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Characters of the Negative Level Highest-Weight Modules for Affine Lie Algebras

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0 Introduction

Our main result in this paper is the following theorem conjectured by G. Lusztig [L].

Let $\mathfrak{g}$ be an affine Lie algebra with Cartan subalgebra $\mathfrak{h}$ and simple coroots $\{h_i\}_{i\in I}$. For $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ (resp. $L(\lambda)$) be the Verma module (resp. the irreducible module) with highest weight λ.

Theorem. For $\lambda \in \mathfrak{h}^*$ such that $(\lambda+\rho)(h_i) \in \mathbb{Z}_{<0}$ for any $i \in I$, we have

$$\operatorname{ch} L\left(w(\lambda+\rho)-\rho\right) = \sum_{y \leqq w} (-1)^{l(w)-l(y)} P_{y,w}(1) \operatorname{ch} M\left(y(\lambda+\rho)-\rho\right) \tag{0.1}$$

for any $w \in W$. Here W is the Weyl group, $\rho \in \mathfrak{h}^*$ is such that $\rho(h_i) = 1$ for any $i \in I$, l is the length function, $\leqq$ is the Bruhat order, $P_{y,w}$ is the Kazhdan-Lusztig polynomial, and ch denotes the character.

This type of character formula was first conjectured by Kazhdan and Lusztig [KL] for finite-dimensional semisimple Lie algebras and was proved by Beilinson-Bernstein [BB] and Brylinski-Kashiwara [BK]. Then its generalization to symmetrizable Kac-Moody Lie algebras concerning the dominant integral weights λ was given by Kashiwara (-Tanisaki) [K2], [KT] and Casian [C1]. This paper is concerned with a different version in the case of the antidominant integral weights for affine Lie algebras.

The schemes of the proofs of those character formulas are all similar. The $\mathfrak{g}$-modules correspond to $\mathcal{D}$-modules on the flag manifold, and the $\mathcal{D}$-modules correspond to perverse sheaves by the Riemann-Hilbert correspondence. Since the perverse sheaf corresponding to a dual Verma module (resp. irreducible highest-weight module) is the zero (resp. minimal) extension of the constant sheaf on a Schubert cell, the proof is

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reduced to the calculation of the local intersection cohomology groups of the Schubert varieties. This last step is now standard by the theory of Weil sheaves (cf. [BBD]) or the theory of Hodge modules (cf. [S]).

However, there are some differences between [K2], [KT] and our case. The natural setting in our case is of the right $\mathcal{D}$-modules supported on the finite-dimensional Schubert varieties, while in [K2], [KT] we used left $\mathcal{D}$-modules supported on the finite-codimensional Schubert varieties. The category of left $\mathcal{D}$-modules and the one of right $\mathcal{D}$-modules are equivalent on finite-dimensional manifolds. The flag manifold in our case is infinite-dimensional, and those two categories are not equivalent. The left $\mathcal{D}$-modules behave well under the pull-back while the right $\mathcal{D}$-modules behave well under the push-forward. This is the reason why we use right $\mathcal{D}$-modules.

For a technical reason, we do not directly treat the right $\mathcal{D}$-modules on the flag manifold itself. Instead, considering the fact that the flag manifold is locally isomorphic to the projective limit of finite-dimensional smooth varieties, we use a "projective limit" of right $\mathcal{D}$-modules on these finite-dimensional varieties as a substitute.

In this paper we will give descriptions of the category of those projective limits of right $\mathcal{D}$-modules, and the functor from this category to the category of $\mathfrak{g}$-modules. Our main result follows from several properties of this functor. Details of the proof will appear elsewhere.

The same result is claimed in the preprint of Casian [C2]. Our method is different, since a functor in the opposite direction is used in [C2].

1 The Kac-Moody Lie algebra

We recall basic facts concerning the Kac-Moody Lie algebra.

Let $\mathfrak{h}$ be a finite-dimensional vector space over $\mathbb{C}$, and let $\{\alpha_i\}_{i\in I}$, $\{h_i\}_{i\in I}$ be linearly independent vectors of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, such that $(\langle h_i, \alpha_j\rangle)_{i,j\in I}$ is a symmetrizable generalized Cartan matrix. The Kac-Moody Lie algebra associated to $(\mathfrak{h}, \{\alpha_i\}, \{h_i\})$ is the Lie algebra $\mathfrak{g}$ generated by the vector space $\mathfrak{h}$ and the elements e_i, f_i ($i \in I$) satisfying the following fundamental relations:

$$[h, h'] = 0 \qquad \text{for } h, h' \in \mathfrak{h}, \tag{1.1}$$

$$[h, e_i] = \alpha_i(h)e_i \qquad \text{for } h \in \mathfrak{h}, \quad i \in I, \tag{1.2}$$

$$[h, f_i] = -\alpha_i(h)f_i \qquad \text{for } h \in \mathfrak{h}, \quad i \in I, \tag{1.3}$$

$$[e_i, f_j] = \delta_{ij}h_i \qquad \text{for } i,\ j \in I, \tag{1.4}$$

$$\mathrm{ad}(e_i)^{1-\alpha_j(h_i)}(e_j) = 0 \qquad \text{for } i, j \in I \text{ with } i \neq j, \tag{1.5}$$

$$\mathrm{ad}(f_i)^{1-\alpha_j(h_i)}(f_j) = 0 \qquad \text{for } i, j \in I \text{ with } i \neq j. \tag{1.6}$$

For $i \in I$, let s_i be the linear automorphism of $\mathfrak{h}^*$ given by

$$s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \qquad \text{for } \lambda \in \mathfrak{h}^*. \tag{1.7}$$

The Weyl group W is by definition the subgroup of $GL(\mathfrak{h}^*)$ generated by $\{s_i\}_{i\in I}$. Then W is a Coxeter group with a canonical system of generators $\{s_i\}_{i\in I}$. We denote the length function by l and the Bruhat order of W by $\leqq$.

Set

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for } h \in \mathfrak{h}\} \qquad \text{for } \alpha \in \mathfrak{h}^*, \tag{1.8}$$

$$\Delta = \left\{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq \{0\}\right\} - \{0\}, \tag{1.9}$$

$$\Delta^+ = \Delta \cap \sum_{i\in I} \mathbb{Z}_{\geqq 0}\alpha_i, \tag{1.10}$$

$$\Delta_{re} = \bigcup_{i\in I} W(\alpha_i), \quad \Delta^+_{re} = \Delta^+ \cap \Delta_{re}, \tag{1.11}$$

and let $\mathfrak{n}$, $\mathfrak{n}^-$, $\mathfrak{b}$, $\mathfrak{b}^-$ be the subalgebras of $\mathfrak{g}$ generated by $\{e_i\}_{i\in I}$, $\{f_i\}_{i\in I}$, $\{e_i\}_{i\in I} \cup \mathfrak{h}$, $\{f_i\}_{i\in I} \cup \mathfrak{h}$, respectively. We then have

$$\Delta = \Delta^+ \cup (-\Delta^+), \quad W(\Delta) = \Delta, \tag{1.12}$$

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \tag{1.13}$$

$$\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}, \tag{1.14}$$

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-. \tag{1.15}$$

For $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$, $N(\lambda)$ be the $\mathfrak{g}$-modules defined by

$$M(\lambda) = U(\mathfrak{g}) \Big/ \left(\sum_{h\in\mathfrak{h}} U(\mathfrak{g})\left(h - \lambda(h)\right) + \sum_{i\in I} U(\mathfrak{g})e_i\right), \tag{1.16}$$

$$N(\lambda) = U(\mathfrak{g}) \Big/ \left(\sum_{h\in\mathfrak{h}} U(\mathfrak{g})\left(h - \lambda(h)\right) + \sum_{i\in I} U(\mathfrak{g})f_i\right), \tag{1.17}$$

where $U(\mathfrak{g})$ denotes the enveloping algebra of $\mathfrak{g}$. Let $M^*(\lambda)$ be the $\mathfrak{g}$-module consisting of $\mathfrak{h}$-finite vectors in $\mathrm{Hom}_{\mathbb{C}}\left(N(-\lambda), \mathbb{C}\right)$. The irreducible $\mathfrak{g}$-module $L(\lambda)$ with highest weight λ is naturally isomorphic to the image of the unique nonzero homomorphism $M(\lambda) \to M^*(\lambda)$.

2 The flag manifold

We fix a $\mathbb{Z}$-lattice P of $\mathfrak{h}^*$ satisfying

$$\alpha_i \in P, \quad \langle h_i, P\rangle \in \mathbb{Z} \qquad \text{for } i \in I. \tag{2.1}$$

For $\alpha = \sum_{i\in I} m_i\alpha_i \in \Delta^+$ set $|\alpha| = \sum_{i\in I} m_i$, and let

$$\mathfrak{n}_k = \bigoplus_{\substack{\alpha\in\Delta^+\\ |\alpha|\geqq k}} \mathfrak{g}_\alpha, \quad \mathfrak{n}_k^- = \bigoplus_{\substack{\alpha\in\Delta^+\\ |\alpha|\geqq k}} \mathfrak{g}_{-\alpha} \qquad \text{for } k \in \mathbb{Z}_{\geqq 0}. \tag{2.2}$$

Define group schemes by

$$T = \mathrm{Spec}(\mathbb{C}[P]), \tag{2.3}$$

$$U = \varprojlim_k \exp(\mathfrak{n}/\mathfrak{n}_k), \tag{2.4}$$

$$U^- = \varprojlim_k \exp(\mathfrak{n}^-/\mathfrak{n}_k^-), \tag{2.5}$$

$$B = (\text{the semidirect product of } T \text{ and } U), \tag{2.6}$$

$$B^- = (\text{the semidirect product of } T \text{ and } U^-). \tag{2.7}$$

Here, for a finite-dimensional nilpotent Lie algebra $\mathfrak{a}$, $\exp(\mathfrak{a})$ denotes the unipotent algebraic group with $\mathfrak{a}$ as its Lie algebra. We have natural isomorphisms

$$\exp : \prod_{\alpha\in\Delta^+} \mathfrak{g}_\alpha \longrightarrow U, \quad \exp : \prod_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha} \longrightarrow U^-. \tag{2.8}$$

In [K1] the first-named author constructed the flag variety of $(\mathfrak{g}, \mathfrak{h}, P)$ as the quotient

$$X = G/B, \tag{2.9}$$

where G is a scheme with locally free right action of B.

Let $\lambda \in P$. We denote the composite of the homomorphisms

$$B \longrightarrow T \xrightarrow{\lambda} \mathbb{G}_m$$

by $b \mapsto b^\lambda$. Let $\pi : G \to X$ be the canonical morphism. An invertible $\mathcal{O}_X$-module $\mathcal{O}_X(\lambda)$ is defined by

$$\Gamma\left(V; \mathcal{O}_X(\lambda)\right) = \left\{\varphi \in \Gamma(\pi^{-1}V; \mathcal{O}_G) \mid \varphi(gb) = \varphi(g)b^{-\lambda} \text{ for } (g,b) \in \pi^{-1}V \times B\right\} \tag{2.10}$$

for any open subset V of X.

We have natural left actions of the group scheme B^- and the braid group W' of W on X. Let Y be a T-stable subset of X. For $w' \in W'$, the set $w'Y$ depends only on the image $w \in W$ of w' under the canonical homomorphism $W' \to W$, and hence we simply denote it by wY.

Let B_0 be the subgroup of B generated by T and the elements of the form $\exp(x)$ with $x \in \mathfrak{g}_\alpha$, $\alpha \in \Delta^+_{\mathrm{re}}$. Note that B_0 is not a scheme but an inductive limit of schemes. We have a natural left action of B_0 on X compatible with those of B^- and W'. Let $x_0 \in X$ be 1 mod B. Then x_0 is a unique B_0-fixed point. We set

$$X^w = B^- w x_0, \quad X_w = B_0 w x_0 \qquad \text{for } w \in W. \tag{2.11}$$

Proposition 1 [K1]. (i) X^w is a locally closed subscheme of X with codimension $l(w)$, and if $\mathfrak{g}$ is not of finite type, it is isomorphic to the infinite-dimensional affine space $\mathbb{A}^\infty$.

(ii) $X = \sqcup_{w \in W} X^w$.

(iii) $\overline{X^w} = \sqcup_{y \geqq w} X^y$.

Moreover we have the following results.

Proposition 2. (i) X_w is a locally closed subscheme of X isomorphic to the affine space $\mathbb{A}^{l(w)}$.

(ii) $\sqcup_{w \in W} X_w$ is naturally isomorphic to the "flag manifold" treated in [KP], [T], and others.

(iii) $\overline{X_w} = \sqcup_{y \leqq w} X_y$.

3 $\mathcal{D}$-modules

For a finite subset F of W set

$$X^F = \bigcup_{w \in F} X^w, \quad X_F = \bigcup_{w \in F} X_w. \tag{3.1}$$

By Propositions 1 and 2, the set X^F (resp. X_F) is open (resp. closed) if and only if F satisfies

$$w \in F, \quad y \in W, \qquad y \leqq w \Longrightarrow y \in F. \tag{3.2}$$

A subset Y (resp. Z) of X is called an admissible open (resp. closed) subset if $Y = X^F$ (resp. $Z = X_F$) for a finite subset F of W satisfying (3.2).

Let Z be an admissible closed subset of X. In this case Z is projective. Take an admissible open subset Y of X containing Z. We can take such Y since we have $X^F \supset X_F$ if F satisfies (3.2). For $k \geqq 0$ set $U_k^- = \exp(\prod_{\substack{\alpha\in\Delta^+\\|\alpha|\geqq k}} \mathfrak{g}_{-\alpha}) \subset U^-$. If k is sufficiently large, then U_k^- acts on Y locally freely, and hence the quotient $U_k^-\backslash Y$ is a finite-dimensional smooth variety. We do not know if $U_k^-\backslash Y$ is separated or not. If k is large enough, the natural morphism $Z \to U_k^-\backslash Y$ is a closed immersion. Fix such Z, Y, and k. For $l \geqq k$ set

$$Y_l = U_l^-\backslash Y \tag{3.3}$$

and let

$$\pi_l : Y \longrightarrow Y_l \tag{3.4}$$

$$p_l : Y_{l+1} \longrightarrow Y_l \tag{3.5}$$

$$i_l : Z \hookrightarrow Y_l \tag{3.6}$$

be the natural morphisms. For $\lambda \in P$, we can naturally define an invertible $\mathcal{O}_{Y_l}$-module $\mathcal{O}_{Y_l}(\lambda)$ satisfying

$$\pi_l^*\mathcal{O}_{Y_l}(\lambda) \cong \mathcal{O}_X(\lambda) \mid Y \quad \text{and} \tag{3.7}$$

$$p_l^*\mathcal{O}_{Y_l}(\lambda) \cong \mathcal{O}_{Y_{l+1}}(\lambda). \tag{3.8}$$

Let $\mathcal{D}_{Y_l}$ be the sheaf of differential operators on Y_l, and set

$$\mathcal{D}_{Y_{l+1}\to Y_l} = \mathcal{O}_{Y_{l+1}} \otimes_{p_l^{-1}\mathcal{O}_{Y_l}} p_l^{-1}\mathcal{D}_{Y_l}, \tag{3.9}$$

$$\mathcal{D}_{Y_l}(\lambda) = \mathcal{O}_{Y_l}(-\lambda) \otimes_{\mathcal{O}_{Y_l}} \mathcal{D}_{Y_l} \otimes_{\mathcal{O}_{Y_l}} \mathcal{O}_{Y_l}(\lambda), \tag{3.10}$$

$$\mathcal{D}_{Y_{l+1}\to Y_l}(\lambda) = \mathcal{O}_{Y_{l+1}}(-\lambda) \otimes_{\mathcal{O}_{Y_{l+1}}} \mathcal{D}_{Y_{l+1}\to Y_l} \otimes_{p_l^{-1}\mathcal{O}_{Y_l}} p_l^{-1}\mathcal{O}_{Y_l}(\lambda). \tag{3.11}$$

Then $\mathcal{D}_{Y_l}(\lambda)$ is a ring acting on $\mathcal{O}_{Y_l}(-\lambda)$ from the left, and on $\Omega_{Y_l} \otimes \mathcal{O}_{Y_l}(\lambda)$ from the right. Here Ω_{Y_l} is the sheaf of differential forms of degree $\dim Y_l$. We have a natural $(\mathcal{D}_{Y_{l+1}}(\lambda), p_l^{-1}\mathcal{D}_{Y_l}(\lambda))$-bimodule structure on $\mathcal{D}_{Y_{l+1}\to Y_l}(\lambda)$.

Let $\mathbb{H}_l$ be the abelian category of right holonomic $\mathcal{D}_{Y_l}(\lambda)$-modules supported in Z. For $\mathcal{M} \in \mathbb{H}_l$ we define $\mathcal{M}^* \in \mathbb{H}_l$ by

$$\mathcal{M}^* = \Omega_{Y_l} \otimes \mathcal{O}_{Y_l}(2\lambda) \otimes \mathcal{E}xt^{\dim Y_l}_{\mathcal{D}_{Y_l}(\lambda)}\left(\mathcal{M}, \mathcal{D}_{Y_l}(\lambda)\right). \tag{3.12}$$

This defines a contravariant exact functor $*: \mathbb{H}_l \to \mathbb{H}_l$ such that $** = \mathrm{id}$. For $\mathcal{M} \in \mathbb{H}_{l+1}$ we define $\int_{p_l} \mathcal{M} \in \mathbb{H}_l$ by

$$\int_{p_l} \mathcal{M} = (p_l)_* \left(\mathcal{M} \otimes_{\mathcal{D}_{Y_{l+1}}(\lambda)} \mathcal{D}_{Y_{l+1}\to Y_l}(\lambda)\right). \tag{3.13}$$

Since $i_{l+1}(Z) \xrightarrow{\sim} i_l(Z)$, $\int_{p_l}$ induces an equivalence $\mathbb{H}_{l+1} \xrightarrow{\sim} \mathbb{H}_l$.

We define an abelian category $\mathbb{H} = \mathbb{H}(Z, \lambda, Y, k)$ as follows. An object is a family $\mathcal{M} = (\mathcal{M}_l)_{l \geqq k} \in \prod_{l \geqq k} \mathrm{Ob}(\mathbb{H}_l)$ together with isomorphisms

$$r_l : \int_{p_l} \mathcal{M}_{l+1} \xrightarrow{\sim} \mathcal{M}_l, \tag{3.14}$$

and morphisms are given by

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{H}}(\mathcal{M}, \mathcal{N}) \\ &= \left\{ (\varphi_l) \in \prod_{l \geqq k} \mathrm{Hom}_{\mathbb{H}_l}(\mathcal{M}_l, \mathcal{N}_l) \;\middle|\; \varphi_l \circ r_l = r_l \circ \left(\int_{p_l} \varphi_{l+1}\right) \text{ for } l \geqq k \right\}. \end{aligned} \tag{3.15}$$

Note that $\mathbb{H}$ is equivalent to $\mathbb{H}_l$. The duality functor $*: \mathbb{H} \to \mathbb{H}$ is defined by

$$\mathcal{M}^* = (\mathcal{M}_l^*)_{l \geqq k}. \tag{3.16}$$

Since $\int_{p_l}$ commutes with the duality, this is well defined. It is a contravariant exact functor such that $** = \mathrm{id}$. Note that the category $\mathbb{H}(Z, \lambda, Y, k)$ does not depend on Y and k. Moreover for $\lambda, \lambda' \in P$, $\mathbb{H}(Z, \lambda, Y, k)$ and $\mathbb{H}(Z, \lambda', Y, k)$ are equivalent by $\mathcal{M}_l \mapsto \mathcal{M}_l \otimes \mathcal{O}_{Y_l}(\lambda' - \lambda)$.

For $\mathcal{M} \in \mathbb{H}$ the homomorphism $\mathcal{O}_{Y_{l+1}} \to \mathcal{D}_{Y_{l+1}\to Y_l}(\lambda)$ induces a homomorphism $p_{l*}\mathcal{M}_{l+1} \to \mathcal{M}_l$, which gives a homomorphism

$$H^i(Y_{l+1}; \mathcal{M}_{l+1}) \longrightarrow H^i(Y_l; \mathcal{M}_l) \qquad \text{for } i \geqq 0. \tag{3.17}$$

Set

$$H^i(Y; \mathcal{M}) = \varprojlim_{l} H^i(Y_l; \mathcal{M}_l) \qquad \text{for } i \geqq 0. \tag{3.18}$$

Proposition 3. For $\mathcal{M} \in \mathbb{H}$, $H^i(Y; \mathcal{M})$ carries a natural structure of a $\mathfrak{g}$-module.

There is no homomorphism from $\mathfrak{g}$ to $\mathcal{D}_{Y_l}(\lambda)$ because there is no action of $\mathfrak{g}$ on Y_l. However, we can construct a section of $\mathcal{D}_{Y_{l+m}\to Y_l}(\lambda)$ corresponding to $A \in \mathfrak{g}$ if $m \gg 0$. This induces an action of $\mathfrak{g}$ on $H^i(Y; \mathcal{M})$.

For $w \in W$ such that $X_w \subset Z$, we define $\mathcal{B}_w$, $\mathcal{M}_w$, $\mathcal{L}_w \in \mathrm{Ob}(\mathbb{H})$ by

$$(\mathcal{B}_w)_l = \mathcal{H}^{\operatorname{codim} i_l(X_w)}_{i_l(X_w)}\left(\Omega_{Y_l} \otimes_{\mathcal{O}_{Y_l}} \mathcal{O}_{Y_l}(\lambda)\right), \tag{3.19}$$

$$\mathcal{M}_w = \mathcal{B}_w^*, \tag{3.20}$$

$$\mathcal{L}_w = \text{the image of the natural morphism } \mathcal{M}_w \to \mathcal{B}_w. \tag{3.21}$$

Then $\mathcal{L}_w$ is a simple object of $\mathbb{H}$ satisfying $\mathcal{L}_w^* \cong \mathcal{L}_w$.

Let $\mathbb{H}_0 = \mathbb{H}_0(Z, \lambda, Y, k)$ be the full subcategory of $\mathbb{H}$ consisting of $\mathcal{M} \in \mathbb{H}$ whose composition factors are isomorphic to $\mathcal{L}_w$ for some $w \in W$ satisfying $X_w \subset Z$.

4 The main result

We fix $\rho \in \mathfrak{h}^*$ satisfying

$$\langle h_i, \rho \rangle = 1 \qquad \text{for any } i \in I. \tag{4.1}$$

Define a shifted action of the Weyl group W on P by

$$w \circ \lambda = w(\lambda + \rho) - \rho. \tag{4.2}$$

Assume that $\lambda \in P$ satisfies the following:

$$\langle h_i, \lambda \rangle < -1 \qquad \text{for any } i \in I; \tag{4.3}$$

$$\text{the Verma module } M(\lambda) \text{ is an irreducible } \mathfrak{g}\text{-module.} \tag{4.4}$$

We keep the notation of Section 3. Let $F \subset W$ be such that $Z = X_F$. Then we have the following theorem.

Theorem.

(i) $H^n(Y; \mathcal{M}) = 0$ for any $\mathcal{M} \in \mathbb{H}_0$ and $n > 0$;

(ii) $H^0(Y; *)$ defines an exact functor from $\mathbb{H}_0$ to the category of $U(\mathfrak{g})$-modules;

(iii) $H^0(Y; \mathcal{B}_w) = \hat{M}^*(w \circ \lambda)$ for any $w \in F$;

(iv) $H^0(Y; \mathcal{M}_w) = \hat{M}(w \circ \lambda)$ for any $w \in F$;

(v) $H^0(Y; \mathcal{L}_w) = \hat{L}(w \circ \lambda)$ for any $w \in F$.

Here $\hat{M}^*(w \circ \lambda)$ is the completion of $M^*(w \circ \lambda)$, etc.

By [KK] the condition (4.4) is satisfied if $\mathfrak{g}$ is an affine Lie algebra and if λ satisfies (4.3). The conjecture of Lusztig is easily derived from this theorem along with the standard arguments.

The proof of this theorem is accomplished by showing the following statements by induction on the dimension of Z.

For $\mathcal{M} \in \mathbb{H}_0$, $\mu \in P$, and $n \in \mathbb{Z}$, the weight space $H^n(Y_l; \mathcal{M}_l)_\mu$ is constant for $l \gg k$. (4.5)

This implies that $H^n(Y; \mathcal{M})$ is isomorphic to $\prod_\mu \big(\varprojlim_l H^n(Y_l; \mathcal{M}_l)_\mu\big)$. We denote $\oplus_\mu \big(\varprojlim_l H^n(Y_l; \mathcal{M}_l)\big)_\mu$ by $\bar{H}^n(Y; \mathcal{M})$. Hence $\bar{H}^n(Y; \mathcal{M})$ is the set of $\mathfrak{h}$-finite vectors of $H^n(Y; \mathcal{M})$.

$$\bar{H}^n(Y; \mathcal{M}) = 0 \qquad \text{for } \mathcal{M} \in \mathbb{H}_0 \text{ and } n > 0. \tag{4.6}$$

$$\mathrm{Hom}_{\mathbb{H}_0}(\mathcal{M}, \mathcal{N}) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(\bar{H}^0(Y; \mathcal{M}), \bar{H}^0(Y; \mathcal{N})\right) \qquad \text{for } \mathcal{M}, \mathcal{N} \in \mathbb{H}_0. \tag{4.7}$$

$$\bar{H}^0(Y; \mathcal{M}) \otimes \mathcal{D}(\lambda) = \mathcal{M} \qquad \text{for } \mathcal{M} \in \mathbb{H}_0. \tag{4.8}$$

Let $\mathcal{M} \in \mathbb{H}_0$ and let N be a $U(\mathfrak{g})$-submodule of $\bar{H}^0(Y; \mathcal{M})$.
If $N \otimes \mathcal{D}(\lambda) = \mathcal{M}$, then $N = \bar{H}^0(Y; \mathcal{M})$. (4.9)

$$\bar{H}^0(Y; \mathcal{B}_w) = M^*(w \circ \lambda) \qquad \text{for } w \in F. \tag{4.10}$$

$$\bar{H}^0(Y; \mathcal{M}_w) = M(w \circ \lambda) \qquad \text{for } w \in F. \tag{4.11}$$

$$\bar{H}^0(Y; \mathcal{L}_w) = L(w \circ \lambda) \qquad \text{for } w \in F. \tag{4.12}$$

In (4.8) and (4.9), for a $U(\mathfrak{g})$-submodule N of $\bar{H}^0(Y; \mathcal{M})$, $N \otimes \mathcal{D}(\lambda)$ denotes the subobject of $\mathcal{M}$ "generated by" N. We remark that we do not know how to construct the functor $M \mapsto M \otimes D(\lambda)$ from a category of $\mathfrak{g}$-modules to $\mathbb{H}_0$, but there is no problem in constructing $N \otimes \mathcal{D}(\lambda) \subset \mathcal{M}$ for $N \subset \bar{H}^0(Y; \mathcal{M})$.

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