x)\, \tilde{v}(y)\, \delta(z - x - y)\, dx\, dy.$$

If we consider the Lie algebra $\mathfrak{g}_t = (V, t\mu)$ i.e. $[x, y]_t = t[x, y]$, the first member of the equality becomes

$$(3.2)\quad \varphi_t(z) = \int \left(\frac{j(tx)\, j(ty)}{j(tz)}\right)^{\frac{1}{2}} \tilde{u}(x)\, \tilde{v}(y)\, \delta\left(z - \frac{1}{t} \log e^{tx} e^{ty}\right) dx\, dy,$$

and this must be equal to the second member which is the value of φ_t for $t=0$. Therefore it is enough to show that φ_t does not depend on t, or equivalently $\frac{\partial}{\partial t}\varphi_t=0$. Let us calculate this derivative.

Lemma 3.2. *Let $F(x,y)$ and $G(x,y)$ be two $\mathfrak{g}$-valued real analytic functions on $(x,y)\in\mathfrak{g}\times\mathfrak{g}$ defined in a neighborhood of the origin. Suppose that $F(0,0)=G(0,0)=0$ and that*

$$x+y-\log e^y e^x=(1-e^{-\operatorname{ad}x})F(x,y)+(e^{\operatorname{ad}y}-1)G(x,y).$$

Then, we have

$$\frac{\partial}{\partial t}\frac{1}{t}\log e^{tx}e^{ty}=\left(\left\langle\left[x,\frac{1}{t}F(tx,ty)\right],\partial_x\right\rangle+\left\langle\left[y,\frac{1}{t}G(tx,ty)\right],\partial_y\right\rangle\right)\frac{1}{t}\log e^{tx}e^{ty}. \tag{3.3}$$

Here $\langle A(x),\partial_x\rangle$ is the derivation defined by

$$(\langle A(x),\partial_x\rangle u)(x)=\frac{d}{d\varepsilon}u(x+\varepsilon A(x))|_{\varepsilon=0}.$$

Proof. Set $F_t=t^{-1}F(tx,ty)$ and $G_t=t^{-1}G(tx,ty)$. Then, the right hand side of (3.3) is the value of

$$t^{-1}\frac{d}{d\varepsilon}\log\exp(tx+\varepsilon[tx,F_t])\exp(ty+\varepsilon[ty,G_t])$$

at $\varepsilon=0$. We shall calculate

$$A=\exp(tx+\varepsilon[tx,F_t])\exp(ty+\varepsilon[ty,G_t])$$

modulo ε^2. We have

$$\begin{aligned}\exp(tx+\varepsilon[tx,F_t])&=e^{tx}\exp\varepsilon\frac{1-e^{-\operatorname{ad}tx}}{\operatorname{ad}(tx)}[tx,F_t]\\&=e^{tx}\exp\varepsilon(1-e^{-\operatorname{ad}tx})F_t\quad\text{modulo }\varepsilon^2,\end{aligned}$$

and similarly $\exp(ty+\varepsilon[ty,G_t])=\exp\varepsilon(e^{\operatorname{ad}ty}-1)G_t\exp ty$ modulo ε^2. Thus, we have

$$\begin{aligned}A&=e^{tx}\exp\varepsilon((1-e^{-\operatorname{ad}tx})F_t+(e^{\operatorname{ad}ty}-1)G_t)e^{ty}\\&=e^{tx}\exp\varepsilon\left(x+y-\frac{1}{t}\log e^{ty}e^{tx}\right)e^{ty}\\&=e^{(t+\varepsilon)x}\exp\varepsilon\left(y-\frac{1}{t}\log e^{ty}e^{tx}\right)e^{ty}\\&=e^{(t+\varepsilon)x}e^{ty}\exp\varepsilon\left(y-\frac{1}{t}\log e^{tx}e^{ty}\right)\\&=e^{(t+\varepsilon)x}e^{(t+\varepsilon)y}\exp-\varepsilon\left(\frac{1}{t}\log e^{tx}e^{ty}\right)\quad\text{modulo }\varepsilon^2.\end{aligned}$$

We have therefore

$$\log A = \log e^{(t+\varepsilon)x}\, e^{(t+\varepsilon)y} - \frac{\varepsilon}{t}\log e^{tx}\, e^{ty}$$

$$= \frac{t}{t+\varepsilon}\log e^{(t+\varepsilon)x}\, e^{(t+\varepsilon)y}.$$

This implies Lemma 3.2. Q.E.D.

This lemma shows in particular

$$\text{(3.4)}\quad \frac{\partial}{\partial t}\delta\left(z - \frac{1}{t}\log e^{tx} e^{ty}\right) = (\langle [x, F_t], \partial_x\rangle + \langle [y, G_t], \partial_y\rangle)\,\delta\left(z - \frac{1}{t}\log e^{tx} e^{ty}\right).$$

Therefore, integrating by parts, we have the equality

$$\text{(3.5)}\quad p_1 = \int \left(\frac{j(tx)\, j(ty)}{j(tz)}\right)^{\frac{1}{2}} \tilde{u}(x)\,\tilde{v}(y)\,\frac{\partial}{\partial t}\delta\left(z - \frac{1}{t}\log e^{tx} e^{ty}\right) dx\, dy$$

$$= -\int \Big\{(\langle [x, F_t], \partial_x\rangle + \langle [y, G_t], \partial_y\rangle + \operatorname{div}_x [x, F_t]$$

$$+ \operatorname{div}_y [y, G_t]) \left(\frac{j(tx)\, j(ty)}{j(tz)}\right)^{\frac{1}{2}} \tilde{u}(x)\,\tilde{v}(y)\Big\}\ \delta\left(z - \frac{1}{t}\log e^{tx} e^{ty}\right) dx\, dy.$$

Here div_x (resp. div_y) signifies the divergent with respect to the variable x (resp. y), i.e. the function $\operatorname{div}_x E(x)$ is the sum of the vector field $\langle E(x), \partial_x\rangle$ and its formal adjoint.

If a function $\varphi(x)$ satisfies $\varphi(\mathrm{Ad}(g)x) = \chi(g)\,\varphi(x)$ with a character $\chi(g)$, then we have

$$\langle [A, x], \partial_x\rangle\, \varphi = (\delta\chi)(A)\,\varphi(x) \qquad \text{for } A \in \mathfrak{g}.$$

Here, $\delta\chi$ is the derivative of χ. Hence, if φ is an absolute invariant, φ and $\langle [A, x], \partial_x\rangle$ commute. Since $(j(x)\, j(y)/j(z))^{\frac{1}{2}}$ is an absolute invariant

$$\langle [x, F_t], \partial_x\rangle + \langle [y, G_t], \partial_y\rangle + \operatorname{div}_x [x, F_t] + \operatorname{div}_y [y, G_t]$$

commutes with this function. Since $\tilde{u}(x)$ is a relative invariant with respect to the character $|\det(\mathrm{Ad}(g); \mathfrak{g})^{-1}|$, we have

$$\langle [A, x], \partial_x\rangle\, \tilde{u}(x) = -\operatorname{tr}(\operatorname{ad} A)\,\tilde{u}(x).$$

Thus, we obtain

$$\text{(3.6)}\quad p_1 = -\int (\operatorname{tr}(\operatorname{ad}(F_t + G_t), \mathfrak{g}) + \operatorname{div}_x [x, F_t]$$
$$+ \operatorname{div}_y [y, G_t])(j(tx)\, j(ty)/j(tz))^{\frac{1}{2}}\, \tilde{u}(x)\,\tilde{v}(y)\ \delta\left(z - \frac{1}{t}\log e^{tx} e^{ty}\right) dx\, dy.$$

Lemma 3.3. $\displaystyle \frac{\partial}{\partial t}\log j(tx) = \operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{t \operatorname{ad} x} - 1} - \frac{1}{t}\right).$

Proof.

$$\frac{\partial}{\partial t}\log\det\frac{1-e^{-\operatorname{ad}tx}}{\operatorname{ad}(tx)}=\operatorname{tr}\frac{\operatorname{ad}tx}{1-e^{-\operatorname{ad}tx}}\frac{\partial}{\partial t}\frac{1-e^{-\operatorname{ad}tx}}{\operatorname{ad}tx}$$
$$=\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}-\frac{1}{t}\right).$$

By this lemma we have

$$\frac{\partial}{\partial t}\left(\frac{j(tx)j(ty)}{j(tz)}\right)^{\frac{1}{2}}=\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)\left(\frac{j(tx)j(ty)}{j(tz)}\right)^{\frac{1}{2}}.$$

We obtain finally

$$\frac{\partial}{\partial t}\varphi_t=-\int\Big\{\operatorname{div}_x[x,F_t]+\operatorname{div}_y[y,G_t]+\operatorname{tr}\operatorname{ad}(F_t+G_t)$$
$$-\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)\Big\}$$
$$\cdot\left(\frac{j(tx)j(ty)}{j(tz)}\right)^{\frac{1}{2}}\tilde{u}(x)\,\tilde{v}(y)\,\delta\left(z-\frac{1}{t}\log e^{tx}e^{ty}\right)dx\,dy.$$

In order to see that $\partial\varphi_t/\partial t$ vanishes, it is enough to show

$$\text{(3.7)}\quad \operatorname{div}_x[x,F_t]+\operatorname{div}_y[y,G_t]+\operatorname{tr}\operatorname{ad}(F_t+G_t)$$
$$-\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)=0$$

when $z=\frac{1}{t}\log e^{tx}e^{ty}$. Since the left hand side of this formula is homogeneous of degree 1 when we assign degree -1 to t and degree 1 to x and y, it is enough to show (3.7) when $t=1$.

For a $\mathfrak{g}$-valued function $A(x)$, let us denote by $\partial_x A$ the endomorphism of $\mathfrak{g}$ defined by $\mathfrak{g}\ni a\mapsto\frac{d}{dt}A(x+ta)|_{t=0}$. Then $\operatorname{div}_x A(x)=\operatorname{tr}\partial_x A(x)$.

Since $\partial_x[x,A(x)]=(\operatorname{ad}x)\,\partial_x A-\operatorname{ad}A$, the formula (3.7) is equivalent to

$$\text{(3.8)}\quad \operatorname{tr}(\operatorname{ad}x)(\partial_x F)+\operatorname{tr}(\operatorname{ad}y)(\partial_y G)=\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1\right)$$

with $z=\log e^x e^y$. This completes the proof of Theorem.

§4. Biinvariant Differential Operators

We consider the algebra $\mathbf{I}(\mathfrak{g})$ of the $\mathbf{G}$-invariant elements of $\mathbf{S}(\mathfrak{g})$. We identify $\mathbf{S}(\mathfrak{g})$ with the algebra of constant coefficient differential operators on $\mathfrak{g}$, hence $\mathbf{I}(\mathfrak{g})$ is identified with the ring of constant coefficient differential operators on $\mathfrak{g}$

invariant by the action of **G**. We consider the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of $\mathfrak{g}$ and its center $\mathbf{Z}(\mathfrak{g})$. We identify $\mathbf{U}(\mathfrak{g})$ with the algebra of the left invariant differential operators, hence $\mathbf{Z}(\mathfrak{g})$ will be identified with the ring of biinvariant differential operators on **G**.

We denote by δ the Dirac function on **G** supported at the unit e, then $u * \delta = \delta * u = u$. On the other hand, we have $P(u * v) = u * Pv$ for $P \in \mathbf{U}(\mathfrak{g})$. This shows that $Pu = u * P\delta$. We shall denote by the same letter δ the Dirac function on $\mathfrak{g}$ supported at the origin. Similarly if $P \in \mathbf{S}(\mathfrak{g})$, $Pu = u * P\delta = P\delta * u$. We shall denote by $(\exp)^*$ (resp. $(\exp)_*$) the pull-back of functions or differential operators on **G** to those on $\mathfrak{g}$ (resp. the inverse of $(\exp)^*$), by the exponential map.

We shall denote by β the linear mapping from $\mathbf{S}(\mathfrak{g})$ onto $\mathbf{U}(\mathfrak{g})$ obtained by symmetrization. We have $(\beta(P)\varphi)(e) = (P\tilde{\varphi})(0)$ with $\tilde{\varphi}(x) = \varphi(e^x)$, hence $(\beta(P)\delta)(e^x) = j(-x)^{-1}(P\delta)(x)$.

For a real analytic function $f(x)$ on $\mathfrak{g}$ defined on a neighborhood of the origin, and $P \in \mathbf{S}(\mathfrak{g})$, we define

$$D(f)P \in \mathbf{S}(\mathfrak{g}) \quad \text{by} \quad ((D(f)P)\delta)(x) = f(-x)\,P\delta(x),$$

or

$$((D(f)P)\varphi)(0) = P(x \mapsto f(x)\,\varphi(x))(0).$$

We shall denote by γ the map from $\mathbf{I}(\mathfrak{g})$ onto $\mathbf{Z}(\mathfrak{g})$ defined by $P \mapsto \beta(D(j^{\frac{1}{2}})P)$.

Duflo [4] has proved that for any Lie algebra $\mathfrak{g}$, γ is an isomorphism of the rings $\mathbf{I}(\mathfrak{g})$ and $\mathbf{Z}(\mathfrak{g})$.

We have seen that for any $P \in \mathbf{I}(\mathfrak{g})$,

$$\chi_0(e^x)(P\delta)(x) = P\delta(x),$$

and hence $\chi_0(e^x)$ and P commute. In fact,

$$\chi_0(e^{x-y})(P\delta)(x-y) = (P\delta)(x-y)$$

and this implies

$$\chi_0(e^x)(P\delta)(x-y) = \chi_0(e^y)(P\delta)(x-y).$$

Let us denote by $\mathfrak{g}_0 = \{A \in \mathfrak{g};\ \operatorname{tr}\operatorname{ad} A = 0\}$, this implies that $P \in \mathbf{S}(\mathfrak{g}_0)$ (see also [3, 13]). In particular, we have $j(x)^{\frac{1}{2}}(P\delta)\ x) = j(-x)^{\frac{1}{2}}(P\delta)_{,}x)$, as $j(x) = (\det e^{-\operatorname{ad} x})\,j(-x)$. So we have $\Phi(\gamma(P)\delta) = P\delta$. If we take $v = \gamma(P)\delta$ then we can get from Theorem the following proposition.

Proposition 4.1. *If* Conjecture *is true for* $\mathfrak{g}$, *then for every* $\tilde{u} \in \tilde{\mathscr{I}}(\mathfrak{g})$ *and* $P \in \mathbf{I}(\mathfrak{g})$

$$((\exp)^*\,\gamma(P))\,\tilde{u} = (j(x)^{-\frac{1}{2}}\,P\,j(x)^{\frac{1}{2}})\,\tilde{u}.$$

(In particular γ *is an isomorphism of the ring* $\mathbf{I}(\mathfrak{g})$ *and* $\mathbf{Z}(\mathfrak{g})$.)

However, we can get a more precise result applying the same method as in the preceeding paragraphs. Let us denote by **D** the ring of the germs of the differential operators at the origin.

Proposition 4.2. *Suppose that* Conjecture *is true for* $\mathfrak{g}$, *then for any* $P \in \mathbf{I}(\mathfrak{g})$

$$j(x)^{\frac{1}{2}}((\exp)^* \gamma(P))\, j(x)^{-\frac{1}{2}} - P$$

is contained in the left ideal of $\mathbf{D}$ *generated by the* $(\langle [A, x], \partial_x \rangle + \operatorname{tr} \operatorname{ad} A)$*'s* $(A \in \mathfrak{g})$.

(As we have $(\langle [A, x], \partial_x \rangle + \operatorname{tr} \operatorname{ad} A)\, \tilde{u}(x) = 0$ for every $\tilde{u} \in \tilde{\mathscr{I}}(\mathfrak{g})$, this implies Proposition 4.1.)

Proof. Remark that for $P \in \mathbf{S}(\mathfrak{g})$, $\exp^*(\beta(P))$ is the differential operator defined by

$$((\exp)^* \beta(P)\, u)(x) = P_y(u(\log e^x e^y))|_{y=0},$$

where P_y means that P operates on the y variable. Hence

$$Q = j(x)^{\frac{1}{2}}((\exp)^* \gamma(P))\, j(x)^{-\frac{1}{2}}$$

is the operator:

$$(Q u)(x) = P_y \left(\frac{j(x)^{\frac{1}{2}} j(y)^{\frac{1}{2}}}{j(\log e^x e^y)^{\frac{1}{2}}} u(\log e^x e^y) \right) \bigg|_{y=0}.$$

As before we introduce the Lie algebra $\mathfrak{g}_t$ and the corresponding operator Q_t, then

$$(Q_t u)(x) = P_y \left(\frac{j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}}}{j(\log e^{tx} e^{ty})^{\frac{1}{2}}} u \left(\frac{1}{t} \log e^{tx} e^{ty} \right) \right) \bigg|_{y=0}.$$

Let us remark that if we define the left ideal $\mathscr{L}_t$ of $\mathbf{D}$ generated by the element $\langle [x, A]_t, \partial_x \rangle + \operatorname{tr}(\operatorname{ad}_t A; \mathfrak{g}_t)$ then for $t \neq 0$ $\mathscr{L}_t = \mathscr{L}$. Hence we have to prove that: $Q_t - P \in \mathscr{L}$. As $Q_0 = P$, it is sufficient to prove that $\frac{\partial}{\partial t} Q_t \in \mathscr{L}$, where

$$\left(\left(\frac{\partial}{\partial t} Q_t \right) u \right)(x) = \frac{\partial}{\partial t} (Q_t u)(x)$$
$$= P_y \left(\frac{\partial}{\partial t} \frac{j(t x)^{\frac{1}{2}} j(t y)^{\frac{1}{2}}}{j(\log e^{tx} e^{ty})^{\frac{1}{2}}} u \left(\frac{1}{t} \log e^{tx} e^{ty} \right) \right) \bigg|_{y=0}.$$

Let F and G be as in Lemma 3.2,

$$F_t(x, y) = \frac{F(t x, t y)}{t}, \qquad G_t(x, y) = \frac{G(t x, t y)}{t},$$

and

$$d(x, y, t) = \tfrac{1}{2} \operatorname{tr} \left(\frac{\operatorname{ad} x}{e^{t \operatorname{ad} x} - 1} + \frac{\operatorname{ad} y}{e^{t \operatorname{ad} y} - 1} - \frac{\operatorname{ad} z}{e^{t \operatorname{ad} z} - 1} - \frac{1}{t} \right)$$
$$- \operatorname{tr}((\operatorname{ad} x)\, \partial_x F_t + (\operatorname{ad} y)\, \partial_y G_t)$$

where $z = \dfrac{\log e^{tx} e^{ty}}{t}$. Then we prove:

$$(4.1)\quad \frac{\partial}{\partial t}\left(j(tx)^{\frac{1}{2}}j(ty)^{\frac{1}{2}}j(\log e^{tx}e^{ty})^{-\frac{1}{2}}u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)\right)$$

$$=d(x,y,t)\,j(tx)^{\frac{1}{2}}j(ty)^{\frac{1}{2}}j(\log e^{tx}e^{ty})^{-\frac{1}{2}}u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$

$$+\sum_{i=1}^{n}\alpha_i(x,y,t)((\langle[e_i,z],\partial_z\rangle+\operatorname{tr}\operatorname{ad}e_i)\cdot u)\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$

$$+\sum_{i=1}^{n}\langle[y,e_j],\partial_y\rangle\cdot\beta_i(x,y,t)\,u\left(\frac{1}{t}\log e^{tx}e^{ty}\right).$$

Here, $e_i\,(i=1,2,\ldots,n)$ is a basis of the Lie algebra $\mathfrak{g}$, $\langle[e_i,z],\partial_z\rangle$ denotes the adjoint field corresponding to e_i, and $\alpha_i(x,y,t)$, $\beta_i(x,y,t)$ are analytic functions defined near the origin.

To prove (4.1), we compute as in Lemma 3.2

$$\frac{1}{t+\varepsilon}\log e^{(t+\varepsilon)x}e^{(t+\varepsilon)y}\qquad\text{modulo }\varepsilon^2$$

$$=\frac{1}{t}\log e^{tx}e^{\varepsilon x}e^{ty}e^{\varepsilon y}e^{-\varepsilon\frac{\log e^{tx}e^{ty}}{t}}$$

$$=\frac{1}{t}\log e^{tx}e^{ty}e^{\varepsilon\left(e^{-t\operatorname{ad}y}\left(x+y-\frac{\log e^{ty}e^{tx}}{t}\right)\right)}$$

$$=\frac{1}{t}\log e^{tx}e^{ty}e^{\varepsilon(e^{-t\operatorname{ad}y}((1-e^{-t\operatorname{ad}x})F_t+(e^{t\operatorname{ad}y}-1)G_t))}.$$

We write

$$e^{-t\operatorname{ad}y}((1-e^{-t\operatorname{ad}x})F_t+(e^{t\operatorname{ad}y}-1)G_t)$$
$$=(1-e^{-\operatorname{ad}(\log e^{tx}e^{ty})})F_t+(1-e^{-t\operatorname{ad}y})(G_t-F_t).$$

So we have

$$\frac{d}{dt}\left(\frac{1}{t}\log e^{tx}e^{ty}\right)=\langle[z,F_t],\partial_z\rangle\left(\frac{1}{t}\log e^{tx}e^{ty}\right)+\langle[y,G_t-F_t],\partial_y\rangle\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$

(if $F_t(x,y)=\sum f_i(x,y,t)\,e_i$, and $\mathbf{I}(x)=x$

$$\langle[F_t,z],\partial_z\rangle\left(\frac{1}{t}\log e^{tx}e^{ty}\right)=\sum f_i(x,y)(\langle[e_i,z],\partial_z\rangle\cdot\mathbf{I})\left(\frac{1}{t}\log e^{tx}e^{ty}\right)\Big).$$

We write

$$\frac{\partial}{\partial t}j(tx)^{\frac{1}{2}}j(ty)^{\frac{1}{2}}j(\log e^{tx}e^{ty})^{-\frac{1}{2}}u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)\Bigg|_{t=t_0}$$

$$=\frac{\partial}{\partial t}j(tx)^{\frac{1}{2}}j(ty)^{\frac{1}{2}}j\left(t\frac{\log e^{tx}e^{ty}}{t}\right)^{-\frac{1}{2}}u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)\Bigg|_{t=t_0}$$

$$=\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)$$

$$\cdot j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx}e^{ty})^{-\frac{1}{2}} u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$

$$+j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}}\frac{\partial}{\partial t}\left(j\left(t_0\frac{\log e^{tx}e^{ty}}{t}\right)^{-\frac{1}{2}} u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)\right)\Bigg|_{t=t_0}$$

by Lemma 3.3.

Now if $(G_t-F_t)(x,y)=\sum\lambda_i(x,y,t)\,e_i$ we have

$$\langle[y,G_t-F_t],\partial_y\rangle=\sum_{i=1}^{n}\langle[y,e_i],\partial_y\rangle\,\lambda_i(x,y,t)-\operatorname{tr}\operatorname{ad}y\,\partial_y(G_t-F_t).$$

As j is an absolute invariant, j commutes with the adjoint fields.

Hence from the preceeding calculation, we obtain that the left hand side of (4.1) is equal to

$$\left(\frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)-\operatorname{tr}\operatorname{ad}y\,\partial_y(G_t-F_t)\right)$$

$$\cdot j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx}e^{ty})^{-\frac{1}{2}} u\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$

$$+j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx}e^{ty})^{-\frac{1}{2}}(\langle[z,F_t],\partial_z\rangle\cdot u)\left(\frac{1}{t}\log e^{tx}e^{ty}\right)$$

$$+\sum_{i=1}^{n}\langle[y,e_i],\partial_y\rangle\cdot\left(\beta_i(x,y,t)\,u\left(\frac{1}{t}\log(e^{tx}e^{ty})\right)\right).$$

But, we have

$$\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{t\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{t\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{t\operatorname{ad}z}-1}-\frac{1}{t}\right)-\operatorname{tr}\operatorname{ad}y\,\partial_y(G_t-F_t)$$

$$=d(x,y,t)+\operatorname{tr}(\operatorname{ad}y\,\partial_y F_t+\operatorname{ad}x\,\partial_x F_t).$$

Let us remark here that if E is in $\hat{\mathbf{L}}$, we have $g\cdot E(x,y)=E(gx,gy)$ for every $g\in\mathbf{G}$. The operator $(\partial_x E)\operatorname{ad}x+(\partial_y E)\operatorname{ad}y$ is the linear operator

$$c\mapsto\frac{d}{d\varepsilon}E(x+\varepsilon[x,c],y+\varepsilon[y,c])|_{\varepsilon=0}$$

$$=\frac{d}{d\varepsilon}E(\exp\varepsilon c\cdot x,\exp\varepsilon c\cdot y)|_{\varepsilon=0}$$

$$=\frac{d}{d\varepsilon}\exp\varepsilon c\cdot E(x,y)|_{\varepsilon=0}$$

$$=-[E(x,y),c]$$

hence is the operator $-\operatorname{ad}E$.

We then obtain that the left side of (4.1) is equal to

$$d(x,y,t)\, j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx} e^{ty})^{-\frac{1}{2}} u\left(\frac{1}{t}\log e^{tx} e^{ty}\right)$$

$$-j(tx)^{\frac{1}{2}} j(ty)^{\frac{1}{2}} j(\log e^{tx} e^{ty})^{-\frac{1}{2}}((\langle [F_t, z], \partial_z\rangle + \operatorname{tr}\operatorname{ad} F_t)\cdot u)\left(\frac{1}{t}\log e^{tx} e^{ty}\right)$$

$$+\sum_{i=1}^{n} \langle [y, e_i], \partial_y\rangle\, \beta_i(x,y,t)\, u\left(\frac{1}{t}\log(e^{tx} e^{ty})\right),$$

which is of the required form.

Now if our conjecture is true for $\mathfrak{g}$, then we can find F and G such that $d(x,y,t)=0$. Now we remark that if $P\in\mathbf{I}(\mathfrak{g})$,

$$P_y\langle [y, e_i], \partial_y\rangle = \langle [y, e_i], \partial_y\rangle P_y$$

hence $(P_y\langle [y, e_i], \partial_y\rangle \psi(y))|_{y=0}=0$. Let $R_i(t)$ denote the differential operator

$$(R_i(t)\varphi)(x) = P_y\left(\alpha_i(x,y,t)\,\varphi\left(\frac{1}{t}\log e^{tx} e^{ty}\right)\right)\Bigg|_{y=0}.$$

We obtain from (4.1)

$$\frac{\partial}{\partial t} Q_t = \sum_{i=1}^{n} R_i(t)(\langle [e_i, x], \partial_x\rangle + \operatorname{tr}\operatorname{ad} e_i),$$

i.e. $\frac{\partial}{\partial t} Q_t \in \mathscr{L}$. Q.E.D.

Remark. The same proof shows the corresponding fact for biinvariant integral operators.

Remark. We will see in the next section that our conjecture is true for $\mathbf{G}$ solvable; we can easily deduce from Proposition 4.1, the fact that every biinvariant operator on $\mathbf{G}$ is locally solvable, which was already obtained by Rouvière [14] and Duflo-Raïs [5]. In fact P being invariant by the action of $\mathbf{G}$ we can find a fundamental solution for P, which is invariant by $\mathbf{G}$. It follows that $(\exp)^*\gamma(P)$ has a local fundamental solution. If $\mathbf{G}$ is exponential solvable, the maps F and G can be constructed in the whole space $\mathfrak{g}$ hence the Propositions 4.1 and 4.2 hold on the whole space $\mathfrak{g}$. So $\exp^*(j(P))$ has a fundamental solution on the space G, (Weita Chang [2] has proven recently that every biinvariant operator on an simply connected solvable group is globally solvable). We recall that M. Duflo has shown that every biinvariant differential operator on a Lie group $\mathbf{G}$ is locally solvable [4].

§ 5. Proof of Proposition 0

First we shall translate our conjecture into another form. Let us write for an $A\in\hat{\mathbf{L}}$

$$2(x+y-\log e^y e^x) = ((x+y-\log e^y e^x)+A)+(x+y-\log e^y e^x)-A.$$

Hence we will consider $A\in\hat{\mathbf{L}}$ such that $(x+y-\log e^y e^x)+A$ is divisible by x (i.e. in $[x,\hat{\mathbf{L}}]$) and $(x+y-\log e^y e^x)-A$ is divisible by y (i.e. in $[y,\hat{\mathbf{L}}]$). As $x+y-\log e^y e^x \equiv \frac{1}{2}[x,y] \bmod [[\hat{\mathbf{L}},\hat{\mathbf{L}}],\hat{\mathbf{L}}]$ and $[x,y]$ is divisible by x and y, we may take A in $[[\hat{\mathbf{L}},\hat{\mathbf{L}}],\hat{\mathbf{L}}]$. We will write $x+y-\log e^y e^x+A=[x,P]$, $A-(x+y-\log e^y e^x)=[y,Q]$, choose $F=\frac{1}{2}\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}P$, $G=-\frac{1}{2}\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}Q$ and translate our conjecture in terms of A.

We shall first give two preliminary lemmata.

Lemma 5.1.

i) $$\partial_x \log e^x e^y = \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\,\frac{e^{\operatorname{ad}x}-1}{\operatorname{ad}x}$$

and

ii) $$\partial_y \log e^x e^y = \frac{\operatorname{ad}z}{1-e^{-\operatorname{ad}z}}\,\frac{1-e^{-\operatorname{ad}y}}{\operatorname{ad}y}.$$

Here $z=\log e^x e^y$.

Proof. We have, modulo ε^2,

$$\log e^{(x+\varepsilon a)}e^y=\log e^{\varepsilon\frac{e^{\operatorname{ad}x}-1}{\operatorname{ad}x}a}e^x e^y=\log e^{\varepsilon\frac{e^{\operatorname{ad}x}-1}{\operatorname{ad}x}a}e^z$$
$$=z+\varepsilon\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\,\frac{e^{\operatorname{ad}x}-1}{\operatorname{ad}x}a.$$

The formula ii) is shown in the same way. Q.E.D.

Lemma 5.2. *Let* $a\in\mathfrak{g}$, $f(\lambda)$ *and* $g(\lambda)$ *two power series on* λ. *Then*

$$\operatorname{tr}(f(\operatorname{ad}x)\,\partial_x(g(\operatorname{ad}x)a))=\operatorname{tr}\left(f(\operatorname{ad}x)\frac{g(0)-g(\operatorname{ad}x)}{\operatorname{ad}x}\operatorname{ad}a;\,\mathfrak{g}\right).$$

Proof. By linearity, we may assume $g(\lambda)=\lambda^n$. If $n=0$, the lemma is evident. Suppose $n\geqq 1$. Then we have

$$\operatorname{ad}(x+\varepsilon c)^n a-(\operatorname{ad}x)^n a=\varepsilon\sum_{k=0}^{n-1}(\operatorname{ad}x)^{n-1-k}(\operatorname{ad}c)(\operatorname{ad}x)^k a$$
$$=-\varepsilon\sum_{k=0}^{n-1}(\operatorname{ad}x)^{n-1-k}\operatorname{ad}((\operatorname{ad}x)^k a)c \quad \text{modulo } \varepsilon^2.$$

Thus we have

$$\partial_x(g(\operatorname{ad}x)a)=-\sum_{k=0}^{n-1}(\operatorname{ad}x)^{n-1-k}\operatorname{ad}((\operatorname{ad}x)^k a).$$

If $k>0$, $\operatorname{tr} f(\operatorname{ad}x)(\operatorname{ad}x)^{n-1-k}\operatorname{ad}((\operatorname{ad}x)^k a)$ vanishes. In fact, if we set $b=(\operatorname{ad}x)^{k-1}a$ and $\varphi(\lambda)=\lambda^{n-1-k}f(\lambda)$, then

$$\operatorname{tr}\varphi(\operatorname{ad}x)\operatorname{ad}((\operatorname{ad}x)b)=\operatorname{tr}\varphi(\operatorname{ad}x)(\operatorname{ad}x\operatorname{ad}b-\operatorname{ad}b\operatorname{ad}x)=0.$$

Therefore, we obtain

$$\operatorname{tr} f(\operatorname{ad}x)\,\partial_x g(\operatorname{ad}x)a=-\operatorname{tr} f(\operatorname{ad}x)(\operatorname{ad}x)^{n-1}(\operatorname{ad}a). \quad \text{Q.E.D.}$$

Proposition 5.3. Conjecture *is implied from the following: For any Lie algebra* $\mathfrak{g}$, *we can find* A *in* $[[\hat{\mathbf{L}}, \hat{\mathbf{L}}], \hat{\mathbf{L}}]$ *satisfying the conditions* i), ii) *and* iii):

i) *There is* P *in* $\hat{\mathbf{L}}$ *such that* $A+x+y-\log e^y e^x=[x, P]$ *and that* P *gives a convergent power series on* $(x, y)\in \mathfrak{g}\times\mathfrak{g}$.

ii) *There is* Q *in* $\hat{\mathbf{L}}$ *such that* $A-(x+y-\log e^y e^x)=[y, Q]$ *and that* Q *gives a convergent power series on* $(x, y)\in \mathfrak{g}\times\mathfrak{g}$.

iii) $\operatorname{tr}\dfrac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}\partial_x A-\operatorname{tr}\dfrac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}\partial_y A=\operatorname{tr}\left(\dfrac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1+\tfrac{1}{2}\operatorname{ad}z\right)$,

where $z=\log e^x e^y$.

Proof. We have $x+y-\log e^y e^x=\frac{1}{2}[x, P]-\frac{1}{2}[y, Q]$. Let $F=\frac{1}{2}\dfrac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}P$ and $G=-\frac{1}{2}\dfrac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}Q$. Then we have

$$x+y-\log e^y e^x=(1-e^{-\operatorname{ad}x})F+(e^{\operatorname{ad}y}-1)G.$$

We have $[x, P]=2(1-e^{-\operatorname{ad}x})F$. Therefore, by Lemma 5.2, we have

$$\begin{aligned}\operatorname{tr}\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}\partial_x[x, P]&=2\operatorname{tr}\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}\frac{0-(1-e^{-\operatorname{ad}x})}{\operatorname{ad}x}\operatorname{ad}F+2\operatorname{tr}(\operatorname{ad}x)\partial_x F\\&=2\operatorname{tr}(\operatorname{ad}x)\partial_x F-2\operatorname{tr}\operatorname{ad}F.\end{aligned}$$

Similarly, we have $-\operatorname{tr}\dfrac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}\partial_y[y, Q]=2\operatorname{tr}(\operatorname{ad}y)\partial_y G-2\operatorname{tr}\operatorname{ad}G$. Set $\tilde{z}=\log e^y e^x$, we have, by Lemma 5.1

$$\partial_x[x, P]=\partial_x(x+y-\log e^y e^x+A)=1-\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\frac{1-e^{-\operatorname{ad}x}}{\operatorname{ad}x}+\partial_x A.$$

Hence, we obtain

$$\begin{aligned}\operatorname{tr}\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}\partial_x[x, P]&=\operatorname{tr}\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}\partial_x A+\operatorname{tr}\left(\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}-\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\right)\\&=\operatorname{tr}\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}\partial_x A+\operatorname{tr}\left(\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}-\frac{\operatorname{ad}z}{1-e^{-\operatorname{ad}z}}\right).\end{aligned}$$

In the same way, we have

$$-\operatorname{tr}\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}\partial_y[y, Q]=-\operatorname{tr}\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}\partial_y A+\operatorname{tr}\left(\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\right).$$

Thus, we obtained

$$\begin{aligned}&\operatorname{tr}(\operatorname{ad}x)(\partial_x F)+\operatorname{tr}(\operatorname{ad}y)(\partial_y G)\\&\quad=\operatorname{tr}(\operatorname{ad}F)+\operatorname{tr}(\operatorname{ad}G)+\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}\partial_x A-\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}\partial_y A\right)\\&\quad+\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-\frac{\operatorname{ad}z}{1-e^{-\operatorname{ad}z}}\right)\end{aligned}$$

$$=\operatorname{tr}(\operatorname{ad}F)+\operatorname{tr}(\operatorname{ad}G)+\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{1-e^{-\operatorname{ad}z}}-1+\tfrac{1}{2}\operatorname{ad}z\right).$$

Since $\lambda/(1-e^{-\lambda})=\lambda/(e^{\lambda}-1)+\lambda$ and $\operatorname{tr}\operatorname{ad}z=\operatorname{tr}(\operatorname{ad}x+\operatorname{ad}y)$, this equals

$$\operatorname{tr}(\operatorname{ad}F)+\operatorname{tr}(\operatorname{ad}G)+\tfrac{1}{4}\operatorname{tr}(\operatorname{ad}x-\operatorname{ad}y)+\tfrac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1\right).$$

Hence, it is enough to show that

(5.1) $\operatorname{tr}(\operatorname{ad}F)+\operatorname{tr}(\operatorname{ad}G)=\tfrac{1}{4}\operatorname{tr}(\operatorname{ad}y-\operatorname{ad}x)$.

However, adding a constant multiple of x (resp. y) to P (resp. Q), we may assume that P (resp. Q) is equal to αy (resp. βx) modulo $[\hat{\mathbf{L}},\hat{\mathbf{L}}]$. However $x+y-\log e^{y}e^{x}\equiv-\tfrac{1}{2}[x,y]$ modulo $[[\hat{\mathbf{L}},\hat{\mathbf{L}}],\hat{\mathbf{L}}]$ and hence $P\equiv\tfrac{1}{2}y$ (resp. $Q\equiv\tfrac{1}{2}x$). Thus, we have $F\equiv\tfrac{1}{4}y$ (resp. $G=-\tfrac{1}{4}x$) modulo $[\hat{\mathbf{L}},\hat{\mathbf{L}}]$. Since $\operatorname{tr}\operatorname{ad}[\hat{\mathbf{L}},\hat{\mathbf{L}}]=0$, (5.1) is satisfied. Q.E.D.

Let A satisfy i), ii), iii), of the Proposition 4.3. We may remark that $A'(x,y)=\tfrac{1}{4}(A(x,y)-A(y,x)-A(-x,-y)+A(-y,-x))$ satisfies also 1), 2), and 3). This follows from the following observations:

a) if $m(x,y)=x+y-\log e^{y}e^{x}$, then $m(x,y)=-m(-y,-x)$;

$$\begin{aligned}m(x,y)-m(y,x)&=\log e^{x}e^{y}-\log e^{y}e^{x}\\&=(e^{\operatorname{ad}x}-1)\log e^{y}e^{x}\\&=(1-e^{\operatorname{ad}y})\log e^{x}e^{y}\end{aligned}$$

hence is divisible by x and y.

b) if $t(x,y)=\operatorname{tr}\left(\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1+\frac{\operatorname{ad}z}{2}\right)$ then $t(x,y)=t(y,x)=t(-x,-y)$.

c) for any $E\in[\hat{\mathbf{L}},\hat{\mathbf{L}}]$,

$$\operatorname{tr}\frac{\operatorname{ad}x}{1-e^{-\operatorname{ad}x}}\partial_xE-\operatorname{tr}\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}\partial_yE=\operatorname{tr}\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}\partial_xE-\operatorname{tr}\frac{\operatorname{ad}y}{1-e^{-\operatorname{ad}y}}\partial_yE.$$

In fact the difference is

$$\begin{aligned}\operatorname{tr}(\operatorname{ad}x\,\partial_xE+\operatorname{ad}y\,\partial_yE)&=\operatorname{tr}(\partial_xE\operatorname{ad}x+\partial_yE\operatorname{ad}y)\\&=-\operatorname{tr}\operatorname{ad}E(x,y)\qquad\text{(see 4.2)}\\&=0\qquad\text{as } E\in[\hat{\mathbf{L}},\hat{\mathbf{L}}].\end{aligned}$$

We will now construct A in $[[\hat{\mathbf{L}},\hat{\mathbf{L}}],\hat{\mathbf{L}}]$ such that

$$A(x,y)=-A(y,x)=-A(-x,-y)$$

and i) $x+y-\log e^{y}e^{x}+A(x,y)=[x,P]$ and P gives a convergent power series on $(x,y)\in\mathfrak{g}\times\mathfrak{g}$. ((ii) follows then). If $\mathfrak{g}$ is solvable we will be able to prove that A satisfies also the condition iii).

We consider now the condition i):

$$x+y-\log e^y e^x + A(x,y) = [x, P(x,y)].$$

Then for every t, we will have

$$tx+ty-\log e^{ty}e^{tx}+A(tx,ty)=t[x,P(tx,ty)].$$

Hence $\frac{\partial}{\partial t}(tx+ty-\log e^{ty}e^{tx})+\frac{\partial}{\partial t}A(tx,ty)\in[x,\hat{\mathbf{L}}]$ and $\frac{\partial}{\partial t}A(tx,ty)$ satisfies the same antisymmetry relation as A.

Let θ be the vector field $\langle x, \partial_x\rangle + \langle y, \partial_y\rangle$ (or the derivation of $\hat{\mathbf{L}}$ defined by $\theta|\mathbf{L}_n = n\,\mathrm{id}\,\mathbf{L}_n$ where $\mathbf{L}_n$ is the space of elements of $\mathbf{L}$ of degree n) then $t\frac{\partial}{\partial t}B(tx,ty)|_{t=1}=\theta B$, for $B\in\hat{\mathbf{L}}$. We compute

$$\theta(x+y-\log e^y e^x) = x+y-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot y-\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\cdot x$$

with $\tilde{z}=\log e^y e^x$ and we will write $\theta(x+y-\log e^y e^x)$ as an antisymmetric element mod $[x,\hat{\mathbf{L}}]$.

For any real analytic function $g(\lambda)$, we have $g(\operatorname{ad}z)=e^{\operatorname{ad}x}g(\operatorname{ad}\tilde{z})\,e^{-\operatorname{ad}x}$, in particular $g(\operatorname{ad}\tilde{z})\cdot x\equiv g(\operatorname{ad}z)\cdot x$ modulo $[x,\hat{L}]$ and

$$g(\operatorname{ad}z)\cdot y\equiv g(\operatorname{ad}\tilde{z})\,e^{-\operatorname{ad}x}y\equiv g(\operatorname{ad}\tilde{z})\,e^{-\operatorname{ad}\tilde{z}}\cdot y \quad \text{modulo } [x,\hat{L}].$$

Hence we write modulo $[x,\hat{L}]$

$$\begin{aligned}\theta(x+y-\log e^y e^x) &= \left(1-\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\right)\cdot x + y - \frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}e^{-\operatorname{ad}\tilde{z}}\cdot y\\ &\equiv f(\operatorname{ad}\tilde{z})\cdot x + f(\operatorname{ad}z)\cdot y, \quad \text{where} \quad f(\lambda)=\left(1-\frac{\lambda}{1-e^{-\lambda}}\right)\\ &\equiv f(\operatorname{ad}\tilde{z})\cdot x - f(\operatorname{ad}z)\cdot y + 2f(\operatorname{ad}z)\cdot y.\end{aligned}$$

We write, as $f(0)=0$,

$$\begin{aligned}f(\operatorname{ad}z)\cdot y &= \frac{f(\operatorname{ad}z)}{e^{\operatorname{ad}z}-1}(e^{\operatorname{ad}z}-1)\,y\\ &= \frac{f(\operatorname{ad}z)}{e^{\operatorname{ad}z}-1}(e^{\operatorname{ad}x}-1)\,y,\end{aligned}$$

therefore $f(\operatorname{ad}z)\cdot y\equiv\left(\frac{f(\operatorname{ad}\tilde{z})}{e^{\operatorname{ad}\tilde{z}}-1}-\frac{f(\operatorname{ad}z)}{e^{\operatorname{ad}z}-1}\right)\cdot y.$

As $\left(\frac{f(\operatorname{ad}\tilde{z})}{e^{\operatorname{ad}\tilde{z}}-1}-\frac{f(\operatorname{ad}z)}{e^{\operatorname{ad}z}-1}\right)\cdot x\equiv 0$ we obtain that

$$\theta(x+y-\log e^y e^x)\equiv f(\operatorname{ad}z)\cdot x - f(\operatorname{ad}z)\cdot y + 2\left(\frac{f(\operatorname{ad}\tilde{z})}{e^{\operatorname{ad}\tilde{z}}-1}-\frac{f(\operatorname{ad}z)}{e^{\operatorname{ad}z}-1}\right)\cdot(x+y).$$

Let us denote by $\alpha(x, y)$ the second member of this equality. We have obviously $\alpha(x, y) = -\alpha(y, x)$, hence if we define $\beta(x, y) = \frac{1}{2}(\alpha(x, y) + \alpha(-y, -x))$, β will satisfy the relation $\beta(x, y) = -\beta(y, x) = -\beta(-x, -y)$ and $\theta(x + y - \log e^y e^x) \equiv \beta(x, y)$ mod $[x, \hat{L}]$. We remark that the function $h(\lambda) = \left(1 - \frac{\lambda}{1 - e^{-\lambda}}\right) \frac{1}{e^\lambda - 1}$ verifies $h(\lambda) = -h(-\lambda) - 1$ as $\frac{1}{1 - e^{-\lambda}} = \frac{1}{e^\lambda - 1} + 1$, hence

$$\beta(x, y) = 2\left(\frac{f(\operatorname{ad}\tilde{z})}{e^{\operatorname{ad}\tilde{z}} - 1} - \frac{f(\operatorname{ad} z)}{e^{\operatorname{ad} z} - 1}\right) \cdot (x + y) + \tfrac{1}{2}(f(\operatorname{ad}\tilde{z}) + f(-\operatorname{ad} z)) \cdot x - \tfrac{1}{2}(f(\operatorname{ad} z) + f(-\operatorname{ad}\tilde{z})) \cdot y.$$

We can therefore define $A(x, y)$ by the differential equation:

$$(5.2) \quad \theta A = 2\left(1 - \frac{\operatorname{ad} z}{1 - e^{-\operatorname{ad} z}}\right) \frac{1}{e^{\operatorname{ad} z} - 1}(x + y) - 2\left(1 - \frac{\operatorname{ad}\tilde{z}}{1 - e^{-\operatorname{ad}\tilde{z}}}\right) \frac{1}{e^{\operatorname{ad}\tilde{z}} - 1} \cdot (x + y) + \tfrac{1}{2}\left(\frac{\operatorname{ad} z}{e^{\operatorname{ad} z} - 1} - 1\right) \cdot x + \tfrac{1}{2}\left(\frac{\operatorname{ad}\tilde{z}}{1 - e^{-\operatorname{ad}\tilde{z}}} - 1\right) \cdot x - \tfrac{1}{2}\left(\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}} - 1} - 1\right) \cdot y - \tfrac{1}{2}\left(\frac{\operatorname{ad} z}{1 - e^{-\operatorname{ad} z}} - 1\right) \cdot y,$$

with the initial condition $A(0, 0) = 0$ ($\tilde{z} = \log e^y e^x$, $z = \log e^x e^y$). As the second member is a convergent power series at the origin, so is $A(x, y)$.

The preceding calculation implies now 1) and 2) of the:

Lemma 5.4.

1) $A(x, y) = -A(y, x) = -A(-x, -y)$,
2) $x + y - \log e^y e^x + A \in [x, \hat{L}]$,
3) $A \in [\hat{L}, [\hat{L}, \hat{L}]]$.

For 3) we remark that $A \in [\hat{L}, \hat{L}]$, and the properties $A(x, y) = -A(-x, -y)$ implies that $A \in [\hat{L}, [\hat{L}, \hat{L}]]$. The lemma is proven. Q.E.D.

Let q be a power series of the two non commutative variables x and y, i.e. q is in the completion of the tensor algebra $\hat{T}(x, y)$ of the vector space $\mathbb{C}x + \mathbb{C}y$. We denote by $c(q)$ the image of q under the map $\hat{T}(x, y) \to \hat{S}(x, y) = C[[x, y]]$, i.e. $c(q)$ is a power series in the commutative variables x and y.

Lemma 5.5. *If* $\mathfrak{g}$ *is solvable,* $\operatorname{tr}(q(\operatorname{ad} x, \operatorname{ad} y))$ *depends only on* $c(q)$.

Proof. There is a basis of $\mathfrak{g}^{\mathbb{C}}$ where the operators $\operatorname{ad} x, \operatorname{ad} y$ are lower triangular, then $\operatorname{ad}[x, y] = \operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x$ have zeros on the diagonal, and the lemma follows.

Let us write $A = p(\operatorname{ad} x, \operatorname{ad} y) \cdot [x, y]$, where p is a convergent power series in the non commutative variables x and y.

Lemma 5.6. *Let* $\mathfrak{g}$ *be solvable, then*

$$\operatorname{tr} \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}} \partial_x A - \operatorname{tr} \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} \partial_y A$$

$$= -\operatorname{tr}\left((e^{\operatorname{ad} z}-1)\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}\right)\left(\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}\right) p(\operatorname{ad} x, \operatorname{ad} y)\right).$$

Proof. Let us consider the endomorphism

$$\mathfrak{g} \ni c \mapsto \frac{d}{d\varepsilon} p(\operatorname{ad} x+\varepsilon \operatorname{ad} c, \operatorname{ad} y) \cdot [x, y]|_{\varepsilon=0};$$

this is a sum of terms of the form

$$\begin{aligned} & p_1(\operatorname{ad} x, \operatorname{ad} y) \operatorname{ad} c\, p_2(\operatorname{ad} x, \operatorname{ad} y) \cdot [x, y] \\ & \quad = -p_1(\operatorname{ad} x, \operatorname{ad} y) \operatorname{ad}(p_2(\operatorname{ad} x, \operatorname{ad} y) \cdot [x, y]) \cdot c. \end{aligned}$$

The trace of the endomorphism $\frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}} p_1(\operatorname{ad} x, \operatorname{ad} y) \operatorname{ad}(p_2(\operatorname{ad} x, \operatorname{ad} y) \cdot [x, y])$ vanishes by the preceding lemma. So the only term appearing in $\operatorname{tr} \frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}} \partial_x A$ will come from the trace of the endomorphism

$$c \mapsto \frac{d}{d\varepsilon} \frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}} p(\operatorname{ad} x, \operatorname{ad} y)[x+\varepsilon c, y]|_{\varepsilon=0}.$$

We obtain that the left side of the equality is:

$$\begin{aligned} & -\operatorname{tr}\left(\frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}} \operatorname{ad} y+\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1} \operatorname{ad} x\right) p(\operatorname{ad} x, \operatorname{ad} y) \\ & \quad = -\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}\right)\left(\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}\right)(e^{\operatorname{ad} z}-1) p(\operatorname{ad} x, \operatorname{ad} y). \end{aligned}$$

If we restrict our attention when $\mathfrak{g}$ is solvable, we have to prove:

$$-\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1} \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}(e^{\operatorname{ad} z}-1) p(\operatorname{ad} x, \operatorname{ad} y)\right)=\operatorname{tr}\left(\frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1}-1+\tfrac{1}{2} \operatorname{ad} z\right).$$

Hence, considering the commutative ring $\mathbb{C}[[x, y]]$ we need only to prove:

$$\mathbf{c}(p)(x, y)=\left(1-\frac{x+y}{2}-\frac{x+y}{e^{x+y}-1}\right) \frac{1}{e^{x+y}-1} \frac{e^x-1}{x} \frac{e^y-1}{y}.$$

We denote by $q(x, y)$ the right hand side.

Let us consider the homomorphism h: $[\hat{\mathbf{L}}, \hat{\mathbf{L}}] \to [\hat{\mathbf{L}}, \hat{\mathbf{L}}]/[[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$ and let us write for $m \in [\hat{\mathbf{L}}, \hat{\mathbf{L}}]$, $m=\varphi(\operatorname{ad} x, \operatorname{ad} y) \cdot [x, y]$ then clearly $\mathbf{h}(m)$ depends only on $\mathbf{c}(\varphi)$. Therefore, for $f(x, y) \in \mathbb{C}[[x, y]]$, we shall write $f(\operatorname{ad} x, \operatorname{ad} y)[x, y]$ for the element $\varphi(\operatorname{ad} x, \operatorname{ad} y)[x, y]$ modulo $[[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$ with $f=\mathbf{c}(\varphi)$.

Remark 5.7. If $f(x, y) \in \mathbb{C}[[x, y]]$ is such that $f(\operatorname{ad} x, \operatorname{ad} y) \cdot [x, y] \equiv 0$ modulo $[[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$, then $f(x, y)=0$. In fact if $\varphi(\operatorname{ad} x, \operatorname{ad} y) \cdot [x, y] \in [[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$, with $f=c(\varphi)$ then $\operatorname{tr}(\partial_x \varphi(\operatorname{ad} x, \operatorname{ad} y) \cdot [x, y]; \mathfrak{g})=0$ for any solvable Lie algebra $\mathfrak{g}$.

On the other hand the same calculation as in Lemma 5.6 shows that

$$\operatorname{tr}(\partial_x(\varphi(\operatorname{ad}x, \operatorname{ad}y)\cdot[x,y]); \mathfrak{g}) = -\operatorname{tr}(\varphi(\operatorname{ad}x, \operatorname{ad}y)\operatorname{ad}y; \mathfrak{g}).$$

Considering the 2 dimension Lie algebra $\mathfrak{g}$ with basis H, A and relation $[H,A] = A$, we have for $x = x_1 H + x_2 A$, $y = y_1 H + y_2 A$,

$$\operatorname{tr}(\varphi(\operatorname{ad}x, \operatorname{ad}y)\operatorname{ad}y; \mathfrak{g}) = f(x_1, y_1)y_1,$$

hence $f(x_1, y_1)y_1 = 0$, and so is f.

Proposition 0 will result from the following lemma.

Lemma 5.8. *Let*

$$\alpha = \left(1 - \frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}\cdot(x+y-\tilde{z}) + \tfrac{1}{2}\tilde{z} - \left(1 - \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\right)\frac{1}{\operatorname{ad}z}\cdot(x+y-z) - \tfrac{1}{2}z$$

then

1) $\mathbf{h}(\alpha) = q(\operatorname{ad}x, \operatorname{ad}y)\cdot[x,y]$,
2) $\mathbf{h}(\alpha) = \mathbf{h}(A)$.

Proof. 1) We have as $(x+y-\tilde{z}) \in [\hat{\mathbf{L}}, \hat{\mathbf{L}}]$,

$$\alpha \equiv \left(1 - \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\right)\frac{1}{\operatorname{ad}z}\cdot(x+y-\tilde{z}) + \tfrac{1}{2}\tilde{z}$$

$$- \left(1 - \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\right)\frac{1}{\operatorname{ad}z}(x+y-z) - \tfrac{1}{2}z \quad \text{modulo } [[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$$

$$\equiv \left(1 - \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\right)\frac{1}{\operatorname{ad}z}(z-\tilde{z}) - \tfrac{1}{2}(z-\tilde{z})$$

and 1) will result from the following formula:

$$(5.3)\quad (z-\tilde{z}) \equiv \frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}\,\frac{e^{\operatorname{ad}x}-1}{\operatorname{ad}x}\,\frac{e^{\operatorname{ad}y}-1}{\operatorname{ad}y}\cdot[x,y] \quad \text{modulo } [[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]].$$

Proof of (5.3). Let

$$\varphi_1(x,y) = (e^x - e^{-y})^{-1}\left(\frac{e^x-1}{x} - \frac{1-e^{-y}}{y}\right)$$

$$\varphi_2(x,y) = (e^y - e^{-x})^{-1}\left(\frac{1-e^{-x}}{x} - \frac{e^y-1}{y}\right).$$

then φ_1 and φ_2 are analytic functions at the origin. We have

a) $x+y-\tilde{z} \equiv \varphi_1(\operatorname{ad}x, \operatorname{ad}y)\cdot[x,y]$,
b) $(x+y-z) \equiv \varphi_2(\operatorname{ad}x, \operatorname{ad}y)\cdot[x,y] \quad \text{mod } [[\hat{\mathbf{L}}, \hat{\mathbf{L}}], [\hat{\mathbf{L}}, \hat{\mathbf{L}}]]$.

For a) we consider

$$e^{\operatorname{ad}x} - e^{-\operatorname{ad}y})(x+y-\tilde{z}) = (e^{\operatorname{ad}x} - e^{-\operatorname{ad}y})(x+y)$$

(as $(e^{\operatorname{ad}x}-e^{-\operatorname{ad}y})(\tilde{z})=e^{-\operatorname{ad}y}(e^{\operatorname{ad}\tilde{z}}-1)\cdot\tilde{z}=0)$ so

$$(e^{\operatorname{ad}x}-e^{-\operatorname{ad}y})(x+y-\tilde{z})=(e^{\operatorname{ad}x}-1)+(1-e^{-\operatorname{ad}y})\cdot(x+y)$$
$$=\left(\frac{e^{\operatorname{ad}x}-1}{\operatorname{ad}x}-\frac{1-e^{-\operatorname{ad}y}}{\operatorname{ad}y}\right)\cdot[x,y]$$

and we obtain the equality a) by Remark 5.7. Now

$$z-\tilde{z}\equiv(\varphi_1-\varphi_2)(\operatorname{ad}x,\operatorname{ad}y)\cdot[x,y]$$

but

$$\varphi_1=(e^z-1)^{-1}\left(\frac{e^z-e^y}{x}-\frac{e^y-1}{y}\right),\quad \varphi_2=(e^z-1)^{-1}\left(\frac{e^x-1}{x}-\frac{e^z-e^x}{y}\right),$$

with $z=x+y$, and

$$\varphi_1-\varphi_2=(e^z-1)^{-1}\left(\frac{(e^x-1)(e^y-1)}{x}+\frac{(e^x-1)(e^y-1)}{y}\right)$$
$$=(e^z-1)^{-1}\frac{z}{xy}(e^x-1)(e^y-1)$$

and this proves Formula (5.3).

Let us prove 2) in Lemma 5.8. We let

$$\zeta(x,y)=\left(1-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}(x+y-\tilde{z})+\tfrac{1}{2}\tilde{z},$$

then $\alpha(x,y)=\zeta(x,y)-\zeta(y,x)$. As $x+y-\tilde{z}\in[\hat{\mathbf{L}},\hat{\mathbf{L}}]$, we have

$$\zeta(tx,ty)\equiv\left(1-\frac{t\operatorname{ad}\tilde{z}}{e^{t\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{t\operatorname{ad}\tilde{z}}(tx+ty-\tilde{z}(tx,ty))+\tfrac{1}{2}\tilde{z}(tx,ty)$$
$$\equiv\left(1-\frac{t\operatorname{ad}\tilde{z}}{e^{t\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}\left(x+y-\frac{1}{t}\tilde{z}(tx,ty)\right)+\tfrac{1}{2}\tilde{z}(tx,ty)$$
$$\text{modulo } [[\hat{\mathbf{L}},\hat{\mathbf{L}}],[\hat{\mathbf{L}},\hat{\mathbf{L}}]].$$

Here $\tilde{z}$ still denotes $\log e^y e^x$ and $\tilde{z}(tx,ty)=\log e^{ty}e^{tx}$. We have

$$\frac{\partial}{\partial t}\left(\left(1-\frac{tz}{e^{tz}-1}\right)\frac{1}{z}\right)_{t=1}=\frac{1}{e^z-1}\left(\frac{z}{1-e^{-z}}-1\right).$$

So

$$(\theta\zeta)(x,y)\equiv\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y-\tilde{z})+\left(1-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}(\tilde{z}-\theta\tilde{z})+\tfrac{1}{2}\theta\tilde{z}$$
$$\equiv\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)-\left(1-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}\cdot\theta\tilde{z}+\tfrac{1}{2}\theta\tilde{z}$$

as

$$\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot\tilde{z}=\tfrac{1}{2}\tilde{z}=\left(1-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\right)\frac{1}{\operatorname{ad}\tilde{z}}\cdot\tilde{z}.$$

Recalling that

$$\theta\tilde{z}=\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot y+\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\cdot x=\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)+\operatorname{ad}\tilde{z}\cdot x,$$

we obtain:

$$\begin{aligned}\theta\zeta(x,y)&\equiv\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)+\left(\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)\\&\quad+\left(\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}-1\right)\cdot x+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot y+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\cdot x\\&\equiv 2\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)-\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)\\&\quad+\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}x-x+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot y+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\cdot x\\&\equiv 2\left(\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\cdot x-x-\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}\cdot y.\end{aligned}$$

After antisymmetrization, we obtain

$$\begin{aligned}\theta\alpha(x,y)&\equiv 2\left(1-\frac{\operatorname{ad}z}{1-e^{-\operatorname{ad}z}}\right)\frac{1}{e^{\operatorname{ad}z}-1}\cdot(x+y)-2\left(1-\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}\right)\frac{1}{e^{\operatorname{ad}\tilde{z}}-1}\cdot(x+y)\\&\quad+\left(\tfrac{1}{2}\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}+\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{1-e^{-\operatorname{ad}\tilde{z}}}-1\right)\cdot x-\left(\tfrac{1}{2}\frac{\operatorname{ad}\tilde{z}}{e^{\operatorname{ad}\tilde{z}}-1}+\tfrac{1}{2}\frac{\operatorname{ad}z}{1-e^{-\operatorname{ad}z}}-1\right)\cdot y\\&\equiv\theta A.\quad \text{c.q.f.d.}\end{aligned}$$

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Received December 15, 1977/April 10, 1978