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ON CRYSTAL BASES OF THE Q-ANALOGUE OF UNIVERSAL ENVELOPING ALGEBRAS

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To the memory of Professor Michio Kuga who taught me the joy of doing mathematics

CONTENTS

§0. Introduction. The notion of the q-analogue of universal enveloping algebras is introduced independently by V. G. Drinfeld and M. Jimbo in 1985 in their study of exactly solvable models in the statistical mechanics. This algebra $U_q(\mathfrak{g})$ contains a parameter q, and, when $q = 1$, this coincides with the universal enveloping algebra. In the context of exactly solvable models, the parameter q is that of temperature, and $q = 0$ corresponds to the absolute temperature zero. For that reason, we can expect that the q-analogue has a simple structure at $q = 0$. In [K1] we named crystallization the study at $q = 0$, and we introduced the notion of crystal bases. Roughly speaking, crystal bases are bases of $U_q(\mathfrak{g})$-modules at $q = 0$ that satisfy certain axioms. There, we proved the existence and the uniqueness of crystal bases of finite-dimensional representations of $U_q(\mathfrak{g})$ when $\mathfrak{g}$ is one of the classical Lie algebras A_n, B_n, C_n and D_n. K. Misra and T. Miwa ([M]) proved the existence of a crystal base of the basic representation of $U_q(A_n^{(1)})$ and gave its combinatorial description.

The aim of this article is to give the proof of the existence and uniqueness theorem of crystal bases for an arbitrary symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. Moreover, we globalize this notion. Namely, with the aid of a crystal base we construct a base named the global crystal base of any highest weight irreducible integrable

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$U_q(\mathfrak{g})$-module. In the case of A_n, D_n, and E_n, this coincides with the canonical base of Lusztig introduced in [L1]. (Cf. [L2].)

Let us explain more precisely our results. Let $U_q(\mathfrak{g})$ be the q-analogue of universal enveloping algebra. (Cf. §1.1.) For an integrable $U_q(\mathfrak{g})$-module M (cf. §1.2), we introduce the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ of M. (Cf. §2.2.) Then we define the notion of crystal base of M. (Cf. Definition 2.3.1.)

For an integral dominant weight λ, let $V(\lambda)$ denote the irreducible $U_q(\mathfrak{g})$-module with highest weight λ. Let u_λ be the highest weight vector of $V(\lambda)$. We denote by A the ring of rational functions in the variable q regular at $q = 0$. Let $L(\lambda)$ be the smallest sub-A-module of $V(\lambda)$ that contains u_λ and that is stable by the actions of $\tilde{f}_i$. Let $B(\lambda)$ be the subset of $L(\lambda)/qL(\lambda)$ consisting of the nonzero vectors of the form $\tilde{f}_{i_1}\dots\tilde{f}_{i_l}u_\lambda$ mod $qL(\lambda)$. Our first main result is an existence theorem.

THEOREM 2 (existence). *$(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.*

Similarly to the case of an integrable $U_q(\mathfrak{g})$-module, we define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ on $U_q^-(\mathfrak{g})$. (Cf. (3.5.1).) They satisfy $\tilde{e}_i\tilde{f}_i = 1$. Here $U_q^-(\mathfrak{g})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i. We denote by $L(\infty)$ the smallest sub-A-module of $U_q^-(\mathfrak{g})$ that contains 1 and that is stable by the actions of $\tilde{f}_i$. We denote by $B(\infty)$ the subset of $L(\infty)/qL(\infty)$ consisting of vectors $\tilde{f}_{i_1}\dots\tilde{f}_{i_l}\cdot 1$ mod $qL(\infty)$. Then $(L(\infty), B(\infty))$ has a similar property to crystal bases.

THEOREM 4. *We have that*

(i) $\tilde{e}_iL(\infty) \subset L(\infty)$, $\tilde{f}_iL(\infty) \subset L(\infty)$, *and* $\tilde{e}_iB(\infty) \subset B(\infty) \cup \{0\}$, $\tilde{f}_iB(\infty) \subset B(\infty)$;
(ii) $B(\infty)$ *is a base of* $L(\infty)/qL(\infty)$; *and*
(iii) *if* $b \in B(\infty)$ *satisfies* $\tilde{e}_ib \neq 0$, *then* $b = \tilde{f}_i\tilde{e}_ib$.

The relations of $(L(\infty), B(\infty))$ and $(L(\lambda), B(\lambda))$ are given by the following theorem.

THEOREM 5. *Let* π_λ: $U_q^-(\mathfrak{g}) \to V(\lambda)$ *be the* $U_q^-(\mathfrak{g})$*-linear homomorphism sending* 1 *to* u_λ. *Then*

(i) $\pi_\lambda(L(\infty)) = L(\lambda)$.
Hence π_λ *induces the surjective homomorphism* $\bar{\pi}_\lambda$: $L(\infty)/qL(\infty) \to L(\lambda)/qL(\lambda)$.
(ii) *By* $\bar{\pi}_\lambda$, $\{b \in B(\infty); \bar{\pi}_\lambda(b) \neq 0\}$ *is isomorphic to* $B(\lambda)$.
(iii) $\tilde{f}_i \circ \bar{\pi}_\lambda = \bar{\pi}_\lambda \circ \tilde{f}_i$.
(iv) *If* $b \in B(\infty)$ *satisfies* $\bar{\pi}_\lambda(b) \neq 0$, *then* $\tilde{e}_i\bar{\pi}_\lambda(b) = \bar{\pi}_\lambda(\tilde{e}_ib)$.

These three theorems are proven simultaneously by the induction on weights. The good behavior of crystal bases under tensor products plays a crucial role in the course of the proof.

Thus, we can construct bases of $U_q^-(\mathfrak{g})$ and $V(\lambda)$ at $q = 0$. Similarly, we can define bases at $q = \infty$. Then we can define bases of $U_q^-(\mathfrak{g})$ and $V(\lambda)$ which give the crystal bases at $q = 0$ or ∞. Let $U_{\mathbb{Z}}^-(\mathfrak{g})$ be the sub-$\mathbb{Z}[q, q^{-1}]$-algebra of $U_q(\mathfrak{g})$ generated by the $f_i^{(n)}$, introduced by Lusztig. Let $-$ be the ring homomorphism of $U_q^-(\mathfrak{g})$ given by $\bar{q} = q^{-1}$, $\bar{f}_i = f_i$. Let us denote by $V_{\mathbb{Z}}(\lambda)$ the $U_{\mathbb{Z}}(\mathfrak{g})$-module $U_{\mathbb{Z}}^-(\mathfrak{g})u_\lambda$ and let $-$ denote the automorphism of $V(\lambda)$ defined by $\overline{Pu_\lambda} = \bar{P}u_\lambda$ for any $P \in U_q^-(\mathfrak{g})$.

THEOREM 6. *$(\mathbb{Q}\otimes_{\mathbb{Z}} U_{\mathbb{Z}}^{-}(\mathfrak{g}))\cap L(\infty)\cap L(\infty)^{-}\cong L(\infty)/qL(\infty)$ and $(\mathbb{Q}\otimes_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda))\cap L(\lambda)\cap L(\lambda)^{-}\cong L(\lambda)/qL(\lambda)$ for any integrable dominant weight λ.*

Let $b\mapsto G(b)$ be the inverse of these isomorphisms. Then we have another theorem.

THEOREM 7. *Let n be a nonnegative integer and $i\in I$.*

(i) *We have*

$$f_i^n U_q^-(\mathfrak{g})\cap U_{\mathbb{Z}}^-(\mathfrak{g})=\bigoplus_{b\in \tilde{f}_i^n B(\infty)}\mathbb{Z}[q,q^{-1}]G(b).$$

(ii) *For any dominant integral weight λ, we have*

$$f_i^n V(\lambda)\cap V_{\mathbb{Z}}(\lambda)=\bigoplus_{b\in B(\lambda)\cap \tilde{f}_i^n B(\lambda)}\mathbb{Z}[q,q^{-1}]G(b).$$

These results were announced in Comptes Rendus ([K2]).

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PART I. CRYSTALLIZATION

§1. The q-analogue of universal enveloping algebra

1.1. *Definition of $U_q(\mathfrak{g})$.* We shall review the definition of $U_q(\mathfrak{g})$. Suppose that the following data are given.

(1.1.1) a finite-dimensional $\mathbb{Q}$-vector space $\mathfrak{t}$,

(1.1.2) a finite index set I (the set of simple roots),

(1.1.3) a *linearly independent* subset $\{\alpha_i\in\mathfrak{t}^*; i\in I\}$ of $\mathfrak{t}^*$ and a subset $\{h_i\in\mathfrak{t}; i\in I\}$ of $\mathfrak{t}$,

(1.1.4) a ($\mathbb{Q}$-valued) symmetric form (,) on $\mathfrak{t}^*$, and

(1.1.5) a lattice P of $\mathfrak{t}^*$.

We assume that they satisfy the following properties.

(1.1.6) $\langle h_i,\alpha_j\rangle$ is a generalized Cartan matrix (i.e. $\langle h_i,\alpha_i\rangle=2$, $\langle h_i,\alpha_j\rangle\in\mathbb{Z}_{\leqslant 0}$ for $i\neq j$ and $\langle h_i,\alpha_j\rangle=0\Leftrightarrow\langle h_j,\alpha_i\rangle=0$).

(1.1.7) $(\alpha_i,\alpha_i)\in\mathbb{Z}_{>0}$.

(1.1.8) $\langle h_i, \lambda\rangle = \dfrac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for any i and $\lambda \in \mathfrak{t}^*$.

(1.1.9) $\alpha_i \in P$ and $h_i \in P^* = \{h \in \mathfrak{t}; \langle h, P\rangle \subset \mathbb{Z}\}$ for any i.

Hence $\{\langle h_i, \alpha_j\rangle\}$ is a symmetrizable generalized Cartan matrix. Let $\mathfrak{g}$ be the associated Kac-Moody Lie algebra; i.e., $\mathfrak{g}$ is the Lie algebra generated by $\mathfrak{t}$, e_i, and f_i $(i \in I)$ with the following fundamental commutation relations.

(1.1.10) $\mathfrak{t}$ is an abelian subalgebra of $\mathfrak{g}$,

(1.1.11) $[h, e_i] = \langle h, \alpha_i\rangle e_i$, $[h, f_i] = -\langle h, \alpha_i\rangle f_i$,

(1.1.12) $[e_i, f_j] = \delta_{ij} h_i$, and

(1.1.13) $(ad e_i)^{1-\langle h_i, \alpha_j\rangle} e_j = (ad f_i)^{1-\langle h_i, \alpha_j\rangle} f_j = 0$ for $i \neq j$.

Then the q-analogue $U_q(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ is by definition the algebra over the rational function field $\mathbb{Q}(q)$ generated by the symbols e_i, f_i $(i \in I)$ and q^h $(h \in P^*)$ with the following fundamental commutation relations.

(1.1.14) $q^h = 1$ for $h = 0$.

(1.1.15) $q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^*$.

(1.1.16) $q^h e_i q^{-h} = q^{\langle h, \alpha_i\rangle} e_i$ and $q^h f_i q^{-h} = q^{-\langle h, \alpha_i\rangle} f_i$

for $h \in P^*$ and $i \in I$.

(1.1.17) Setting $q_i = q^{(\alpha_i, \alpha_i)}$ and $t_i = q^{(\alpha_i, \alpha_i) h_i}$, $[e_i, f_j] = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$.

(1.1.18) For $i \neq j$, setting $b = 1 - \langle h_i, \alpha_j\rangle$,

$$\sum_{n=0}^{b} (-1)^n e_i^{(n)} e_j e_i^{(b-n)} = \sum_{n=0}^{b} (-1)^n f_i^{(n)} f_j f_i^{(b-n)} = 0.$$

Here we set

(1.1.19)
$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \ [n]_i! = \prod_{k=1}^{n} [k]_i,$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_i = \frac{[n]_i!}{[m]_i! [n-m]_i!} \quad \text{for } n \geqslant m \geqslant 0 \text{ and}$$

$$e_i^{(n)} = e_i^n/[n]_i!, f_i^{(n)} = f_i^n/[n]_i!.$$

We understand $e_i^{(n)} = f_i^{(n)} = 0$ for $n < 0$.

Note that we have

(1.1.20) $$q_i^{\langle h_i, \alpha_j\rangle} = q_j^{\langle h_j, \alpha_i\rangle} = q^{2(\alpha_i, \alpha_j)}.$$

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the sub-$\mathbb{Q}(q)$-algebra of $U_q(\mathfrak{g})$ generated by the e_i (resp. f_i). Then we have (cf. [L1], [L2], [L3])

(1.1.21) $$U_q(\mathfrak{g}) \cong U_q^-(\mathfrak{g}) \bigotimes_{\mathbb{Q}(q)} \mathbb{Q}(q)[P^*] \bigotimes_{\mathbb{Q}(q)} U_q^+(\mathfrak{g}).$$

Here $\mathbb{Q}(q)[P^*]$ is the group ring $\oplus_{h \in P^*} \mathbb{Q}(q) q^h$. We set

(1.1.22) $$Q = \sum Z\alpha_i, Q_+ = \sum \mathbb{Z}_{\geqslant 0}\alpha_i \quad \text{and} \quad Q_- = -Q_+.$$

We use frequently the formula

(1.1.23) $$t_i e_j t_i^{-1} = q_i^{\langle h_i, \alpha_j\rangle} e_j, \ t_i f_j t_i^{-1} = q_i^{-\langle h_i, \alpha_j\rangle} f_j \quad \text{and}$$

$$e_i^{(n)} f_i^{(m)} = \sum_{k \geqslant 0} f_i^{(m-k)} e_i^{(n-k)} \begin{Bmatrix} q_i^{n-m} t_i \\ k \end{Bmatrix}_i.$$

Here we use the notations

(1.1.24) $$\{x\}_i = (x - x^{-1})/(q_i - q_i^{-1}) \quad \text{and} \quad \begin{Bmatrix} x \\ n \end{Bmatrix}_i = \frac{\prod_{k=1}^{n} \{q_i^{1-k} x\}_i}{[n]_i!}.$$

Hence we have

(1.1.25) $$\begin{Bmatrix} q_i^m \\ n \end{Bmatrix}_i = \begin{cases} \begin{bmatrix} m \\ n \end{bmatrix}_i & \text{for } m \geqslant n \geqslant 0, \\ 0 & \text{for } n > m \geqslant 0, \\ (-1)^n \begin{bmatrix} n-1-m \\ n \end{bmatrix}_i & \text{for } n \geqslant 0 > m, \\ 1 & \text{for } n = 0. \end{cases}$$

Note also that

(1.1.26) $$[n]_i! \in q_i^{-n(n-1)/2}(1 + qA) \quad \text{and}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_i \in q_i^{-n(m-n)}(1 + qA) \quad \text{for} \quad m \geqslant n \geqslant 0.$$

Here A is the subring of $\mathbb{Q}(q)$ consisting of rational functions without poles at $q = 0$.

We have

(1.1.27) $$2(P, Q) \subset Z,$$

(1.1.28) $$(\lambda, \lambda) \in \mathbb{Z} \quad \text{for any} \quad \lambda \in Q,$$

(1.1.29)
$$(\lambda, \lambda) - (\mu, \mu) \in \mathbb{Z} \quad \text{for any} \quad \lambda, \mu \in P \quad \text{such that} \quad \lambda - \mu \in Q, \quad \text{and}$$

(1.1.30) $$2(\lambda_1, \lambda_2) - 2(\mu_1, \mu_2) \in \mathbb{Z}$$

$$\text{for any} \quad \lambda_j, \mu_j \in P \quad \text{such that} \quad \lambda_j - \mu_j \in Q\ (j = 1, 2).$$

In fact, (1.1.27) follows from (1.1.7) and $2(\lambda, \alpha_i) = (\alpha_i, \alpha_i)\langle h_i, \lambda\rangle$, (1.1.28) follows from (1.1.7) and $2(Q, Q) \subset \mathbb{Z}$, (1.1.29) follows from $(\lambda, \lambda) - (\mu, \mu) = (\lambda - \mu, \lambda - \mu) + 2(\mu, \lambda - \mu)$, and finally (1.1.30) follows from $2(\lambda_1, \lambda_2) - 2(\mu_1, \mu_2) = 2(\lambda_1 - \mu_1, \lambda_2) + 2(\mu_1, \lambda_2 - \mu_2)$ and (1.1.27).

Remark 1.1.1. We may replace the inner product $(\ ,\ )$ on $\mathfrak{t}^*$ with $c(\ ,\ )$ for a positive integer c. This gives the same effect as replacing q with q^c.

1.2. *Integrable representations.* Let M be a $U_q(\mathfrak{g})$-module. For any $\lambda \in P$, we set

(1.2.1) $$M_\lambda = \{u \in M;\ q^h u = q^{\langle h, \lambda\rangle} u \quad \text{for any} \quad h \in P^*\}.$$

We say that M is *integrable* if M satisfies the conditions that

(1.2.2) $$M = \bigoplus_{\lambda \in P} M_\lambda,$$

(1.2.3) $$\dim M_\lambda < \infty \quad \text{for any} \quad \lambda, \quad \text{and}$$

(1.2.4) for any i, M is a union of finite-dimensional $U_q(\mathfrak{g}_i)$-modules.

Here $U_q(\mathfrak{g}_i)$ is the subalgebra generated by e_i and f_i. In this paper we consider only integrable representations. Note that the condition (1.2.3) is less important and that most of our results hold without this condition.

Remark that for any $a \in P/Q$, letting p be the projection $P \to P/Q$,

(1.2.5) $$M_{[a]} = \bigoplus_{\lambda \in p^{-1}(a)} M_\lambda$$

is a $U_q(\mathfrak{g})$-module and $M = \oplus_a M_{[a]}$.

We set

(1.2.6) $$P_+ = \{\lambda \in P;\ \langle h_i, \lambda\rangle \geqslant 0 \quad \text{for any} \quad i \in I\}.$$

Let $\lambda \in P_+$ and let $V(\lambda)$ be the irreducible $U_q(\mathfrak{g})$-module with highest weight λ. Let u_λ be its highest weight vector. Then we have (cf. [L1], [L2], [L3])

$$(1.2.7)\quad V(\lambda) = U_q(\mathfrak{g})\Big/\Big(\sum_i U_q(\mathfrak{g})e_i + \sum_h U_q(\mathfrak{g})(q^h - q^{\langle h,\lambda\rangle}) + \sum_i U_q(\mathfrak{g})f_i^{1+\langle h_i,\lambda\rangle}\Big)$$

$$\cong U_q^-(\mathfrak{g})\Big/\sum_i U_q^-(\mathfrak{g})f_i^{1+\langle h_i,\lambda\rangle}.$$

Let $\mathcal{O}_{\text{int}}$ denote the category of integrable $U_q(\mathfrak{g})$-modules M such that there exists a finite subset F of P with $M = \bigoplus_{\lambda\in F+Q_-} M_\lambda$. Then it is known (cf. [L1], [L2], [L3], [R]) that $\mathcal{O}_{\text{int}}$ is a semisimple category and that its irreducible objects are isomorphic to some $V(\lambda)$.

1.3. *Automorphisms of* $U_q(\mathfrak{g})$. We denote by $*$ the antiautomorphism of $U_q(\mathfrak{g})$ as $\mathbb{Q}(q)$-algebra given by

$$(1.3.1)\qquad e_i^* = e_i, f_i^* = f_i \quad \text{and} \quad (q^h)^* = q^{-h}.$$

We denote by $-$ the automorphism of $U_q(\mathfrak{g})$ given by

$$(1.3.2)\qquad \overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q^h} = q^{-h}$$

$$(1.3.3)\qquad \overline{a(q)u} = a(q^{-1})\overline{u} \quad \text{for any} \quad a(q) \in \mathbb{Q}(q) \quad \text{and} \quad u \in U_q(\mathfrak{g}).$$

We can check easily that they are well defined. They preserve $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$. Moreover, we have

$$(1.3.4)\qquad -\,- = ** = id \quad \text{and} \quad *- = -\,*.$$

1.4. *Comultiplications.* We shall define two comultiplications $\Delta_\pm\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ that satisfy the coassociative law:

$$(1.4.1)\qquad \begin{array}{ccc} U_q(\mathfrak{g}) & \xrightarrow{\Delta_\pm} & U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \\ \downarrow{\scriptstyle \Delta_\pm} & & \downarrow{\scriptstyle \Delta_\pm\otimes id} \\ U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) & \xrightarrow{id\otimes\Delta_\pm} & U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \end{array}$$

is a commutative diagram.

$$(1.4.2)\qquad \Delta_+(q^h) = q^h \otimes q^h,$$

$$\Delta_+(e_i) = e_i \otimes 1 + t_i \otimes e_i,$$

$$\Delta_+(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i;$$

$$\Delta_-(q^h) = q^h \otimes q^h, \tag{1.4.3}$$

$$\Delta_-(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i,$$

$$\Delta_-(f_i) = f_i \otimes 1 + t_i \otimes f_i.$$

The well-definedness of $\Delta_\pm$ and (1.4.1) can be easily verified. These two comultiplications are related as follows. Via $\Delta_\pm$, the tensor product $M \otimes N$ of $U_q(\mathfrak{g})$-modules M and N has two structures of $U_q(\mathfrak{g})$-module. We denote by $M \otimes_\pm N$ the $U_q(\mathfrak{g})$-module $M \otimes N$ via $\Delta_\pm$. Now assume that M and N have weight decomposition

$$M = \bigoplus_{\lambda \in P} M_\lambda, \qquad N = \bigoplus_{\lambda \in P} N_\lambda. \tag{1.4.4}$$

Assume that $2(P, P) \subset \mathbb{Z}$ for the sake of simplicity. Then we define

$$\varphi_{M,N}\colon M \otimes_- N \to M \otimes_+ N \tag{1.4.5}$$

by $\varphi_{MN}(u \otimes v) = q^{2(\lambda,\mu)}(u \otimes v)$ for $u \in M_\lambda$ and $v \in N_\mu$.
Then we can check easily that $\varphi_{M,N}$ is a $U_q(\mathfrak{g})$-linear isomorphism. Moreover, if $(\lambda, \lambda) \in \mathbb{Z}$ for any $\lambda \in P$, we define $\psi_M \in \operatorname{Aut}(M)$ by

$$\psi_M(u) = q^{-(\lambda,\lambda)}u \qquad \text{for} \qquad u \in M_\lambda; \tag{1.4.6}$$

then the following diagram commutes:

$$\begin{array}{ccc} M \otimes_- N & \xrightarrow{\varphi_{M,N}} & M \otimes_+ N \\ \Big\downarrow \psi_{M \otimes_- N} & \searrow^{\psi_M \otimes \psi_N} & \Big\downarrow \psi_{M \otimes_+ N} \\ M \otimes_- N & \xrightarrow[\varphi_{M,N}]{} & M \otimes_+ N. \end{array} \tag{1.4.7}$$

We leave the verification to the reader. Note that we can endow the structures of Hopf algebra on $U_q(\mathfrak{g})$ with $\Delta_\pm$ as comultiplication.

Remark 1.4.1. If $2(P, P) \subset \mathbb{Z}$ is not satisfied, then, assuming $M = M_{[a]}$ and $N = N_{[b]}$ ($a, b \in P/Q$ and $\lambda_0 \in p^{-1}(a)$, $\mu_0 \in p^{-1}(b)$, see (1.2.5)), replace $2(\lambda, \mu)$ in the definition of $\varphi_{M,N}$ by $2(\lambda, \mu) - 2(\lambda_0, \mu_0)$ and replace $-(\lambda, \lambda)$ in the definition of ψ_M by $-(\lambda, \lambda) + (\lambda_0, \lambda_0)$. Then $2(\lambda, \mu) - 2(\lambda_0, \mu_0)$ and $-(\lambda, \lambda) + (\lambda_0, \lambda_0)$ are integers by (1.1.29) and (1.1.30), and hence $\varphi_{M,N}$ and ψ_M are well defined.

§2. Crystal base

2.1. *Upper and lower crystal bases.* In [K1] we introduced the notion of crystal base. We shall call it upper crystal base, and we shall introduce here lower crystal base. We shall see later that they are related as follows: (L, B) is a lower crystal base of M if and only if $\psi_M(L, B)$ is an upper crystal base.

2.2. *Operators $\tilde{e}_i$ and $\tilde{f}_i$.* Let M be an integral $U_q(\mathfrak{g})$-module. Then by the theory of integrable representations of $U_q(sl_2)$, we have

(2.2.1) $$M = \bigoplus_{0 \leqslant n \leqslant \langle h_i, \lambda \rangle} f_i^{(n)}(\operatorname{Ker} e_i \cap M_\lambda).$$

We define the endomorphisms $\tilde{e}_i$, $\tilde{f}_i$ of M by

(2.2.2) $$\tilde{f}_i(f_i^{(n)}u) = f_i^{(n+1)}u \quad \text{and} \quad \tilde{e}_i(f_i^{(n)}u) = f_i^{(n-1)}u$$

$$\text{for } u \in \operatorname{Ker} e_i \cap M_\lambda \text{ with } 0 \leqslant n \leqslant \langle h_i, \lambda \rangle.$$

Similarly, we have

(2.2.3) $$M = \bigoplus_{0 \leqslant n \leqslant -\langle h_i, \mu \rangle} e_i^{(n)}(\operatorname{Ker} f_i \cap M_\mu).$$

These two decompositions are connected as follows:

(2.2.4) if $0 \leqslant n \leqslant \langle h_i, \lambda \rangle$ and $u \in \operatorname{Ker} e_i \cap M_\lambda$,

then $v = f_i^{(\langle h_i, \lambda \rangle)}u$ belongs to $\operatorname{Ker} f_i \cap M_{s_i(\lambda)}$ and $f_i^{(n)}u = e_i^{(\langle h_i, \lambda \rangle - n)}v$.

Here $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$. Hence we obtain

(2.2.5) $$\tilde{f}_i(e_i^{(n)}v) = e_i^{(n-1)}v \quad \text{and} \quad \tilde{e}_i(e_i^{(n)}v) = e_i^{(n+1)}v$$

$$\text{for } v \in \operatorname{Ker} f_i \cap M_\mu \text{ with } 0 \leqslant n \leqslant -\langle h_i, \mu \rangle.$$

Note that $\tilde{f}_i(f_i^{(n)}u) = f_i^{(n+1)}u$ and $\tilde{e}_i(e_i^{(n)}v) = e_i^{(n+1)}v$ hold whenever $e_i u = 0$ and $f_i v = 0$.

2.3. *Crystal base.* Let M be an integrable $U_q(\mathfrak{g})$-module. Let A be the subring of $\mathbb{Q}(q)$ consisting of rational functions regular at $q = 0$.

Definition 2.3.1. A pair (L, B) is called a lower crystal base of M if it satisfies the following conditions:

(2.3.1) L is a free sub-A-module of M such that $M \cong \mathbb{Q}(q) \otimes_A L$,

(2.3.2) B is a base of the $\mathbb{Q}$-vector space L/qL,

(2.3.3) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for any i.

By this $\tilde{f}_i$ and $\tilde{e}_i$ act on L/qL.

(2.3.4) $$\tilde{e}_i B \subset B \cup \{0\} \quad \text{and} \quad \tilde{f}_i B \subset B \cup \{0\}.$$

(2.3.5) $$L = \bigoplus_{\lambda \in P} L_\lambda \quad \text{and} \quad B = \bigsqcup_{\lambda \in P} B_\lambda$$

$$\text{where } L_\lambda = L \cap M_\lambda \quad \text{and} \quad B_\lambda = B \cap (L_\lambda / qL_\lambda).$$

(2.3.6) For $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$.

Let us study elementary properties of crystal bases.

PROPOSITION 2.3.2. (i) *For $(\lambda, n) \in P \times \mathbb{Z}$ with $0 \leqslant n \leqslant \langle h_i, \lambda \rangle$, let $a^i_{\lambda,n}(q)$, $b^i_{\lambda,n}(q)$ be an element of $1 + qA$. We define endomorphisms $\tilde{e}'_i$ and $\tilde{f}'_i$ of an integrable $U_q(\mathfrak{g})$-module M by*

(2.3.7) $$\tilde{f}'_i(f_i^{(n)} u) = a^i_{\lambda,n}(q) f_i^{(n+1)} u$$

$$\tilde{e}'_i(f_i^{(n)} u) = b^i_{\lambda,n}(q) f_i^{(n-1)} u$$

$$\textit{for } u \in \operatorname{Ker} e_i \cap M_\lambda \textit{ with } 0 \leqslant n \leqslant \langle h_i, \lambda \rangle.$$

Then the definition of lower crystal base obtained by replacing $\tilde{e}_i$ and $\tilde{f}_i$ with $\tilde{e}'_i$ and $\tilde{f}'_i$ is equivalent to the original one.

(ii) *Let (L, B) be a crystal base. Let $\lambda \in P$ and let $u = \sum f_i^{(n)} u_n$ be an element of L_λ with $u \in \operatorname{Ker} e_i \cap M_{\lambda + n\alpha_i}$, $0 \leqslant n \leqslant \langle h_i, \lambda + n\alpha_i \rangle$. Then*

(a) *all u_n belong to L,*
(b) *if $u \bmod qL$ belongs to B, then there is n_0 such that $u_n \in qL$ for $n \neq n_0$, $u_{n_0} \bmod qL$ belongs to B and $u \equiv f_i^{(n_0)} u_{n_0} \bmod qL$, and*
(c) *$\tilde{e}'_i = \tilde{e}_i$ and $\tilde{f}'_i = \tilde{f}_i$ on L/qL.*

Proof. Let L be a sub-A-module of M such that $\tilde{e}'_i L \subset L$, $\tilde{f}'_i L \subset L$ and $L = \oplus_{\lambda \in P} L_\lambda$. We shall show first that, if $u = \sum_{n=0}^N f_i^{(n)} u_n$ belongs to L_λ, $e_i u_n = 0$ and $u_n = 0$ except when $0 \leqslant n \leqslant \langle h_i, \lambda + n\alpha_i \rangle$, then u_n belongs to L. We argue by the induction on N. If $N = 0$, this is trivial. If $N > 0$, then

$$\tilde{e}'_i u = \sum_{n=1}^{N} a^i_{\lambda + n\alpha_i, n} f_i^{(n-1)} u_n \in L.$$

Hence, by the hypothesis of the induction, $a^i_{\lambda + n\alpha_i, n} u_n$ belongs to L for $n \geqslant 1$. Since $a^i_{\lambda + n\alpha_i, n}$ is an invertible element of A, u_n belongs to L for $n \geqslant 1$. Then $\tilde{f}_i'^n u_n$ is a multiple of $f_i^{(n)} u_n$ by an invertible element of A. Therefore, $f_i^{(n)} u_n$ belongs to L for $n \geqslant 1$. Hence u_0 belongs to L. Thus we have proven that all u_n belong to L. The rest of the statements are its direct consequence except (ii)(b). We shall prove (ii)(b) by the induction on N. If $N = 0$, then it is trivial. If $\tilde{e}_i u = \sum_{n=1}^{N} f_i^{(n-1)} u_n \in qL$, then $u_n \in qL$ for $n \geqslant 1$ and $u \equiv u_0 \bmod qL$. If $\tilde{e}_i u \notin qL$, then $\tilde{e}_i u \bmod qL$ belongs to B. Hence there is $n_0 \geqslant 1$ such that $u_n \in qL$ for $n \neq n_0$ by the hypothesis of induction. Hence $\tilde{e}_i u \equiv f_i^{(n_0-1)} u_{n_0}$. By (2.3.6), $u \equiv \tilde{f}_i \tilde{e}_i u \equiv f_i^{(n_0)} u_{n_0}$. Q.E.D.

2.4. *Upper crystal base.* For any integrable $U_q(\mathfrak{g})$-module M, we define $\tilde{e}_i'$ and $\tilde{f}_i'$ as in [K1]:

$$\Delta_i = q_i^{-1} t_i + q_i t_i + (q_i - q_i^{-1})^2 e_i f_i - 2,\ \tilde{e}_i' = (q_i t_i \Delta_i)^{-1/2} e_i \text{ and } \tilde{f}_i' = (q_i t_i^{-1} \Delta_i)^{-1/2} f_i.$$

We say that (L, B) is an *upper crystal base* if (L, B) satisfies the conditions in Definition 2.3.1 with $\tilde{e}_i'$ and $\tilde{f}_i'$ instead of $\tilde{e}_i$ and $\tilde{f}_i$. Then for $\lambda \in P$ and n with $0 \leqslant n \leqslant \langle h_i, \lambda \rangle$ we have

$$\tilde{e}_i' f_i^{(n)} u = q_i^{2n - \langle h_i, \lambda \rangle - 1} (1 - q_i^{1 + \langle h_i, \lambda \rangle})^{-1} (1 - q_i^{2(\langle h_i, \lambda \rangle - n + 1)}) (1 - q_i^2)^{-1} f_i^{(n-1)} u,$$

$$\tilde{f}_i' f_i^{(n)} u = q_i^{-1 + \langle h_i, \lambda \rangle - 2n} (1 - q_i^{1 + \langle h_i, \lambda \rangle})^{-1} (1 - q_i^{2n+2}) (1 - q_i^2)^{-1} f_i^{(n+1)} u$$

for $u \in \operatorname{Ker} e_i \cap M_\lambda$. Hence we have, assuming $(\mu, \mu) \in \mathbb{Z}$ for any $\mu \in P$,

$$\psi_M^{-1} \tilde{e}_i' \psi_M f_i^{(n)} u = (1 - q_i^{1 + \langle h_i, \lambda \rangle})^{-1} (1 - q_i^{2(\langle h_i, \lambda \rangle - n + 1)}) (1 - q_i^2)^{-1} f_i^{(n-1)} u$$

and

$$\psi^{-1} M \tilde{f}_i' \psi_M f_i^{(n)} u = (1 - q_i^{1 + \langle h_i, \lambda \rangle})^{-1} (1 - q_i^{2n+2}) (1 - q_i^2)^{-1} f_i^{(n+1)} u.$$

Hence, by Proposition 2.3.2 we obtain the following lemma.

LEMMA 2.4.1. *(L, B) is a lower crystal base if and only if $\psi_M(L, B)$ is an upper crystal base.*

Moreover, Proposition 6 in [K1] and (1.4.7) imply the following theorem.

THEOREM 1. *Let M_1 and M_2 be integrable $U_q(\mathfrak{g})$-modules and let (L_j, B_j) be a lower crystal base of M_j $(j = 1, 2)$. Set $L = L_1 \otimes_A L_2 \subset M_1 \otimes M_2$ and $B = \{b_1 \otimes b_2; b_j \in B_j$ $(j = 1, 2)\} \subset L/qL$. Then we have the following.*

(i) *(L, B) is a lower crystal base of $M_1 \otimes_- M_2$.*
(ii) *For $b_1 \in B_1$, $b_2 \in B_2$ and $i \in I$, we have*

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if there exists } n \geqslant 1 \text{ such that } \tilde{f}_i^n b_1 \neq 0 \text{ and } \tilde{e}_i^n b_2 = 0; \\ b_1 \otimes \tilde{f}_i b_2 & \text{otherwise.} \end{cases}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i b_2 & \text{if there exists } n \geqslant 1 \text{ such that } \tilde{e}_i^n b_2 \neq 0 \text{ and } \tilde{f}_i^n b_1 = 0; \\ \tilde{e}_i b_1 \otimes b_2 & \text{otherwise.} \end{cases}$$

We can rewrite the formulas in Theorem 1 (ii) as follows. For a lower crystal base (L, B) and $b \in B$, we set

(2.4.1) $$\varepsilon_i(b) = \max\{n; \tilde{e}_i^n b \neq 0\} = \max\{n; b \in \tilde{f}_i^n B\}$$

$$\varphi_i(b) = \max\{n; \tilde{f}_i^n b \neq 0\} = \max\{n; b \in \tilde{e}_i^n B\}.$$

Then we have

(2.4.2) $$\langle h_i, \lambda\rangle = \varphi_i(b) - \varepsilon_i(b) \qquad \text{for} \qquad b \in B_\lambda.$$

In fact, by Proposition 2.3.2 there exists $n \geqslant 0$ and $u \in L_{\lambda+n\alpha_i}$ such that $b = f_i^{(n)}u$ mod qL and $e_i u = 0$. Hence, if we set $b' = u$ mod qL, then $\tilde{e}_i b' = 0$, $b = \tilde{f}_i^n b'$, $b' \neq 0$ and $b' = \tilde{e}_i^n b$. Hence $n = \varepsilon_i(b)$. Set $l = \langle h_i, \lambda + n\alpha_i\rangle \geqslant 0$. Then $u = e_i^{(l)} f_i^{(l)} u$, $f_i^{(l+1)}u = 0$. Hence, $\tilde{f}_i^{l+1} b' = 0$, $b' = \tilde{e}_i^l \tilde{f}_i^l b'$, and $\tilde{f}_i^l b' \neq 0$. This shows $\varphi_i(b) = l - n$. Therefore, we have $\varphi_i(b) = \langle h_i, \lambda\rangle + n = \langle h_i, \lambda\rangle + \varepsilon_i(b)$, which shows (2.4.2).

Now (ii) can be rewritten as

(2.4.3) $$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2); \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2). \end{cases}$$

$$\tilde{e}_i(b_i \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2); \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2). \end{cases}$$

In particular, for $i \in I$, integrable $U_q(\mathfrak{g})$-modules M_1, M_2 and $u_j \in (M_j)_{\lambda_j}$ such that $e_i u_j = 0$ $(j = 1, 2)$, let L be the A-module generated by $f_i^{(n)}u_1 \otimes f_i^{(m)}u_2 (n, m \geqslant 0)$.

Then we have, modulo qL

(2.4.4) $$\tilde{f}_i(f_i^{(n)}u_1 \otimes f_i^{(m)}u_2) \equiv \begin{cases} f_i^{(n+1)}u_1 \otimes f_i^{(m)}u_2 & \text{for } \langle h_i, \lambda_1\rangle - n > m; \\ f_i^{(n)}u_1 \otimes f_i^{(m+1)}u_2 & \text{for } \langle h_i, \lambda_1\rangle - n \leqslant m. \end{cases}$$

$$\tilde{e}_i(f_i^{(n)}u_1 \otimes f_i^{(m)}u_2) \equiv \begin{cases} f_i^{(n-1)}u_1 \otimes f_i^{(m)}u_2 & \text{for } \langle h_i, \lambda_1\rangle - n \geqslant m; \\ f_i^{(n)}u_1 \otimes f_i^{(m-1)}u_2 & \text{for } \langle h_i, \lambda_1\rangle - n < m. \end{cases}$$

Here we assumed $0 \leqslant n \leqslant \langle h_i, \lambda_1\rangle$ and $0 \leqslant m \leqslant \langle h_i, \lambda_2\rangle$. This is obtained by applying Theorem 1 to the sl_2-case. (Cf. [K1].)

2.5. *Inner product.* Let M be a $U_q(\mathfrak{g})$-module. Let $(,)$ be a bilinear symmetric form on M satisfying the property that

(2.5.1) $$(q^h u, v) = (u, q^h v),$$

$$(f_i u, v) = (u, q_i^{-1} t_i e_i v) \qquad \text{and}$$

$$(e_i u, v) = (u, q_i t_i^{-1} f_i v).$$

LEMMA 2.5.1. *Let M_j ($j = 1, 2$) be two $U_q(\mathfrak{g})$-modules and let (,) be a bilinear symmetric form satisfying* (2.5.1). *Define the bilinear symmetric form* (,) *on $M_1 \otimes_- M_2$ by*

$$(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, u_2)(v_1, v_2) \quad \text{for} \quad u_j, v_j \in M_j. \tag{2.5.2}$$

Then (,) *on $M_1 \otimes_- M_2$ satisfies* (2.5.1).

The proof is straightforward.

For $\lambda \in P_+$ there exists a unique bilinear symmetric form (,) on $V(\lambda)$ satisfying (2.5.1) and

$$(u_\lambda, u_\lambda) = 1. \tag{2.5.3}$$

This is an easy consequence of (1.2.6) and the fact that $q^h \mapsto q^h$, $f_i \mapsto q_i^{-1} t_i e_i$, $e_i \mapsto q_i t_i^{-1} f_i$ defines the antiautomorphism of $U_q(\mathfrak{g})$.

Let $\lambda, \mu \in P_+$ and let $\Phi(\lambda, \mu)$: $V(\lambda+\mu) \to V(\lambda) \otimes_- V(\mu)$ and $\Psi(\lambda, \mu)$: $V(\lambda) \otimes_- V(\mu) \to V(\lambda + \mu)$ be the unique $U_q(\mathfrak{g})$-linear homorphisms such that

$$\Phi(\lambda, \mu)(u_{\lambda+\mu}) = u_\lambda \otimes u_\mu \tag{2.5.4}$$

$$\Psi(\lambda, \mu)(u_\lambda \otimes u_\mu) = u_{\lambda+\mu}.$$

Then we have

$$\Psi(\lambda, \mu) \circ \Phi(\lambda, \mu) = \mathrm{id}_{V(\lambda+\mu)}. \tag{2.5.5}$$

Let (,) be the bilinear symmetric forms on $V(\lambda + \mu)$ and $V(\lambda) \otimes V(\mu)$ defined as above. Then we have

$$(\Psi(\lambda, \mu)(w), u) = (w, \Phi(\lambda, \mu)(u)) \quad \text{for} \quad w \in V(\lambda) \otimes V(\mu) \quad \text{and} \quad u \in V(\lambda + \mu). \tag{2.5.6}$$

This follows easily from the uniqueness of a bilinear form (,) on $(V(\lambda) \otimes V(\mu)) \times V(\lambda + \mu)$ satisfying (2.5.1) and $(u_\lambda \otimes u_\mu, u_{\lambda+\mu}) = 1$.

2.6. *Existence and uniqueness theorems.* Hereafter, crystal base means lower crystal base. Let $\lambda \in P_+$ and let $V(\lambda)$ be the irreducible $U_q(\mathfrak{g})$-module with a highest weight vector u_λ with weight λ as in §1.2. Let $L(\lambda)$ be the A-module generated by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda$. Let $B(\lambda)$ be the subset of $L(\lambda)/qL(\lambda)$ consisting of the nonzero vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda$.

THEOREM 2. *$(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.*

The proof will be given in §4.

The following theorem is proven in [K1] under the assumption that Theorem 2 holds.

THEOREM 3 (uniqueness). *Let $M \in \mathcal{O}_{int}$ and let (L, B) be a crystal base of M. Then there exists an isomorphism $M \cong \oplus_j V(\lambda_j)$ by which (L, B) is isomorphic to $\oplus_j(L(\lambda_j), B(\lambda_j))$.*

We shall give here a simpler proof of this theorem admitting Theorem 2.

LEMMA 2.6.1. *Let $\lambda \in P_+$. Then*

(i) $\{u \in L(\lambda)/qL(\lambda); \tilde{e}_i u = 0$ *for any* $i\} = V(\lambda)_\lambda$, *and*
(ii) $\{u \in V(\lambda); \tilde{e}_i u \in L(\lambda)$ *for any* $i\} = L(\lambda) + V(\lambda)_\lambda$.

Proof. (i) It is enough to show that for $\mu \neq \lambda$ and $u \in (L(\lambda)/qL(\lambda))_\mu$, if $\tilde{e}_i u = 0$ for any i, then $u = 0$. Let us write $u = \sum_{b \in B(\lambda)_\mu} a_b b$ with $a_b \in \mathbb{Q}$. Then, for any i, $\{b \in B(\lambda)_\mu; \tilde{e}_i b \neq 0\} \hookrightarrow B(\lambda)_{\mu+\alpha_i}$ by $b \mapsto \tilde{e}_i b$. Hence $\sum a_b \tilde{e}_i b = 0$ implies $a_b = 0$ when $\tilde{e}_i b \neq 0$. Since all $b \in B(\lambda)_\mu$ have some i with $\tilde{e}_i b \neq 0$, all a_b vanish.

(ii) For $\mu \neq \lambda$ and $u \in V(\lambda)_\mu$ with $\tilde{e}_i u \in L(\lambda)$ for any i, we shall show $u \in L(\lambda)$. Let us take the smallest $n \geqslant 0$ such that $u \in q^{-n}L(\lambda)$. Assuming $n > 0$, let us derive the contradiction. Set $b = q^n u \bmod qL(\lambda)$. Then $\tilde{e}_i b = 0$ for any i. Hence $b = 0$ by (i). Therefore $u \in q^{1-n}L(\lambda)$, which contradicts the choice of n. Q.E.D.

LEMMA 2.6.2. *Let $\lambda \in P_+$ and L be a sub-A-module of $V(\lambda)$ such that $L = \oplus_{\mu \in P} L_\mu$ and $L_\lambda = Au_\lambda$.*

(i) *If $\tilde{f}_i L \subset L$ for any i, then $L(\lambda) \subset L$.*
(ii) *If $\tilde{e}_i L \subset L$ for any i, then $L \subset L(\lambda)$.*

Proof. Part (i) is obvious. In order to prove (ii) let us show $L_\mu \subset L(\lambda)_\mu$. By the induction on μ, we may assume that $\mu \neq \lambda$ and $L_{\mu+\alpha_i} \subset L(\lambda)_{\mu+\alpha_i}$ for any i. Hence $\tilde{e}_i L_\mu \subset L(\lambda)$ for any i. Then the preceding lemma implies the desired result $L_\mu \subset L(\lambda)_\mu$. Q.E.D.

Theorem 3 is easily reduced to the following lemma.

LEMMA 2.6.3. *Let $M \in \mathrm{Ob}(\mathcal{O}_{\mathrm{int}})$ and $\lambda \in P_+$ such that $M_{\lambda+\alpha_i} = 0$ for any i. Let (L, B) be a crystal base of M. Let $M = N_1 \oplus N_2$ with $N_1 = U_q(\mathfrak{g})M_\lambda$. Set $L_j = N_j \cap L$, $B_j = B \cap (L_j/qL_j)$. Then we have*

$$L = L_1 \oplus L_2, \qquad B = B_1 \sqcup B_2, \tag{2.6.1}$$

$$(L_1, B_1) \cong (L(\lambda), B(\lambda))^{\oplus \dim M_\lambda}. \tag{2.6.2}$$

Proof. Since $N_1 \cong V(\lambda)^{\oplus B_\lambda}$ and $(N_1)_\lambda = M_\lambda$, N_1 has a crystal base $(\tilde{L}, \tilde{B})$ such that $\tilde{L}_\lambda = L_\lambda$, $\tilde{B}_\lambda = B_\lambda$ and $(\tilde{L}, \tilde{B}) \cong (L(\lambda), B(\lambda))^{\oplus B_\lambda}$. Then the preceding lemma holds by replacing $(L(\lambda), B(\lambda))$ and $V(\lambda)$ with $(\tilde{L}, \tilde{B})$ and N_1. Hence $L_1 = \tilde{L}$. Moreover, p is the projection $M \to N_{1,p}(L) = \tilde{L}$. Then they imply $L = L_1 \oplus L_2$. Now, we shall show $B_\mu = \tilde{B}_\mu \cup (B_2)_\mu$ for any $\mu \in P$. If μ is not a weight of $V(\lambda)$, then this is trivial. Hence we may assume $\mu \in \lambda + Q_-$. If $\mu = \lambda$, this is also trivial. Hence by the induction of μ, we may assume $\mu \neq \lambda$ and $B_{\mu+\alpha_i} \subset \tilde{B} \sqcup B_2$ for any i. For $b \in B_\mu$ write $b = u_1 + u_2$ with $u_j \in L_j/qL_j$. If $u_1 = 0$, then there is nothing to prove. If $u_1 \neq 0$, then there exists i such that $\tilde{e}_i u_1 \neq 0$ by Lemma 2.6.1. Since $\tilde{e}_i b = \tilde{e}_i u_1 + \tilde{e}_i u_2 \in B_{\mu+\alpha_i} = \tilde{B}_{\mu+\alpha_i} \sqcup (B_2)_{\mu+\alpha_i}$, we obtain $\tilde{e}_i b \in \tilde{B}_{\mu+\alpha_i}$. Hence $b = \tilde{f}_i \tilde{e}_i b \in \tilde{B}$. Thus we obtain $B = \tilde{B} \sqcup B_2$. Since $\tilde{B} \subset B_1$ and $B_1 \cap B_2 = \phi$, we have $B_1 = \tilde{B}$. Now the rest of the steps are straightforward.

Thus Theorem 3 is proven under the assumption that Theorem 2 holds.

§3. Crystal base of $U_q^-(\mathfrak{g})$. In this section we shall define the crystal base of $U_q^-(\mathfrak{g})$. We regard $U_q^-(\mathfrak{g})$ as the projective limit of $V(\lambda)$. Then the endomorphism $t_i e_i$ on $V(\lambda)$ converges to an operator on $U_q^-(\mathfrak{g})$ with respect to the q-adic topology. With this operator we can define the notion of crystal base on $U_q^-(\mathfrak{g})$.

3.1. *Q-analogue of boson.* Let $\mathscr{B}$ be the algebra over $\mathbf{Q}(q)$ generated by two elements e and f with fundamental relations

(3.1.1) $$ef = q^{-2}fe + 1.$$

If we put $q = 1$, then this is a commutation relation of boson. The commutation relation (3.1.1) implies

(3.1.2) $$e^n f^{(m)} = \sum_{\nu=0}^{n} q^{-2nm+(n+m)\nu-(\nu(\nu-1)/2)} \begin{bmatrix} n \\ \nu \end{bmatrix} f^{(m-\nu)} e^{n-\nu}.$$

Here we set

$$f^{(m)} = \begin{cases} f^m/[m]! & \text{for } m \geqslant 0; \\ 0 & \text{for } m < 0. \end{cases}$$

3.2. *Decomposition of $\mathscr{B}$-module.* Let M be a $\mathscr{B}$-module such that

(3.2.1) for any $u \in M$ there is $n \geqslant 1$ such that $e^n u = 0$.

We define the endomorphism P on M by

(3.2.2) $$P = \sum (-1)^n q^{-(n(n-1)/2)} f^{(n)} e^n.$$

PROPOSITION 3.2.1. *Let M be a $\mathscr{B}$-module satisfying* (3.2.1).

(a) *For any $u \in M$ there exist unique $u_n \in M$ $(n \geqslant 0)$ such that*

(3.2.3) $$eu_n = 0 \quad \textit{for any } n,$$

(3.2.4) $$u_n = 0 \qquad \text{for } n \gg 0,$$

(3.2.5) $$u = \sum f^{(n)} u_n .$$

(b) *We have* $u_n = q^{(n(n-1)/2)} P e^n u$.
(c) $M = \operatorname{Im} f \oplus \operatorname{Ker} e$.
(d) P *is the projector onto* Ker e *according to the direct sum decomposition in* (c).

Proof. We shall prove first

(3.2.6) $$Pf = eP = 0.$$

We have

$$\begin{aligned} Pf &= \sum (-1)^n q^{-(n(n-1)/2)} f^{(n)} e^n f \\ &= \sum_n (-1)^n q^{-(n(n-1)/2)} f^{(n)} (q^{-2n} f e^n + q^{1-n}[n] e^{n-1}) \\ &= \sum_{n \geqslant 0} (-1)^n q^{-(n(n-1)/2)-2n} [n+1] f^{(n+1)} e^n + \sum_{n \geqslant 0} (-1)^{n+1} q^{-(n(n+1)/2)-n} [n+1] f^{(n+1)} e^n \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} eP &= \sum (-1)^n q^{-(n(n-1)/2)} (q^{-2n} f^{(n)} e + q^{1-n} f^{(n-1)}) e^n \\ &= \sum (-1)^n q^{-(n(n-1)/2)-2n} f^{(n)} e^{n+1} + \sum (-1)^{n+1} q^{-((n+1)n/2)-n} f^{(n)} e^{n+1} \\ &= 0. \end{aligned}$$

Hence we have (3.2.6). Now we shall show

(3.2.7) $$1 = \sum q^{(n(n-1)/2)} f^{(n)} P e^n .$$

We have

$$\begin{aligned} f^{(n)} P e^n &= \sum_m (-1)^m q^{-(m(m-1)/2)} \begin{bmatrix} n+m \\ n \end{bmatrix} f^{(n+m)} e^{n+m} \\ &= \sum_{m \geqslant n} (-1)^{m+n} q^{-((m-n)(m-n-1)/2)} \begin{bmatrix} m \\ n \end{bmatrix} f^{(m)} e^m \\ &= \sum_{m \geqslant n} (-1)^{m+n} q^{-(n(n-1)/2)+(m-1)n-(m(m-1)/2)} \begin{bmatrix} m \\ n \end{bmatrix} f^{(m)} e^m . \end{aligned}$$

Hence (3.2.7) follows from $\sum_{n=0}^{m}(-1)^n\left[\begin{smallmatrix}m\\n\end{smallmatrix}\right]q^{(m-1)n}=0$, which is a consequence of

$$\sum_{n=0}^{m}(-1)^n\begin{bmatrix}m\\n\end{bmatrix}x^n=\prod_{\nu=1}^{m}(1-q^{-1-m+2\nu}x). \tag{3.2.8}$$

Thus it remains to prove the uniqueness of u_n and (b). Assume $eu_n=0$ and $\sum_n f^{(n)}u_n=u$. Then we have for any n

$$\begin{aligned}e^nu&=\sum_m e^nf^{(m)}u_m\\&=\sum q^{-2nm+\nu(n+m)-(\nu(\nu-1)/2)}\begin{bmatrix}n\\\nu\end{bmatrix}f^{(m-\nu)}e^{n-\nu}u_m.\end{aligned}$$

Since $Pf^{(m-\nu)}e^{n-\nu}u_m=0$ except $n=m=\nu$, we obtain $Pe^nu=q^{-(n(n-1)/2)}u_n$.

Q.E.D.

3.3. *The reduced q-analogue.* Let $\mathscr{B}_q(\mathfrak{g})$ be the algebra generated by e_i', f_i ($i\in I$) with the commutation relations

$$e_i'f_j=q_i^{-\langle h_i,\alpha_j\rangle}f_je_i'+\delta_{ij}. \tag{3.3.1}$$

(3.3.2) For $i\neq j$, setting $b=1-\langle h_i,\alpha_j\rangle$,

$$\sum_{n=0}^{b}(-1)^n\begin{bmatrix}b\\n\end{bmatrix}_i e_i'^n e_j' e_i'^{b-n}=\sum_{n=0}^{b}(-1)^n\begin{bmatrix}b\\n\end{bmatrix}_i f_i^nf_jf_i^{b-n}=0.$$

We call $\mathscr{B}_q(\mathfrak{g})$ the reduced q-analogue. Then $\mathscr{B}_q(\mathfrak{g})$ has the antiautomorphism a defined by

$$a(f_i)=e_i' \quad\text{and}\quad a(e_i')=f_i. \tag{3.3.3}$$

3.4. *$\mathscr{B}_q(\mathfrak{g})$-module structure on $U_q^-(\mathfrak{g})$.* Let $U_q^-(\mathfrak{g})$ be the subalgebra over $\mathbb{Q}(q)$ of $U_q(\mathfrak{g})$ generated by f_i. Then by [L1], [L2], [L3] the fundamental relations of f_i are

$$\sum_n(-1)^n\begin{bmatrix}1-\langle h_i,\alpha_j\rangle\\n\end{bmatrix}_i f_i^nf_jf_i^{1-\langle h_i,\alpha_j\rangle-n}=0 \quad\text{for}\quad i\neq j. \tag{3.4.1}$$

LEMMA 3.4.1. *For any $P\in U_q^-(\mathfrak{g})$ there exist unique Q, $R\in U_q^-(\mathfrak{g})$ such that*

$$[e_i,P]=\frac{t_iQ-t_i^{-1}R}{q_i-q_i^{-1}}. \tag{3.4.2}$$

Proof. The uniqueness follows from (1.1.21). Since $U_q^-(\mathfrak{g})$ is generated by the f_j and the lemma is true for $P=1$, it is enough to show that, if the lemma is true for

P, then the lemma is true for f_jP. Assume (3.4.2). Then

$$[e_i, f_jP] = [e_i, f_j]P + f_j[e_i, P] = \frac{\delta_{ij}(t_i - t_i^{-1})}{q_i - q_i^{-1}}P + \frac{f_j(t_iQ - t_i^{-1}R)}{q_i - q_i^{-1}}.$$

Hence we obtain

$$(3.4.3) \qquad [e_i, f_jP] = \frac{t_i(q_i^{\langle h_i, \alpha_j\rangle}f_jQ + \delta_{ij}P) - t_i^{-1}(q_i^{-\langle h_i, \alpha_j\rangle}f_jR + \delta_{ij}P)}{q_i - q_i^{-1}}. \qquad \text{Q.E.D.}$$

By this lemma, if we set $Q = e_i''(P)$ and $R = e_i'(P)$, then e_i' and e_i'' are endomorphisms of $U_q^-(\mathfrak{g})$. Moreover, (3.4.3) gives

$$(3.4.4) \qquad e_i''f_j = q_i^{\langle h_i, \alpha_j\rangle}f_je_i'' + \delta_{ij} \qquad \text{and} \qquad e_i'f_j = q_i^{-\langle h_i, \alpha_j\rangle}f_je_i' + \delta_{ij}.$$

Here f_j acts on $U_q^-(\mathfrak{g})$ by the left multiplication.

LEMMA 3.4.2. *$U_q^-(\mathfrak{g})$ is a left $\mathscr{B}_q(\mathfrak{g})$-module.*

Proof. It remains to prove that for $i \neq j$, $S = \sum(-1)^n \begin{bmatrix} b \\ n \end{bmatrix}_i e_i'^n e_j' e_i'^{b-n}$ vanishes as an endomorphism of $U_q^-(\mathfrak{g})$. Here $b = 1 - \langle h_i, \alpha_j\rangle$. In order to see this we shall calculate the commutation relation between S and f_k. We have by (3.1.2)

$$e_i'^n f_k = q_i^{-n\langle h_i, \alpha_k\rangle}f_k e_i'^n + \delta_{ik}q_i^{1-n}[n]_i e_i'^{n-1}.$$

Hence we have (see (1.1.20) and note $q_j^{-\langle h_j, \alpha_i\rangle} = q_i^{b-1}$)

$$\begin{aligned}
Sf_k &= \sum(-1)^n \begin{bmatrix} b \\ n \end{bmatrix}_i e_i'^n e_j'(q_i^{-(b-n)\langle h_i, \alpha_k\rangle}f_k e_i'^{b-n} + \delta_{ik}q_i^{1-b+n}[b-n]_i e_i'^{b-n-1}) \\
&= \sum(-1)^n \begin{bmatrix} b \\ n \end{bmatrix}_i e_i'^n(q_i^{-(b-n)\langle h_i, \alpha_k\rangle}(q_j^{-\langle h_j, \alpha_k\rangle}f_k e_j' + \delta_{jk})e_i'^{b-n} \\
&\quad + \delta_{ik}q_i^{1-b+n}[b-n]_i e_j' e_k'^{b-n-1}) \\
&= \sum(-1)^n \begin{bmatrix} b \\ n \end{bmatrix}_i \{q_i^{-(b-n)\langle h_i, \alpha_k\rangle}q_j^{-\langle h_j, \alpha_k\rangle}(q_i^{-n\langle h_i, \alpha_k\rangle}f_k e_i'^n + \delta_{ik}q_i^{1-n}[n]_i e_i'^{n-1})e_j' e_i'^{b-n} \\
&\quad + \delta_{jk}q_i^{-(b-n)\langle h_i, \alpha_k\rangle}e_i'^b + \delta_{ik}q_i^{1-b+n}[b-n]_i e_i'^n e_j' e_k'^{b-n-1}\}
\end{aligned}$$

$$= q_k^{-b\langle h_k,\alpha_i\rangle-\langle h_k,\alpha_j\rangle} f_k S + \delta_{ik}\sum(-1)^n\begin{bmatrix} b \\ n \end{bmatrix}_i q_i^{n-b}[n]_i e_i'^{n-1}e_j'e_i'^{b-n}$$

$$+ \delta_{ik}\sum(-1)^n\begin{bmatrix} b \\ n \end{bmatrix}_n q_i^{1-b+n}[b-n]_i e_i'^n e_j' e_i'^{b-n-1}$$

$$+ \delta_{jk}\sum(-1)^n\begin{bmatrix} b \\ n \end{bmatrix}_i q_i^{-(b-n)\langle h_i,\alpha_j\rangle} e_i'^b .$$

Since $\begin{bmatrix} b \\ n+1 \end{bmatrix}_i q_i^{n-b+1}[1+n]_i = \begin{bmatrix} b \\ n \end{bmatrix}_i q_i^{1-b+n}[b-n]_i$, the second term and the third cancel out. The last term vanishes by (3.2.8). Thus we obtain

$$Sf_k = q_k^{-b\langle h_k,\alpha_i\rangle-\langle h_k,\alpha_j\rangle} f_k S. \tag{3.4.5}$$

Then $S = 0$ follows from (3.4.5) and $S\cdot 1 = 0$. Q.E.D.

The following lemma makes explicit a $\mathscr{B}_q(\mathfrak{g})$-module structure on $U_q^-(\mathfrak{g})$.

LEMMA 3.4.3. $U_q^-(\mathfrak{g}) \cong \mathscr{B}_q(\mathfrak{g})/\sum_i \mathscr{B}_q(\mathfrak{g})e_i'$.

Proof. Since 1 is annihilated by e_i', we have a surjective morphism

$$\mathscr{B}_q(\mathfrak{g})\Big/\sum_i \mathscr{B}_q(\mathfrak{g})e_i' \to U_q^-(\mathfrak{g}).$$

If C is the subalgebra of $\mathscr{B}_q(\mathfrak{g})$ generated by f_i, then we have

$$C \xrightarrow{\varphi} \mathscr{B}_q(\mathfrak{g})\Big/\sum_i \mathscr{B}_q(\mathfrak{g})e_i' \xrightarrow{\psi} U_q^-(\mathfrak{g}).$$

It is clear that ψ and φ are surjective. By the fact that (3.4.1) is the fundamental relations of $U_q^-(\mathfrak{g})$, $\psi\circ\varphi$ is an isomorphism. Hence φ and ψ are isomorphisms. Q.E.D.

PROPOSITION 3.4.4. *There is a unique symmetric form* (,) *on* $U_q^-(\mathfrak{g})$ *such that*

$$(f_i u, v) = (u, e_i' v), \tag{3.4.6}$$

$$(1, 1) = 1.$$

Proof. The uniqueness is clear. We shall prove the existence. Let us endow $M = \mathrm{Hom}(U_q^-(\mathfrak{g}), \mathbb{Q}(q))$ with the structure of a left $\mathscr{B}_q(\mathfrak{g})$-module via a; i.e., we have

$$(f_i\varphi)(u) = \varphi(e_i' u) \tag{3.4.7}$$

$$(e_i'\varphi)(u) = \varphi(f_iu)$$

$$\text{for } u \in U_q^-(\mathfrak{g}) \quad \text{and} \quad \varphi \in M.$$

Let φ_0 be an element of M such that

$$\varphi_0(1) = 1 \quad \text{and} \quad \varphi_0\Big(\sum_i f_iU_q^-(\mathfrak{g})\Big) = 0. \tag{3.4.8}$$

Since $e_i'\varphi_0 = 0$ for any i, we have a homomorphism

$$\psi\colon U_q^-(\mathfrak{g}) \cong \mathscr{B}_q(\mathfrak{g})\Big/\sum_i \mathscr{B}_q(\mathfrak{g})e_i' \to M \tag{3.4.9}$$

which sends 1 to φ_0.

Now, we define a bilinear form (,) on

$$(u, v) = (\psi(u))(v) \qquad \text{for } u, v \in U_q^-(\mathfrak{g}). \tag{3.4.10}$$

Then we have

$$(1, 1) = 1 \tag{3.4.11}$$

$$(f_iu, v) = (u, e_i'v) \quad \text{and} \quad (e_i'u, v) = (u, f_iv).$$

One can see easily that such a bilinear form is unique. Since $(u, v)' = (v, u)$ satisfies the same condition, (,) is symmetric. Q.E.D.

For $\xi \in Q_-$, we set

$$U_q^-(\mathfrak{g})_\xi = \{P \in U_q^-(\mathfrak{g});\ q^hPq^{-h} = q^{\langle h,\xi\rangle}P \qquad \text{for any } h \in P^*\}. \tag{3.4.12}$$

If P is an element of $U_q^-(\mathfrak{g})_\xi$, then we say that ξ is the weight of P.

PROPOSITION 3.4.5. *For $i, j \in I$, we have*

$$e_i'e_j'' = q_i^{\langle h_i,\alpha_j\rangle}e_j''e_i' \qquad in \qquad \mathrm{End}(U_q^-(\mathfrak{g})).$$

Proof. For $k \in I$ we have

$$\begin{aligned} e_i'e_j''f_k &= e_i'(q_j^{\langle h_j,\alpha_k\rangle}f_ke_j'' + \delta_{jk}) \\ &= q_k^{\langle h_k,\alpha_j\rangle}(q_i^{-\langle h_i,\alpha_k\rangle}f_ke_i' + \delta_{ki})e_j'' + \delta_{jk}e_i' \\ &= q_k^{-\langle h_k,\alpha_i\rangle+\langle h_k,\alpha_j\rangle}f_ke_i'e_j'' + \delta_{ki}q_i^{\langle h_i,\alpha_j\rangle}e_j'' + \delta_{jk}e_i'. \end{aligned}$$

Similarly, we have

$$e_j'' e_i' f_k = q_k^{\langle h_k, \alpha_j\rangle - \langle h_k, \alpha_i\rangle} f_k e_j'' e_i' + \delta_{kj} q_j^{-\langle h_j, \alpha_i\rangle} e_i' + \delta_{ik} e_j'.$$

Hence, if we set $S = e_i' e_j'' - q_i^{\langle h_i, \alpha_j\rangle} e_j'' e_i$, then

$$S f_k = q_k^{\langle h_k, \alpha_j\rangle - \langle h_k, \alpha_i\rangle} f_k S.$$

Then $S \cdot 1 = 0$ gives $S = 0$. Q.E.D.

COROLLARY 3.4.6. *Let $i \in I$ and let P be an element of $U_q^-(\mathfrak{g})$ of weight $\xi \in Q_-$ which satisfies $e_i' P = 0$. Then for any element u with weight $\lambda \in P$ of a $U_q(\mathfrak{g})$-module such that $e_i u = 0$, we have*

$$t_i^n e_i^n P u = \frac{q_i^{n(2\langle h_i, \lambda+\xi\rangle + 3n+1)}}{(q_i - q_i^{-1})^n} (e_i''^n P) u.$$

Proof. We shall prove it by the induction on n. We have

$$\begin{aligned} t_i^{n+1} e_i^{n+1} P u &= t_i t_i^n e_i e_i^n P u \\ &= q_i^{2n} t_i e_i t_i^n e_i^n P u \\ &= q_i^{n(2\langle h_i, \lambda+\xi\rangle + 3n+1)} (q_i - q_i^{-1})^{-n} q_i^{2n} t_i e_i (e_i''^n P) u. \end{aligned}$$

Since

$$\begin{aligned} t_i e_i (e_i''^n P) u &= t_i [e_i, e_i''^n P] u \\ &= \frac{t_i^2 e_i''^{n+1} P - e_i' e_i''^n P}{q_i - q_i^{-1}} u. \end{aligned}$$

By the preceding lemma we have $e_i' e_i''^n P = 0$. Hence we obtain

$$t_i^{n+1} e_i^{n+1} P u = q_i^{n(2\langle h_i, \lambda+\xi\rangle + 3n+3)} (q_i - q_i^{-1})^{-n-1} q_i^{2\langle h_i, \lambda+\xi+(n+1)\alpha_i\rangle} (e_i''^{n+1} P) u.$$

Then the assertion follows from

$$\begin{aligned} & n(2\langle h_i, \lambda+\xi\rangle + 3n + 3) + 2(\langle h_i, \lambda+\xi\rangle + 2(n+1)) \\ &= (n+1)(2\langle h_i, \lambda+\xi\rangle + 3n + 4). \end{aligned}$$

Q.E.D.

We shall prove that the inner product on $U_q^-(\mathfrak{g})$ is nondegenerate.

LEMMA 3.4.7. *Let $P \in U_q^-(\mathfrak{g})$. Then, if $e_i' P = 0$ for any i, then P is a constant multiple of 1.*

Proof. We may assume $P \in U_q^-(\mathfrak{g})_\xi$. We shall prove it by the induction of $|\xi|$. Here $|\xi| = \sum |n_i|$ for $\xi = \sum n_i \alpha_i$. We may assume $\xi \neq 0$.

(a) Case $|\xi| = 1$. In this case, P has the form cf_i for some i and $c \in \mathbb{Q}(q)$. Therefore, $c = e_i' P = 0$.

(b) Case $|\xi| > 1$. For any $j \in I$, we have $e_i' e_j'' P = q_i^{\langle h_i, \alpha_j \rangle} e_j'' e_i' P = 0$. Hence $e_j'' P = 0$ by the hypothesis of the induction. Hence $e_j P = P e_j$ for any j. Now let $\lambda \in P_+$ satisfy $\langle h_j, \lambda \rangle \gg 0$ so that $U_q^-(\mathfrak{g})_\xi \xrightarrow{\sim} V(\lambda)_{\lambda+\xi}$ by the homomorphism $U_q^-(\mathfrak{g}) \ni Q \mapsto Q u_\lambda$. Then $e_j(Pu_\lambda) = 0$ for any j. Since $V(\lambda)$ is irreducible and $U_q(\mathfrak{g}) P u_\lambda$ does not contain u_λ, $Pu_\lambda = 0$ and hence $P = 0$. Q.E.D.

COROLLARY 3.4.8. *(,) is nondegenerate.*

Proof. We shall prove that (,) is nondegenerate on $U_q^-(\mathfrak{g})_\xi$ by the induction on $|\xi|$. If $\xi = 0$, this is trivial. Assume $|\xi| > 0$. If $P \in U_q^-(\mathfrak{g})_\xi$ satisfies $(P, U_q^-(\mathfrak{g})_\xi) = 0$, then $(e_i' P, U_q^-(\mathfrak{g})_{\xi+\alpha_i}) = (P, f_i U_q^-(\mathfrak{g})_{\xi+\alpha_i}) = 0$, and hence $e_i' P = 0$ for any i by the hypothesis of induction. It remains to apply the preceding lemma. Q.E.D.

COROLLARY 3.4.9. *$U_q^-(\mathfrak{g})$ is a simple $\mathscr{B}_q(\mathfrak{g})$-module.*

Proof. Let M be a nonzero submodule of $U_q^-(\mathfrak{g})$. Taking a highest weight vector of M, M contains a nonzero element P such that $e_i' P = 0$ for any i. Then P is a constant multiple of 1, and hence $M = U_q^-(\mathfrak{g})$. Q.E.D.

Remark 3.4.10. Let $\mathscr{O}(\mathscr{B}_q(\mathfrak{g}))$ be the category of $\mathscr{B}_q(\mathfrak{g})$-modules M such that for any element u of M there exists an integer l such that $e_{i_1}' e_{i_2}' \dots e_{i_l}' u = 0$ for any $i_l, \dots, i_l \in I$. Then it is not difficult to prove that $\mathscr{O}(\mathscr{B}_q(\mathfrak{g}))$ is semisimple and $U_q^-(\mathfrak{g})$ is a unique isomorphic class of simple objects of $\mathscr{O}(\mathscr{B}_q(\mathfrak{g}))$. Since we do not use this result, we leave the proof to the reader.

Remark 3.4.11. $\mathscr{B}_q(\mathfrak{g})$ has a similar structure to Hopf algebra. Let us define the comultiplication

$$\Delta: \mathscr{B}_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes \mathscr{B}_q(\mathfrak{g})$$

by

$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i, \tag{3.4.13}$$

$$\Delta(e_i') = (q_i^{-1} - q_i) t_i e_i \otimes 1 + t_i.$$

Then Δ is a well-defined $\mathbb{Q}(q)$-algebra homomorphism, and it satisfies the coassociative law:

$$\begin{array}{ccc} \mathscr{B}_q(\mathfrak{g}) & \xrightarrow{\Delta} & U_q(\mathfrak{g}) \otimes \mathscr{B}_q(\mathfrak{g}) \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta_- \otimes \mathrm{id}} \\ U_q(\mathfrak{g}) \otimes \mathscr{B}_q(\mathfrak{g}) & \xrightarrow{\mathrm{id} \otimes \Delta} & U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \otimes \mathscr{B}_q(\mathfrak{g}) \end{array}$$

is commutative. Hence, for a left $U_q(\mathfrak{g})$-module M and a left $\mathscr{B}_q(\mathfrak{g})$-module L, $M \otimes L$ has the structure of a $\mathscr{B}_q(\mathfrak{g})$-module, and there is a natural isomorphism

$$(M \otimes N) \otimes L \simeq M \otimes (N \otimes L)$$

for a $\mathscr{B}_q(\mathfrak{g})$-module L and $U_q(\mathfrak{g})$-modules M and N.

3.5. *Crystal base of* $U_q^-(\mathfrak{g})$. Let M be a $\mathscr{B}_q(\mathfrak{g})$-module in $\mathcal{O}(\mathscr{B}_q(\mathfrak{g}))$. (Cf. Remark 3.4.11.) Let i be an element of I. Then we have by Proposition 3.2.1

$$M = \bigoplus_{n \geq 0} f_i^{(n)} \operatorname{Ker} e_i'.$$

We define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ by

(3.5.1)
$$\tilde{e}_i(f_i^{(n)}u) = f_i^{(n-1)}u \quad \text{and}$$
$$\tilde{f}_i(f_i^{(n)}u) = f_i^{(n+1)}u \quad \text{for } u \in \operatorname{Ker} e_i'.$$

Note that

(3.5.2)
$$\tilde{e}_i \tilde{f}_i = 1.$$

Moreover, $\tilde{f}_i \tilde{e}_i$ is the projector to $f_i M$ with respect to $M = \operatorname{Ker} e_i' \oplus f_i M$.

A crystal base of M is by definition a pair (L, B) satisfying the following properties.

(3.5.3) L is a free sub-A-module of M such that $M \cong \mathbb{Q}(q) \otimes L$.

(3.5.4) B is a base of the $\mathbb{Q}$-vector space L/qL.

(3.5.5) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for any i.

By this $\tilde{f}_i$ and $\tilde{e}_i$ act on L/qL.

(3.5.6) $\tilde{e}_i B \subset B \cup \{0\}$ and $\tilde{f}_i B \subset B$.

(3.5.7) For $b \in B$ such that $\tilde{e}_i b \in B$, $b = \tilde{f}_i \tilde{e}_i b$.

Let $L(\infty)$ be the sub-A-module of $U_q^-(\mathfrak{g})$ generated by $\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} \cdot 1$. Let $B(\infty)$ be the subset of $L(\infty)/qL(\infty)$ consisting of the vectors of the form $\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} \cdot 1$.

THEOREM 4. $(L(\infty), B(\infty))$ *is a crystal base of* $U_q^-(\mathfrak{g})$.

This theorem will be proven in the next section.

The relations of $(L(\infty), B(\infty))$ and $(L(\lambda), B(\lambda))$ are given by the following theorem.

THEOREM 5. *Let π_λ: $U_q^-(\mathfrak{g}) \to V(\lambda)$ be the $U_q^-(\mathfrak{g})$-linear homomorphism sending 1 to u_λ. Then*

(i) $\pi_\lambda(L(\infty)) = L(\lambda)$.
Hence π_λ induces the surjective homomorphism $\bar{\pi}_\lambda$: $L(\infty)/qL(\infty) \to L(\lambda)/qL(\lambda)$.
(ii) *By $\bar{\pi}_\lambda$, $\{b \in B(\infty); \bar{\pi}_\lambda(b) \neq 0\}$ is isomorphic to $B(\lambda)$.*
(iii) $\tilde{f}_i \circ \bar{\pi}_\lambda = \bar{\pi}_\lambda \circ \tilde{f}_i$.
(iv) *If $b \in B(\infty)$ satisfies $\bar{\pi}_\lambda(b) \neq 0$, then $\tilde{e}_i \bar{\pi}_\lambda(b) = \bar{\pi}_\lambda(\tilde{e}_i b)$.*

The proof of Theorem 5 will be also given in the next section.

Remark 3.5.1. We can prove the following theorems (cf. Theorems 3, 1), but we omit their proofs.

THEOREM. *Let (L, B) be a crystal base of a $\mathscr{B}_q(\mathfrak{g})$-module M in $\mathcal{O}(\mathscr{B}_q(\mathfrak{g}))$. Then (L, B) is a direct sum of copies of $(L(\infty), B(\infty))$.*

THEOREM. *Let (L_1, B_1) be a crystal base of an integrable $U_q(\mathfrak{g})$-module M_1 and (L_2, B_2) a crystal base of a $\mathscr{B}_q(\mathfrak{g})$-module M_2 in $\mathcal{O}(\mathscr{B}_q(\mathfrak{g}))$. Then $(L_1, B_1) \otimes (L_2, B_2)$ is a crystal base of $M_1 \otimes M_2$ in $\mathcal{O}(\mathscr{B}_q(\mathfrak{g}))$, and the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $B_1 \otimes B_2 \sqcup \{0\}$ are described by the same formula as in Theorem 1.*

§4. Grand loop

4.1. *Preliminaries.* We shall prove Theorems 2, 4, and 5 at once by the induction on weights. For $\lambda, \mu \in P_+$ we denote as in §2.5 by $\Phi(\lambda, \mu)$: $V(\lambda + \mu) \to V(\lambda) \otimes_- V(\mu)$ and $\Psi(\lambda, \mu)$: $V(\lambda) \otimes_- V(\mu) \to V(\lambda + \mu)$ the $U_q(\mathfrak{g})$-linear homomorphisms such that $\Phi(\lambda, \mu)(u_{\lambda+\mu}) = u_\lambda \otimes u_\mu$ and $\Psi(\lambda, \mu)(u_\lambda \otimes u_\mu) = u_{\lambda+\mu}$. Hence we have

$$\Psi(\lambda, \mu) \circ \Phi(\lambda, \mu) = \mathrm{id}_{V(\lambda+\mu)}. \tag{4.1.1}$$

Therefore, we have

$$V(\lambda) \otimes_- V(\mu) = \mathrm{Im}\, \Phi(\lambda, \mu) \oplus \mathrm{Ker}\, \Psi(\lambda, \mu). \tag{4.1.2}$$

Since $\Psi(\lambda, \mu)$ and $\Phi(\lambda, \mu)$ are $U_q(\mathfrak{g})$-linear, they commute with $\tilde{e}_i$ and $\tilde{f}_i$. We also define the homomorphism $S(\lambda, \mu)$: $V(\lambda) \otimes_- V(\mu) \to V(\lambda)$ as

$$S(\lambda, \mu)(u \otimes v_\mu) = u \quad \text{for} \quad u \in V(\lambda) \quad \text{and} \tag{4.1.3}$$
$$S(\lambda, \mu)\Big(V(\lambda) \otimes \sum_i f_i V(\mu)\Big) = 0.$$

By the definition of Δ_-, we have $f_i(u \otimes v) = f_i u \otimes v + t_i u \otimes f_i v$, and the last terms are sent to zero by $S(\lambda, \mu)$. Hence we have that

$$S(\lambda, \mu) \quad \text{is } U_q^-(\mathfrak{g})\text{-linear}. \tag{4.1.4}$$

Therefore, $S(\lambda, \mu) \circ \Phi(\lambda, \mu)$: $V(\lambda + \mu) \to V(\lambda)$ is a unique $U_q^-(\mathfrak{g})$-linear homomorphism that sends $u_{\lambda+\mu}$ to u_λ.

Hereafter, we denote $\otimes_-$ by $\otimes$.

4.2. *Induction hypotheses.* For $\xi \in Q_-$ we write $\xi = \sum n_i \alpha_i$, and we set

(4.2.1) $$|\xi| = \sum |n_i|.$$

We also set

(4.2.2) $$Q_-(l) = \{\xi \in Q_-; |\xi| \le l\}.$$

If $|\xi| = 0$, then $\xi = 0$, and, if $|\xi| = 1$, then ξ coincides with some $-\alpha_i$. Let π_λ: $U_q^-(\mathfrak{g}) \to V(\lambda)$ be the $U_q^-(\mathfrak{g})$-linear homomorphism sending 1 to u_μ. Let C_l be the collection of following statements.

(C_l.1) For $\xi \in Q_-(l)$, $\tilde{e}_i L(\infty)_\xi \subset L(\infty)$.
(C_l.2) For $\xi \in Q_-(l)$ and $\lambda \in P_+$, $\tilde{e}_i L(\lambda)_{\lambda+\xi} \subset L(\lambda)$.
(C_l.3) For $\xi \in Q_-(l)$ and $\lambda \in P_+$, $\pi_\lambda(L(\infty)_\xi) = L(\lambda)_{\lambda+\xi}$.
(C_l.4) For $\xi \in Q_-(l)$, $B(\infty)_\xi$ is a base of $L(\infty)_\xi/qL(\infty)_\xi$.
(C_l.5) For $\xi \in Q_-(l)$ and $\lambda \in P_+$, $B(\lambda)_{\lambda+\xi}$ is base of $L(\lambda)_{\lambda+\xi}/qL(\lambda)_{\lambda+\xi}$.
(C_l.6) For $\xi \in Q_-(l-1)$ and $\lambda \in P_+$, $\tilde{f}_i(Pu_\lambda) \equiv (\tilde{f}_i P)u_\lambda \bmod qL(\lambda)$ for $P \in L(\infty)_\xi$.
(C_l.7) For $\xi \in Q_-(l)$ and $\lambda \in P_+$, we have $\tilde{e}_i B(\infty)_\xi \subset B(\infty) \cup \{0\}$ and $\tilde{e}_i B(\lambda)_{\lambda+\xi} \subset B(\lambda) \sqcup \{0\}$.
(C_l.8) For $\xi \in Q_-(l)$ and $\lambda, \mu \in P_+$, we have $\Phi(\lambda, \mu)(L(\lambda + \mu)_{\lambda+\mu+\xi}) \subset L(\lambda) \otimes L(\mu)$.
(C_l.9) For $\xi \in Q_-(l)$ and $\lambda, \mu \in P_+$, we have $\Psi(\lambda, \mu)((L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}) \subset L(\lambda + \mu)$.
(C_l.10) For $\xi \in Q_-(l)$ and $\lambda, \mu \in P_+$, $\Psi(\lambda, \mu)((B(\lambda) \otimes B(\mu))_{\lambda+\mu+\xi}) \subset B(\lambda + \mu) \sqcup \{0\}$.
(C_l.11) For $\xi \in Q_-(l)$ and $\lambda \in P_+$,

$$\{b \in B(\infty)_\xi; \bar{\pi}_\lambda(b) \ne 0\} \xrightarrow{\sim} B(\lambda)_{\lambda+\xi}.$$

Here $\bar{\pi}_\lambda$: $(L(\infty)/qL(\infty))_\xi \to (L(\lambda)/qL(\lambda))_{\lambda+\xi}$ is the homomorphism induced by π_λ. (Cf. (C_l.3).)
(C_l.12) For $\xi \in Q_-(l)$, $\lambda \in P_+$ and $b \in B(\infty)_\xi$ such that $\bar{\pi}_\lambda(b) \ne 0$, we have $\tilde{e}_i \bar{\pi}_\lambda(b) = \bar{\pi}_\lambda(\tilde{e}_i b)$.
(C_l.13) For $\xi \in Q_-(l)$, $\lambda \in P_+$ and $b \in B(\lambda)_{\lambda+\xi}$ and $b' \in B(\lambda)_{\lambda+\xi+\alpha_i}$, $b = \tilde{f}_i b'$ if and only if $b' = \tilde{e}_i b$.
(C_l.14) For $\xi \in Q_-(l)$ and $b \in B(\infty)$, if $\tilde{e}_i b \ne 0$, then $b = \tilde{f}_i \tilde{e}_i b$.

We remark that these statements are not independent. For example (C_l.10) has meaning only under the hypothesis (C_l.9), etc.

We shall prove C_l by the induction on l. We may assume that $\{h_i; i \in I\}$ is linearly independent by taking an extension of $\mathfrak{t}$ if necessary. For $i \in I$ let Λ_i be an element of $\mathfrak{t}^*$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for any j. We may assume that P contains Λ_i

without loss of generality. In fact, $P + \sum \mathbb{Z}\Lambda_i$ satisfies the properties (1.1.9) and $(P + \sum \mathbb{Z}\Lambda_i)^* \subset P^*$.

4.3. *Consequences of* C_{l-1}. Now assuming C_{l-1}, let us prove C_l. Since C_0 and C_1 are almost trivial, we may assume

$$l \geq 2. \tag{4.3.1}$$

Hereafter, C_{l-1} *is assumed.*

LEMMA 4.3.1. *Let* $\xi \in Q_-(l-1)$, $\lambda \in P_+$, *and* $u \in L(\infty)_\xi$ *(resp.* $L(\lambda)_{\lambda+\xi}$*). If* $u = \sum f_i^{(n)} u_n$ *and if* $e_i' u_n = 0$ *(resp.* $u_n \in V(\lambda)_{\lambda+\xi+n\alpha_i}$, $e_i u_n = 0$, *and* $u_n = 0$ *except when* $\langle h_i, \lambda + \xi + n\alpha_i \rangle \geq n \geq 0$*), then all* u_n *belong to* $L(\infty)$ *(resp.* $L(\lambda)$*). If moreover* $u \bmod qL(\infty)$ *(resp.* $qL(\lambda)$*) belongs to* $B(\infty)$ *(resp.* $B(\lambda)$*), then there exists* n *such that* $u \equiv f_i^{(n)} u_n$ *modulo* $qL(\infty)$ *(resp.* $qL(\lambda)$*).*

Since the proof is similar to that of Proposition 2.3.2, we omit it. We remark that we need only $(C_{l-1}.1)$ and $(C_{l-1}.2)$ in order to prove the first statement.

For $\xi \in Q_-(l-1)$ and $b \in B(\lambda)_{\lambda+\xi}$ (resp. $\in B(\infty)_\xi$), we set

$$\varepsilon_i(b) = \max\{n;\ \tilde{e}_i^n b \neq 0\}. \tag{4.3.2}$$

By Lemma 4.3.1, for $\xi \in Q_-(l-1)$ and $b \in B(\lambda)_{\lambda+\xi}$ (resp. $B(\infty)_\xi$), there exists $u \in L(\lambda)_{\lambda+\xi+\varepsilon_i(b)\alpha_i}$ (resp. $L(\infty)_{\xi+\varepsilon_i(b)\alpha_i}$) such that $e_i u = 0$ (resp. $e_i' u = 0$) and $b \equiv f_i^{(\varepsilon_i(b))} u \bmod qL(\lambda)$ (resp. $\bmod qL(\infty)$). Note that $u \bmod qL(\lambda)$ (resp. $qL(\infty)$) belongs to $B(\lambda)$ (resp. $B(\infty)$).

LEMMA 4.3.2. *Let* $\xi, \xi' \in Q_-(l-1)$, $\lambda, \mu \in P_+$, *and* $i \in I$.

(i) $\tilde{f}_i(L(\lambda)_{\lambda+\xi} \otimes L(\mu)_{\mu+\xi'}) \subset L(\lambda) \otimes L(\mu)$ *and* $\tilde{e}_i(L(\lambda)_{\lambda+\xi} \otimes L(\mu)_{\mu+\xi'}) \subset L(\lambda) \otimes L(\mu)$.

(ii) *If* $b \in B(\lambda)_{\lambda+\xi}$ *and* $b' \in B(\mu)_{\mu+\xi'}$, *then we have*

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \langle h_i, \lambda + \xi\rangle + \varepsilon_i(b) > \varepsilon_i(b'), \\ b \otimes \tilde{f}_i b' & \text{if } \langle h_i, \lambda + \xi\rangle + \varepsilon_i(b) \leq \varepsilon_i(b'); \end{cases}$$

$$\tilde{e}_i(b \otimes b') = \begin{cases} b \otimes \tilde{e}_i b' & \text{if } \langle h_i, \lambda + \xi\rangle + \varepsilon_i(b) < \varepsilon_i(b'), \\ \tilde{e}_i b \otimes b' & \text{if } \langle h_i, \lambda + \xi\rangle + \varepsilon_i(b) \geq \varepsilon_i(b'), \end{cases}$$

Here the equalities are those in $L(\lambda) \otimes L(\mu)/qL(\lambda) \otimes L(\mu)$.

(iii) *For* $b \otimes b' \in B(\lambda)_{\lambda+\xi'} \otimes B(\mu)_{\mu+\xi''}$, $\tilde{e}_i(b \otimes b') \neq 0$ *implies* $b \otimes b' = \tilde{f}_i \tilde{e}_i(b \otimes b')$.

(iv) *For* $b \in B(\lambda)_{\lambda+\xi}$ *and* $b' \in B(\mu)_{\mu+\xi'}$, *if* $\tilde{e}_i(b \otimes b') = 0$ *for any* i, *then* $\xi = 0$ *and* $b = u_\lambda$.

(v) *For* $b \in B(\lambda)_\xi$, $\tilde{f}_i(b \otimes u_\mu) = \tilde{f}_i b \otimes u_\mu$ *or* $\tilde{f}_i b = 0$.

Proof. (i) By Lemma 4.3.1 it is enough to show that, for $u \in L(\lambda)_{\lambda+\xi+n\alpha_i}$ and $v \in L(\mu)_{\mu+\xi'+m\alpha_i}$ such that $e_i u = e_i v = 0$, $\langle h_i, \lambda + \xi + n\alpha_i\rangle \geq n \geq 0$, and

$\langle h_i, \mu + \xi' + m\alpha_i \rangle \geq m \geq 0,$

$$\tilde{f}_i(f_i^{(n)}u \otimes \tilde{f}_i^{(m)}v) \in L(\lambda) \otimes L(\mu) \tag{4.3.3}$$

$$\tilde{e}_i(f_i^{(n)}u \otimes \tilde{f}_i^{(m)}v) \in L(\lambda) \otimes L(\mu).$$

Let M be the A-modules generated by $f_i^{(\nu)}u \otimes f_i^{(\nu')}v$. Then M is stable by $\tilde{e}_i$ and $\tilde{f}_i$ by Theorem 1. Then (4.3.3) follows from $M \subset L(\lambda) \otimes L(\mu)$.
(ii), (iii), and (iv) We may assume $b \equiv f_i^{(n)}u \bmod qL(\lambda)$ and $b' \equiv f_i^{(m)}v \bmod qL(\mu)$ as above. Then $\varepsilon_i(b) = n$ and $\varepsilon_i(b') = m$. Set $a = \langle h_i, \lambda + \xi + n\alpha_i \rangle$ and let M be the A-module generated by $f_i^{(\nu)}u \otimes f_i^{(\nu')}v$. Then by Theorem 4 (see also (2.4.1)–(2.4.4)), we have mod $q_i M$

$$\tilde{f}_i(f_i^{(n)}u \otimes f_i^{(m)}v) \equiv \begin{cases} f_i^{(n+1)}u \otimes f_i^{(m)}v & \text{for } a - n > m, \\ f_i^{(n)}u \otimes f_i^{(m+1)}v & \text{for } a - n \leq m; \end{cases} \tag{4.3.4}$$

$$\tilde{e}_i(f_i^{(n)}u \otimes f_i^{(m)}v) \equiv \begin{cases} f_i^{(n)}u \otimes f_i^{(m-1)}v & \text{for } a - n < m, \\ f_i^{(n-1)}u \otimes f_i^{(m)}v & \text{for } a - n \geq m. \end{cases}$$

Since $M \subset L(\lambda) \otimes L(\mu)$, the second assertions hold, (iii) follows from this formula, and (iv) follows from the fact that $b = u_\lambda$ if $\tilde{e}_i b = 0$ for any i.
Part (v) also follows from (4.3.4). Q.E.D.

Now we shall give several corollaries of this lemma.

COROLLARY 4.3.3. *For $\xi, \xi' \in Q_-(l-1)$ and $\lambda, \mu \in P_+$, $\tilde{f}_i(B(\lambda)_{\lambda+\xi} \otimes B(\mu)_{\mu+\xi'})$ and $\tilde{e}_i(B(\lambda)_{\lambda+\xi} \otimes B(\mu)_{\mu+\xi'})$ are contained in $B(\lambda) \otimes B(\mu) \sqcup \{0\}$.*

COROLLARY 4.3.4. *For $\xi \in Q_-(l)$ and $\lambda, \mu \in P_+$,*

$$\Phi(\lambda, \mu)(L(\lambda + \mu)_{\lambda+\mu+\xi}) \subset L(\lambda) \otimes L(\mu).$$

In fact, this follows from $(C_{l-1}.8)$, Lemma 4.3.2, and $L(\lambda + \mu)_{\lambda+\mu+\xi} = \sum \tilde{f}_i L(\lambda + \mu)_{\lambda+\mu+\xi+\alpha_i}$ for $\xi \neq 0$.

COROLLARY 4.3.5. *For $i_1, \ldots, i_l \in I$ and $\mu \in P_+$, set $\lambda = \Lambda_{i_{l-1}}$. Then*

$$\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu) = v \otimes w \quad \text{in} \quad L(\lambda) \otimes L(\mu)/qL(\lambda) \otimes L(\mu).$$

Here, $v \in B(\lambda)_{\lambda+\xi}$, $w \in B(\mu)_{\mu+\xi'} \cup \{0\}$ for some $\xi, \xi' \in Q_-(l-1) \setminus \{0\}$.

Proof. Assume first $i_l \neq i_{l-1}$. Then $f_{i_l} u_\lambda = 0$ implies

$$\tilde{f}_{i_l}(u_\lambda \otimes u_\mu) = f_{i_l}(u_\lambda \otimes u_\mu) = t_{i_l} u_\lambda \otimes f_{i_l} u_\mu = u_\lambda \otimes (\tilde{f}_{i_l} u_\mu).$$

Since $\tilde{e}_{i_{l-1}} \tilde{f}_{i_l} u_\mu = e_{i_{l-1}} f_{i_l} u_\mu = 0$ and $\tilde{f}_{i_{l-1}} u_\lambda = f_{i_{l-1}} u_\lambda \neq 0$, we have

$$\tilde{f}_{i_{l-1}} \tilde{f}_{i_l}(u_\lambda \otimes u_\mu) \equiv (\tilde{f}_{i_{l-1}} u_\lambda) \otimes (\tilde{f}_{i_l} u_\mu) \bmod qL(\lambda) \otimes L(\mu). \tag{4.3.5}$$

If $i_l = i_{l-1}$, then

$$\tilde{f}_{i_l}(u_\lambda \otimes u_\mu) \equiv (\tilde{f}_{i_l} u_\lambda) \otimes u_\mu,$$

and, since $f_{i_l}^2 u_\lambda = 0$, $\tilde{f}_{i_l}^2(u_\lambda \otimes u_\mu) \equiv \tilde{f}_{i_l} u_\lambda \otimes \tilde{f}_{i_l} u_\mu$, Hence in the both cases, (4.3.5) holds. Then the assertion follows from Lemma 4.3.2. Q.E.D.

COROLLARY 4.3.6. *Let $\lambda, \mu \in P_+$ and $\xi \in Q_-(l)$. Then*

$$(L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi} = \sum_i \tilde{f}_i (L(\lambda) \otimes L(\mu))_{\lambda+\xi+\alpha_i} + u_\lambda \otimes L(\mu)_{\mu+\xi}.$$

Proof. Let L be the left-hand side and $\tilde{L}$ the right-hand side. We already know $\tilde{L} \subset L$. For $\xi' \in Q_-(l-1)\backslash\{0\}$ and $b \in B(\lambda)_{\lambda+\xi'} \otimes B(\mu)_{\mu+\xi-\xi'}$, there exists i such that $\tilde{e}_i b \neq 0$ by Lemma 4.3.2(iv). Then Lemma 4.3.2(iii) implies $b = \tilde{f}_i \tilde{e}_i b$. Therefore, we obtain $L(\lambda)_{\lambda+\xi'} \otimes L(\mu)_{\mu+\xi-\xi'} \subset \tilde{L} + qL$. Hence we have

$$L \subset \tilde{L} + L(\lambda) \otimes u_\mu + qL.$$

For $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda \in B(\lambda)_{\lambda+\xi}$, we have

$$(\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda) \otimes u_\mu \equiv \tilde{f}_{i_1}((\tilde{f}_{i_2} \cdots f_{i_l} u_\lambda) \otimes u_\mu) \bmod qL(\lambda) \otimes L(\mu)$$

by Lemma 4.3.2(v). Thus we obtain $L \subset \tilde{L} + qL$. Then Nakayama's lemma implies the desired result. Q.E.D.

COROLLARY 4.3.7. *For $\lambda, \mu \in P_+$ and $i_1, \ldots, i_l \in I$, we have one of the following two cases.*

(i) $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda \in qL(\lambda)$.
(ii) $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu) \equiv (\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda) \otimes u_\mu \bmod qL(\lambda) \otimes L(\mu)$.

This follows immediately from Lemma 4.3.2(v)

LEMMA 4.3.8. *Let $\lambda, \mu \in P_+$.*

(i) $S(\lambda, \mu)(L(\lambda) \otimes L(\mu)) = L(\lambda)$.
(ii) *For $\xi \in Q_-(l-1)$,*

$$\begin{array}{ccc} (L(\lambda) \otimes L(\mu)/qL(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi} & \xrightarrow{S(\lambda,\mu)} & (L(\lambda)/qL(\lambda))_{\lambda+\xi} \\ \downarrow \tilde{f}_i & & \downarrow \tilde{f}_i \\ (L(\lambda) \otimes L(\mu)/qL(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi-\alpha_i} & \xrightarrow{S(\lambda,\mu)} & (L(\lambda)/qL(\lambda))_{\lambda+\xi-\alpha_i} \end{array}$$

commutes.

Proof. Part (i) follows immediately from $L(\mu)_\mu = Au_\mu$. Let us prove (ii). For $w \in (L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}$, we shall show $\tilde{f}_i S(\lambda, \mu)w \equiv S(\lambda, \mu)\tilde{f}_i w \bmod qL(\lambda)$. $(L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}$ is generated by vectors of the form $f_i^{(n)}u \otimes f_i^{(m)}v$ with $u \in L(\lambda)$, $v \in L(\mu)$ and $e_i u = 0$, $e_i v = 0$. Hence we may assume $w = f_i^{(n)}u \otimes f_i^{(m)}v$. Let M be the A-module generated by $f_i^{(k)}u \otimes f_i^{(k')}v$. Then $M \subset L(\lambda) \otimes L(\mu)$. Then $\tilde{f}_i w \equiv f_i^{(m+l)}u \otimes f_i^{(m)}v$ or $f_i^{(n)}u \otimes f_i^{(m+1)}v \bmod qM$. Hence $S(\lambda, \mu)(\tilde{f}_i w)$ and $\tilde{f}_i(S(\lambda, \mu)w)$ belong to $qL(\lambda)$ except when $v \in L(\mu)_\mu$ and $m = 0$. Now assume $v = u_\mu$. Then $\tilde{f}_i(f_i^{(n)}u \otimes u_\mu) \equiv f_i^{(n+1)}u \otimes u_\mu$ or $f_i^{(n)}u \otimes f_i u_\mu \bmod qM$ according to whether $f_i^{(n+1)}u \neq 0$ or $f_i^{(n+1)}u = 0$. Hence $S(\lambda, \mu)\tilde{f}_i(f_i^{(n)}u \otimes u_\mu) \equiv f_i^{(n+1)}u = \tilde{f}_i f_i^{(n)}u$. Q.E.D.

LEMMA 4.3.9. *Let $\xi \in Q_-(l)$ and $u \in V(\lambda)_{\lambda+\xi}$, and $n, k \in \mathbb{Z}_{\geq 0}$ with $n + k \geq 1$. Assume*

$$t_i^\nu e_i^{(\nu)}u \in q_i^{\nu(\nu+n+k)}qL(\lambda) \qquad \text{for any } \nu \text{ such that } 1 \leq \nu \leq n+k. \tag{4.3.6}$$

Then we have

$$\tilde{f}_i^n f_i^{(k)}u \equiv f_i^{(k+n)}u \bmod qL(\lambda), \tag{4.3.7}$$

$$\tilde{e}_i^n f_i^{(k)}u \equiv f_i^{(k-n)}u \bmod qL(\lambda). \tag{4.3.8}$$

Proof. We write

$$u = \sum f_i^{(m)}u_m \tag{4.3.9}$$

with $u_m \in \operatorname{Ker} e_i \cap V(\lambda)_{\lambda+\xi+m\alpha_i}$, $\langle h_i, \lambda + \xi + m\alpha_i \rangle \geq m \geq 0$. Then we have, setting $a = \langle h_i, \lambda + \xi \rangle$,

$$\begin{aligned} t_i^\nu e_i^{(\nu)} f_i^{(m)} u_m &= t_i^\nu f_i^{(m-\nu)} \begin{bmatrix} \nu - m + \langle h_i, \lambda + \xi + m\alpha_i \rangle \\ \nu \end{bmatrix}_i u_m \\ &= q_i^{\nu(a+2\nu)} \begin{bmatrix} \nu + a + m \\ \nu \end{bmatrix}_i f_i^{(m-\nu)} u_m. \end{aligned}$$

Hence by Lemma 4.3.1 we obtain (see (1.1.26))

$$q_i^{\nu(a+2\nu)} q_i^{-\nu(a+m)} u_m \in q_i^{\nu(\nu+n+k)} qL(\lambda) \quad \text{for} \quad m \geq \nu \quad \text{and} \quad l \leq \nu \leq n+k.$$

Hence we obtain

$$q_i^{\nu(\nu-n-m-k)} u_m \in qL(\lambda) \quad \text{for} \quad m \geq \nu \quad \text{and} \quad 1 \leq \nu \leq n+k.$$

Hence setting $\nu = n + k$ when $m \geq n + k$ and $\nu = m$ when $0 < m < n + k$, we obtain

$$q_i^{-m(n+k)} u_m \in qL(\lambda) \qquad \text{for} \qquad m > 0. \tag{4.3.10}$$

Now we have

$$\tilde{f}_i^n f_i^{(k)} u = \sum \begin{bmatrix} m+k \\ m \end{bmatrix}_i f_i^{(n+k+m)} u_m$$

and

$$f_i^{(n+k)} u = \sum \begin{bmatrix} m+n+k \\ m \end{bmatrix}_i f_i^{(n+k+m)} u_m .$$

Therefore, (4.3.10) implies that both $\tilde{f}_i^n f_i^{(k)} u$ and $f_i^{(n+k)} u$ are equal to $f_i^{(n+k)} u_0$ modulo $qL(\lambda)$. This proves (4.3.7). We have

$$\tilde{e}_i^n f^{(k)} u = \sum_{m \geq n-k} f_i^{(m+k-n)} \begin{bmatrix} m+k \\ m \end{bmatrix}_i u_m \equiv f_i^{(k-n)} u_0 \bmod qL(\lambda),$$

and, when $k \geq n$,

$$f_i^{(k-n)} u = \sum \begin{bmatrix} m+k-n \\ m \end{bmatrix}_i f_i^{(m+k-n)} u_m .$$

Hence both $\tilde{e}_i^n f_i^{(k)} u$ and $f_i^{(k-n)} u$ are equal to $f_i^{(k-n)} u_0$ modulo $qL(\lambda)$. Q.E.D.

4.4. *Proof of* (C_l.3) *and* (C_l.6). We shall first prove (C_l.6) when $\langle h_i, \lambda \rangle \gg 0$.

LEMMA 4.4.1. *Let* $i \in I$, $\xi \in Q_-(l)$ *and* $P \in U_q^-(\mathfrak{g})_\xi$. *Then for* $\lambda \in P_+$, *with* $\langle h_i, \lambda \rangle \gg 0$,

$$(\tilde{f}_i P) u_\lambda \equiv \tilde{f}_i (P u_\lambda) \quad \textit{and}$$

$$(\tilde{e}_i P) u_\lambda \equiv \tilde{e}_i (P u_\lambda) \quad \textit{modulo} \quad qL(\lambda).$$

Proof. We may assume $P = f_i^{(k)} Q$ with $e_i' Q = 0$ and $Q \in U_-(\mathfrak{g})_{\xi + k\alpha_i}$. Then $(\tilde{f}_i P) u_\lambda = f_i^{(k+1)} Q u_\lambda$ and $(\tilde{e}_i P) u_\lambda = f_i^{(k-1)} Q u_\lambda$. By Corollary 3.4.6 we have

$$t_i^\nu e_i^{(\nu)} Q u_\lambda \in q_i^{\nu(\nu+k+1)} qL(\lambda) \quad \text{for} \quad 1 \leq \nu \leq 1+k .$$

Then the lemma follows from Lemma 4.3.9. Q.E.D.

Now we shall show (C_l.6) for arbitrary λ.

PROPOSITION 4.4.2. *For* $\xi \in Q_-(l-1)$ *and* $P \in L(\infty)_\xi$, *we have for any* $\lambda \in P_+$

$$(\tilde{f}_i P) u_\lambda \equiv \tilde{f}_i (P u_\lambda) \bmod qL(\lambda). \tag{4.4.1}$$

Proof. Let us take μ such that $\langle h_i, \mu \rangle \gg 0$. Then by the preceding lemma

$$(\tilde{f}_i P) u_{\lambda+\mu} \equiv \tilde{f}_i (P u_{\lambda+\mu}) \bmod qL(\lambda+\mu). \tag{4.4.2}$$

Hence by applying $\Phi(\lambda, \mu)$ to (4.4.2), Corollary 4.3.4 implies

$$(\tilde{f}_i P)(u_\lambda \otimes u_\mu) \equiv \tilde{f}_i(P(u_\lambda \otimes u_\mu)) \bmod qL(\lambda) \otimes L(\mu).$$

Then applying $S(\lambda, \mu)$, Lemma 4.3.8 implies

$$(\tilde{f}_i P)u_\lambda \equiv \tilde{f}_i(Pu_\lambda) \bmod qL(\lambda). \qquad \text{Q.E.D.}$$

COROLLARY 4.4.3. *For any $\lambda \in P_+$ and $\xi \in Q_-(l)$, we have*

$$\pi_\lambda(L(\infty)_\xi) = L(\lambda)_{\lambda+\xi}.$$

Proof. By the preceding proposition we have

$$\pi_\lambda(L(\infty)_\xi) \subset L(\lambda)_{\lambda+\xi}$$

and

$$L(\lambda)_{\lambda+\xi} \subset \pi_\lambda(L(\infty)_\xi) + qL(\lambda)_{\lambda+\xi}.$$

Then Nakayama's lemma implies the desired result. Q.E.D.

By this proposition π_λ induces a surjective homomorphism $\bar{\pi}_\lambda$: $(L(\infty)/qL(\infty))_\xi \to (L(\lambda)/qL(\lambda))_{\lambda+\xi}$.

COROLLARY 4.4.4. *For $\xi \in Q_-(l)$ and $\lambda \in P_+$, we have*

$$(\bar{\pi}_\lambda B(\infty)_\xi)\backslash\{0\} = B(\lambda)_{\lambda+\xi}.$$

This follows immediately from Proposition 4.4.2.

COROLLARY 4.4.5. *If $\lambda \in P_+$ satisfies $\langle h_i, \lambda\rangle \gg 0$ for any i, then for any $\xi \in Q_-(l)$, $L(\infty)_\xi \xrightarrow{\sim} L(\lambda)_{\lambda+\xi}$ and $B(\infty)_\xi\backslash\{0\} \xrightarrow{\sim} B(\lambda)_{\lambda+\xi}$.*

This follows from $U_q^-(\mathfrak{g})_\xi \xrightarrow{\sim} V(\lambda)_{\lambda+\xi}$.

4.5. *Small loop.* We shall show $\tilde{e}_i L(\infty)_\xi \subset L(\infty)$ and $\tilde{e}_i L(\lambda)_{\lambda+\xi} \subset L(\lambda)$. We fix $\xi \in Q_-$ with $|\xi| = l$. Take a finite set T of P_+ such that $T \in \Lambda_j$ for any j. We shall show

$$(4.5.1)_n \quad \tilde{e}_i L(\infty)_\xi \subset q^{-n}L(\infty) \quad \text{and} \quad \tilde{e}_i L(\lambda)_\xi \subset q^{-n}L(\lambda) \quad \text{for} \quad \lambda \in T$$

by the descending induction on $n \geq 0$. If $n \gg 0$, then $(4.5.1)_n$ is obvious. Now assuming $(4.5.1)_n$ for $n > 0$, we shall derive $(4.5.1)_{n-1}$. By Lemma 4.4.1, $(4.5.1)_n$ and Corollary 4.4.3 imply

$$(4.5.2) \quad \tilde{e}_i L(\lambda)_{\lambda+\xi} \subset q^{-n}L(\lambda) \quad \text{for} \quad \lambda \in P_+ \quad \text{with} \quad \langle h_i, \lambda\rangle \gg 0.$$

LEMMA 4.5.1 *For $\lambda \in T$ and $\mu \in P_+$ with $\langle h_i, \mu\rangle \gg 0$,*

$$\tilde{e}_i((L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}) \subset q^{-n}L(\lambda) \otimes L(\mu).$$

Proof. Let $u \in L(\lambda)_{\lambda+\xi'}$ and $v \in L(\mu)_{\mu+\xi''}$ with $\xi = \xi' + \xi''$. We shall show that $\tilde{e}_i(u \otimes v)$ belongs to $q^{-n}L(\lambda) \otimes L(\mu)$. When $|\xi'|$ and $|\xi''|$ are less than l, it is already proven (Lemma 4.3.2). Hence we may assume either $\xi' = 0, \xi'' = \xi$ or $\xi' = \xi, \xi'' = 0$.

(a) $\xi' = 0$ and $\xi'' = \xi$. We may assume $u = u_\lambda$. Write $v = \sum f_i^{(m)} v_m$ with $e_i v_m = 0$. Here the summation runs over m such that $\langle h_i, \lambda + \xi + m\alpha_i\rangle \geq m \geq 0$. Then $\tilde{e}_i v = \sum f_i^{(m-1)} v_m \in q^{-n}L(\mu)$ by (4.5.2), and hence $v_m \in q^{-n}L(\mu)$ for $m \geq 1$. Since

$$\tilde{e}_i(u \otimes v) = \sum_{m \geq 1} \tilde{e}_i(u_\lambda \otimes f_i^{(m)} v_m),$$

this is contained in the A-module M generated by $f_i^{(m')} u_\lambda \otimes f_i^{(m'')} v_m$ with $m \geq 1$ by Theorem 1. Then the result follows from $M \subset q^{-n}L(\lambda) \otimes L(\mu)$

(b) $\xi' = \xi$ and $\xi'' = 0$. The proof is similar to the case (a) by using $(4.5.1)_n$ instead of (4.5.2). Q.E.D.

Now, we shall show another lemma.

LEMMA 4.5.2. *If $\lambda \in P_+$ satisfies $\langle h_j, \lambda\rangle \gg 0$ for any j, then*

$$\tilde{e}_i L(\lambda)_{\lambda+\xi} \subset q^{1-n}L(\lambda).$$

Proof. It is enough to show that

$$\tilde{e}_i(\tilde{f}_{i_1} \cdots \tilde{f}_{i_1} u_\lambda) \in q^{1-n}L(\lambda). \tag{4.5.3}$$

Set $\lambda_0 = \Lambda_{i_{l-1}}$ and $\mu = \lambda - \lambda_0$. Then by Corollary 4.3.5 we have

$$w = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_{\lambda_0} \otimes u_\mu) \equiv v \otimes v' \bmod qL(\lambda_0) \otimes L(\mu) \tag{4.5.4}$$

with $\xi', \xi'' \in Q_-(l-1) \setminus \{0\}$ and $v \in L(\lambda_0)_{\lambda_0+\xi'}$, $v' \in L(\mu)_{\mu+\xi''}$. By Lemma 4.3.2 we have $\tilde{e}_i(v \otimes v') \subset L(\lambda_0) \otimes L(\mu)$. Hence, $\tilde{e}_i w$ belongs to $\tilde{e}(qL(\lambda_0) \otimes L(\mu)) + L(\lambda_0) \otimes L(\mu)$. Then the preceding lemma implies

$$\tilde{e}_i w \in q^{1-n}(L(\lambda_0) \otimes L(\mu))_{\lambda+\mu+\xi+\alpha_i}. \tag{4.5.5}$$

Applying $\Psi(\lambda_0, \mu)$ to (4.5.5), we obtain by $(C_{l-1}.9)$

$$\tilde{e}_i \tilde{f}_{i_0} \cdots \tilde{f}_{i_l} u_{\lambda_0+\mu} \in q^{1-n}L(\lambda_0 + \mu). \tag{4.5.6}$$

Q.E.D.

Take $\lambda \in P_+$ such that $\langle h_j, \lambda\rangle \gg 0$ for any j. Then Lemma 4.4.1 implies that $\tilde{e}_i(Pu_\lambda) \equiv (\tilde{e}_i P)u_\lambda \bmod qL(\lambda)$ for $P \in L(\infty)_\xi$, and hence $(\tilde{e}_i P)u_\lambda \in q^{1-n}L(\lambda)$ by

Corollary 4.4.3 and the preceding lemma. Thus we obtain by Corollary 4.4.5

(4.5.7) $$\tilde{e}_i L(\infty)_\xi \subset q^{1-n} L(\infty).$$

Now it remains to prove

(4.5.8) $$\tilde{e}_i L(\lambda)_{\lambda+\xi} \subset q^{1-n} L(\lambda) \quad \text{for} \quad \lambda \in T.$$

For $w = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda$, we shall show $\tilde{e}_i w \in q^{1-n} L(\lambda)$. If $w \in qL(\lambda)$, then we have $\tilde{e}_i w \in q^{1-n} L(\lambda)$. When $w \notin qL(\lambda)$, take $\mu \in P_+$ with $\langle h_j, \mu \rangle \gg 0$ for any j. Then we have

(4.5.9) $$\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu) \equiv w \otimes u_\mu \bmod qL(\lambda) \otimes L(\mu)$$

by Corollary 4.3.7. On the other hand, Lemma 4.5.2 implies $\tilde{e}_i \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_{\lambda+\mu} \in q^{1-n} L(\lambda + \mu)$, and hence $\tilde{e}_i \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu) \in q^{1-n} L(\lambda) \otimes L(\mu)$ by $(C_{l-1}.8)$. This implies, along with Lemma 4.5.1 and (4.5.9), that

$$\tilde{e}_i(w \otimes u_\mu) \in q^{1-n} L(\lambda) \otimes L(\mu) + q\tilde{e}_i(L(\lambda) \otimes L(\mu)) \subset q^{1-n} L(\lambda) \otimes L(\mu).$$

Now write $w = \sum f_i^{(m)} w_m$ with $e_i w_m = 0$. Then $\tilde{e}_i w = \sum f_i^{(m-1)} w_m \in q^{-n} L(\lambda)$ implies $w_m \in q^{-n} L(\lambda)$ for $m > 0$. Letting M be the A-module generated $f_i^{(\nu)} w_m \otimes f_i^{(\nu')} u_\mu$ $(m > 0)$, we have

$$\tilde{e}_i(w \otimes u_\mu) = \sum_{m>0} \tilde{e}_i(f_i^{(m)} w_m \otimes u_\mu) \equiv \sum_{m>0} f_i^{(m-1)} w_m \otimes u_\mu = \tilde{e}_i w \otimes u_\mu \bmod qM.$$

Since $qM \subset q^{1-n} L(\lambda) \otimes L(\mu)$, we obtain $\tilde{e}_i w \otimes u_\mu \in q^{1-n} L(\lambda) \otimes L(\mu)$. By applying $S(\lambda, \mu)$, we obtain $\tilde{e}_i w \in q^{1-n} L(\lambda)$.

In both cases we have $\tilde{e}_i w \in q^{1-n} L(\lambda)$. Therefore, we obtain (4.5.8). Thus the induction proceeds, and we can conclude $\tilde{e}_i L(\infty)_\xi \subset L(\infty)$ and $\tilde{e}_i L(\lambda)_{\lambda+\xi} \subset L(\lambda)$ for $\xi \in Q_-(l)$ and $\lambda \in P_+$. Thus $(C_l.1)$ and $(C_l.2)$ are established.

Then the following statements are similarly proven as in Lemmas 4.3.1 and 4.3.2.

(4.5.10) For $u = \sum f_i^{(n)} u_n \in L(\lambda)_{\lambda+\xi}$ such that

$\lambda \in P_+, \xi \in Q_-(l), u_n \in V(\lambda)_{\lambda+\xi+n\alpha_i}, e_i u_n = 0$

and $u_n = 0$ except $\langle h_i, \lambda + \xi + n\alpha_i \rangle \geq n$,

we have $u_n \in L(\lambda)$.

(4.5.11) For $\xi', \xi'' \in Q_-(l)$ and $\lambda, \mu \in P_+$,

$$\tilde{e}_i(L(\lambda)_{\lambda+\xi'} \otimes L(\mu)_{\mu+\xi''}) \subset L(\lambda) \otimes L(\mu).$$

4.6. *Proof of* $(C_l.7)$ *and* $(C_l.12)$. We have already shown $(C_l.1)$, $(C_l.2)$, $(C_l.3)$, $(C_l.6)$, and $(C_l.8)$. We shall now prove $(C_l.7)$ and $(C_l.12)$.
The following lemma can be proven as in Lemma 4.3.2.

LEMMA 4.6.1 *Let* $\lambda, \mu \in P_+$, $\xi \in Q_-(l)$. *Then for any* $u \in L(\lambda)_{\lambda+\xi}$,

$$\tilde{e}_i(u \otimes u_\mu) \equiv \tilde{e}_i u \otimes u_\mu \text{ modulo } qL(\lambda) \otimes L(\mu).$$

Proof. Write $u = \sum f_i^{(n)} u_n$ as in (4.5.10). Then all u_n belong to $L(\lambda)$ by (4.5.10). Hence, we may assume $u = f_i^{(n)} w$ with $e_i w = 0$, and $w \in L(\lambda)_{\lambda+\xi+n\alpha_i}$. Let M be the A-module generated by $f_i^{(\nu)} w \otimes f_i^{(\nu')} u_\mu$. Then by Theorem 1 we have $\tilde{e}_i(f_i^{(n)} w \otimes u_\mu) \equiv f_i^{(n-1)} w \otimes u_\mu \bmod qM$. Then the lemma follows from $M \subset L(\lambda) \otimes L(\mu)$. Q.E.D.

Let $w = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \cdot 1$. Then, taking $\lambda = \Lambda_{i_{l-1}}$, $\mu \in P_+$ with $\langle h_j, \mu \rangle \gg 0$ for any j, Corollary 4.3.4 implies

$$\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu) \equiv v \otimes w \bmod qL(\lambda) \otimes L(\mu) \tag{4.6.1}$$

with $v \in L(\lambda)_{\lambda+\xi'}$ and $w \in L(\mu)_{\mu+\xi''}$ and $\xi', \xi'' \in Q_-(l-1)$. Moreover, v and w belong to $B(\lambda)$ and $B(\mu) \cup \{0\}$ at $q = 0$. Hence, we obtain

$$\tilde{e}_i \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu) \equiv \tilde{e}_i(v \otimes w) \equiv \tilde{e}_i v \otimes w \quad \text{or} \quad v \otimes \tilde{e}_i w \bmod qL(\lambda) \otimes L(\mu). \tag{4.6.2}$$

Therefore, applying $\Psi(\lambda, \mu)$, we obtain by $(C_{l-1}.7)$, $(C_{l-1}.9)$ and $(C_{l-1}.10)$

$$\tilde{e}_i \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_{\lambda+\mu}) \in B(\lambda + \mu) \sqcup \{0\}. \tag{4.6.3}$$

Hence, by Corollary 4.4.5 and Lemma 4.4.1 we obtain

$$\tilde{e}_i \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \cdot 1 \in B(\infty) \sqcup \{0\}. \tag{4.6.4}$$

This proves $\tilde{e}_i B(\infty)_\xi \subset B(\infty) \sqcup \{0\}$, which is a half of $(C_l.7)$.

Now we shall show $(C_l.12)$. Let $\lambda \in P_+$ and $P = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \cdot 1$ and $w = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda \equiv P u_\lambda \bmod qL(\lambda)$. Assume that w does not belong to $qL(\lambda)$. Then, for $\mu \in P_+$ with $\langle h_j, \mu \rangle \gg 0$ for any j, Corollary 4.3.7 implies $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu) \equiv w \otimes u_\mu \bmod qL(\lambda) \otimes L(\mu)$.

By Lemma 4.4.1 and $(C_l.2)$ we have

$$\tilde{e}_i(\tilde{f}_{i_1} \ldots \tilde{f}_{i_l} u_{\lambda+\mu}) \equiv \tilde{e}_i(P u_{\lambda+\mu}) \equiv (\tilde{e}_i P) u_{\lambda+\mu} \bmod qL(\lambda + \mu).$$

Hence, applying $\Phi(\lambda, \mu)$, we have

$$\tilde{e}_i(\tilde{f}_{i_1} \ldots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu)) \equiv (\tilde{e}_i P)(u_\lambda \otimes u_\mu) \bmod qL(\lambda) \otimes L(\mu).$$

Thus, we obtain by Lemma 4.6.1

$$\tilde{e}_i w \otimes u_\mu \equiv \tilde{e}_i(w \otimes u_\mu) \equiv \tilde{e}_i(\tilde{f}_{i_1} \dots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu)) \equiv (\tilde{e}_i P)(u_\lambda \otimes u_\mu) \bmod qL(\lambda) \otimes L(\mu).$$

Hence, by applying $S(\lambda, \mu)$ we obtain $\tilde{e}_i w \equiv (\tilde{e}_i P) u_\lambda \bmod qL(\lambda)$. Thus we proved $(C_l.12)$. Then $\tilde{e}_i B(\lambda)_{\lambda+\xi} \subset B(\lambda) \sqcup \{0\}$ follows from $\tilde{e}_i B(\infty)_\xi \subset B(\infty) \sqcup \{0\}$ because we already know $\bar{\pi}_\lambda B(\infty) \backslash \{0\} = B(\lambda)$ by Corollary 4.4.4. This completes the proof of $(C_l.7)$.

4.7. *Partial proof of* $(C_l.9)$. Let us denote by $L(\infty)^*$ and $L(\lambda)^*$ the dual lattice of $L(\infty)$ and $L(\infty)$ with respect to the inner product introduced in Proposition 3.4.4 and §2.5, respectively. This means

(4.7.1) $$L(\infty)^* = \{P \in U_q^-(\mathfrak{g}); (P, L(\infty)) \subset A\} \qquad \text{and}$$

$$L(\lambda)^* = \{u \in V(\lambda); (u, L(\lambda)) \subset A\}.$$

We shall see later (Propositions 5.1.1. and 5.1.2) that they coincide with $L(\infty)$ and $L(\lambda)$. The following lemma shows the relation of the inner products on $U_q^-(\mathfrak{g})$ and $V(\lambda)$.

LEMMA 4.7.1. *For* $\xi = -\sum n_i \alpha_i \in Q_-$ *and* $P, Q, \in U_q^-(\mathfrak{g})_\xi$, *there exists a polynomial* $f(x_1, \dots, x_n)$ *in* $x = (x_i)_{i \in I}$ *with coefficients in* $Q(q)$ *such that*

(4.7.2) $$(Pu_\lambda, Qu_\lambda) = f(x) \qquad \textit{with} \qquad x_i = q_i^{2\langle h_i, \lambda \rangle}.$$

(4.7.3) $$f(0) = \left(\prod_i (1 - q_i^2)^{-n_i}\right)(P, Q).$$

Proof. We shall prove by the induction on $|\xi|$. If $|\xi| = 0$, it is obvious. When $|\xi| > 0$, we may assume $Q = f_i R$ with $R \in U_q^-(\mathfrak{g})_{\xi+\alpha_i}$.

$$\begin{aligned}(Pu_\lambda, Qu_\lambda) &= q_i^{-1}(t_i e_i P u_\lambda, Ru_\lambda) \\ &= q_i^{-1}\left(\frac{t_i^2 e_i''(P) - e_i'(P)}{q_i - q_i^{-1}} u_\lambda, Ru_\lambda\right) \\ &= (1 - q_i^2)^{-1}(e_i'(P)u_\lambda, Ru_\lambda) - q_i^{2\langle h_i, \lambda+\xi+\alpha_i \rangle}(1 - q_i^2)^{-1}(e_i''(P)u_\lambda, Ru_\lambda).\end{aligned}$$

Hence (4.7.2) follows. The last equality follows from $(P, Q) = (e_i' P, Q)$. Q.E.D.

LEMMA 4.7.2. *If* $\lambda \in P_+$ *satisfies* $\langle h_i, \lambda \rangle \gg 0$ *for any* i, *then* $\pi_\lambda(L(\infty)_\xi^*) = L(\lambda)_\xi^*$ *for any* $\xi \in Q_-(l)$.

Proof. Since $\pi_\lambda(L(\infty)) = L(\lambda)$ and $U_q^-(\mathfrak{g})_\xi \cong V(\lambda)_{\lambda+\xi}$, this follows immediately from the preceding lemma.

PROPOSITION 4.7.3. *Let $\lambda \in P_+$ and $\xi \in Q_-(l)$. If $\mu \in P_+$ satisfies $\langle h_i \mu \rangle \gg 0$ for any i, then we have*

$$\Psi(\lambda, \mu)((L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}) = L(\lambda+\mu)_{\lambda+\mu+\xi}. \tag{4.7.4}$$

Proof. We may assume $\pi_{\lambda+\mu}(L(\infty)^*_\xi) = L(\lambda+\mu)^*_{\lambda+\mu+\xi}$ and $\pi_\mu(L(\infty)^*_\xi) = L(\mu)^*_{\mu+\xi}$. By Corollary 4.3.6 we have

$$(L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi} = \sum \tilde{f}_i(L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi+\alpha_i} + u_\lambda \otimes L(\mu)_{\mu+\xi}. \tag{4.7.5}$$

On the other hand, for $u \in L(\lambda+\mu)^*_{\lambda+\mu+\xi}$ we have, by (2.5.6) and $(C_{l-1}.10)$,

$$\begin{aligned}(\Phi(\lambda, \mu)(u), \tilde{f}_i(L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi+\alpha_i}) \\ = (u, \tilde{f}_i \Psi(\lambda, \mu)(L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi+\alpha_i}) \subset (u, \tilde{f}_i L(\lambda+\mu)_{\lambda+\mu+\xi+\alpha_i}) \subset A.\end{aligned} \tag{4.7.6}$$

Let us write $u = Pu_{\lambda+\mu}$ with $P \in L(\infty)^*_\xi$. Then, writing $\xi = -\sum n_i \alpha_i$, we have

$$\Delta_-(P) \equiv \left(\prod_i t_i^{n_i}\right) \otimes P \bmod \left(\sum_i f_i U_q(\mathfrak{g})\right) \otimes U_q^-(\mathfrak{g}). \tag{4.7.7}$$

Hence,

$$\begin{aligned}\Phi(\lambda, \mu)(Pu_{\lambda+\mu}) &\equiv \left(\prod_i t_i^{n_i} u_\lambda\right) \otimes Pu_\mu \\ &= \left(\prod_i q_i^{n_i \langle h_i, \lambda \rangle}\right) u_\lambda \otimes Pu_\mu \bmod \left(\sum f_i V(\lambda)\right) \otimes V(\mu).\end{aligned}$$

Therefore, we obtain

$$(\Phi(\lambda, \mu)(u), u_\lambda \otimes L(\mu)_{\mu+\xi}) \subset \left(\prod_i q_i^{n_i \langle h_i, \lambda \rangle}\right)(Pu_\mu, L(\mu)_{\mu+\xi}) \subset A.$$

Thus, we obtain

$$(\Phi(\lambda, \mu)(u), (L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}) \subset A \qquad \text{for any} \qquad u \in L(\lambda+\mu)^*_{\lambda+\mu+\xi}.$$

This implies

$$\begin{aligned}(L(\lambda+\mu)^*_{\lambda+\mu+\xi}, \Psi(\lambda, \mu)(L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}) \\ = (\Phi(\lambda, \mu)L(\lambda+\mu)^*_{\lambda+\mu+\xi}, (L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}) \subset A,\end{aligned}$$

and hence $\Psi(\lambda, \mu)((L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}) \subset L(\lambda + \mu)_{\lambda+\mu+\xi}$. The other inclusion follows from $L(\lambda + \mu)_{\lambda+\mu+\xi} = \Psi(\lambda, \mu)\Phi(\lambda, \mu)L(\lambda + \mu)_{\lambda+\mu+\xi}$ and Corollary 4.3.4. Q.E.D.

4.8. *Proof of* $(C_l.13)$ *and* $(C_l.14)$. First, let us prove $(C_l.14)$. Let $b \in B(\infty)_\xi$ with $\tilde{e}_i b \neq 0$. Set $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \cdot 1$. Then, for $\lambda = i_{l-1}$ and μ with $\langle h_j, \mu\rangle \gg 0$ for any j,

(4.8.1) $$\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu) \equiv v \otimes w \bmod qL(\lambda) \otimes L(\mu)$$

with $\xi', \xi'' \in Q_-(l-1)$, $v \in L(\lambda)_{\xi'}$, $w \in L(\mu)_{\xi''}$. Moreover, $v \bmod qL(\lambda) \in B(\lambda)$ and $w \bmod qL(\mu) \in B(\mu) \sqcup 0$. We have by (4.5.11)

$$\tilde{e}_i \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\lambda) \equiv \tilde{e}_i(v \otimes w) \bmod qL(\lambda) \otimes L(\mu),$$

and hence $\tilde{e}_i \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_{\lambda+\mu} \equiv \tilde{e}_i(\Psi(\lambda, \mu)(v \otimes w)) \bmod qL(\lambda + \mu)$ by $(C_{l-1}.10)$. Since $\bar{\pi}_{\lambda+\mu}(\tilde{e}_i b) = \tilde{e}_i \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_{\lambda+\mu} \neq 0$ by Lemma 4.4.1 and Corollary 4.4.5, $\tilde{e}_i(v \otimes w)$ does not belong to $qL(\lambda) \otimes L(\mu)$. Thus, we obtain $w \bmod qL(\mu) \in B(\mu)$. Therefore, we have by Lemma 4.3.2 (iii)

(4.8.2) $$\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu) \equiv \tilde{f}_i \tilde{e}_i(\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes u_\mu)) \bmod qL(\lambda) \otimes L(\mu).$$

Then Proposition 4.7.3 implies

$$\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_{\lambda+\mu} \equiv \tilde{f}_i \tilde{e}_i \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_{\lambda+\mu} \bmod qL(\lambda + \mu).$$

Then Lemma 4.4.1 implies

$$\bar{\pi}_{\lambda+\mu}(b) = \bar{\pi}_{\lambda+\mu}(\tilde{f}_i \tilde{e}_i b).$$

Thus $b = \tilde{f}_i \tilde{e}_i b$ follows from Corollary 4.4.5. This proves $(C_l.14)$.

Let us prove $(C_l.13)$. Let $b \in B(\lambda)_{\lambda+\xi}$ such that $\tilde{e}_i b \neq 0$. Then there exist $\tilde{b} \in B(\infty)_\xi$ such that $b = \bar{\pi}_\lambda(\tilde{b})$. Then $(C_l.12)$ implies $\bar{\pi}_\lambda(\tilde{e}_i \tilde{b}) = \tilde{e}_i b \neq 0$. Hence $\tilde{e}_i \tilde{b} \neq 0$. Now $(C_l.14)$ implies $\tilde{b} = \tilde{f}_i \tilde{e}_i \tilde{b}$. Finally, we have

$$\tilde{f}_i \tilde{e}_i b = \tilde{f}_i \bar{\pi}_\lambda(\tilde{e}_i \tilde{b}) = \bar{\pi}_\lambda(\tilde{f}_i \tilde{e}_i \tilde{b}) = \bar{\pi}_\lambda(\tilde{b}) = b \qquad \text{by} \qquad (C_l.6).$$

Now assume that $b \in B(\lambda)_{\lambda+\xi+\alpha_i}$ satisfies $\tilde{f}_i b \neq 0$. Let $\tilde{b} \in B(\infty)_{\xi+\alpha_i}$ such that $\bar{\pi}_\lambda(\tilde{b}) = b$. Then $\bar{\pi}_\lambda(\tilde{f}_i \tilde{b}) = \tilde{f}_i b \neq 0$, and hence $(C_l.12)$ implies

$$\tilde{e}_i \tilde{f}_i b = \tilde{e}_i \bar{\pi}_\lambda(\tilde{f}_i \tilde{b}) = \bar{\pi}_\lambda(\tilde{e}_i \tilde{f}_i \tilde{b}) = \bar{\pi}_\lambda(\tilde{b}) = b.$$

This completes the proof of $(C_l.13)$.

4.9. *Proof of* $(C_l.4)$ *and* $(C_l.5)$. The proof of $(C_l.4)$ being similar, we only give the proof of $(C_l.5)$. Assuming $\sum_{b \in B(\lambda)_{\lambda+\xi}} a_b b = 0$, let us show $a_b = 0$. For any i we have

$$\sum_b a_b \tilde{e}_i b = 0.$$

Since $\tilde{e}_i b \neq 0$ implies $b = \tilde{f}_i \tilde{e}_i b$ by $(C_l.7)$ and $(C_l.13)$, $\{\tilde{e}_i b; b \in B(\lambda)_{\lambda+\xi}, \tilde{e}_i b \neq 0\}$ is linearly independent by $(C_{l-1}.5)$. Hence $a_b = 0$ if $\tilde{e}_i b \neq 0$. Since there exists i such that $\tilde{e}_i b \neq 0$ for any b, all a_b vanish.

4.10. *End of proof.* We have proven C_l except $(C_l.9)$, $(C_l.10)$, and $(C_l.11)$. We shall show the remaining statements. First, we shall prove a lemma.

LEMMA 4.10.1. *For $\xi \in Q_-(l)\backslash\{0\}$ and $\lambda \in P_+$, we have*

(4.10.1) $$\{u \in (L(\infty)/qL(\infty))_\xi; \tilde{e}_i u = 0 \quad \textit{for any} \quad i\} = 0,$$

(4.10.2) $$\{u \in (L(\lambda)/qL(\lambda))_{\lambda+\xi}; \tilde{e}_i u = 0 \quad \textit{for any} \quad i\} = 0,$$

(4.10.3) $$\{u \in U_q^-(\mathfrak{g})_\xi; \tilde{e}_i u \in L(\mathfrak{g}) \quad \textit{for any} \quad i\} = L(\infty)_\xi, \qquad \textit{and}$$

(4.10.4) $$\{u \in V(\lambda)_{\lambda+\xi}; \tilde{e}_i u \in L(\lambda) \quad \textit{for any} \quad i\} = L(\lambda)_{\lambda+\xi}.$$

Proof. The proof being similar, we shall prove only (4.10.2) and (4.10.4). Assume that $u \in (L(\lambda)/qL(\lambda))_\xi$ satisfies $\tilde{e}_i u = 0$ for any i. Write $u = \sum_{b \in B(\lambda)_{\lambda+\xi}} a_b b$. Then $\sum a_b \tilde{e}_i b = 0$. Hence $a_b = 0$ if $\tilde{e}_i b \neq 0$ for some i. Therefore, all a_b vanish.

Let us prove (4.10.4). Let $u \in V(\lambda)_{\lambda+\xi}$ and assume $\tilde{e}_i u \in L(\lambda)$ for any i. If $u \in q^{-n}L(\lambda)$ for $n > 0$, then $\tilde{e}_i(q^n u) \in qL(\lambda)$ for any i. Hence (4.10.1) implies $u \in q^{1-n}L(\lambda)$. This shows $u \in L(\lambda)$ by the induction on n. Q.E.D.

Now we shall prove $(C_l.9)$.

COROLLARY 4.10.2. *For $\xi \in Q_-(l)$, λ, $\mu \in P_+$,*

$$\Psi(\lambda, \mu)((L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}) \subset L(\lambda + \mu).$$

Proof. We may assume $|\xi| = l \geqslant 2$. By (4.5.11) we have $\tilde{e}_i((L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}) \subset L(\lambda) \otimes L(\mu)$. Hence

$$\begin{aligned}&\tilde{e}_i \Psi(\lambda, \mu)((L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi}) \\ &\quad \subset \Psi(\lambda, \mu)((L(\lambda) \otimes L(\mu))_{\lambda+\mu+\xi+\alpha_i}) \subset L(\lambda + \mu).\end{aligned}$$

Then the preceding lemma implies the desired result. Q.E.D.

Let us prove $(C_l.11)$. Since we know already $B(\lambda)_\xi = \bar{\pi}_\lambda B(\infty)_\xi \backslash \{0\}$, it remains to prove that, for $b, b' \in B(\infty)_\xi$, $\bar{\pi}_l(b) = \bar{\pi}_\lambda(b') \neq 0$ implies $b = b'$. There exists i such that $\tilde{e}_i \bar{\pi}_\lambda(b) \neq 0$. Hence, by $(C_l.12)$, $\bar{\pi}_l(\tilde{e}_i b) = \bar{\pi}_\lambda(\tilde{e}_i b') \neq 0$. Thus, $\tilde{e}_i b = \tilde{e}_i b' \neq 0$ by $(C_l.7)$ and $(C_{l-1}.11)$. Then $(C_l.14)$ implies $b = b'$.

Finally, we shall prove $(C_l.10)$. First, note that, C_l being all proven except $(C_l.10)$, Lemma 4.3.2 is still valid with ξ, $\xi' \in Q_-(l)$. In particular we have that

(4.10.5) for $\xi \in Q_-(l)$, $\tilde{e}_i((B(\lambda) \otimes B(\mu))_{\lambda+\mu+\xi} \subset B(\lambda) \otimes B(\mu) \sqcup \{0\}$, and

(4.10.6) if $b \in (B(\lambda) \otimes B(\mu))_{\lambda+\mu+\xi}$ and $\tilde{e}_i b \neq 0$, then $b = \tilde{f}_i \tilde{e}_i b$.

Now let $b \in (B(\lambda) \otimes B(\mu))_{\lambda+\mu+\xi}$. If there is i such that $\tilde{e}_i b \in B(\lambda) \otimes B(\mu)$, then by (4.10.5), (4.10.6), and $(C_{l-1}.10)$, $\Psi(\lambda, \mu)(b) = \Psi(\lambda, \mu)(\tilde{f}_i \tilde{e}_i b) = \tilde{f}_i \Psi(\lambda, \mu)(\tilde{e}_i b)$ belongs to $B(\lambda + \mu) \sqcup \{0\}$. If $\tilde{e}_i b = 0$ for any i, then $\tilde{e}_i \Psi(\lambda, \mu)(b) = 0$ for any i. Hence (4.10.2) implies $\Psi(\lambda, \mu)(b) = 0$.

Thus we have proven $(C_l.1)$–$(C_l.14)$, and the induction proceeds. This completes the proof of the Theorems 2, 4, and 5.

PART II. MELTING THE CRYSTAL BASE

§5. Polarization

5.1 *Inner product.* In this section we shall investigate the properties of crystal bases with respect to the inner products on $V(\lambda)$ and $U_q^-(\mathfrak{g})$. (Cf. §2.5 and Proposition 3.4.4.)

PROPOSITION 5.1.1. *Let $\lambda \in P_+$.*

(i) $(L(\lambda), L(\lambda)) \subset A$.

Let $(\ , \ )_0$ be the $\mathbb{Q}$-valued inner product on $L(\lambda)/qL(\lambda)$ induced by $(\ , \ )|_{q=0}$ on $L(\lambda)$.

(ii) $(\tilde{e}_i u, v)_0 = (u, \tilde{f}_i v)$ *for $u, v \in L(\lambda)/qL(\lambda)$.*

(iii) *$B(\lambda)$ is an orthonormal base with respect to $(\ , \ )_0$. In particular, $(\ , \ )_0$ is positive definite.*

(iv) $L(\lambda) = \{u \in V(\lambda); (u, L(\lambda)) \subset A\}$.

Proof. (i) We shall prove $(L(\lambda)_{\lambda+\xi}, L(\lambda)_{\lambda+\xi}) \subset A$ by the induction on $|\xi|$. If $|\xi| = 0$, then this is trivial. Assume $|\xi| > 0$. Since $L(\lambda)_{\lambda+\xi} = \sum \tilde{f}_i L(\lambda)_{\lambda+\xi+\alpha_i}$, it is enough to show

$$(\tilde{f}_i u, v) \equiv (u, \tilde{e}_i v) \bmod qA \tag{5.1.1}$$

$$\text{for} \quad u \in L(\lambda)_{\lambda+\xi+\alpha_i} \quad \text{and} \quad v \in L(\lambda)_{\lambda+\xi}.$$

We may assume $u = f_i^{(n)} u_0$ and $v = f_i^{(m)} v_0$ with $e_i u_0 = e_i v_0 = 0$, $\langle h_i, \lambda + \xi + (n + 1)\alpha_i \rangle \geqslant n$ and $\langle h_i, \lambda + \xi + m\alpha_i \rangle \geqslant m$.
Then, we have

$$(f_i^{(n+1)} u_0, f_i^{(m)} v_0) = \frac{1}{[m]_i!}((q_i^{-1} t_i e_i)^m f_i^{(n+1)} u_0, v_0).$$

Since $(q_i^{-1} t_i e_i)^m = q_i^{-m} q_i^{-m(m-1)} t_i^m e_i^m = q_i^{-m^2} t_i^m e_i^m$, we have, setting $\mu = \lambda + \xi$,

$$\begin{aligned}(f_i^{(n+1)} u_0, f_i^{(m)} v_0) &= q_i^{-m2} (t_i^m e_i^{(m)} f_i^{(n+1)} u_0, v_0) \\ &= \delta_{n+1,m} q_i^{-m^2} \left(t_i^m \begin{bmatrix} \langle h_i, \mu + (n+1)\alpha_i \rangle \\ m \end{bmatrix}_i u_0, v_0 \right)\end{aligned}$$

$$= \delta_{n+1,m} q_i^{-m^2+m\langle h_i,\mu+m\alpha_i\rangle} \begin{bmatrix} \langle h_i, \mu\rangle + 2m \\ m \end{bmatrix} (u_0, v_0)$$

$$= \delta_{n+1,m} q_i^{m(\langle h_i,\mu\rangle+m)} \begin{bmatrix} \langle h_i, \mu\rangle + 2m \\ m \end{bmatrix}_i (u_0, v_0).$$

Since $(u_0, v_0) \in A$ by the hypothesis of induction and $q_i^{m(\langle h_i,\mu\rangle+m)}\left[{\langle h_i,\mu\rangle+2m \atop m}\right]_i$ belongs to $1 + qA$ (cf. (1.1.26)), we obtain

$$(5.1.2) \qquad (f_i^{(n+1)}u_0, f_i^{(m)}v_0) \equiv \delta_{n+1,m}(u_0, v_0) \bmod qA.$$

Similar arguments show that

$$(5.1.3) \qquad (f_i^{(n)}u_0, f_i^{(m-1)}v_0) \equiv \delta_{n+1,m}(u_0, v_0) \bmod qA.$$

Hence, we obtain (5.1.1.). Thus, we obtain (i) and (ii).

Let us prove (iii). We shall show $(b, b')_0 = \delta_{b,b'}$ for $b, b' \in B(\lambda)_{\lambda+\xi}$ by the induction on $|\xi|$. If $|\xi| = 0$, this is obvious, and if $|\xi| > 0$, taking i such that $\tilde{e}_i b \in B(\lambda)$, we have

$$(b, b')_0 = (\tilde{f}_i\tilde{e}_i b, b')_0 = (\tilde{e}_i b, \tilde{e}_i b')_0 = \delta_{\tilde{e}_i b, \tilde{e}_i b'} = \delta_{bb'}.$$

Part (iv) follows easily from (i) and (iii). Q.E.D.

Similar arguments show the following proposition.

PROPOSITION 5.1.2.

(i) $$(L(\infty), L(\infty)) \subset A.$$

Let $(\ ,\ )_0$ *denote the* $\mathbb{Q}$*-valued inner product on* $L(\infty)/qL(\infty)$ *induced by* $(\ ,\ )|_{q=0}$ *on* $L(\infty)$.

(ii) $(\tilde{e}_i u, v)_0 = (u, \tilde{f}_i v)_0$ *for* $u, v \in L(\infty)/qL(\infty)$.

(iii) $B(\infty)$ *is an orthonormal base of* $(\ ,\ )_0$. *In particular,* $(\ ,\ )_0$ *is positive definite.*

(iv) $B(\infty) = \{P \in U_q^-(\mathfrak{g}); (P, L(\infty)) \subset A\}$.

Now the following is the consequence of the positivity of $(\ ,\ )_0$.

PROPOSITION 5.1.3. *For* $\lambda \in P_+$, *we have*

$$(5.1.4) \qquad L(\infty) = \{u \in U_q^-(\mathfrak{g}); (u, u) \in A\},$$

$$(5.1.5) \qquad L(\lambda) = \{u \in V(\lambda); (u, u) \in A\}.$$

Proof. The proof of (5.1.4) being similar, we shall only prove (5.1.5). For $u \in V(\lambda)$, with $(u, u) \in A$ let us take the smallest $n \geq 0$ such that $u \in q^{-n}L(\lambda)$. If $n > 0$, $(q^n u, q^n u) \in qA$. Hence, $v = q^n u \bmod qL(\lambda)$ satisfies $(v, v)_0 = 0$. Then the positive definiteness of $(\ , \ )_0$ implies $v = 0$, or equivalently $u \in q^{1-n}L(\lambda)$. This is a contradiction. Therefore u belongs to $L(\lambda)$. Q.E.D.

5.2. *The $*$-operator.* In this section we shall prove the following proposition and its consequences.

PROPOSITION 5.2.1. *For $P, Q \in U_q^-(\mathfrak{g})$ we have*

$$(P^*, Q^*) = (P, Q). \tag{5.2.1}$$

Here $*$ is the antiautomorphism defined in §1.3.

In order to prove this we shall prepare several lemmas.

LEMMA 5.2.2. (i) *For any i, j we have*

$$(Ad(t_i)e_i'') \circ e_j' = e_j' \circ Ad(t_i)e_i''. \tag{5.2.2}$$

(ii) *We have*

$$(Pf_i, Q) = (P, Ad(t_i)e_i''Q) \quad \text{for any} \quad P, Q \in U_q^-(\mathfrak{g}). \tag{5.2.3}$$

Proof. Part (i) follows immediately from Proposition 3.4.5.

Let us prove (ii). When $P = 1$, $(f_i, f_i) = (1, Ad(t_i)e_i''f_i)$ implies (5.2.3) for any Q. Hence it is enough to show that, if P satisfies (5.2.3) for any Q, then we have

$$(f_jPf_i, Q) = (f_jP, Ad(t_i)e_i''Q). \tag{5.2.4}$$

By using (5.2.2) we have

$$\begin{aligned}(f_jPf_i, Q) &= (Pf_i, e_j'Q)\\ &= (P, Ad(t_i)e_i''e_j'Q)\\ &= (P, e_j'(Ad(t_i)e_i'')Q)\\ &= (f_jP, Ad(t_i)e_i''Q).\end{aligned}$$

Q.E.D.

LEMMA 5.2.3. *We have*

$$(e_i'(P^*))^* = Ad(t_i)e_i''P \quad \text{for any} \quad P \in U_q^-(\mathfrak{g}). \tag{5.2.5}$$

Proof. We have

$$\begin{aligned}[e_i, P] &= \frac{(t_ie_i''P - t_i^{-1}e_i'P)}{q_i - q_i^{-1}}\\ &= \frac{((Adt_i)e_i''P)t_i - ((Adt_i^{-1})e_i'P)t_i^{-1}}{q_i - q_i^{-1}}.\end{aligned}$$

Hence taking $*$, we obtain

$$[P^*, e_i] = \frac{t_i^{-1}((Adt_i)e_i''P) - t_i((Adt_i^{-1})e_i'P)^*}{q_i - q_i^{-1}}.$$

Thus we obtain the desired result. Q.E.D.

Now we are ready to prove Proposition 5.2.1. Since (5.2.1) is true for $P = 1$, it is enough to prove that (5.2.1) implies

(5.2.6) $$((Pf_i)^*, Q^*) = (Pf_i, Q).$$

We have, by (5.2.3) and (5.2.5),

$$\begin{aligned}((Pf_i)^*, Q^*) &= (f_iP^*, Q^*) = (P^*, e_i'Q^*)\\ &= (P, (e_i'Q^*)^*) = (P, Ad(t_i)e_i''Q)\\ &= (Pf_i, Q).\end{aligned}$$

This completes the proof of Proposition 5.2.1.

Then Proposition 5.2.1 and Proposition 5.1.3 immediately imply the following result.

PROPOSITION 5.2.4. $L(\infty)^* = L(\infty)$.

Here $*$ is the antiautomorphism of $U_q^-(\mathfrak{g})$.

§6. Global crystal bases

6.1. *$\mathbb{Z}$-forms.* Let us denote by $U_q^{\mathbb{Z}}(\mathfrak{g})$ the sub-$\mathbb{Z}[q, q^{-1}]$-algebra of $U_q(\mathfrak{g})$ generated by $f_i^{(n)}$, $e_i^{(n)}$, and q^h, $\left\{\begin{smallmatrix} q^h \\ n \end{smallmatrix}\right\}$ $(h \in P^*)$. Let $U_{\mathbb{Z}}^-(\mathfrak{g})$ denote the sub-$\mathbb{Z}[q, q^{-1}]$-algebra of $U_q(\mathfrak{g})$ generated by $f_i^{(n)}$. Then $U_q^{\mathbb{Z}}(\mathfrak{g})$ and $U_{\mathbb{Z}}^-(\mathfrak{g})$ are stable by the automorphisms $*$ and $-$. By the commutation relation (3.1.2)

(6.1.1) $U_{\mathbb{Z}}^-(\mathfrak{g})$ is stable by e_i'.

Thus, Proposition 3.2.1 implies that

(6.1.2)
if $\sum f_i^{(n)}u_n$ belongs to $U_{\mathbb{Z}}^-(\mathfrak{g})$ and if $e_i'u_n = 0$, then all u_n belong to $U_{\mathbb{Z}}^-(\mathfrak{g})$, and

(6.1.3) $U_{\mathbb{Z}}^-(\mathfrak{g})$ is stable by $\tilde{e}_i$ and $\tilde{f}_i$.

We set

(6.1.4) $$(f_i^nU_q^-(\mathfrak{g}))^{\mathbb{Z}} = f_i^nU_q^-(\mathfrak{g}) \cap U_{\mathbb{Z}}^-(\mathfrak{g}).$$

Then (6.1.2) implies that

$$(f_i^n U_q^-(\mathfrak{g}))^{\mathbb{Z}} = \sum_{k \geqslant n} f_i^{(k)} U_{\mathbb{Z}}^-(\mathfrak{g}). \tag{6.1.5}$$

In fact, $\sum f_i^{(n)} u_n (e_i' u_n = 0)$ belongs to $f_i^n U_q^-(\mathfrak{g})$ if and only if $u_k = 0$ for $k < n$. Let us set

$$L_{\mathbb{Z}}(\infty) = L(\infty) \cap U_{\mathbb{Z}}^-(\mathfrak{g}). \tag{6.1.6}$$

Then, by (6.1.3), $L_{\mathbb{Z}}(\infty)$ is stable by $\tilde{f}_i$ and $\tilde{e}_i$.
We have therefore

$$B(\infty) \subset L_{\mathbb{Z}}(\infty)/qL_{\mathbb{Z}}(\infty) \subset L(\infty)/qL(\infty). \tag{6.1.7}$$

Let $A_{\mathbb{Z}}$ be the sub–$\mathbb{Z}$-algebra of $\mathbb{Q}(q)$ generated by q and $(1 - q^{2n})^{-1} (n \geqslant 1)$. Let $K_{\mathbb{Z}}$ be the subalgebra generated by $A_{\mathbb{Z}}$ and q^{-1}. Then we have

$$A_{\mathbb{Z}} = A \cap K_{\mathbb{Z}}. \tag{6.1.8}$$

We can easily see

$$(U_{\mathbb{Z}}^-(\mathfrak{g}), U_{\mathbb{Z}}^-(\mathfrak{g})) \subset K_{\mathbb{Z}}, \tag{6.1.9}$$

and hence

$$(L_{\mathbb{Z}}(\infty), L_{\mathbb{Z}}(\infty)) \subset A_{\mathbb{Z}}. \tag{6.1.10}$$

Since $f(0)$ is an integer for any $f \in A_{\mathbb{Z}}$, we obtain

$$(\ ,\ )_0 \quad \text{is } \mathbb{Z}\text{-valued on } L_{\mathbb{Z}}(\infty)/qL_{\mathbb{Z}}(\infty). \tag{6.1.11}$$

PROPOSITION 6.1.1. (i) *$L_{\mathbb{Z}}(\infty)/qL_{\mathbb{Z}}(\infty)$ is a free $\mathbb{Z}$-module with $B(\infty)$ as a base.*
(ii) $B(\infty) \cup (-B(\infty)) = \{u \in L_{\mathbb{Z}}(\infty)/qL_{\mathbb{Z}}(\infty); (u, u)_0 = 1\}$.

Proof. (i) If $\sum a_b b$ belongs to $L_{\mathbb{Z}}(\infty)/qL_{\mathbb{Z}}(\infty)$, then, for any b', $(\sum a_b b, b')_0 = a_{b'}$ belongs to $\mathbb{Z}$.
(ii) If $u = \sum a_b b \in L_{\mathbb{Z}}(\infty)/qL_{\mathbb{Z}}(\infty)$ satisfies $(u, u)_0 = 1$, then $\sum a_b^2 = 1$.
Since a_b are integers, there exists b_0 such that $a_{b_0} = \pm 1$ and $a_b = 0$ for $b \neq b_0$.
Q.E.D.

COROLLARY 6.1.2. *$L_{\mathbb{Z}}(\infty)^* = L_{\mathbb{Z}}(\infty)$ and $B(\infty)^* \subset B(\infty) \cup (-B(\infty))$. Here, $*$ is the antiautomorphism of $U_q(\mathfrak{g})$ defined in* §1.3.

This follows from Propositions 6.1.1 and 5.2.4.
We conjecture that $B(\infty)^* = B(\infty)$. This is shown by Lusztig [L1], [L2], [L3] in the A_n, D_n, E_n case.

We set, for $\lambda \in P_+$,

$$V_{\mathbb{Z}}(\lambda) = U_{\mathbb{Z}}^-(\mathfrak{g})u_\lambda. \tag{6.1.12}$$

Then $V_{\mathbb{Z}}(\lambda)$ is a $U_q^{\mathbb{Z}}(\mathfrak{g})$-module by (1.1.23). Note that $V_{\mathbb{Z}}(\lambda)$ is not stable by $\tilde{e}_i$ and $\tilde{f}_i$ in general. We set also

$$(f_i^n V(\lambda))^{\mathbb{Z}} = (f_i^n U_q^-(\mathfrak{g}))^{\mathbb{Z}} u_\lambda = \sum_{k\geqslant n} f_i^{(k)} V_{\mathbb{Z}}(\lambda), \tag{6.1.13}$$

$$L_{\mathbb{Z}}(\lambda) = V_{\mathbb{Z}}(\lambda) \cap L(\lambda). \tag{6.1.14}$$

Let $-$ be the automorphism of $V(\lambda)$ defined by

$$(Pu_\lambda)^- = \bar{P}u_\lambda \quad \text{for} \quad P \in U_q^-(\mathfrak{g}). \tag{6.1.15}$$

This is well defined by (1.2.6).
Then $V_{\mathbb{Z}}(\lambda)$ and $(f_i^n V(\lambda))^{\mathbb{Z}}$ are stable by $-$.
Since $L(\lambda) = \pi_\lambda(L(\infty))$, we obtain

$$\pi_\lambda(L_{\mathbb{Z}}(\infty)) \subset L_{\mathbb{Z}}(\lambda), \tag{6.1.16}$$

and hence

$$B(\lambda) \subset L_{\mathbb{Z}}(\lambda)/qL_{\mathbb{Z}}(\lambda) \subset L(\lambda)/qL(\lambda). \tag{6.1.17}$$

As seen later (or proven similarly as in Proposition 6.1.1), $L_{\mathbb{Z}}(\lambda)/qL_{\mathbb{Z}}(\lambda)$ is a free $\mathbb{Z}$-module with $B(\lambda)$ as a base.

PROPOSITION 6.1.3. *Let M be an integrable $U_q(\mathfrak{g})$-module and let $M_{\mathbb{Z}}$ be a sub–$U_q^{\mathbb{Z}}(\mathfrak{g})$-module of M. Let $\lambda \in P_+$ and $i \in I$. Assume that $n = -\langle h_i, \lambda\rangle \geqslant 0$. Then*

$$(M_{\mathbb{Z}})_\lambda = \sum_{k\geqslant n} f_i^{(k)}(M_{\mathbb{Z}})_{\lambda+k\alpha_i}. \tag{6.1.18}$$

This follows immediately from the following lemma.

LEMMA 6.1.4. *When $n \geqslant 1$, we have $u = \sum_{k\geqslant n}(-1)^{k-n}\begin{bmatrix} k-1 \\ k-n\end{bmatrix}_i f_i^{(k)}e_i^{(k)}u$ for any $u \in M_\lambda$.*

Proof. We may assume $u = f_i^{(m)}v$ with $v \in \operatorname{Ker} e_i \cap M_{\lambda+m\alpha_i}$ with $m \geqslant n$. Then we have

$$\sum (-1)^{k-n}\begin{bmatrix} k-1 \\ k-n\end{bmatrix}_i f_i^{(k)}e_i^{(k)}u$$

$$= \sum_{k=n}^{m} (-1)^{k-n} \begin{bmatrix} k-1 \\ k-n \end{bmatrix}_i f_i^{(k)} \begin{bmatrix} (k-m)+(2m-n) \\ k \end{bmatrix}_i f_i^{(m-k)} v$$

$$= \sum_{k=n}^{m} (-1)^{k-n} \begin{bmatrix} k-1 \\ k-n \end{bmatrix}_i \begin{bmatrix} k+m-n \\ k \end{bmatrix}_i \begin{bmatrix} m \\ k \end{bmatrix}_i f_i^{(m)} v.$$

Hence this lemma follows from the identity

(6.1.19)
$$\sum_{k=0}^{m} (-1)^k \begin{bmatrix} k+n-1 \\ k \end{bmatrix} \begin{bmatrix} k+m+n \\ m \end{bmatrix} \begin{bmatrix} m+n \\ k+n \end{bmatrix} = 1 \quad \text{for} \quad m \geqslant 0, n \geqslant 1.$$

Proof of (6.1.19). The following formula is known (e.g., see [A], p. 37, (3.3.11)).

(6.1.20)
$$\sum_{k \geqslant 0} \frac{[a_1+a_2+b_1+b_2+k]!}{[k]![a_1-k]![a_2-k]![b_1+k]![b_2+k]!}$$
$$= \frac{[a_1+a_2+b_1+b_2]![a_1+a_2+b_1]![a_1+a_2+b_2]!}{[a_1]![a_2]![a_1+b_1]![a_2+b_1]![a_1+b_2][a_2+b_2]!}.$$

Here $a_j, b_j \geqslant 0$, and we understand $1/[n]! = 0$ for $n < 0$. If we set $a_2 = m, b_1 = b_2 = n$, and $x = q^{a_1}$, it reduces to

(6.1.21)
$$\sum_{k=0}^{m} \begin{Bmatrix} x \\ k \end{Bmatrix} \begin{bmatrix} m+n \\ k+n \end{bmatrix} \begin{Bmatrix} q^{m+2n}x \\ n+k \end{Bmatrix} = \begin{Bmatrix} q^{m+2n}x \\ m+n \end{Bmatrix} \begin{Bmatrix} q^{m+n}x \\ m \end{Bmatrix}.$$

(See §1.1 for the notation.)

Then setting $x = q^{-n}$ and using $\begin{Bmatrix} x \\ k \end{Bmatrix} = (-1)^k \begin{bmatrix} k+n-1 \\ k \end{bmatrix}$, (6.1.21) reduces to (6.1.19). Q.E.D.

§7. Proof of Theorems 6 and 7

7.1. *Triviality of vector bundles over* $\mathbb{P}^1$. We shall give some preparatory lemmas for the proof of Theorems 6 and 7. Remember that A is the ring of rational functions regular at $q = 0$. Hence, $\bar{A}$ is the ring of rational functions regular at $q = \infty$. Here $-$ is the automorphism $q \mapsto q^{-1}$.

LEMMA 7.1.1. *Let V be a finite-dimensional vector space over $\mathbb{Q}(q)$, M a sub-$\mathbb{Z}[q, q^{-1}]$-module of V, L_0 a free sub-A-module of V, and L_∞ a free sub-$\bar{A}$-module of V such that $V \cong \mathbb{Q}(q) \otimes_A L_0 \cong \mathbb{Q}(q) \otimes_{\bar{A}} L_\infty$.*

(i) *Assume that* $M \cap L_0 \cap L_\infty \to (M \cap L_0)/(M \cap qL_0)$ *is an isomorphism. Then*

$$M \cap L_0 \cong \mathbb{Z}[q] \otimes_{\mathbb{Z}} (M \cap L_0 \cap L_\infty),$$

$$M \cap L_\infty \cong \mathbb{Z}[q^{-1}] \otimes_{\mathbb{Z}} (M \cap L_0 \cap L_\infty),$$

$$M \cong \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} (M \cap L_0 \cap L_\infty),$$

$$M \cap L_0 \cap L_\infty \xrightarrow{\sim} M \cap L_\infty/M \cap q^{-1}L_\infty \qquad \textit{and}$$

$$(\mathbb{Q} \otimes M) \cap L_0 \cap L_\infty \simeq \mathbb{Q} \otimes_{\mathbb{Z}} (M \cap L_0/M \cap qL_0)$$

$$\simeq (\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} M) \cap L_0/(\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} M) \cap qL_0 .$$

(ii) *Let* E *be a* $\mathbb{Z}$*-module and* φ: $E \to M \cap L_0 \cap L_\infty$ *a homomorphism. Assume that*

(a) $M = \mathbb{Z}[q, q^{-1}]\varphi(E)$ *and*
(b) $E \to L_0/qL_0$ *and* $E \to L_\infty/q^{-1}L_\infty$ *are injective.*

Then, $E \to M \cap L_0 \cap L_\infty \to M \cap L_0/M \cap qL_0$ *are isomorphisms.*

Proof. Note that L_0 is finitely generated over A.
(i) Set $E = M \cap L_0 \cap L_\infty$. Then $E \hookrightarrow L_0/qL_0$ implies that E is a torsion free $\mathbb{Z}$-module. Moreover, $A \otimes_{\mathbb{Z}} E \hookrightarrow L_0$ and $\mathbb{Q}(q) \otimes_{\mathbb{Z}} E \hookrightarrow V$.
By the assumption we have $M \cap L_0 \subset E + M \cap qL_0$. Hence, we obtain easily by the induction on $n \geqslant 0$

$$M \cap L_0 \subset \sum_{k=0}^{n} \mathbb{Z}q^kE + M \cap q^{n+1}L_0 . \tag{7.1.1}$$

Now we shall show

$$M \cap L_0 \cap q^nL_\infty = \sum_{k=0}^{n} \mathbb{Z}q^kE. \tag{7.1.2}$$

By (7.1.1), we have

$$M \cap L_0 \cap q^nL_\infty = \left(\sum_{k=0}^{n} \mathbb{Z}q^kE + M \cap q^{n+1}L_0\right) \cap q^nL_\infty$$

$$= \sum_{k=0}^{n} \mathbb{Z}q^kE + q^n(M \cap qL_0 \cap L_\infty).$$

Since $M \cap qL_0 \cap L_\infty = 0$, we obtain (7.1.2). This implies the first isomorphism. Then the third follows from $M \cong \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}[q]} (M \cap L_0)$. By (7.1.2) we have $M \cap L_\infty =$

$\bigcup_n q^{-n}(M \cap L_0 \cap q^n L_\infty) = \mathbb{Z}[q^{-1}]E \cong \mathbb{Z}[q^{-1}] \otimes E$. This implies the second isomorphism and $M \cap L_\infty / M \cap q^{-1}L_\infty \cong E$ gives the fourth isomorphism. The last isomorphism follows from $(\mathbb{Q} \otimes_{\mathbb{Z}} M) \cap L_0 \cap L_\infty \cong \mathbb{Q} \otimes_{\mathbb{Z}} (M \cap L_0 \cap L_\infty) \cong \mathbb{Q} \otimes_{\mathbb{Z}} E$ and $(\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} M) \cap L_0 = S^{-1}M \cap L = S^{-1}(M \cap L) = A \otimes_{\mathbb{Z}} E$. Here $S = \{f(q) \in \mathbb{Z}[q]; f(0) \neq 0\}$.
(ii) Note that E is torsion free. Condition (b) implies $\mathbb{Q}(q) \otimes_{\mathbb{Z}} E \subset V$ and $\mathbb{Q}(q) \otimes_{\mathbb{Z}} E \cap L_0 = A \otimes_{\mathbb{Z}} E$. Hence (a) implies $M \cong \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} E$. Therefore, $M \cap L_0 \subset \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} E \cap A \otimes_{\mathbb{Z}} E = (\mathbb{Z}[q, q^{-1}] \cap A) \otimes_{\mathbb{Z}} E = \mathbb{Z}[q] \otimes_{\mathbb{Z}} E$, which implies $M \cap L_0 \cong \mathbb{Z}[q] \otimes_{\mathbb{Z}} E$. Similarly, $M \cap L_\infty \cong \bar{A} \otimes_{\mathbb{Z}} E$. Therefore, we have $M \cap L_0 \cap L_\infty \cong (\mathbb{Z}[q, q^{-1}] \cap A \cap \bar{A}) \otimes_{\mathbb{Z}} E \cong E$ and $M \cap L_0 / M \cap qL_0 \cong E$.

Q.E.D.

LEMMA 7.1.2. *Let V, M, L_0, and L_∞ be as in the general assumption of the preceding lemma. Let N be a sub-$\mathbb{Z}[q, q^{-1}]$-module of M. Assume the following conditions.*

(i) $N \cap L_0 \cap L_\infty \xrightarrow{\sim} N \cap L_0 / N \cap qL_0$.
(ii) *There exist a $\mathbb{Z}$-module F and a homomorphism $\varphi: F \to M \cap (L_0 + N) \cap (L_\infty + N)$ such that*
   (i) $M = \mathbb{Z}[q, q^{-1}]\varphi(F) + N$ *and*
   (ii) *two homomorphisms induced by φ, $F \xrightarrow{\psi} (L_0 + \mathbb{Q} \otimes N)/(qL_0 + \mathbb{Q} \otimes N)$ and $F \to (L_\infty + \mathbb{Q} \otimes N)/(q^{-1}L_\infty + \mathbb{Q} \otimes N)$ are injective.*
   Then we have

(i) $M \cap L_0 \cap L_\infty \to M \cap L_0 / M \cap qL_0$ *is an isomorphism.*
(ii) $0 \to N \cap L_0 / N \cap qN_0 \to M \cap L_0 / M \cap qL_0 \xrightarrow{g} (L_0 + \mathbb{Q} \otimes N)/(qL_0 + \mathbb{Q} \otimes N)$ *is exact and $g(M \cap L_0 / M \cap qL_0) = \psi(F)$.*

Proof. Replacing F with a finitely generated sub-$\mathbb{Z}$-module F' and M with $\mathbb{Z}[q, q^{-1}]\varphi(F') + N$, we may assume from the beginning that F is finitely generated. Since F is torsion free, F is a free A-module. Since $N = N \cap L_0 + N \cap L_\infty$ by the preceding lemma, we have

$$\begin{aligned} M \cap (L_0 + N) \cap (L_\infty + N) &= N + M \cap L_0 \cap (L_\infty + N) \\ &= N + M \cap L_0 \cap (L_\infty + N \cap L_0) \\ &= N + M \cap L_0 \cap L_\infty. \end{aligned}$$

Hence, by changing φ we may assume from the beginning that $\varphi(F) \subset M \cap L_0 \cap L_\infty$. In the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & N \cap L_0 \cap L_\infty & \longrightarrow & F \oplus (N \cap L_0 \cap L_\infty) & \longrightarrow & F & \longrightarrow 0 \\ & \downarrow \beta & & \downarrow \alpha & & \downarrow \psi & \\ 0 \longrightarrow & \mathbb{Q} \otimes (N \cap L_0 \cap L_\infty) & \longrightarrow & L_0 / qL_0 & \longrightarrow & (L_0 + \mathbb{Q} \otimes N)/(qL_0 + \mathbb{Q} \otimes N) & \end{array}$$

the rows are exact by the preceding lemma. Then the injectivity of β and ψ shows that α is injective. Similarly $F \oplus (N \cap L_0 \cap L_\infty) \to L_\infty/q^{-1}L_\infty$ is injective. Hence, applying Lemma 7.1.1 (ii) with $E = F \oplus (N \cap L_0 \cap L_\infty)$, we obtain (i) and $F \oplus (N \cap L_0 \cap L_\infty) \xrightarrow{\sim} M \cap L_0 \cap L_\infty \xrightarrow{\sim} M \cap L_0/M \cap qL_0$. Q.E.D.

We remark that Lemma 7.1.1 and Lemma 7.1.2 can be translated by the language of vector bundles on $\mathbb{P}^1$ as follows. Let X be the $\mathbb{Z}$-scheme $\mathbb{P}^1$ and $U_0 = \operatorname{Spec} \mathbb{Z}[q] \subset X$ and $U_\infty = \operatorname{Spec} \mathbb{Z}[q^{-1}] \subset X$ so that $X = U_0 \cup U_1$. Let i_0: $\operatorname{Spec}(\mathbb{Z}) \to X$ be the section given by $q = 0$. Let $\mathscr{F}$ be a torsion free coherent $\mathscr{O}_X$-module given by $\Gamma(U_0; \mathscr{F}) = L_0 \cap M$, $\Gamma(U_\infty; \mathscr{F}) = L_\infty \cap M$. Then $M \cap L_0 \cap L_\infty = \Gamma(X; \mathscr{F})$ and $M \cap L_0/M \cap qL_0 \cong \Gamma(\operatorname{Spec}(\mathbb{Z}), i_0^*\mathscr{F})$. Therefore, for example Lemma 7.1.1 (i) is translated to the statement that $\Gamma(X; \mathscr{F}) \xrightarrow{\sim} \Gamma(\operatorname{Spec} \mathbb{Z}, i_0^*\mathscr{F})$ implies $\mathscr{F} \cong \Gamma(X; \mathscr{F}) \otimes \mathscr{O}_X$.

7.2. *Induction hypothesis.* Let us consider the following collection (G_l) of statements for $l \geqslant 0$. (Cf. (4.2.2).)

$(G_l.1)$ For any $\xi \in Q_-(l)$,

$$U_{\mathbb{Z}}^-(\mathfrak{g})_\xi \cap L_{\mathbb{Z}}(\infty) \cap L_{\mathbb{Z}}(\infty)^- \to L_{\mathbb{Z}}(\infty)_\xi/qL_{\mathbb{Z}}(\infty)_\xi$$

is an isomorphism.

$(G_l.2)$ For any $\xi \in Q_-(l)$, and $\lambda \in P_+$,

$$V_{\mathbb{Z}}(\lambda)_{\lambda+\xi} \cap L_{\mathbb{Z}}(\lambda) \cap L_{\mathbb{Z}}(\lambda)^- \to L_{\mathbb{Z}}(\lambda)_{\lambda+\xi}/qL_{\mathbb{Z}}(\lambda)_{\lambda+\xi}$$

is an isomorphism.

Let us denote by $b \mapsto G(b)$ and $b \mapsto G_\lambda(b)$ the inverse homomorphisms of these isomorphisms.

$(G_l.3)$ For $\xi \in Q_-(l)$, $n \geqslant 0$, and $b \in \tilde{f}_i^n(B(\infty)_{\xi+n\alpha_i})$,

$$G(b) \in f_i^n U_q^-(\mathfrak{g}).$$

7.3. *Consequences of G_{l-1}.* We shall prove G_l by the induction on l. Since G_l is obvious for $l = 0$, let us assume $l > 0$ and G_{l-1}. Then we shall prove G_l.

LEMMA 7.3.1. *For $\xi \in Q_-(l-1)$ we have*

$$U_{\mathbb{Z}}^-(\mathfrak{g})_\xi \cap L_{\mathbb{Z}}(\infty) = \bigoplus_{b \in B(\infty)_\xi} \mathbb{Z}[q]G(b), \tag{7.3.1}$$

$$U_{\mathbb{Z}}^-(\mathfrak{g})_\xi = \bigoplus_{b \in B(\infty)_\xi} \mathbb{Z}[q, q^{-1}]G(b), \tag{7.3.2}$$

$$V_{\mathbb{Z}}(\lambda)_{\lambda+\xi} \cap L(\lambda) = \bigoplus_{b \in B(\lambda)_{\lambda+\xi}} \mathbb{Z}[q]G_\lambda(b), \quad \text{and} \tag{7.3.3}$$

$$V_{\mathbb{Z}}(\lambda)_{\lambda+\xi} = \bigoplus_{b \in B(\lambda)_{\lambda+\xi}} \mathbb{Z}[q, q^{-1}]G_\lambda(b). \tag{7.3.4}$$

Proof. They follow easily from Lemma 7.1.1, $(G_{l-1}.1)$, and $(G_{l-1}.2)$.

The following lemma also follows easily from (G_{l-1}).

LEMMA 7.3.2. *For* $\xi \in Q_-(l-1)$, $b \in L_{\mathbb{Z}}(\infty)_\xi/qL_{\mathbb{Z}}(\infty)_\xi$, *and* $\lambda \in P_+$, $G(b)u_\lambda = G_\lambda(\bar{\pi}_\lambda b)$.

LEMMA 7.3.4. *For* $\xi \in Q_-(l-1)$ *and* $b \in L_{\mathbb{Z}}(\infty)_\xi/qL_{\mathbb{Z}}(\infty)_\xi$, *we have*

$$\overline{G(b)} = G(b). \tag{7.3.5}$$

Proof. Set $Q = (G(b) - \overline{G(b)})/(q - q^{-1})$. Then Q belongs to $U_{\mathbb{Z}}^-(\mathfrak{g})_\xi \cap qL(\infty) \cap L(\infty)^-$, and hence it vanishes. Q.E.D.

7.4. *Triviality of* $f_i^n V(\lambda)$ *for* $n \geqslant 1$. The first step is to prove the following proposition.

PROPOSITION 7.4.1. *For* $\xi \in Q_-(l)$, $\lambda \in P_+$, $n \geqslant 1$, *and* $i \in I$,

$$(f_i^n V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} \cap L(\lambda) \cap L(\lambda)^- \overset{\sim}{\to} (f_i^n V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} \cap L(\lambda)/q((f_i^n V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} \cap L(\lambda)). \tag{7.4.1}$$

$$\simeq \bigoplus_{b \in B(\lambda)_{\lambda+\xi} \cap \tilde{f}_i^n B(\lambda)} \mathbb{Z}b.$$

Proof. We shall show this by the descending induction on n. Remark that $(f_i^n V(\lambda))_{\lambda+\xi} = 0$ and $B(\lambda)_{\lambda+\xi} \cap \tilde{f}_i^n B(\lambda) = \phi$ for $n > l$. Hence, we may assume

(7.4.2)
$$(f_i^{n+1} V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} \cap L(\lambda) \cap L(\lambda)^- \overset{\sim}{\to} (f_i^{n+1} V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} \cap L(\lambda)/((f_i^{n+1} V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} \cap qL(\lambda))$$

$$\simeq \bigoplus_{b \in B(\lambda)_{\lambda+\xi} \cap \tilde{f}_i^{n+1} B(\lambda)} \mathbb{Z}b.$$

When $n + \langle h_i, \lambda + \xi \rangle < 0$, $B(\lambda)_{\lambda+\xi} \cap \tilde{f}_i^n B(\lambda) = B(\lambda)_{\lambda+\xi} \cap \tilde{f}_i^{-\langle h_i, \lambda+\xi \rangle} B(\lambda)$, and Proposition 7.1.3 implies

$$(f_i^n V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} = (f_i^{-\langle h_i, \lambda+\xi \rangle} V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} V(\lambda)^{\mathbb{Z}}_{\lambda+\xi}.$$

Therefore, we can reduce to the case $n = -\langle h_i, \lambda + \xi \rangle$. Hence we may assume from the beginning

$$n + \langle h_i, \lambda + \xi \rangle \geqslant 0. \tag{7.4.3}$$

By the definition we have

$$(f_i^n V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} = f_i^{(n)}(V_{\mathbb{Z}}(\lambda)_{\lambda+\xi+n\alpha_i}) + (f_i^{n+1} V(\lambda))^{\mathbb{Z}}_{\lambda+\xi}.$$

Since $n \geqslant 1$, (G_{l-1}) gives

$$V_{\mathbb{Z}}(\lambda)_{\lambda+\xi+n\alpha_i} = \bigoplus_{\substack{b \in B(\infty)_{\xi+n\alpha_i} \\ \pi_\lambda(b) \neq 0}} \mathbb{Z}[q, q^{-1}]G(b)u_\lambda .$$

If $\tilde{e}_i b \neq 0$, then $G(b) \in (f_i U_q^-(\mathfrak{g}))^{\mathbb{Z}}$ by $(G_{l-1}.3)$, and hence

$$(f_i^n V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} = \sum_{b \in S} \mathbb{Z}[q, q^{-1}] f_i^{(n)} G(b) u_\lambda + (f_i^{n+1} V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} .$$

Here, $S = \{b \in B(\infty)_{\xi+n\alpha_i};\ \bar{\pi}_\lambda(b) \neq 0, \tilde{e}_i b = 0\} \simeq \{b \in B(\lambda)_{\lambda+\xi+n\alpha_i};\ \tilde{e}_i b = 0\}$. Now let us prove (7.4.1) by using Lemma 7.1.2 with $V = V(\lambda)_{\lambda+\xi}$, $M = (f_i^n V(\lambda))^{\mathbb{Z}}_{\lambda+\xi}$, $N = (f_i^{n+1} V(\lambda))^{\mathbb{Z}}_{\lambda+\xi}$, $L_0 = L(\lambda)_{\lambda+\xi}$, $L_\infty = L(\lambda)^-_{\lambda+\xi}$ and $F = \bigoplus_{b \in S} \mathbb{Z} f_i^{(n)} G(b) u_\lambda$. We have (see (6.1.2))

$$f_i^{(n)} G(b) \equiv f_i^{(n)} P_i G(b) \bmod (f_i^{n+1} U_q^-(\mathfrak{g}))^{\mathbb{Z}}$$

where P_i is the projector to $\operatorname{Ker} e_i'$ with respect to the decomposition $U_q^-(\mathfrak{g}) = \operatorname{Ker} e_i' \oplus f_i U_q^-(\mathfrak{g})$. Moreover, $\tilde{f}_i^n b \equiv f_i^{(n)} P_i G(b) \bmod qL(\infty)$. Hence we have

$$M \cap L / M \cap qL \supset \bigoplus_{b \in S} \mathbb{Z}\bar{\pi}_\lambda(\tilde{f}_i^n b), \tag{7.4.4}$$

and $f_i^{(n)} G(b) u_\lambda \in M \cap (L_0 + N)$. Set $H = (L_0 + \mathbb{Q} \otimes N)/(qL_0 + \mathbb{Q} \otimes N) = (L_0/qL_0)/(\mathbb{Q} \otimes N \cap L_0/\mathbb{Q} \otimes N \cap qL_0)$. By (7.4.2) and Lemma 7.1.1 (i), we have

$$(\mathbb{Q} \otimes N) \cap L_0 / (\mathbb{Q} \otimes N) \cap qL_0 = \bigoplus_{b \in B(\lambda)_{\lambda+\xi} \cap \tilde{f}_i^{n+1} B(\lambda)} \mathbb{Q} b .$$

Hence $H = \bigoplus_{b \in B(\lambda)_{\lambda+\xi} \setminus \tilde{f}_i^{n+1} B(\lambda)} \mathbb{Q} b$. Moreover, the image of $f_i^{(n)} G(b) u_\lambda$ to H is $\bar{\pi}_\lambda(\tilde{f}_i^n b)$. By (7.4.3), S is isomorphic to $B(\lambda)_{\lambda+\xi} \cap \tilde{f}_i^n B(\lambda) \setminus \tilde{f}_i^{n+1} B(\lambda)$ by $b \mapsto \bar{\pi}_\lambda(\tilde{f}_i^n b)$. Hence, $F \to H$ is injective, and (7.4.1) follows from Lemma 7.4.2 and (7.4.4) because the condition at $q = \infty$ can be verified by taking $-$. Q.E.D.

COROLLARY 7.4.2. *For $\xi \in Q_-(l)$, $n \geqslant 1$ and $i \in I$, we have*

$$(f_i^n U_q^-(\mathfrak{g}))^{\mathbb{Z}}_\xi \cap L(\infty) \cap L(\infty)^- \xrightarrow{\sim} (f_i^n U_q^-(\mathfrak{g}))^{\mathbb{Z}}_\xi \cap L(\infty) / (f_i^n U_q^-(\mathfrak{g}))^{\mathbb{Z}}_\xi \cap qL(\infty) \tag{7.4.5}$$

$$\xrightarrow{\sim} \bigoplus_{b \in \tilde{f}_i^n B(\infty) \cap B(\infty)_\xi} \mathbb{Z} b .$$

Proof. It is enough to remark that, for λ with $\langle h_j, \lambda \rangle \gg 0$ for any j, we have

$$U_q^-(\mathfrak{g})_\xi \xrightarrow{\sim} V(\lambda)_{\lambda+\xi},\ (f_i^n U_q^-(\mathfrak{g}))^{\mathbb{Z}}_\xi \xrightarrow{\sim} (f_i^n V(\lambda))^{\mathbb{Z}}_{\lambda+\xi},$$

$$L(\infty)_\xi \xrightarrow{\sim} L(\lambda)_{\lambda+\xi},\ L(\infty)^-_\xi \xrightarrow{\sim} L(\lambda)^-_{\lambda+\xi}$$

and

$$\bigoplus_{b \in \tilde{f}_i^n B(\infty) \cap B(\infty)_\xi} \mathbb{Z}b \simeq \bigoplus_{b \in B(\lambda)_{\lambda+\xi} \cap \tilde{f}_i^n B(\lambda)} \mathbb{Z}b. \qquad \text{Q.E.D.}$$

7.5. *End of proof.* For $\xi \in Q_-(l)$ and $i \in I$, let us denote by G_i the inverse of the isomorphism

$$(7.5.1) \qquad (f_i U_q^-(\mathfrak{g}))_\xi^{\mathbb{Z}} \cap L(\infty) \cap L(\infty)^- \simeq \bigoplus_{b \in \tilde{f}_i B(\infty) \cap B(\infty)_\xi} \mathbb{Z}b.$$

We have by Proposition 7.4.1

(7.5.2) $(f_i^n U_q^-(\mathfrak{g}))_\xi^{\mathbb{Z}} = \bigoplus \mathbb{Z}[q, q^{-1}] G_i(b)$ for $n \geqslant 1$, where the direct sum ranges over $b \in \tilde{f}_i^n B(\infty) \cap B(\infty)_\xi$.

The next step is to prove the following lemma.

LEMMA 7.5.1. *Let $i, j \in I$, $\xi \in Q_-(l)$ and $b \in \tilde{f}_i B(\infty) \cap \tilde{f}_j B(\infty) \cap B(\infty)_\xi$. Then we have $G_i(b) = G_j(b)$.*

Proof. Let us write $b = \tilde{f}_{i_1} \cdots \tilde{f}_k \cdot 1$.
Let us take $\lambda \in P_+$ with $\langle h_k, \lambda \rangle = 0$ and $\langle h_\nu, \lambda \rangle \gg 0$ for $\nu \in I \setminus \{k\}$. Then

$$(7.5.3) \qquad V(\lambda)_{\lambda+\xi} \simeq U_q^-(\mathfrak{g})_\xi / U_q^-(\mathfrak{g})_{\xi+\alpha_k} f_k.$$

Now $\langle h_k, \lambda \rangle = 0$ implies $\tilde{f}_k u_\lambda = 0$ and hence $\bar{\pi}_\lambda(b) = 0$. Therefore, $G_i(b) u_\lambda \in qL(\lambda)$. Hence, $G_i(b) u_\lambda$ belongs to $(f_i V(\lambda))_{\lambda+\xi}^{\mathbb{Z}} \cap qL(\lambda) \cap L(\lambda)^-$, which is zero by Proposition 7.4.1. Thus, we obtain $G_i(b) u_\lambda = 0$. Hence, (7.5.3) implies $G_i(b) \in U_q^-(\mathfrak{g}) f_k$. Similarly, $G_j(b) \in U_q^-(\mathfrak{g}) f_k$. Therefore, $Q = G_i(b) - G_j(b)$ belongs to $U_q^-(\mathfrak{g}) f_k \cap qL_{\mathbb{Z}}(\infty) \cap L_{\mathbb{Z}}(\infty)^-$. Proposition 5.2.4 implies $Q^* \in f_k U_q^-(\mathfrak{g}) \cap qL_{\mathbb{Z}}(\infty) \cap L_{\mathbb{Z}}(\infty)^-$. Then it remains to apply Corollary 7.4.2. Q.E.D.

Thus we can define G: $L(\infty)_\xi / qL(\infty)_\xi \to U_{\mathbb{Z}}^-(\mathfrak{g})_\xi \cap L(\infty) \cap L(\infty)^-$ by $G(b) = G_i(b)$ for $b \in \tilde{f}_i B(\infty) \cap B(\infty)_\xi$. Then we have

$$(7.5.4) \qquad b \equiv G(b) \bmod qL(\infty),$$

$$(7.5.5) \qquad (f_i^n U_q^-(\mathfrak{g}))_\xi^{\mathbb{Z}} = \bigoplus_{b \in \tilde{f}_i^n B(\infty) \cap B(\infty)_\xi} \mathbb{Z}[q, q^{-1}] G(b) \quad \text{for } n \geqslant 1.$$

Since $U_{\mathbb{Z}}^-(\mathfrak{g})_\xi = \sum_i (f_i U_q^-(\mathfrak{g}))_\xi^{\mathbb{Z}}$, we obtain

$$(7.5.6) \qquad U_{\mathbb{Z}}^-(\mathfrak{g})_\xi = \sum_{b \in B(\infty)_\xi} \mathbb{Z}[q, q^{-1}] G(b).$$

Then $(G_l.1)$ follows from Lemma 7.1.1 (ii), and $(G_l.3)$ follows from (7.5.5). Finally, let us show $(G_l.2)$.

LEMMA 7.5.2. *Let* $\xi \in Q_-(l)$, $b \in B(\infty)_\xi$, *and* $\lambda \in P_+$. *If* $\bar{\pi}_\lambda(b) = 0$, *then* $G(b)u_\lambda^- = 0$.

Proof. Take i such that $\tilde{e}_i b \neq 0$. Then $G(b)u_\lambda \in (f_i V(\lambda))^{\mathbb{Z}}_{\lambda+\xi} \cap qL(\lambda) \cap L(\lambda)^- = 0$.
Q.E.D.

By this lemma we have

$$V_{\mathbb{Z}}(\lambda)_{\lambda+\xi} = \sum_{\substack{b \in B(\infty)_\xi \\ \bar{\pi}_\lambda(b) \neq 0}} G(b)u_\lambda .$$

Then, $(G_l.2)$ follows from Lemma 7.2.1 (ii), and $\{b \in B(\infty)_\xi; \bar{\pi}_\lambda(b) \neq 0\} \simeq B(\lambda)_{\lambda+\xi}$. Thus, the induction proceeds, and (G_l) is valid for any $l \geqslant 0$. Now Theorems 6 and 7 follow from (G_l), Lemma 7.2.1, Proposition 7.4.1, and Corollary 7.4.2.

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